Computing The Cubical Cohomology Ring

(Extended Abstract)

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Abstract The goal of this work is to establish a new algorithm for computing the cohomology ring of cubical complexes. The cubical structure enables an explicit recurrence formula for the cup product. We derive this formula and, next, show how to extend the Mrozek and Batko [7] homology coreduction algorithm to the cohomology ring structure. The implementation of the algorithm is a work in progress. This research is aimed at applications in electromagnetism and in image processing, among other fields.

Keywords computation; cohomology; cup product; cubical complex; coreductions.

1 Introduction

In the past two decades, the homology and cohomology theories gained a vivid attention outside of the mathematics community prompted by its modern applications in sciences and engineering. The development of computational approach to these theories is motivated, among others, by problems in dynamical systems, material science, electromagnetism, geometric modeling, image understanding, and digital image processing. Conversely, that development is enabled by the progress in computer science. Although algebraic topology has raised from applications and has been thought of as a computable tool at its early stage, its practical implementation had to wait until the modern generation of powerful computers due to a very high complexity of operations, involved especially in high-dimensional problems.

Until recently, the main progress has been done in computation of homology groups of finitely representable structures. The software library CHomP [2] is an example of a systematic approach to computing homology in arbitrary dimensions and for any structures leading to a finitely generated chain complex. Many applied problems are related to the topological analysis of data which express well in cubical grids. One may think about pixels and voxels represented by unit squares or cubes in digital images [1]. This naturally leads to the framework of cubical sets and cubical homology presented in [6]. In this paper, we work in the same framework but extending the material to dual cochain groups.

Cohomology theory, not less important than homology from the point of view of applications but intrinsically harder, had to wait longer for computer implementations. Wherever a mathematical model was making it possible as, for example, in the case of orientable manifolds, the duality has been used to avoid explicitly working with cohomology. This approach is observable in works on computational electromagnetism such as [5, 10, 3]. However, among features distinguishing cohomology from homology is the cup product, which renders a ring structure on the cohomology. The cup product is a difficult concept which has been more challenging to make it

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explicit enough for computer programs than homology or cohomology groups. Some of significant application-oriented work on computing the cohomology ring of simplicial complexes is done by Real *at. al.* [4]. The concept of cross product is easier and more natural in the context of cubical sets than for simplical or singular complexes, because the cartesian product of generating cubes is again a generating cube. This is not true for simplices.

In order to make the formula for the cup product explicit, we derive an explicit formula for a chain map induced by the diagonal map. Actually, this task is more complex than it may seem at the first glance and some effort is devoted to the related constructions.

2 Cubical cohomology groups

Recall from [6, Chapter 2] that $X \subset \mathbb{R}^d$ is a *cubical set* if it is a finite union of *elementary cubes* $Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d$, where I_i is an interval of the form $I = [k, k+1]$ (non-degenerate) or $I = [k, k]$ (degenerate) for some $k \in \mathbb{Z}$. For short, $[k] := [k, k]$. The set of all elementary cubes in $\mathbb{R}^d$ is denoted by $\mathcal{K}(\mathbb{R}^d)$ and those of dimension k by $\mathcal{K}_k(\mathbb{R}^d)$. Those which are contained in X are denoted by $\mathcal{K}(X)$, respectively, $\mathcal{K}_k(X)$.

The group $C_k(\mathbb{R}^d)$ of *cubical k-chains* is the free abelian group generated by $\mathcal{K}_k(\mathbb{R}^d)$, its *canonical basis.* For $k < 0$ and $k > d$, we set $C_k(\mathbb{R}^d) := 0$. We abandon here the distinction made in [6] between the geometric cubes $Q \in \mathcal{K}(\mathbb{R}^d)$ and their algebraic dual chains $\widehat{Q}$ so to focus on the duality in the sense of cochains. The boundary map $\partial_k : C_k(\mathbb{R}^d) \to C_{k-1}(\mathbb{R}^d)$ is defined for each $Q \in \mathcal{K}_k(\mathbb{R}^d)$ as an alternating sum of its $(k-1)$–dimensional faces. The exact alternation formula is given in [6, Chapter 2] by induction on the *embedding number d* of Q with the help of the *cubical cross product*

$$\times : C_p(\mathbb{R}^n) \times C_q(\mathbb{R}^m) \to C_{p+q}(\mathbb{R}^{n+m}),$$

which is defined on the canonical basis elements $P \in \mathcal{K}_p^n$ and $Q \in \mathcal{K}_q^m$ as the cartesian product $P \times Q$ and extended on all pairs of chains (c, c') by bilinearity.

