Improved Locally Adaptive Sampling Criterion for Topology Preserving Reconstruction of Multiple Regions

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Abstract Volume based digitization processes often deal with non-manifold shapes. Even though many reconstruction algorithms have been proposed for non-manifold surfaces, they usually don't preserve topological properties. Only recently, methods were presented which—given a finite set of surface sample points—result in a mesh representation of the original boundary preserving all or certain neighbourhood relations, even if the sampling is sparse and highly noise corrupted.

We show that the required sampling conditions of the algorithm called "refinement reduction" limit the guaranteed correctness of the outcome to a small class of shapes. We define new locally adaptive sampling conditions that depend on our new pruned medial axis and finally prove without any restriction on shapes that under these new conditions, the result of "refinement reduction" corresponds to a superset of a topologically equivalent mesh.

1 Introduction

Surface reconstruction from boundary points is a well known problem for which many algorithms have been proposed. We are interested in algorithms for which topological correctness of the result can be guaranteed. Recent development has been headed toward guarantees under less strict assumptions on the sampling points.

In [1] the concept of ϵ-samples is introduced in order to give a provably correct algorithm for reconstructing smooth surfaces. A finite set S of sampling points on the boundary is an ϵ-sampling if every boundary point b has a sampling point in a distance of at most $\epsilon\,\mathrm{lfs}(b)$, where the *local feature size* (lfs) denotes the distance from b to the medial axis (MA). Similar restrictions by the maximal value of the ratio of the sampling density and local curvature have been made in [3, 2] in order to prove the topological correctness of the resulting reconstructions. The results proposed in [7] and independently in [12] allow reconstructions of non-smooth surfaces from noise-corrupted samples assuming a known global bound on the sampling density. This is based on the *weak feature size* (wfs), which denotes the distance between the boundary of the shape and the set of criticals on the distance transform. In [12], this was explicitly extended to handle the reconstruction of non-manifold surfaces. Later [11], the assumptions on sampling density were based on a *local region size* (lrs), which has the advantage to be a locally adaptive measure based on the boundary points and the local maxima on the distance transform. The associated algorithm termed "refinement reduction" deals with highly noise corrupted samplings of non-manifold boundaries and results in a refinement of the original boundary leading to the well known concept called *oversegmentation* in the 2D case.

In this work, we show that the sampling density based on lrs limits the guaranteed correctness of the reconstruction to shapes having only one maximum on the distance transform in each connected region. We introduce a new pruned medial axis called *homotopical axis* and use it to propose new sampling conditions. Finally, we prove that under these new sampling conditions the above-mentioned "refinement reduction" algorithm results in a superset of a topologically equivalent approximation of the original non-manifold boundary. In this way, we extend the guaranteed correctness of the reconstruction to any kind of shapes.

The paper is structured as follows: After introducing some preliminary concepts in Section 2, we propose our new homotopical axis in Section 3 and give new sampling conditions in Section 4. In Section 5, we recall the "refinement reduction" algorithm [11]. Finally, we discuss the above-mentioned limitation of the lrs-based approach in Section 6 and give the central proof that our new sampling conditions lead to a superset of a topological equivalent of the original boundary, even for arbitrary non-manifolds.

2 Basic Definitions

The focus of this work is the reconstruction of surfaces of multiple regions, given a sampling of their boundary. So, we do not only consider the 2-manifold surface of a single solid, but a partition of the 3D space into different regions.

The following definition is adapted from [11]:

Definition 2.1 (Space Partition) *In 3D, a* space partition $\mathcal{R}$ *is defined by a finite set of pair-wise disjoint regions* $\mathcal{R} = \{R_i \subset \mathbb{R}^3\}$ *such that each region* $R_i \in \mathcal{R}$ *is a connected open set and the union of the closures of the regions covers the whole space, i.e.* $\bigcup_i \overline{R_i} = \mathbb{R}^3$*. The* boundary *of the partition is* $\partial\mathcal{R} := \bigcup_i \partial R_i$*.*

In order to investigate the volumetric information about the space partition, we make use of the well known concept of the *distance transform*. The *reversed distance transform* delivers for each input point the touching points of the maximal inscribed ball with the boundary.

