

Strong separating (k, \bar{k}) –surfaces on \mathbb{Z}^3 *

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Abstract For each adjacency pair $(k, \bar{k}) \neq (6, 6)$, $k, \bar{k} \in \{6, 18, 26\}$, we introduce a new family $\mathcal{S}_{k\bar{k}}$ of surfaces in the discrete space \mathbb{Z}^3 that strictly contains several families of surfaces previously defined, and other objects considered as surfaces, in the literature. Actually, $\mathcal{S}_{k\bar{k}}$ characterizes the strongly \bar{k} –separating objects of the family of digital surfaces, defined by means of continuous analogues, of the universal (k, \bar{k}) –spaces introduced in [6].

Keywords discrete surface; continuous analogue, strong separation.

1 Introduction

In the graph–theoretical approach to Digital Topology, the search for a definition of digital surfaces as subsets of voxels is still a work in progress since it was started in the early 1980’s. Despite the interest of the applications in which it is involved (ranging from visualization to image segmentation and graphics), there is not yet a well established general notion of digital surface that naturally extends to higher dimensions. The fact is that, after the first definition of surface, proposed by Morgenthaler [11] for the grid \mathbb{Z}^3 with the usual adjacency pairs $(26, 6)$ and $(6, 26)$, each new contribution has either increased the number of surfaces (strong surfaces [3] and simplicity surfaces [7]) or extended the definition to other adjacency pairs [8], but still leaving out some objects considered as surfaces for practical purposes [10].

For each adjacency pair $(k, \bar{k}) \neq (6, 6)$, $k, \bar{k} \in \{6, 18, 26\}$, and within the framework for Digital Topology in [2], we have recently found [6] a homogeneous (k, \bar{k}) –space $(R^3, f_{k\bar{k}})$ whose set of digital surfaces is the largest in that class of digital spaces. Moreover, these sets of surfaces contain all those quoted above. Of course a Jordan separation property holds for them, but some do not satisfy the strong separation property usually required to discrete surfaces in \mathbb{Z}^3 . On the other hand they are defined by means of continuous analogues, and thus it might not be considered as a completely discrete construction.

Our goal in this paper is twofold. Firstly we provide a completely discrete characterization of the digital surfaces in each (k, \bar{k}) –space $(R^3, f_{k\bar{k}})$, by extending Kong’s method [8] based on plates and graphs. Then, we find in §5 a local characterization for the strong separating condition of digital surfaces of $(R^3, f_{k\bar{k}})$ which is used to derive a genuine notion of (k, \bar{k}) –surface.

This work contains an extension of previous results for the $(26, 6)$ –adjacency in [4].

2 A set of (k, \bar{k}) –Jordan objects

In this section we introduce, for each of the usual adjacency pairs $(k, \bar{k}) \neq (6, 6)$, $k, \bar{k} \in \{6, 18, 26\}$, defined on \mathbb{Z}^3 , a family of objects that satisfies a Jordan property. Actually, these objects could be considered as a starting definition of a family of (k, \bar{k}) –surfaces since they are made of small

*This work has been partially supported by project MTM2007-65726 MICINN, Spain.

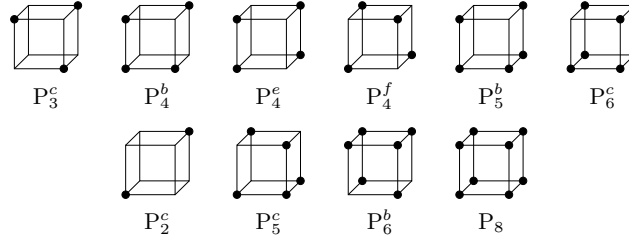


Figure 1: Some patterns that may appear in a digital object $O \subseteq \mathbb{Z}^3$. Each picture represents a unit cube C of \mathbb{Z}^3 , and the black dots are the set of voxels in $O \cap C$. The upper row contains the six non-square plates that may appear in a digital object. The patterns in the lower row cannot appear in a $(26, 6)$ -presurface.

surface pieces, called *plates*, which are adequately glued to each other in the way defined by an *assembly graph*. To introduce these notions we firstly recall some basic definitions from the graph-theoretical approach to Digital Topology.

Two distinct voxels $\sigma = (x_1^\sigma, x_2^\sigma, x_3^\sigma), \tau = (x_1^\tau, x_2^\tau, x_3^\tau) \in \mathbb{Z}^3$ are said to be 6-, 18- or 26-adjacent if $\max\{|x_i^\sigma - x_i^\tau|; 1 \leq i \leq 3\} \leq 1$ and they differ in, at most, one, two or three of their coordinates, respectively. Moreover, we say that two n -adjacent voxels are *strictly n -adjacent* if they are not m -adjacent for any $m < n$, where $n, m \in \{6, 18, 26\}$. A *unit cube* of \mathbb{Z}^3 is any subset C of eight mutually 26-adjacent voxels. Similarly, a *unit square* of \mathbb{Z}^3 is a subset of four mutually 18-adjacent voxels that is actually the intersection of two distinct unit cubes.

For $n \in \{6, 18, 26\}$ the transitive closure of the n -adjacency relation defines an equivalence relation on each subset $A \subseteq \mathbb{Z}^3$, whose classes are called the n -components of A . Moreover, A is said to be n -connected if it has only one n -component.