In [6], this operation is called cubical product and denoted by $c \diamond c'$ so to distinguish it from the cartesian product but we abandon this notation here so to emphasize its equivalence to the cross product in homological algebra. The pair $(\mathcal{C}(\mathbb{R}^d), \partial) := \{\big(C_k(\mathbb{R}^d), \partial_k\big)\}_{k\in\mathbb{Z}}$ is called the *cubical chain complex of* $\mathbb{R}^d$. The cubical cross product induces the isomorphism of the tensor product of chain complexes [9, Chapter 7]

$$C_p(\mathbb{R}^n) \otimes C_q(\mathbb{R}^m) \cong C_{p+q}(\mathbb{R}^{n+m}).$$

Given a cubical set X, the *cubical chain complex of* X is the restriction of $\mathcal{C}(\mathbb{R}^d)$ to the chains supported in X, notation $\mathcal{C}(X)$.

The *cubical cochain complex* $\big(\mathcal{C}^*(\mathbb{R}^d), \delta\big)$ is defined as follows. For any $k \in \mathbb{Z}$, the k–dimensional cochain group

$$C^k(\mathbb{R}^d) = Hom(C_k(\mathbb{R}^d), \mathbb{Z})$$

is the dual group of $C_k(\mathbb{R}^d)$, that is, the group of all homomomorphisms from $C_k(\mathbb{R}^d)$ to $\mathbb{Z}$. The value of a cochain c^k on a chain c_k is denoted by $\langle c^k, c_k\rangle$. The k'th coboundary map $\delta^k : C^k(\mathbb{R}^d) \to C^{k+1}(\mathbb{R}^d)$ is the dual map of ∂_{k+1} defined by

$$\langle \delta^k c^k, c_{k+1}\rangle := \langle c^k, \partial_{k+1} c_{k+1}\rangle.$$

Note that $C^k(\mathbb{R}^d)$ is the free abelian group generated by the dual canonical basis $\{Q^* | Q \in \mathcal{K}_k(\mathbb{R}^d)\}$. The coboundary maps are determined by their values $\langle \delta^k Q^*, P\rangle$,

Given a cubical set $X \in \mathbb{R}^d$, the *cubical cochain complex* $(\mathcal{C}^*(X), \delta)$ of X is the subcomplex of $\big(\mathcal{C}^*(\mathbb{R}^d), \delta\big)$ whose groups $C^k(X)$ consist of those homomorphisms $c^k : C^k(\mathbb{R}^d) \to \mathbb{Z}$ which are *supported* in $C_k(X)$, that is, if $P \in \mathcal{K}(\mathbb{R}^d) \setminus \mathcal{K}(X)$, then $\langle c^k, P\rangle = 0$. The group of k–dimensional *cocycles* of X is $Z^k(X) := \ker \delta^k$, and the group of k–dimensional *coboundaries* of X is $B^k := \operatorname{im} \delta^{k-1}$. The k'th *cohomology group* of X is the quotient group $H^k(X) := Z^k(X)/B^k(X)$.

3 Cubical cup product

Here is an extrapolation of classical definitions of a cross product on cochains:

Definition 3.1 The *cross product* of cochains $c^p \in C^p(X)$ and $c^q \in C^q(Y)$ is a cochain in $C^{p+q}(X \times Y)$ defined on any elementary cube $R \times S \in \mathcal{K}_{p+q}(X \times Y)$, where $R \in \mathcal{K}(X)$ and $S \in \mathcal{K}(Y)$, as follows.

$$\langle c^p \times c^q, Q\rangle := \begin{cases} \langle c^p, R\rangle \cdot \langle c^q, S\rangle & \text{if } \dim R = p \text{ and } \dim S = q, \\ 0 & \text{otherwise.} \end{cases}$$