Definition 2.2 (Distance Transform) *The* distance transform d_B *of a set* $B \subset \mathbb{R}^3$ *is defined as* $\mathrm{d}_B(x) = \min_{y \in B} \|x - y\|$*. The distance transform is called* continuous *if B is infinite, and* discrete *otherwise. The* reversed distance transform *is defined as* $\mathrm{rd}_B(x) = \{y \in \partial\mathcal{R} | \|x-y\| = \mathrm{d}_B(x)\}$*. x is a* local maximum *of the* distance transform *iff* $\exists \epsilon > 0 \, \forall x' : (\|x' - x\| < \epsilon) \rightarrow (\mathrm{d}_B(x') < \mathrm{d}_B(x))$*.*

Since the distance transform is a non-smooth function in general, regular gradient methods cannot be applied to define the critical points and the steepest ascent on d_B. Lieutier [9] extends the definition of gradients:

Definition 2.3 (Gradient and Criticals [9]) *Let* $\Theta(x)$ *be the center of the smallest closed ball enclosing* $\mathrm{rd}_B(x)$*. Then the* gradient *on x is defined as*

$$\nabla(x) = \frac{x - \Theta(x)}{\mathrm{d}_B(x)}$$

and the set of critical points *of* ∇ *is given by* $\mathbf{F}(\mathcal{R}) = \{x \in \mathcal{R} \mid \|\nabla(x)\| = 0\}$*. More generally,* $\mathbf{F}_\beta(\mathcal{R}) = \{x \in \mathcal{R} \mid \|\nabla(x)\| \leq \beta\}$*.*

Obviously, $\lim_{\beta \to 0} (\mathbf{F}_\beta(\mathcal{R})) = \mathbf{F}_0(\mathcal{R}) = \mathbf{F}(\mathcal{R})$ and $\beta \leq \beta' \Rightarrow \mathbf{F}_\beta(\mathcal{R}) \subset \mathbf{F}_{\beta'}(\mathcal{R})$.

Then ∇ gives the direction of the *steepest ascent*, i.e. the direction which maximizes the growth of d_B. Note that ∇ is not continuous. However, Lieutier [9] proves that Euler schemes using the vector field ∇ converge uniformly when the integration step decreases. Integrating ∇ then results in a continuous flow

$$\mathfrak{C} : \mathbb{R}^+ \times \mathcal{R} \mapsto \mathcal{R} \quad \text{with} \quad \mathfrak{C}(t, x) = x + \int_0^t \nabla(\mathfrak{C}(\tau, x)) \, d\tau$$

Throughout our research, the study of volumetric conditions inside the regions is done by following *simple paths*. A simple path is characterized by means of the values of the distance transform along the path:

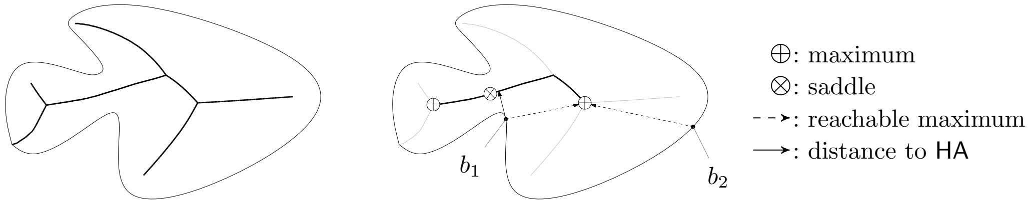


Figure 1: left: medial axis, right: our new homotopical axis (illustrated in 2D for simplicity)

Definition 2.4 (Simple Path) *A continuous map* $\pi : [0,1] \to \mathbb{R}^3$ *is also called a* simple path. *Further,* π *is an* increasing (strictly increasing, decreasing, strictly decreasing) path *on the distance transform iff* $\mathrm{d}_B \circ \pi$ *is* increasing (strictly increasing, decreasing, strictly decreasing) *on* $[0,1]$ *respectively.* π *with* $\pi(0) = x$ *is a* steepest path (starting at x) *iff* $\forall t \in [0,1]\, \exists t' \in \mathbb{R}^+ : \pi(t) = \mathfrak{C}(t', x)$.

In order to investigate which volumetric information about the space partition can be used to define the new sampling criteria of the boundary, we refer to the well known concept of a complete shape descriptor, the *medial axis*:

Definition 2.5 (Medial Axis [4]) *The* medial axis *of a set* $B \subset \mathbb{R}^3$ *is defined as* $\mathsf{MA} = \{x \in \mathbb{R}^3 \mid \#(\mathrm{rd}_B(x)) > 1\}$.