Definition 2.1. Let $O \subseteq \mathbb{Z}^3$ be a digital object. A subset $p \subseteq O$ is said to be a \bar{k} -plate in O if either p is a unit square of \mathbb{Z}^3 or $p = C \cap O$ corresponds (up to rotations and symmetries) to one of the patterns in the set $\mathbb{P}_{\bar{k}}$, where C is a unit cube of \mathbb{Z}^3 and $\mathbb{P}_6 = \{P_3^c, P_4^b, P_4^e, P_4^f, P_5^b, P_6^c\}$, $\mathbb{P}_{18} = \{P_6^c\}$ and $\mathbb{P}_{26} = \emptyset$; see Fig. 1. For any voxel $\sigma \in O$ we denote by $\mathbb{P}_{\bar{k}}(O, \sigma)$ the set of all \bar{k} -plates in O containing σ , while $\mathbb{P}_{\bar{k}}(O)$ is the set of all \bar{k} -plates in O .

Remarks 2.2. Notice that the square plates are the only 26-plates in any object. Notice also that given a unit cube C and a digital object O the set $C - O \neq \emptyset$ is trivially 26-connected, it is 18-connected iff $C \cap O \notin \mathbb{P}_{18}$ and it is 6-connected iff $C \cap O \notin \mathbb{P}_6 \cup \{P_5^b, P_6^b\}$; see Fig. 1.

Besides the \bar{k} -plates, we consider the following bipartite graph, termed the \bar{k} -assembly graph, for any digital object $O \subseteq \mathbb{Z}^3$.

Definition 2.3. The nodes of the \bar{k} -assembly graph of $O \subseteq \mathbb{Z}^3$, $G_{\bar{k}}(O)$, are the elements of $\mathbb{P}_{\bar{k}}(O) \cup E_{\bar{k}}(O)$, where $E_{\bar{k}}(O)$ is the set of all pairs of voxels $\sigma, \tau \in O$ satisfying one of the two following properties:

1. σ and τ are 6-adjacent;
2. σ and τ are strictly 18-adjacent, no voxel in O is 6-adjacent to both of them and, moreover, $\{\sigma, \tau\}$ is in the intersection of two \bar{k} -plates of O .

And two nodes $p \in \mathbb{P}_{\bar{k}}(O)$ and $e \in E_{\bar{k}}(O)$ define an edge of $G_{\bar{k}}(O)$ if and only if $e \subseteq p$.

The \bar{k} -assembly graph of O around a voxel $\sigma \in O$ is the subgraph $G_{\bar{k}}(O, \sigma)$ of $G_{\bar{k}}(O)$ induced by the set of nodes $\mathbb{P}_{\bar{k}}(O, \sigma) \cup E_{\bar{k}}(O, \sigma)$, where $E_{\bar{k}}(O, \sigma) = \{e \in E_{\bar{k}}(O); \sigma \in e\}$.

Remark 2.4. For $\bar{k} \in \{18, 26\}$ the set $E_{\bar{k}}(O)$ consists entirely of pairs of 6-adjacent voxels since the square plates and the P_6^c plates cannot share just two strictly 18-adjacent voxels.

Definition 2.5. A digital object $S \subseteq \mathbb{Z}^3$ is said to be a (k, \bar{k}) -presurface if the following conditions hold for each voxel $\sigma \in S$:

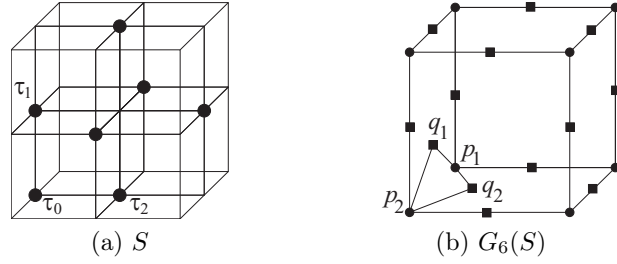


Figure 2: A $(18, 6)$ –presurface S and its 6 –assembly graph $G_6(S)$. Each dot in (b) represents one of the eight unit cubes shown in (a), which are actually 6 –plates of S . Each square in (b) represents a pair of strictly 18 – or 6 –adjacent voxels in S that, together with plates, define the 6 –assembly graph of S .

