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Set-valued TU-games

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Abstract

The goal of this paper is to explore solution concepts for set-valued TU-games. Several stability conditions can be defined since one can have various interpretations of an improvement within the multicriteria framework. We present two different core solution concepts and explore the relationships among them. These concepts generalize the classic core solution for scalar games and can be considered under different preference structures. We give characterizations for the non-emptiness of these core sets and apply the results to four multiobjective operational research games.

Keywords: Multiobjective analysis; Game theory; Core

1. Introduction

It is currently accepted that real-world decision processes are multivalued. This assertion means that decision-making is actually based on several (more than one) criteria. Obviously, using several criteria implies the non-existence of a total order among the evaluation of the different alternatives. Thus, regarding the scalar case, where all the optimal decisions share the same evaluation, in multicriteria decision-making the above property does not make sense. In the latter case, the decision-maker may accept many different alternatives provided that their evaluations are non-dominated componentwise.

Modelling conflict situations where several criteria must be considered simultaneously leads in a natural way to multiobjective game theory (see e.g. Bergstresser and Yu, 1977; Blackwell, 1956; Hwang and Lin, 1987; Shapley, 1959). In this framework the evaluation given to the alternatives considered by the agents is not a unique value but a set of non-dominated vectors (see Fernández et al., 1998; Fernández and Puerto, 1996; Puerto and Fernández, 1995).

The discussion above leads us to consider the class of the multiobjective cooperative TU-games. Within this class any coalition S of player is given a characteristic set of vectors. These vectors represent the non-

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dominated payoffs that the members of a coalition can ensure by themselves. Notice that different from the classic scalar case, in this framework, coalitions may support any of their admissible payoffs in their characteristic set of vectors. Hence, in this class of TU-games one looks not only for fair allocations of the grand coalition's payoffs but for which of the grand coalition's payoffs the above question can be answered in an affirmative way.

When the characteristic set of vectors are singletons, we obtain the class of vector-valued games (see Fernández et al., 2002). In addition, if the number of criteria considered by the agents is only one we obtain the standard theory of cooperative TU-games.

It is also worth noting that with this class we can model any game whose characteristic set of vectors is given implicitly as the set of non-dominated vectors of a multiobjective program. In particular Operation Research games (see Borm et al., 2001) may be analyzed within this new framework when more than one objective is simultaneously considered in the optimization process. Examples are multiobjective flow games, multiobjective minimum spanning tree games, multiobjective combinatorial optimization games, etc.

In order to illustrate the discussion above, we describe in detail three different classes of set-valued TU-games: the multiobjective linear production game, the multiobjective continuous single facility location game and the multiobjective minimum cost spanning tree game. It is worth noting that the two former games come from a continuous multiobjective OR problem (the scalar version of these games were introduced by Owen (1975) and Puerto et al. (2001), respectively) while the latter does from a combinatorial one (the scalar version of this game was introduced by Bird, 1976).

1.1. The multiobjective linear production game

Consider the multiobjective linear production problem:

$$[P]\begin{array}{ll} v\text{-max} & Cx \\ \text{s.t.:} & x \in F(P) := \{x \in \mathbb{R}^p : Ax \leqslant b, x \geqslant 0\}, \end{array}$$

where $C \in \mathbb{R}^{k\times p}$ is the matrix whose rows represent the k different objectives of the problem; $A \in \mathbb{R}^{m\times p}$ is the technological matrix; $b \in \mathbb{R}^m$ is the resource vector; x is the production vector and $F(P)$ is the decision set for the problem $[P]$.