Let $\Omega(x, \epsilon)$ *be the intersection of an open ball placed on* x *with radius* ϵ *and* MA. x *is a* local maximum on MA *iff* $\exists \epsilon > 0\, \forall x' \in \Omega(x, \epsilon) : \mathrm{d}_B(x') > \mathrm{d}_B(x)$.

3 Homotopical Axis

We introduce the *homotopical axis* as a subset of the medial axis (cf. Fig. 1), bounded by i) criticals of the distance transform (⊕ and ⊗) and ii) points which can be reached by steepest ascent starting on infinitesimal environments of criticals (points on thick line —):

Definition 3.1 (Homotopical Axis) *The* homotopical axis*(*HA*) is defined as:*

$$\mathsf{HA} = \lim_{\beta \to 0^+} \mathbf{G}_\beta(\mathcal{R}) \quad \text{where}$$

$$\mathbf{G}_\beta(\mathcal{R}) = \{x \in \mathcal{R} \mid \exists t \in \mathbb{R}^+ \exists y \in \mathbf{F}_\beta(\mathcal{R}) : x = \mathfrak{C}(t, y)\}.$$

$\mathbf{G}_\beta(\mathcal{R})$ is the smallest superset of $\mathbf{F}_\beta$ that contains all points reachable via the flow $\mathfrak{C}$; this concept and notation has been introduced by Chazal and Lieutier [5] together with the proof of the following lemma:

Lemma 3.2 (Homotopy Type of $\mathbf{G}_\beta$ [5]) *Let* $\mathcal{O}$ *be a bounded open set. Then for any* $\beta > 0$, $\mathbf{G}_\beta(\mathcal{O})$ *has the same homotopy type as* $\mathcal{O}$.

This has the following implications relevant for our work:

Corollary 3.3 (Homotopy Type of HA on $\mathcal{O}$) *Let* $\mathsf{HA}_\mathcal{O}$ *be the homotopical axis of a bounded open set* $\mathcal{O}$. *Then, since* $\mathsf{HA}_\mathcal{O}$ *is defined with* $\beta > 0$, $\mathsf{HA}_\mathcal{O}$ *has the same homotopy type as* $\mathcal{O}$.

Corollary 3.4 (Homotopy Type of $\mathbf{G}_\beta(\mathcal{R})$) *Since* $\mathcal{R}$ *is the union of pairwise disjoint bounded open sets, then for any* $\beta > 0$, $\mathbf{G}_\beta(\mathcal{R})$ *has the same homotopy type as* $\mathcal{R}$

Corollary 3.5 (Homotopy Type of HA) *Since* HA *is defined with* $\beta > 0$ *and* $\mathbf{G}_\beta(\mathcal{R})$ *has the same homotopy type as* $\mathcal{R}$ *(corrolary 3.4),* HA *has the same homotopy type as* $\mathcal{R}$.

Notice that while the medial axis (MA) is a complete shape descriptor, an infinite class of different smooth and non-smooth shapes can have the same homotopical axis.

If β is small enough, the closed set $\mathbf{F}_\beta(\mathcal{R})$ can also be seen as the union of connected components $\mathbf{F}_\beta(x)$ containing at least one critical $x \in \mathbf{F}_0(\mathcal{R})$ each.

The gradient on criticals is zero, consequently the steepest path starting on an arbitrary point in space stays in the critical reached first. For the definition of our new feature size, we will need the set of local maxima that are reachable by steepest ascent. In order to be able to escape critical points (where the gradient vanishes) and include maxima that can only be reached by passing other criticals, we use the following recursive definition, starting in the ϵ environment of an arbitrary point $x \in \mathbb{R}^3$:

$$\mathbf{F}^0_\beta(\mathcal{R}, x) = \{y \in \mathbf{F}_0(\mathcal{R}) \mid \forall \epsilon \in \mathbb{R}^+ \, \exists y' \in \mathbb{R}^3 : \|y' - x\| < \epsilon \wedge y = \lim_{t\to\infty} \mathfrak{C}(t, y')\}$$

$$\mathbf{F}^i_\beta(\mathcal{R}, x) = \{y \in \mathbf{F}_0(\mathcal{R}) \mid \exists y' \in \mathbf{F}^{i-1}_\beta(\mathcal{R}, x) \, \exists y'' \in \mathbf{F}_\beta(y') : y = \lim_{t\to\infty} \mathfrak{C}(t, y'')\}$$

Finally, *the set of all criticals reachable by steepest paths* starting on an arbitrary point $x \in \mathbb{R}^3$ is given by $\mathbf{F}^\infty(\mathcal{R}, x) = \lim_{\beta\to 0^+} \mathbf{F}^\infty_\beta(\mathcal{R}, x)$.