1. For each unit cube $C_\sigma \subseteq \mathbb{Z}^3$ containing σ , the intersection $C_\sigma \cap S$ does not correspond (up to rotations and symmetries) to any pattern in $\mathbb{FP}_{\bar{k}\bar{k}}$, where $\mathbb{FP}_{6,26} = \mathbb{FP}_{18,26} = \mathbb{FP}_{6,18} = \mathbb{FP}_{18,18} = \{P_8\}$, $\mathbb{FP}_{26,26} = \mathbb{FP}_{26,18} = \{P_2^c, P_8\}$, $\mathbb{FP}_{18,6} = \{P_5^c, P_6^b, P_8\}$ and $\mathbb{FP}_{26,6} = \{P_2^c, P_5^c, P_6^b, P_8\}$; see Fig. 1.
2. Given a voxel $\tau \in S$ strictly 18 –adjacent to σ and such that no other voxel in S is 6 –adjacent to both σ and τ , let C_1 and C_2 be the two only unit cubes of \mathbb{Z}^3 containing $\{\sigma, \tau\}$. If $\bar{k} = 6$ then $C_i \cap S \in \mathbb{P}_6(O)$ for at least one index $i \in \{1, 2\}$, while if $k, \bar{k} \in \{18, 26\}$ then $\{\sigma, \tau\}$ is contained in a 6 –component of $S \cap (C_1 \cup C_2)$.
3. If $\tau \in S$ is 6 –adjacent to σ then $\mathbb{P}_{\bar{k}}(S, \sigma) \cap \mathbb{P}_{\bar{k}}(S, \tau)$ consists exactly of two plates.
4. $\mathbb{P}_{\bar{k}}(S, \sigma) \neq \emptyset$ and $G_{\bar{k}}(S, \sigma)$ is a cycle.

Remark 2.6. Notice that condition (2) in the definition above is void if $k = 6$.

Example 2.7. Figure 2 depicts a $(18, 6)$ –presurface S , made of eight plates, and its 6 –assembly graph $G_6(S)$. Notice that the voxel $\tau_0 \in S$ belongs to exactly two plates $p_1, p_2 \in \mathbb{P}_6(O)$ and it is 6 –adjacent to both τ_1 and τ_2 . Hence, the pairs of voxels $q_i = \{\tau_0, \tau_i\}$, $i = 1, 2$, belong to $E_6(S, \tau_0)$ and the 6 –assembly graph of S around τ_0 , $G_6(S, \tau_0)$, is the cycle defined by the vertices p_1, p_2, q_1 and q_2 in Fig. 2(b).

The assembly graph endows each presurface with the combinatorial structure of a surface. More precisely, we will show in Th. 4.8 below that (k, \bar{k}) –presurfaces characterize the digital surfaces of the universal (k, \bar{k}) –space $(R^3, f_{k\bar{k}})$ defined in [6] within the approach to Digital Topology in [2]. In particular, we obtain, as a corollary of this characterization and Th. 3.3, that all (k, \bar{k}) –presurfaces are Jordan objects; that is, each k –connected (k, \bar{k}) –presurface S separates its complement $\mathbb{Z}^3 - S$ into two \bar{k} –components.

In order to provide the appropriate context for Th. 4.8 we collect the basic elements of this framework in next section.

3 Universal (k, \bar{k}) –spaces and digital surfaces

In this section we recall the definitions and main results from [6] needed in this paper, which were established within the framework for Digital Topology introduced in [2]. In this approach a *digital space* is a pair (K, f) , where K is a polyhedral complex and f is a *lighting function* from which we associate to each digital image an Euclidean polyhedron called its *continuous analogue*.

In this paper we will only deal with the *universal (k, \bar{k}) –spaces* $(R^3, f_{k\bar{k}})$ defined in [6]. The complex R^3 is determined by the collection of unit cubes in the Euclidean space \mathbb{R}^3 centered at points of integer coordinates. Each 3-cell in R^3 represents a voxel, and so any digital object is a subset of the set $\text{cell}_3(R^3)$ of 3-cells in R^3 . The lower dimensional cells of R^3 (actually, d –cubes, $0 \leq d < 3$) are used to describe the various possible ways voxels link to each other. Notice that each d –cell $\sigma \in R^3$ can be associated to its center $c(\sigma)$. In particular, if $\dim \sigma = 3$ then $c(\sigma) \in \mathbb{Z}^3$

and so every digital object in R^3 can be naturally identified with a subset of the discrete space \mathbb{Z}^3 . Henceforth we shall use this identification without further comment.

Lighting functions are maps of the form $\mathcal{P}(\text{cell}_3(R^3)) \times R^3 \rightarrow \{0, 1\}$, where $\mathcal{P}(\text{cell}_3(R^3))$ stands for the family of all subsets of $\text{cell}_3(R^3)$; i.e., all digital objects. In order to introduce the lighting functions $f_{k\bar{k}}$ we need some more notation.

As usual, given two cells $\alpha, \beta \in R^3$ we write $\alpha \leq \beta$ if α is a face of β , and $\alpha < \beta$ if in addition $\alpha \neq \beta$. Given a digital object $O \subseteq \text{cell}_3(R^3)$ the *star of a cell α in O* is the set $\text{st}_3(\alpha; O) = \{\sigma \in O; \alpha \leq \sigma\}$ of 3-cells (voxels) in O having α as a face. Similarly, the *extended star of α in O* is the set $\text{st}_3^*(\alpha; O) = \{\sigma \in O; \alpha \cap \sigma \neq \emptyset\}$. Finally, the *support* of O is the set $\text{supp}(O)$ of cells of R^3 (not necessarily voxels) that are the intersection of 3-cells in O ; that is, $\alpha \in \text{supp}(O)$ if and only if $\alpha = \cap\{\sigma; \sigma \in \text{st}_3(\alpha; O)\}$. To ease the writing, we use the following notation: $\text{st}_3(\alpha; R^3) = \text{st}_3(\alpha; \text{cell}_3(R^3))$ and $\text{st}_3^*(\alpha; R^3) = \text{st}_3^*(\alpha; \text{cell}_3(R^3))$.