The solution concept for this problem is the set of efficient solutions:

$$\mathscr{E}(P) = \{x \in \mathbb{R}^p : \nexists y \in F(P) \text{ verifying } Cy \geqslant Cx, Cy \neq Cx\}$$

and the set of values of the efficient solutions is:

$$Z(P) = \{z(x) : z(x) = Cx, x \in \mathscr{E}(P)\}.$$

This model can be considered as a game when the pool of resources is controlled by n different agents (players). Let us assume that player i holds a resource vector $b^i = (b^i_1, b^i_2, \ldots, b^i_m)^t$, $i = 1, 2, \ldots, n$. Thus, if coalition S of players is to form it controls a bundle of resources $b(S) = \sum_{i\in S} b^i$. This vector of resources makes possible for the coalition S to produce goods according to the following linear production problem:

$$[P_S]\begin{array}{ll} v\text{-max} & Cx \\ \text{s.t.:} & x \in F(P_S) := \{x \in \mathbb{R}^p : Ax \leqslant b(S), x \geqslant 0\}. \end{array}$$

Finding the set of efficient solutions $\mathscr{E}(P_S)$ of this problem, coalition S obtains payoff vectors in the set $Z(P_S) = \{z \in \mathbb{R}^k : z = Cx, x \in \mathscr{E}(P_S)\}$.

This framework leads naturally to introduce the multiobjective linear production game with n players (agents) and where each coalition, S, can guarantee vectors in $Z(P_S)$.

1.2. The multiobjective continuous single facility location game

A continuous single facility location problem is a set of n users of a certain facility, placed in n different points in the space $\mathbb{R}^m$ with $m \geqslant 1$. The problem consists of finding a location for the facility which minimizes the transportation cost (which depends on the distances from the users to the facility) plus the setup cost. Formally, a continuous single facility location problem is a 4-tuplet (N, Φ, d, K) where:

- $N = \{a_1, \dots, a_n\}$ is a set of n different points in $\mathbb{R}^m$ (with $n \geqslant 2$).
- $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous globalizing function satisfying that: (1) Φ is definite, i.e. $\Phi(x) = 0$ if and only if $x = 0$; (2) Φ is monotone, i.e. $\Phi(x) \leqslant \Phi(y)$ whenever $x \leqslant y$.
- $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a measure of distance, satisfying that, for every $r, s \in \mathbb{R}^m$, $d(r, s) = f(\|r - s\|)$, where f is a lower semicontinuous, non-decreasing and non-negative map from $\mathbb{R}$ to $\mathbb{R}$ with $f(0) = 0$, and $\|\|$ is a norm on $\mathbb{R}^m$.
- K is the setup cost. This cost is independent of the number of users and of the location of the facility; it is mostly installation cost.

Solving the continuous single facility location problem (N, Φ, d, K) for $S \subset N$ means to find an $\bar{x} \in \mathbb{R}^m$ minimizing $\Phi(d^S(x))$, where $d^S(x)$ is the vector in $\mathbb{R}^n$ whose ith component is equal to $d(x, a_i)$ if $a_i \in S$, and equal to zero otherwise. We denote $L(S) = \min_{x \in \mathbb{R}^m} \Phi(d^S(x))$. We impose to simplify the analysis that the setup cost must be greater than or equal to the total transportation cost, i.e. $K \geqslant L(N)$.

This is the classical version of the continuous single facility location problem. Here we consider a natural variant of this problem in which the users in N are interested not only in finding an optimal location of the facility, but also in sharing the corresponding total costs.

Therefore we can associate with (N, Φ, d, K) a cost TU-game (N, v) whose characteristic function v is defined, for every $S \subset N = \{a_1, \dots, a_n\}$, by:

$$v(S) = \begin{cases} K + L(S) & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Every cost TU-game defined in this way is what we call a continuous single facility location game. If several (more than one) globalizing functions Φ_j, $j = 1, \dots, k$ are simultaneously considered then we get a set-valued TU-game. It is worth noting that in this situation $L(S) = v\text{-}\min_{x \in \mathbb{R}^m}(\Phi_1(d^S(x)), \dots, \Phi_k(d^S(x)))$. Thus the set-valued TU game (N, V) is given by $V(S) = K + L(S)$ for any $S \subset N$, and $V(\emptyset) = \{0\}$.