The most important step towards an explicit formula for the cup product is the construction of a homology chain map $\operatorname{diag}_\# : \mathcal{C}(X) \to \mathcal{C}(X \times X)$ induced by the *diagonal map* $\operatorname{diag} : X \to X \times X$ given by $\operatorname{diag}(x) := (x, x)$. This is done using the Acyclic Selector Theorem [6, Theorem 6.22], because our map admits an acyclic-valued representation $\operatorname{Diag} : X \rightrightarrows (X \times X)$ given by $Diag(x) := Q \times Q$, where $Q = \operatorname{ch}(x) \in \mathcal{K}(X)$ is the *cubical enclosure* of x, that is, the smallest cubical set containing x. We use here the notation $F : X \rightrightarrows Y$ for set-valued maps from X to Y introduced in [6, Chapter 6]. The construction of its chain selector goes by induction on the embedding number d of $X \subset \mathbb{R}^d$.

Case $d = 1$: If $Q = [v] \in \mathcal{K}_0(X)$, we put

$$\operatorname{diag}_0([v]) := [v] \times [v]. \tag{1}$$

If $Q = [v_0, v_1] \in \mathcal{K}_1(X)$, we put

$$\operatorname{diag}_1([v_0, v_1]) := [v_0] \times [v_0, v_1] + [v_0, v_1] \times [v_1]. \tag{2}$$

Induction step: Suppose that $\operatorname{diag}_\#$ is defined for cubical sets of embedding numbers $n = 1, \ldots, d-1$ and let's construct it for a cubical set X of the embedding number d. Note that

$$\begin{aligned} \operatorname{diag}(x_1, \ldots, x_d) &= (x_1, \ldots, x_d, x_1, x_2, \ldots, x_d) \\ &= \tau(x_1, x_1, x_2, \ldots, x_d, x_2, \ldots, x_d) \\ &= \tau(\operatorname{diag}(x_1), \operatorname{diag}(x_2, \ldots, x_d)), \end{aligned}$$

where $\tau : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is the permutation of coordinates which transposes the $(d+1)$'st coordinate with the precessing $d-1$ coordinates. This permits expressing the diagonal map as the composition

$$\operatorname{diag} = \tau \circ (diag_{\mathbb{R}^1}, diag_{\mathbb{R}^{d-1}}). \tag{3}$$

The formula for the chain map induced by τ is provided by

Lemma 3.2 *Consider the permutation of cubical sets*

$$\tau : (X_1 \times Y_1) \times (X_2 \times Y_2) \to (X_1 \times X_2) \times (Y_1 \times Y_2)$$

given by $\tau(x_1, y_1, x_2, y_2) = (x_1, x_2, y_1, y_2)$. *Then the homology map of* τ *is induced by the chain map*

$$\tau_\# : \mathcal{C}(X_1 \times Y_1 \times X_2 \times Y_2) \to \mathcal{C}(X_1 \times X_2 \times Y_1 \times Y_2)$$

defined on products of $P_1 \in \mathcal{K}(X_1)$, $P_2 \in \mathcal{K}(Y_1)$, $Q_1 \in \mathcal{K}(X_2)$, *and* $Q_2 \in \mathcal{K}(Y_1)$ *by the formula*

$$\tau_\# ((P_1 \times P_2) \times (Q_1 \times Q_2)) = (-1)^{\dim P_2 \dim Q_1} (P_1 \times Q_1) \times (P_2 \times Q_2).$$

The composition of the acyclic-valued representations of maps involved in the formula (3) has acyclic values. From here we conclude that the composition of the respective chain selections is a chain map induced by diag.