Remark 3.1. The identification between cells in R^3 and their centers gives us a one-to-one correspondence between 0-cells (1-cells) and unit cubes (squares, respectively) of \mathbb{Z}^3 . Namely, $\text{st}_3(\alpha; R^3)$ is a unit cube (square) for each 0-cell (1-cell) $\alpha \in R^3$. Thus, if p is plate in a given object O , then $p = \text{st}_3(\alpha; O) = \text{st}_3(\alpha; R^3) \cap O$ for some cell α with $\dim \alpha \leq 1$, which is called the *center of p* . Similarly, if $e = \{\sigma, \tau\}$ is a node of $G_{\bar{k}}(O)$ in the set $E_{\bar{k}}(O)$, the cell $\delta = \sigma \cap \tau$ is a 2-cell or a 1-cell, depending on whether σ and τ are 6-adjacent or strictly 18-adjacent, which will be also called the *center of the node e* .

For $(k, \bar{k}) \neq (6, 6)$, $k, \bar{k} \in \{6, 18, 26\}$, the lighting functions $f_{k\bar{k}}$ are defined as follows. Given a digital object $O \subseteq \text{cell}_3(R^3)$ and a cell $\delta \in R^3$, $f_{k\bar{k}}(O, \delta) = 1$ if and only one of the following conditions hold:

1. $\dim \delta \geq 2$ and $\delta \in \text{supp}(O)$
2. $\dim \delta = 0$ and $\text{st}_3(\delta; O)$ corresponds (up to rotations and symmetries) to some pattern in the set $\mathbb{P}_{\bar{k}} \cup \mathbb{FP}_{k, \bar{k}}$ (see Defs. 2.1 and 2.5)
3. $\dim \delta = 1$ and $\text{st}_3(\delta; O) = \text{st}_3(\delta; R^3)$ (i.e., δ is the center of a square plate in O), or
4. $\dim \delta = 1$, $\text{st}_3(\delta; O) = \{\sigma, \tau\}$, with $\delta = \sigma \cap \tau$, and one of the next further conditions also holds: (a) for $\bar{k} = 6$, and $k \neq 6$, $f_{k,6}(O, \alpha_1) = f_{k,6}(O, \alpha_2)$, where α_1, α_2 are the two vertices of the 1-cell δ ; or (b) σ and τ belong to distinct 6-components of $\text{st}_3^*(\delta; O)$, for $k, \bar{k} \in \{18, 26\}$.

Each of these maps and, more generally, any lighting function f may be regarded as a “face membership rule”, in the sense of Kovalevsky [9], that assigns to each digital object O the set of cells $f_O = \{\alpha \in R^3; f(O, \alpha) = 1\}$. This set yields a continuous analogue as a natural counterpart of O in ordinary topology. Namely, the *continuous analogue* of O is the polyhedron $|\mathcal{A}_O^f| \subseteq \mathbb{R}^3$ triangulated by the subcomplex of the first derived subdivision of R^3 , \mathcal{A}_O^f , consisting of all simplexes whose vertices are centers $c(\sigma)$ of cells $\sigma \in f_O$.¹

Regarding continuous analogues as a “continuous interpretation” of digital images, we introduce digital notions in terms of the corresponding continuous ones. For example, we say that an object $O \subseteq \text{cell}_3(R^3)$ is *connected* if its continuous analogue $|\mathcal{A}_O|$ is a connected polyhedron. And, in the same way, the background of O , $\text{cell}_3(R^3) - O$, is said to be *co-connected* if $|\mathcal{A}_{R^3}| - |\mathcal{A}_O|$ is connected. Moreover, we call $C \subseteq \text{cell}_3(R^3)$ a (co-)component of O ($\text{cell}_3(R^3) - O$, respectively) if it consists of all voxels σ whose centroids $c(\sigma)$ belong to a component of $|\mathcal{A}_O|$ ($|\mathcal{A}_{R^3}| - |\mathcal{A}_O|$, respectively). Similarly, an object $S \subseteq \text{cell}_3(R^3)$ is a digital surface in the space (R^3, f) if its continuous analogue $|\mathcal{A}_S|$ is a combinatorial surface; that is, if the link $\text{lk}(v; \mathcal{A}_S) = \{A \in \mathcal{A}_S; v, A < B \in \mathcal{A}_S \text{ and } v \notin A\}$ is a 1-sphere for each vertex $v \in \mathcal{A}_S$. See [2] for more details on these notions of connectedness defined in a much more general context and for a definition of digital manifold in arbitrary dimension.

In certain digital spaces these notions are closely related to the usual ones defined on \mathbb{Z}^3 be means of adjacency pairs. More precisely, given an adjacency pair (k, \bar{k}) we say that (R^3, f) is a (k, \bar{k}) -space if the two following properties hold for any digital object $O \subseteq \text{cell}_3(R^3)$:

¹We often drop the “ f ” from the notation and also write \mathcal{A}_{R^3} instead $\mathcal{A}_{\text{cell}_3(R^3)}^f$.

1. C is a component of O iff it is a k -component of O ; and,
2. C is a co-component of the background of O iff it is a \bar{k} -component of $\mathbb{Z}^3 - O$.