4 Sampling Criteria

The proof of correctness of previous surface reconstruction algorithms [1, 2] demanded a local sampling density based on the so-called *local feature size (*lfs*)*. The local feature size lfs(b) of a boundary point b is simply its shortest distance to the medial axis. Since the local feature size is zero at non-smooth boundary points (e.g. corners), all reconstruction algorithms which require a sampling density based on the lfs can only be applied to smooth surfaces, as they need an *infinite* number of sampling points at non-smooth surface parts.

A weaker condition on sampling density has been proposed to recover topological properties of a bounded set [6]. The so-called *weak feature size* is defined as the distance between the boundary and the set of criticals. Even though this feature size is suitable for non-smooth boundaries, the definition is still global for the whole boundary.

In [11], the sampling constraints are based on volumetric conditions of every region yielding local, variable feature sizes for boundary points:

Definition 4.1 (Local Region Size [11]) *Let $b \in \partial\mathcal{R}$ be a boundary point of $\mathcal{R}$. Let $\mathbf{H}(\mathcal{R}, b) \subseteq \mathbf{F}^\infty(\mathcal{R}, b)$ contain all local maxima of $\mathbf{F}^\infty(\mathcal{R}, b)$. Then the* local region size (lrs) *of a boundary point b is defined as:*

$$\text{lrs}(b) = \min_{y \in \mathbf{H}(\mathcal{R}, b)} \text{d}_B(y)$$

The local region size is based on the nearest center of the greatest of all maximal inscribed balls in each adjacent region. This has two advantages. First, the sampling density of corners is no longer infinite. Second, since any number of surrounding regions are taken into account, this definition is also suitable for non-manifold surfaces (i.e. junctions).

Definition 4.2 (Stable Sampling [11]) *Let $\partial\mathcal{R}$ be the boundary of a space partition $\mathcal{R} = \{R_i \subset \mathbb{R}^3\}$ and $S \subset \mathbb{R}^3$ be a finite set of points. Then S is said to be a* stable sampling *of $\partial\mathcal{R}$, if $\forall b \in \partial\mathcal{R} \, \exists s \in S : ||b - s|| < \frac{1}{2}\,\text{lrs}(b)$ and $\forall s \in S \, \exists b \in \partial\mathcal{R} : ||b - s|| < \frac{1}{2}\,\text{lrs}(b)$.*

Obviously, the volumetric information between the local maxima in each region is lost in samplings based on the local region size. Consequently, the surface parts corresponding to narrowings of the solid are undersampled.

We extend lrs and introduce the *local homotopical feature size* (lhfs) based on the homotopical axis. In particular, we take the smallest distance value of the *reachable* local maxima and the distance to HA into account (cf. arrows in Fig. 1):

Definition 4.3 (Local Homotopical Feature Size) *Let $b \in \partial\mathcal{R}$ be a boundary point of $\mathcal{R}$ and $x \in \mathsf{HA}$ be its nearest point in* HA*. Then the* local homotopical feature size *of a boundary point b is defined as*

$$\text{lhfs}(b) = \min\left(\|b - x\|, \min_{y \in \mathbf{H}(\mathcal{R},b)} (\mathrm{d}_B(y))\right)$$

Reconstruction algorithms which require a sampling based on the local homotopical feature size will then be able to handle a locally adaptive sampling density and non-smooth shapes.

Now we define the sampling conditions in such a way that all sampling points are covered by the lhfs-*dilation* of the boundary. The lower bounds on the sampling density affect the mesh construction and obviously restrict the edges to a certain size.