1.3. The multiobjective minimum spanning tree game

Consider a set of N users of some good that is supplied by a common supplier 0 ($N_0 = N \cup \{0\}$). There is a multiobjective cost associated to the distribution system that has to be divided among the users. This situation can be formulated as a set-valued game with N players and a characteristic function that associates to each coalition S a set $V(S)$ that represent the Pareto-minimum cost of constructing a distribution system among the users in S from the source 0.

Let $G = (N_0, E)$ be the complete graph with set of nodes N_0 and set of edges (links) denoted by E. There is a vector of costs associated with the use of each link. Let $e^{ij} = e^{ji} = (e_1^{ij}, e_2^{ij}, \dots, e_k^{ij})$ denote the vector-valued cost of using the link $\{i, j\} \in E$. A tree is a connected graph which contains no cycles. A Pareto-minimum cost spanning tree for a given connected graph, with costs on the edges, is a spanning tree which has Pareto-minimum costs among all spanning trees (see Ehrgott, 2000).

A Pareto-minimum cost spanning tree game, associated to the complete graph $G = (N_0, E)$, is a pair (N, V) where N is the set of player and V is the characteristic function defined by:

1. $V(\emptyset) = \{0\}$.
2. For each non-empty coalition $S \subseteq N$,

$$V(S) = \begin{array}{l} v\text{-}\min \\ T_{S_0} : \text{spanning tree} \end{array} \sum_{\{i,j\} \in E_{T_{S_0}}} e^{ij},$$

where $E_{T_{S_0}}$ is the set of edges of the spanning tree, T_{S_0}, that contains $S_0 = S \cup \{0\}$; and v-min stands for Pareto-minimization.

Remark that the resulting spanning tree T_{S_0} must contain S_0 but it may also contain some additional nodes.

To analyze multiobjective games we extend the classical individual and collective rationality principles using two different orderings in the payoff space. The first one corresponds with a compromise attitude towards negotiation where coalitions admit payoffs that are not worse in all the components than any payoffs that they can ensure by themselves. The second one, is a more restrictive ordering that only accept payoffs that get more in all the components than all payoffs that they can guarantee by themselves. Similar approaches to these two analysis have been done in Fernández et al. (2002), Jörnsten et al. (1995) and Nouweland et al. (1989) and an application can be seen in Fernández et al. (2001).

The paper is organized as follows. In the second section we introduce the definition of set-valued TU-game and the concept of allocation for those games. Moreover, we analyze two different domination relationships that extend the classic domination concept in the scalar case. In Section 3, we introduce the non-dominated allocations sets, NDA sets, and we show the relationship with the core concepts. In Section 2 we study existence theorems for these solution concepts. All the results are illustrated with three different classes of games.

2. Basic concepts

A set-valued TU-game is a pair (N, V), where $N = \{1, 2, \ldots, n\}$ is the set of players and V is a function which assigns to each coalition $S \subseteq N$ a compact subset $V(S)$ of $\mathbb{R}^k$, the *characteristic set* of coalition S, such that $V(\emptyset) = 0$.

Vectors in $V(S)$ represent the worths that the members of coalition S can guarantee by themselves. Notice that the characteristic function in these games are set-to-set maps instead of the usual set-to-point maps.

We denote by G^V the family of all the set-valued TU-games, by G^v the class of vector-valued TU-games and by g^v the family of all the scalar TU-games.