In particular, it is not difficult to show that the digital spaces $(R^3, f_{k\bar{k}})$ defined above are actually homogeneous (k, \bar{k}) -spaces, in the sense that, in addition, the continuous analogue they provide for each digital object is invariant under isometries of the Euclidean space preserving \mathbb{Z}^3 .

On the other hand, in [1, 2, 5] it can be found several homogeneous (k, \bar{k}) -spaces whose sets of digital surfaces contain the families of (k, \bar{k}) -surfaces quoted in the introduction, which are also digital surfaces in the corresponding universal (k, \bar{k}) -space as a consequence of the following

Theorem 3.2 (Th. 20 in [6]). *Any digital surface S in an arbitrary homogeneous (k, \bar{k}) -space is also a digital surface in the universal (k, \bar{k}) -space $(R^3, f_{k\bar{k}})$.*

Finally, and concerning the Jordan property, we have the following separation theorem for digital surfaces in $(R^3, f_{k\bar{k}})$ as a corollary of a Jordan–Brouwer Theorem for fairly general digital spaces in [2].

Theorem 3.3. *Each k -connected digital surface in $(R^3, f_{k\bar{k}})$ separates its background $\text{cell}_3(R^3) - S$ into two \bar{k} -components.*

4 (k, \bar{k}) -presurfaces are digital surfaces in $(R^3, f_{k\bar{k}})$

In this section we will show that the notions of (k, \bar{k}) -presurface and digital surface are equivalent in the universal (k, \bar{k}) -space $(R^3, f_{k\bar{k}})$ for each adjacency pair $(k, \bar{k}) \neq (6, 6)$, $k, \bar{k} \in \{6, 18, 26\}$. This way, continuous analogues and even the lighting function $f_{k\bar{k}}$ are no longer needed to determine whether a given object is a digital surface in the universal (k, \bar{k}) -space. Moreover, (k, \bar{k}) -presurfaces are Jordan objects as a consequence of the separation property stated in Th. 3.3 above.

The characterization of digital surfaces as (k, \bar{k}) -presurfaces relies on the crucial fact that, for any digital object $O \subseteq \mathbb{Z}^3$ satisfying conditions (1) to (3) in Def. 2.5, the \bar{k} -assembly graph $G_{\bar{k}}(O)$ encodes the continuous analogue \mathcal{A}_O in the universal (k, \bar{k}) -space. In the proof of this result we will use the next lemmas, that state almost immediate properties of the lighting function $f_{k\bar{k}}$ in relation to the conditions defining (k, \bar{k}) -presurfaces. The first two lemmas show that $f_{k\bar{k}}(O, \delta) = 1$ for any cell $\delta \in R^3$ which is the center of a node of the \bar{k} -assembly graph of O (see Remark 3.1).

Lemma 4.1. *Let $O \subseteq \mathbb{Z}^3$ be a digital object satisfying condition (1) in Def. 2.5. The two following properties hold for a cell $\delta \in R^3$:*

1. *If $\dim \delta = 0$ then $f_{k\bar{k}}(O, \delta) = 1$ iff δ is the center of a \bar{k} -plate in O .*
2. *If $\dim \delta = 1$ and δ is the center of a square plate in O then $f_{k\bar{k}}(O, \delta) = 1$ and $f_{k\bar{k}}(O, \alpha_i) = 0$ for the two vertices $\alpha_1, \alpha_2 < \delta$. Moreover, if $\gamma > \delta$ is a 2-cell then also $f_{k\bar{k}}(O, \gamma) = 1$.*

Lemma 4.2. *Let $\delta_e = \sigma \cap \tau$ be the center of a node $e = \{\sigma, \tau\}$ of the \bar{k} -assembly graph of O in the set $E_{\bar{k}}(O)$. Then $f_{k\bar{k}}(O, \delta) = 1$.*

Proof. If σ is 6-adjacent to τ then $\dim \delta_e = 2$ and the result follows directly from the definition of $f_{k\bar{k}}$. Otherwise, $\dim \delta_e = 1$. Then, necessarily the vertices $\alpha_1, \alpha_2 < \delta_e$ are the centers of the two \bar{k} -plates of O containing e . Thus, by the definition of the lighting function $f_{k\bar{k}}(O, \alpha_i) = 1$, $i = 1, 2$, and also $f_{k\bar{k}}(O, \delta_e) = 1$ since $\bar{k} = 6$ by Remark 2.4. ■

Lemma 4.3. *Let $O \subseteq \mathbb{Z}^3$ be a digital object and p a \bar{k} -plate in O with center at the cell α_p . A cell $\gamma \in R^3$ belongs to $\text{supp}(p)$ if and if $\alpha_p < \gamma$ and $\gamma \in \text{supp}(O)$.*

Lemma 4.4. *Let $O \subseteq \mathbb{Z}^3$ be a digital object satisfying conditions (1) and (2) in Def. 2.5. If $\beta \in R^3$ is an edge which is not the center of a square plate in O and $f_{k\bar{k}}(O, \beta) = 1$ then $\bar{k} = 6$ necessarily and the two following properties also hold:*

1. The set $A = \{\alpha < \beta; f_{k,6}(O, \alpha) = 1\}$ consists of the two vertices of β which are actually centers of 6-plates in O ; moreover, β is the center of a node $e = \{\sigma, \tau\} \in E_6(O)$ of $G_6(O)$.
2. $f_{k,6}(O, \gamma) = 0$ for any 2-cell $\gamma > \beta$.