Definition 4.4 (Locally Stable Sampling) *Let $\partial\mathcal{R}$ be the boundary of a space partition and $S \subset \mathbb{R}^3$ be a finite set of points. Then S is said to be a* locally stable sampling *of $\partial\mathcal{R}$, iff $\forall b \in \partial\mathcal{R}\, \exists s \in S : \; ||b - s|| < \frac{1}{2}\,\text{lhfs}(b)$ and $\forall s \in S\, \exists b \in \partial\mathcal{R} : \; ||b - s|| < \frac{1}{2}\,\text{lhfs}(b)$.*

5 Refinement Reduction

In this section we recall the definition of the "refinement reduction" algorithm introduced in [11]. Let $S \subset \mathbb{R}^3$ be a finite set of points. Then the convex hull of up to four points $s_0, ..., s_n \in S$ is called an *n-simplex*. Any simplex σ_1 based on the convex hull of a subset of the points defining the simplex σ_2, is called a *face* of σ_2 and σ_2 is called a *coface* of σ_1. A face and a coface a called *proper* if their dimensions differ by exactly one. A simplex is called *centered*, if it contains its circumcenter. A simplex is called *equivocal*, if its circumball contains an other point of S. Notice, these definitions are equivalent to the definitions given in [8]. If r_σ and r_τ are the circumradii of σ and τ then we write $\sigma < \tau$ iff $r_\sigma < r_\tau$. Now a *(simplicial) complex* $\mathcal{K}$ is a set of simplices such that any face of a simplex in $\mathcal{K}$ is also a simplex in $\mathcal{K}$.

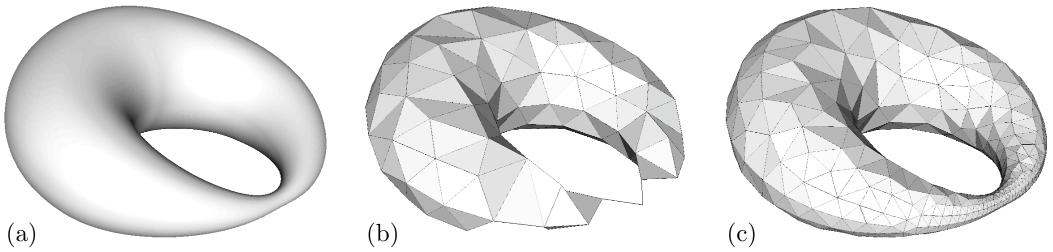
Now, if each region of the space partition represents an object of the real world, the reconstruction task is to reconstruct a second space partition from a discrete set of sampling points of the first partition, such that the two partitions share as much as possible properties of the objects and the relations between them. The following definitions are adapted from [11].

Definition 5.1 (Reconstruction) *Let $\mathcal{K}$ be a simplicial complex based on a set of points $S \in \mathbb{R}^3$. Then a* simplicial complex partition *$\mathcal{D}$ is a set of disjoint subsets D_i of $\mathcal{K}$, such that the regions $|D_i|$ covered by the sets D_i define a space partition $|\mathcal{D}| := \{|D_i|\}$. In case of $\mathcal{K}$ being a Delaunay triangulation, the subcomplex $\partial\mathcal{D} \subset D$, $\partial\mathcal{D} := \mathcal{K} \setminus \bigcup_i D_i$ is called the* result of a reconstruction*. Then, $|\partial\mathcal{D}|$ is called the* reconstructed boundary*, and the pairwise disjoint components D_i* interiors of reconstructed regions*. For each D_i, the underlying space $|D_i|$ is the* reconstructed region*. A simplex σ is called a* boundary simplex *if at least two of its cofaces lie in different interiors of reconstructed regions. Given the radii r_i and r_j of the greatest simplices in the regions $|D_i|$ and $|D_j|$ we write $|D_i| < |D_j|$ iff $r_i < r_j$,*

In order to avoid degenerate cases of the Delaunay triangulation, the points in S are assumed to be *in general position*, which means that no three points are collinear, no four points are cocircular and no five points are cospherical. In addition to that, we assume that no two triangles of the Delaunay triangulation have the same circumradius.

In order to investigate the properties of the resulting reconstruction let us first recall the refinement reduction algorithm proposed in [11]. Let a simplex be called *simple* if it has exactly one proper coface. Let a simplex σ be called *critical* if more than one proper face contains a further point of σ in its circumball. We call such faces the *critical faces*.

Let σ be a boundary simplex between two neighbouring reconstructed regions D_i, D_j, and let r_σ denote its circumradius. Further let $\tau_i \in D_i$ and $\tau_j \in D_j$ be the centered tetrahedra with greatest circumradius r_{τ_i} and r_{τ_j} respectively in the interiors of the reconstructed regions. Then σ is called an *undersampled simplex* if $2r_\sigma \geq r_{\tau_i}$ or $2r_\sigma \geq r_{\tau_j}$.