The following definition is inspired by [9, Chapter 7, Theorem 61.3]

Definition 3.3 Let X be a cubical set. The *cubical cup product*

$$\smile : C^p(X) \times C^q(X) \to C^{p+q}(X)$$

of cochains c^p and c^q is defined on generators $Q \in \mathcal{K}_{p+q}(X)$ by the formula:

$$\langle (c^p \smile c^q), Q \rangle := \langle \operatorname{diag}^{p+q}(c^p \times c^q), Q \rangle = \langle c^p \times c^q, \operatorname{diag}_{p+q}(Q) \rangle \tag{4}$$

The *cup product* $\smile : H^p(X) \times H^q(X) \to H^{p+q}(X)$ is defined on cohomology classes of cocycles as follows.

$$[z^p] \smile [z^q] := [z^p \smile z^q]. \tag{5}$$

We have proved that all standard algebraic and bounding properties of cup product hold for the cubical cup product. The alter-commutative law only holds on the level of cohomology classes:

Theorem 3.4 [Graded commutative law]
If z^p and z^q are cocycles, then

$$[z^q \smile z^p] = (-1)^{pq}[z^p \smile z^q].$$

We give a direct formula for the cubical cochain product (4) of the generating cochains $P^* \in \mathcal{K}^p(X)$ and $Q^* \in \mathcal{K}^q(X)$, and extend it to all cochains by bilinearity. The construction goes by induction on $d = \operatorname{emb}(X)$.

Case $d = 1$: For $k = p + q = 0$, let $R = [v] \in \mathcal{K}_0(X)$. Then

$$\begin{aligned} \langle P^* \smile Q^*, R \rangle &= \langle P^* \times Q^*, [v] \times [v] \rangle \\ &= \begin{cases} 1 & \text{if } P = Q = [v], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $k = p + q = 1$, let $R = [v_0, v_1] \in \mathcal{K}_1(X)$. Then

$$\begin{aligned} \langle P^* \smile Q^*, R \rangle &= \langle P^* \times Q^*, [v_0] \times [v_0, v_1] + [v_0, v_1] \times [v_1] \rangle \\ &= \begin{cases} 1 & \text{if } P = [v_0] \text{ and } Q = [v_0, v_1], \\ 1 & \text{if } P = [v_0, v_1] \text{ and } Q = [v_1], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We rephrase these results as follows:

Theorem 3.5 *Let $P, Q \in \mathcal{K}(X)$, $P = [a, b]$ and $Q = [c, d]$. Then*

$$P^* \smile Q^* = \begin{cases} [a]^* & \text{if } a = b = c = d, \\ [c, d]^* & \text{if } a = b = c = d - 1, \\ [a, b]^* & \text{if } b = c = d = a + 1, \\ 0 & \text{otherwise.} \end{cases}$$

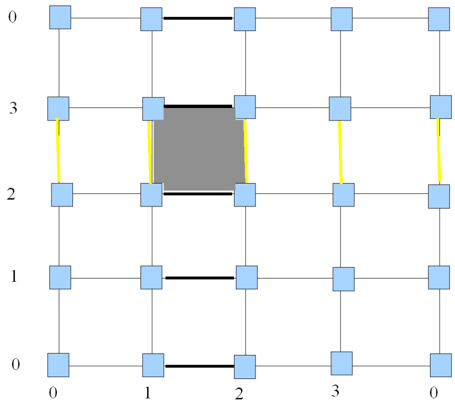
Induction step: Suppose that the formula for $\smile$ is given for cochains on cubical sets of embedding numbers $n = 1, \ldots, d - 1$ and let's extend it to cubical sets X of the embedding number d. This is provided by the following:

Theorem 3.6 *Consider the decomposition of elementary cubes $P = P_1 \times P_2 \in \mathcal{K}_p(X)$ and $Q = Q_1 \times Q_2 \in \mathcal{K}_q(X)$ with $\operatorname{emb} P_1 = \operatorname{emb} Q_1 = 1$ and $\operatorname{emb} P_2 = \operatorname{emb} Q_2 = d - 1$. Then*

$$P^* \smile Q^* = (-1)^{\dim P_2 \dim Q_1} (P_1^* \smile Q_1^*) \times (P_2^* \smile Q_2^*)$$

where $(P_1^* \smile Q_1^*)$ and $(P_2^* \smile Q_2^*)$ are computed using the induction hypothesis.

Example 3.7 We illustrate the cup-product formula in the cubical torus $T := \Gamma^1 \times \Gamma^1 \subset \mathbb{R}^4$, where $\Gamma^1 = \partial[0,1]^2$ is the contour of the square. We parameterize Γ^1 by the interval $[0,4]/0 \sim 4$, which permits visualizing T as the square $[0,4]^2$ with identified edges as shown in the figure below.