Lemma 4.5. Let $O \subseteq \mathbb{Z}^3$ be a digital object satisfying conditions (1) and (3) in Def. 2.5. If $\gamma \in R^3$ is a 2-cell such that $f_{k\bar{k}}(O, \gamma) = 1$ then $\text{st}_3(\gamma; O) = \{\sigma, \tau\}$ and the set $A = \{\alpha < \gamma; f_{k\bar{k}}(O, \alpha) = 1\}$ consists of two elements which are centers of \bar{k} -plates in O . Therefore, γ is the center of the node $\{\sigma, \tau\} \in E_{\bar{k}}(O)$.

Lemma 4.6. Let $O \subseteq \mathbb{Z}^3$ be a digital object satisfying conditions (1), (2) and (3) in Def. 2.5. If $\delta_1 < \delta_2$ are two cells in R^3 with $\dim \delta_i \leq 2$ and $f_{k\bar{k}}(O, \delta_i) = 1$, $i = 1, 2$, then δ_1 is the center of a \bar{k} -plate in O while δ_2 is not.

Proposition 4.7. Let $O \subseteq \mathbb{Z}^3$ be a digital object satisfying conditions (1), (2) and (3) in Def. 2.5, and let \mathcal{A}_O be its continuous analogue in the universal (k, \bar{k}) -space $(R^3, f_{k\bar{k}})$. Then, there exists a simplicial isomorphism $\varphi : G_{\bar{k}}(O) \rightarrow \widehat{\mathcal{A}}_O = \mathcal{A}_O - \{c(\sigma); \sigma \in O\}$, where the simplicial complex $\widehat{\mathcal{A}}_O$ is the subcomplex of \mathcal{A}_O consisting of all simplices $A \in \mathcal{A}_O$ such that $c(\sigma) \notin A$ for any voxel $\sigma \in O$. Moreover, for each $\sigma \in O$ the isomorphism φ restricts to an isomorphism $\varphi_\sigma : G_{\bar{k}}(O, \sigma) \rightarrow \text{lk}(c(\sigma); \mathcal{A}_O)$.

Proof. According to Remark 3.1 let $\alpha_n \in R^3$ be the center of a node $n \in \mathbb{P}_{\bar{k}}(O) \cup E_{\bar{k}}(O)$ of the \bar{k} -assembly graph of O . Since $\dim \alpha_n \leq 2$, the map $n \mapsto \varphi(n) = c(\alpha_n)$ between the set of nodes of $G_{\bar{k}}(O)$ and the vertices of the simplicial complex $\widehat{\mathcal{A}}_O$ is well-defined by Lemmas 4.1 and 4.2. Moreover, φ is a bijection by Lemmas 4.4 and 4.5. This map naturally extends also to edges. Recall that a \bar{k} -plate $p \in \mathbb{P}_{\bar{k}}(O)$ and a pair of voxels $e = \{\sigma_1, \sigma_2\} \in E_{\bar{k}}(O)$ define an edge in $G_{\bar{k}}(O)$ iff $e \subseteq p$. Then $\alpha_p < \alpha_e = \sigma_1 \cap \sigma_2$ by Lemma 4.3 and thus their images determine the 1-cell $\langle c(\alpha_p), c(\alpha_e) \rangle \in \widehat{\mathcal{A}}_O$. Finally we check that φ is actually a simplicial isomorphism; that is, for any edge $\langle c(\gamma_1), c(\gamma_2) \rangle \in \widehat{\mathcal{A}}_O$ the nodes $\varphi^{-1}(c(\gamma_i))$, $i = 1, 2$, determine an edge in $G_{\bar{k}}(O)$. Indeed, if $\gamma_1 < \gamma_2$, then γ_1 is the center of a \bar{k} -plate $p \in \mathbb{P}_{\bar{k}}(O)$ while $\gamma_2 = \sigma_1 \cap \sigma_2$ is the center of a pair of voxels $e = \{\sigma_1, \sigma_2\} \in E_{\bar{k}}(O)$ (here we use again Lemmas 4.4 and 4.5), and the result follows since $\sigma_i \in \text{st}_3(\gamma_1; O) = p$, $i = 1, 2$.

Finally, by using Lemma 4.3 it is immediate to check that $\varphi(\mathbb{P}_{\bar{k}}(O, \sigma) \cup E_{\bar{k}}(O, \sigma)) = \{c(\alpha) \in \mathcal{A}_O; \alpha < \sigma\}$ for each voxel $\sigma \in O$, and therefore the isomorphism φ identifies the graph $G_{\bar{k}}(O, \sigma)$ with $\text{lk}(c(\sigma); \mathcal{A}_O)$. \blacksquare

Theorem 4.8. A digital object $S \subseteq \mathbb{Z}^3$ is a (k, \bar{k}) -presurface if and only if it is a digital surface in the universal (k, \bar{k}) -space $(R^3, f_{k\bar{k}})$.