Example 2.1. Consider the following two-objective linear production problem with three decision makers (players) in which the matrix that represents the two objectives is

$$C = \begin{pmatrix} 2 & 4 \\ 1.5 & 1 \end{pmatrix}$$

and the technological matrix is

$$A = \begin{pmatrix} 1 & 7 & 7 \\ 4 & 8 & 8 \end{pmatrix}^t.$$

The resource vectors for each player are $b^1 = (14, 14, 13)^t$, $b^2 = (18, 9, 22)^t$ and $b^3 = (11, 18, 22)^t$. Then, the characteristic sets for every coalition S ($S \subset N$) are $V(S) = Z(P_S) = \text{conv}(z_1^S, z_2^S)$ (conv(A) means the convex hull of the set A):

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
z_1^S	(6.5,1.625)	(9,2.225)	(8.25,3.89)	(16.75,4,68)	(15,5.41)	(17.38,6.27)	(25,8.58)
z_2^S	(3.71,2.78)	(9,2.25)	(8,4)	(15.14,5.35)	(11.28,6.96)	(17.38,6.27)	(28.14,9.36)

Example 2.2. Let $N = \{a_1, a_2, a_3\}$ be a set of players located at the points 0, 1, 2 on the real line and assume that $0 < \varepsilon$. We consider two globalizing functions Φ_1, Φ_2 given by:

$$\Phi_1(d^N(x)) = \left(\tfrac{1}{2} - \varepsilon\right)|x - 0| + \left(\tfrac{1}{4} + \varepsilon\right)|x - 1| + \tfrac{1}{4}|x - 2|,$$
$$\Phi_2(d^N(x)) = \tfrac{1}{4}|x - 0| + \left(\tfrac{1}{4} - \varepsilon\right)|x - 1| + \left(\tfrac{1}{2} + \varepsilon\right)|x - 2|.$$

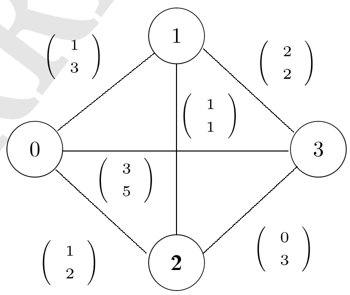
The multiobjective continuous single facility location game is given by the characteristic set $V(S) = K + L(S)$, for any $S \subseteq N$ where:

S	{1}, {2}, {3}	{1,2}	{1,3}
$L(S)$	$\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} \frac{1}{4} + \varepsilon \\ \frac{1}{4} - \varepsilon \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} (\frac{1}{4} - \varepsilon)x + \frac{1}{2} \\ (-\frac{1}{4} - \varepsilon)x + 1 + 2\varepsilon \end{pmatrix}\right\}$ for all $x \in [0,2]$

S	{2,3}	N
(S)	$\left\{\begin{pmatrix} \varepsilon x + \frac{1}{4} - \varepsilon \\ (-\frac{1}{4} - 2\varepsilon)x + \frac{3}{4} + 3\varepsilon \end{pmatrix}\right\}$ for all $x \in [1,2]$	$\left\{\left(\frac{1}{2}x + \frac{1}{4} - \varepsilon 2\varepsilon x + \frac{3}{4} + 3\varepsilon\right)\right\}$ for all $x \in [1,2]$

The reader may notice that $L(S)$ are the non-dominated values of the corresponding bicriteria location problems, i.e. $L(S) = v - \min(\Phi_1(d^S(x)), \Phi_2(d^S(x)))$.

Example 2.3. Consider the complete graph below.



The bi-criteria Pareto-minimum cost spanning tree game associated to the graph is:

S	{1}	{2}	{3}	{2,3}	{1,2}	{1,3}	N
$V(S)$	$\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$		$\left\{\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix}\right\}$

If a set-valued TU-game is played then an interesting question is how an achievable vector $v_N \in V(N)$ should be divided among the players. It is worth noting that this is the same situation that appears in scalar TU-games, where the worth of $v(N) \in \mathbb{R}$ has to be allocated among the players. Nevertheless, in the set-valued case there are many elements that can be considered to be divided among the players.