Consider the cocycle x^1 generated by the sum of four yellow (or light gray) vertical edges at the level $[2,3]$, and y^1 by the sum of black horizonal edges at the level $[1,2]$. Only the edges of the parametric square $[1,2] \times [2,3]$ may contribute to non-zero terms of $x^1 \smile y^1$. Thus,

$$\begin{aligned} x^1 \smile y^1 &= \{([1] \times [2,3])^* + ([2] \times [2,3])^*\} \smile \{([1,2] \times [2])^* + ([1,2] \times [3])^*\} \\ &= 0 - ([1,2] \times [2,3])^* + 0 + 0 = -([1,2] \times [2,3])^*. \end{aligned}$$

In this example, $y^1 \smile x^1 = -x^1 \smile y^1$ but this only is a coincidence. The cohomology classes of cochains x^1 and y^1 generate $H^1(T)$, and $[Q^*]$, where $Q := ([1,2] \times [2,3])$, generates $H^2(T)$.

4 Reduction algorithms for cohomology computing

Reduction procedures have been fruitfully used to speed up the homology computation [8, 7]. We refer to definitions of an *S-complex* (basically a chain complex with a prescribed basis), an *elementary reduction pair*, and an *elementary coreduction pair* $(a,b) \in S \times S$ given in [7]. The extension of discussed methods to cohomology computation is based on the following theorem.

Theorem 4.1 *If (a,b) is an elementary reduction or coreduction pair in $S \times S$, then $H^*(S)$ and $H^*(S \setminus \{a,b\})$ are isomorphic.*

The process of removing reduction pairs can be iterated as long as one is able to find reduction pairs. If we get a complex S^α for which all coboundary maps are null, we are done, since then $H^*(S) \cong H^*(S^\alpha) = S^\alpha$. A reduction of S to S^α is called a shaving if the embedding of cohomology basis of S^α to S is a cohomology basis of S. We show, that arbitrary sequence of coreduction pairs is a shaving. We also show that embedding of the cup product computed for the shaved complex into the initial complex is equal to the cup product computed in the initial complex. Theorem 3.5 gives rise to the algorithm in Table 1.

The cup product on cochains is computed by linearity. One may want to get its representation in a fixed basis of cohomology. This is achieved by an algorithm `findRepresentation` (not presented in this abstract). Since that algorithm requires solving a large number of linear equations, the following comes with help:

Theorem 4.2 *It is sufficient to run the algorithm `findRepresentation` in the shaved complex S^α.*

The proofs of presented results, discussion of the complexity issues, and a more ample literature review will be presented in the full version of the paper. A practical implementation of these algorithms is our forthcoming project. Long-term goals are related to applications of this work to electromagnetism and in image processing.

(**int** , cube) **function** cupProductOfBasicCochains($P^* \in \mathcal{K}^q, Q^* \in \mathcal{K}^p$)
begin
Let $P = [a_1, b_1] \times \ldots \times [a_n, b_n]$ and $Q = [c_1, d_1] \times \ldots \times [c_n, d_n]$;
int coef := 1;
cube $Q = \emptyset$;
for int $i = 1$ to n **do begin**
$coef := coef * \dim([c_i, d_i]) * \dim([a_{i+1}, b_{i+1}] \times \ldots \times [a_n, b_n])$;
cube $Q_{loc} = \emptyset$;
if $(a_i = b_i = c_i = d_i)$ **then** $Q_{loc} = [a_i, a_i]$;
if $(a_i = b_i = c_i = d_i - 1)$ **then** $Q_{loc} = [c_i, d_i]$;
if $(a_i + 1 = b_i = c_i = d_i)$ **then** $Q_{loc} = [a_i, b_i]$;
if $(Q_{loc} = \emptyset)$ **then return** $(0, \emptyset)$;
if $(Q = \emptyset)$ **then** $Q := Q_{loc}$;
else $Q := Q \times Q_{loc}$;
end ;
return (coef,Q^*);
end ;

Table 1: Algorithm computing cup product of elementary cubes

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