(a) (b) (c)

Figure 2: (a): original surface; (b): irreducible refinement; (c): stable refinement

The *refinement reduction* algorithm
Given a sampling S of the boundary of a partition of the space,
1. compute the 3D Delaunay triangulation $\mathcal{K}$ of S
2. Delete all centered tetrahedra from $\mathcal{K}$.
3. Delete all simple equivocal, not critical simplices.
4. Delete all simple criticals in lexicographic order with respect to their circumradius.
5. Delete all undersampled simplices lexicographically according to pairs $(r_\tau, -r_\sigma)$ in increasing order, where r_σ is the circumradius of the undersampled simplex σ, and τ is the tetrahedron with the greatest circumcenter r_τ in the reconstructed region.

6 Stability of Reconstruction

The contribution in [11] is that the refinement reduction algorithm results in a *refinement* of the original space partition:

Definition 6.1 (Refinement [11]) *Given the space partition $\mathcal{R}$, the continuous distance transform $\mathrm{d}_{\partial\mathcal{R}}$ on $\mathcal{R}$, the stable sampling S, and the discrete distance transform d_S on S, let x be the local maximum of $\mathrm{d}_{\partial\mathcal{R}}$ and $\mathbf{H}(S,x)$ be its set of reachable local maxima on d_S. Then, we call $y = \arg\max_{y' \in \mathbf{H}(S,x)} \mathrm{d}_S(y')$ the* associated discrete maximum *of x.*

The discrete complex partition $\mathcal{D}$ is called a refinement *of $\mathcal{R}$, if for any two local maxima x_1, x_2 of $\mathrm{d}_{\partial\mathcal{R}}$ lying inside different regions R_{i_1}, R_{i_2} of $\mathcal{R}$, the discrete maxima y_1, y_2 being associated to x_1, x_2 lie in different reconstructed regions $D_{i'_1}, D_{i'_2}$ of $\mathcal{D}$.*

As we may see on Fig. 2 (b), the reconstruction is a refinement of the original region Fig. 2 (a), but its topology deviates from the original. Any further deletion of a simplex destroys the neighbourhood relation. The narrowing was so sparsely sampled that the resulting mesh intersects the homotopical axis. The goal of a topologically correct reconstruction is to preserve all topological properties. So the *refinement* of a space partition must be reducible to a topological equivalent of the original space partition. The reconstruction of locally stable sampled surface based on lhfs is demonstrated in Fig. 2 (c).

Definition 6.2 (Stable Refinement) *Given a space partition $\mathcal{R}$ with boundary $\partial\mathcal{R}$ and a sampling S of $\partial\mathcal{R}$. Let HA be the homotopical axis of $\partial\mathcal{R}$. Then a refinement $\mathcal{D}$ built on S is called a* stable refinement *of $\mathcal{R}$, if the underlying space of its boundary $\partial\mathcal{D}$ does not cut HA.*

In the following, we show that the result of the refinement reduction algorithm is reducible to stable refinement if the boundary sampling is locally stable as defined in 4.4 and that a stable refinement is reducible to a topologically correct reconstruction.

The main idea of the proof is based on two facts. First, we already have a refinement which means that all maxima are correctly separated and all decreasing paths starting on the homotopical axis meet the reconstructed boundary. Second, increasing paths starting in the homotopical axis stay in the homotopical axis. The outline of the resulting proof is as follows: After we have treated the arising trivial cases in **I**, in **II** we cover the case of the homotopical axis cutting the reconstructed boundary. We show that all increasing paths from the cutting point to maxima of different reconstructed regions belong to the homotopical axis of one original region.

So, removing the cutting simplex will preserve the topological properties of the original region. Assuming the opposite will lead to a contradiction. The result of **III** is that every path to the different original region goes through the boundary and needs to be partly decreasing. In contradiction, **IV** shows that there is an increasing path between the cutting point and the maximum of neighbouring region, which falsifies the assumption.

Theorem 6.3 (Stability of the Minimal Refinement) *Let S be a locally stable sampling of $\partial\mathcal{R}$ and $\mathcal{D}$ be the result of refinement reduction with no undersampled simplices in the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. Then the boundary $\partial\mathcal{D}$ contains the boundary of a stable refinement.*

Proof: We need to show that removing all simplices of $\partial\mathcal{D}$ which cut or touch the homotopical axis result in a space partition which is still a refinement.