Proof. Assume $S \subseteq \mathbb{Z}^3$ is a (k, \bar{k}) -presurface. It will suffice to check that the link $L_\delta = \text{lk}(c(\delta); \mathcal{A}_S)$ is a 1-sphere for each cell $\delta \in R^3$ such that $f_{k\bar{k}}(S, \delta) = 1$. For each voxel $\delta \in S$, L_δ can be identified with the \bar{k} -assembly graph $G_{\bar{k}}(S, \delta)$ around δ , by Prop. 4.7, and hence it is a 1-sphere by condition (4) in Def. 2.5. If $\dim \delta = 2$ the result is an immediate consequence of Lemma 4.5. Similarly, if $\dim \delta = 1$ the result follows from Lemma 4.4 in case δ is not the center of a plate, and from Lemma 4.1(2) otherwise.

Finally, if δ is a vertex then it is the center of a plate $p \in \mathbb{P}_{\bar{k}}(S)$ by Lemma 4.1. By the definition of the lighting function $f_{k\bar{k}}$ we know that $f_{k\bar{k}}(S, \sigma \cap \tau) = 1$ for each pair of 6-adjacent voxels $\sigma, \tau \in p$ and, in particular, it is readily checked that L_δ is a 1-sphere whenever p is a P_6^c plate. If p is not a P_6^c plate then $\bar{k} = 6$. Moreover, if p is not a C_4^f plate, $\sigma_1, \sigma_2 \in p$ are strictly 18-adjacent and no other voxel in p is 6-adjacent to both of them, we derive from the fact that $c(\delta) \in L_{\sigma_i}$, $i = 1, 2$, which has been proved to be a 1-sphere, that $f_{k\bar{k}}(S, \sigma_1 \cap \sigma_2) = 1$. Therefore L_δ is also a 1-sphere in these cases by the definition of $f_{k\bar{k}}$. In case $p = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a C_4^f plate we have some choices to make. As $c(\delta)$ is in the 1-sphere L_{σ_i} , $1 \leq i \leq 4$, it contains exactly two of the three centers $c_i^j = c(\sigma_1 \cap \sigma_i)$, $1 \leq i \neq j \leq 4$. If $c_2^1, c_3^1 \in L_{\sigma_1}$ then $c_1^4 \notin L_{\sigma_4}$ and thus $c_2^4, c_3^4 \in L_{\sigma_4}$. Therefore $c_1^2, c_2^4 \in L_{\sigma_2}$, $c_1^3, c_3^4 \in L_{\sigma_3}$ and so L_δ is also a cycle.

Conversely, assume S is a digital surface in $(R^3, f_{k\bar{k}})$. It will be enough to check conditions (1) to (3) in Def. 2.5 for S since, under the assumption of these properties, we get (4) as an immediate consequence of Prop. 4.7.

If $C \cap O$ corresponds to some pattern in the set $\mathbb{F}\mathbb{P}_{k\bar{k}}$ for some unit cube C , it can be readily checked from the definition of $f_{k\bar{k}}$ that the object O is not a digital surface. Hence condition (1) holds for S . To check condition (2), let $\sigma, \tau \in S$ be two strictly 18-adjacent voxels and assume that they are not 6-connected by a third voxel in S . For the edge $\beta = \langle \alpha_1, \alpha_2 \rangle = \sigma \cap \tau$ we consider the two possible cases:

Case $f_{k\bar{k}}(S, \beta) = 0$. If $\bar{k} = 6$ the definition of $f_{k\bar{k}}$ shows that $f_{k\bar{k}}(S, \alpha) = 1$ for a vertex $\alpha < \beta$. Then by condition (1), already proved, and Lemma 4.1 it follows that α is the center of a \bar{k} -plate in $\mathbb{P}(S, \sigma) \cap \mathbb{P}(S, \tau)$. On the other hand, if $k, \bar{k} \in \{18, 26\}$ then σ, τ are 6-connected in $\text{st}_3^*(\beta; S)$, by the definition of $f_{k\bar{k}}$, which is just condition (2).

Case $f_{k\bar{k}}(S, \beta) = 1$. Then it can be readily checked that $f_{k\bar{k}}(S, \alpha_i) = 1$ for the two vertices α_1, α_2 of β , since S is a digital surface in $(R^3, f_{k\bar{k}})$. Therefore the vertices α_i are centers of \bar{k} -plates in $\mathbb{P}(S, \sigma) \cap \mathbb{P}(S, \tau)$ by Lemma 4.1. Notice that this case is only possible if $\bar{k} = 6$.

Finally we prove condition (3). For this let $\sigma, \tau \in S$ be two 6-adjacent voxels and let $\gamma = \sigma \cap \tau$. Then $f_{k\bar{k}}(S, \gamma) = 1$ by definition of $f_{k\bar{k}}$. As $\text{lk}(c(\gamma); \mathcal{A}_S)$ is a 1-sphere there exist exactly two faces $\alpha_1, \alpha_2 < \gamma$ with $f_{k\bar{k}}(S, \alpha_i) = 1$. If $\dim \alpha_i = 0$ then α_i is the center of a \bar{k} -plate by Lemma 4.1. Similarly, if $\dim \alpha_i = 1$ then the definition of $f_{k\bar{k}}$ yields that $\text{st}_3(\alpha_i; S) = \text{st}_3(\alpha_i; R^3)$ since it contains the two 6-adjacent voxels σ and τ , and hence α_i is the center of a square plate. Therefore $\mathbb{P}(S, \sigma) \cap \mathbb{P}(S, \tau)$ contains at least two \bar{k} -plates. But given the center α_p of any plate $p \in \mathbb{P}(S, \sigma) \cap \mathbb{P}(S, \tau)$ we know that $\alpha_p < \gamma$ by Lemma 4.3 and, moreover, $f_{k\bar{k}}(S, \alpha_p) = 1$ by Lemma 4.1. This way $\alpha_p \in \{\alpha_1, \alpha_2\}$ and $\mathbb{P}(S, \sigma) \cap \mathbb{P}(S, \tau)$ consists of exactly two \bar{k} -plates. ■

5 (k, \bar{k}) -surfaces

As a consequence of Th. 3.3 and Th. 4.8 we get that each k -connected (k, \bar{k}) -presurface S separates its background $\mathbb{Z}^3 - S$ into two \bar{k} -components. In addition to this Jordan property, discrete surfaces are usually required to be strongly \bar{k} -separating; that is, each voxel $\sigma \in S$ should be \bar{k} -adjacent to both \bar{k} -components of $\mathbb{Z}^3 - S$ (see [3]). However, it is easy to check that this global property fails for the voxel τ_0 in the $(18, 6)$ -presurface shown in Fig. 2(a).

Our goal in this section is to find further local conditions characterizing the strong separation property within the class of (k, \bar{k}) -presurfaces in order to obtain genuine discrete surfaces according to the following

Definition 5.1. A (k, \bar{k}) -presurface is said to be a (k, \bar{k}) -surface if it is a strongly \bar{k} -separating object.

In [4, §7] we found the local conditions characterizing the subset of strongly 6-separating $(26, 6)$ -presurfaces. The same conditions, and the proof as well, works for the case $(18, 6)$. For the remaining cases we get the following.

Definition 5.2. Let $S \subseteq \mathbb{Z}^3$ be a (k, \bar{k}) -presurface, $\bar{k} \in \{18, 26\}$. A voxel $\sigma \in S$ is said to be a \bar{k} -surface voxel if for each unit cube $C_\sigma \subseteq \mathbb{Z}^3$ containing σ there exists a voxel $\tau \in C_\sigma - S$ which is \bar{k} -adjacent to σ . Notice that every voxel in a $(k, 26)$ -presurface S is a 26-surface voxel since S cannot contain the pattern P_8 in Fig. 1.

Theorem 5.3. A (k, \bar{k}) -presurface S is a (k, \bar{k}) -surface iff each $\sigma \in S$ is a \bar{k} -surface voxel.

Proof. Recall that S is a digital surface in the universal (k, \bar{k}) -space $(R^3, f_{k\bar{k}})$ and so $|\mathcal{A}_S|$ is a combinatorial surface. As a consequence of the Jordan-Brouwer Theorem the difference $D = |\text{lk}(c(\sigma; \mathcal{A}_{R^3})| - |\text{lk}(c(\sigma; \mathcal{A}_S)|$ consists of two components, each contained in a component of $\mathbb{R}^3 - |\mathcal{A}_S|$. Moreover, these components characterize the \bar{k} -components of $\mathbb{Z}^3 - S$ since $(R^3, f_{k\bar{k}})$ is a (k, \bar{k}) -space (see §3). Let $\delta_1, \delta_2 < \sigma$ be two cells with their centers $c(\delta_i)$ in each of the components

of D . Notice that $f_{k\bar{k}}(S, \delta_i) = 0$. Then, if σ is \bar{k} -surface voxel, the definition of $f_{k\bar{k}}$ gives us two voxels $\tau_i \notin S$ which are \bar{k} -adjacent to σ and such that $\delta_i < \tau_i$, $i = 1, 2$. Therefore, $c(\tau_1)$ and $c(\tau_2)$ are in distinct components of $\mathbb{R}^3 - |\mathcal{A}_S|$ and the result follows.

Conversely, assume $\bar{k} = 18$ (there is nothing to prove if $\bar{k} = 26$). If σ is not a 18-surface voxel then there exists a unit cube C , with center at a vertex $\alpha \in R^3$, such that $C - S$ consists of a single voxel τ which is strictly 26-adjacent to σ . Then, we derive from the definitions that $f_{k,18}(S, \delta) = 1$ for each face $\alpha < \delta < \sigma$. Moreover, the centers of these cells determine a cycle in $\text{lk}(c(\sigma); \mathcal{A}_S)$, and then $f_{k,18}(S, \gamma) = 0$ for any other face $\gamma < \sigma$, in particular $f_{k,18}(S, \alpha) = 0$. This way $\{c(\alpha)\}$ is a component of the difference D above, and hence it follows that σ is 18-adjacent to just one 18-component of $\mathbb{Z}^3 - S$. ■

Remark 5.4. It is worth pointing out that the set of (k, \bar{k}) -surfaces still contains strictly the sets of simplicity and strong surfaces quoted in the introduction since each one of them is a strongly separating object [3, 7].

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