I The result of refinement reduction is a minimal refinement $\mathcal{D}$ without undersampled simplices. Obviously, if $\mathcal{D}$ is a stable refinement, the theorem holds. So let $\mathcal{D}$ not be a stable refinement. Let $|\partial\mathcal{D}|$ be the underlying space of $\partial\mathcal{D}$ and HA be the homotopical axis of $\partial\mathcal{R}$, then $|\partial\mathcal{D}| \cap \mathsf{HA} = X \neq \varnothing$.

Let $|D|$ and $|D'|$ be two reconstructed regions such that there is an $x \in X$ in the common boundary of $|D|$ and $|D'|$. There are two cases to consider: First, at least one reconstructed region contains no continuous maximum, then merging the reconstructed regions does not destroy the refinement condition. Thus, we only have to consider the second case: each reconstructed region contains at least one continuous maximum.

II All continuous local maxima are also local maxima of MAT ([13] Observation 2.6) and so are in HA. Let $y \in |D|$ and $y' \in |D'|$ be two nearest local maxima on HA reachable by steepest paths starting on x. We have to show that there is a path π in $|D|$ between x and y and a path π' in $|D'|$ between x and y' with π, π' entirely contained in HA.

III Let us assume that there is no such π. Since for each continuous region R the intersection $\mathsf{HA} \cap R$ is continuous (corrolary 3.5), x and y must belong to different continuous regions. It follows that any path between x and y must cross $\partial\mathcal{R}$. Therefore, for all paths $\pi_{x,y}$ in $|D|$ between x and y there is a t such that $\pi_{x,y}(t) = b \in \partial\mathcal{R}$ and $\mathrm{d}_S(b) < \frac{1}{2}\,\mathrm{lhfs}(b)$, since by definition of the locally stable sampling $\forall b' \in \partial\mathcal{R}\, \exists s \in S : \|b' - s\| < \frac{1}{2}\,\mathrm{lhfs}(b')$.

IV Let b be the nearest boundary point to x, then, since $x \in \mathsf{HA}$, $\mathrm{d}_{\partial\mathcal{R}}(x) \geq \mathrm{lhfs}(b)$ and $\mathrm{d}_S(x) \geq \frac{1}{2}\,\mathrm{lhfs}(b)$. But by construction of the refinement reduction algorithm, the circumradius of the previously deleted simplices in D are greater than $\mathrm{d}_S(x)$, and so there exists a path $\pi_{x,y}$ between x and y through the circumcenters of the deleted simplices which fulfills $\forall t \in [0,1]$: $\mathrm{d}_S(\pi(t)) \geq \mathrm{d}_S(x) \geq \frac{1}{2}\,\mathrm{lhfs}(b)$, which contradicts the previous paragraph.

Obviously the same is valid for π' and for all $x \in X$ which are also in the boundary of $|D|$ and $|D'|$. Since π and π' exist, there is a continuous path between y and y' in HA. Consequently, the local maxima of the continuous distance transform lying inside D and D' lie in the same continuous region. Then, after removing the simplex containing x, no local maxima lying in different continuous regions will lie in one reconstructed region, and the resulting discrete space partition is still a refinement. □

The consequence of theorem 6.3 is that neither elementary thinning nor undersampled merge result in a reconstructed region containing points of the homotopical axis belonging to different continuous regions.

In fact the stable refinement does not make any conclusions about the reconstructed regions which do not contain any points of HA. Such regions are often called voids, since they do not contribute to the topologically correct separation of the space. It is possible to get rid of these voids by reducing the refinement incrementally into the desired result using so-called "Euler Operators" [10].

7 Conclusions

We illustrated that the result of the "refinement reduction" algorithm does *not* correspond to a superset of a mesh which is topologically equivalent to the original boundary, even if the sampling conditions based on lrs are fulfilled. We introduced the concept of a *homotopical axis* that is a homotopy equivalent subset of the medial axis and used this to propose the *local homotopical feature size* (lhfs), for the first time supporting locally adaptive sampling of arbitrary non-manifold surfaces. Finally, we proved that the "refinement reduction" can be reduced to a topologically equivalent approximation of the original boundary under new sampling conditions based on the lhfs.

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