The extension of the idea of allocation used in scalar games to set-valued TU-games consists of using a payoff matrix (an element of $\mathbb{R}^{k\times n}$) whose rows are allocations of the criteria. Since the payoffs are vectors, the allocations in these games are matrices X with k rows (criteria) and n columns (players). The ith column, X^i, in matrix X represents the payoffs of ith player for each criteria; therefore $X^i = (x_1^i, x_2^i, \dots, x_k^i)^t$ are the payoffs for player i. The jth row, X_j, in matrix X is an allocation among the players of the total amount obtained in each criteria; $X_j = (x_j^1, x_j^2, \dots, x_j^n)$ are the payoffs corresponding to criteria j for each player. The sum $X^S = \sum_{i\in S} X^i$ is the overall payoff obtained by coalition S.

Matrix X is an allocation of the game $(N, V) \in G^V$ if $X^N = \sum_{i\in N} X^i \in V(N)$. The set of the allocations of the game is denoted by $I^*(N, V)$.

3. Dominance and core concepts

An important point in the development of set-valued TU-games is the use of the new orderings defined in the set of allocations. To this end, we must replace the complete order "$\leqslant$" in $\mathbb{R}$, for the comparison between allocations and the characteristic sets, by the considered orderings in $\mathbb{R}^k$, that is, "be better or equal componentwise", denoted by "$\geqq$", and "not be worse", denoted by "$\nleq$".

To simplify the presentation in the following, $X^S \nleq V(S)$ means $X^S \nleq v^S \ \forall v^S \in V(S)$, that is, there does not exist $v^S \in V(S)$ such that $X^S \leqq v^S$, $X^S \neq v^S$. Analogously $X^S \geqq V(S)$ means $X^S \geqq v^S \ \forall v^S \in V(S)$, that is, $X_j^S \geqslant v_j^S \ \forall j = 1, 2, \dots, k, \ \forall v^S \in V(S)$.

These orderings, above defined, lead us to two different core concepts in set-valued TU-games. When the ordering is defined as "$\nleq$", we have the following definition of core:

Definition 3.1. The *dominance core* of a game $(N, V) \in G^V$ is the set of allocations, $X \in I^*(N, V)$, such that $X^S \nleq V(S) \ \forall S \subset N$. We will denote this set as $C(N, V; \nleq)$.

Nevertheless, it may happen that in some situations the preference structure assumed by the agents is stronger, and coalitions only accept allocations if they get more than the worth given by the characteristic set. This assumption modifies the rationale of the decision process under the game and, therefore, the core concept will be modified accordingly. Proceeding similarly, we introduce now the concept of core with respect to the strong ordering, that we will call the *preference core*.

Definition 3.2. The *preference core* of a game $(N, V) \in G^V$ is the set of allocations, $X \in I^*(N, V)$, such that $X^S \geqq V(S) \ \forall S \subset N$. We will denote this set as $C(N, v; \geqq)$.

The preference core is always included in the dominance core. Thus, it may happen that the former set is empty while the latter set is not. Nevertheless, if the preference core is non-empty then the players will only agree on allocations within this set because all the players will be better off without assuming any compromise. Therefore, this solution concept must be considered in any set-valued game provided that we are given tools to check whether it is non-empty.

The dominance core defined above coincides with the set of stable outcomes (SO) introduced by van den Nouweland et al. (1989). Thus, our treatment is similar to that of these authors although our characterization is different. In addition, we characterize the preference core, a concept not considered in the above mentioned paper.

Example 3.1. Let us assume a production situation where three agents can produce, using three different technologies A, B, C, two types of goods. The characteristic set of any coalition S is given by the production levels of each good using the existing technologies, i.e. $V(S)$ is a set of three vectors (technologies) with two components each one (goods). The following table defines the characteristic set-valued map of the game (N, V).

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
A		(1/2,1)		(5/2,3/2)	(1,2)		(5,4)
B		(1,1/2)		(2,2)	(2,1)		(6,3)
C		(4/5,3/4)		(3,1)	(3/2,4/3)		(3,6)

If the agents decide to cooperate and to produce with the technology A they must allocate the vector of goods (5,4), the allocation

$$X = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

is in the preference core, while

$$Y = \begin{pmatrix} 1 & 3/2 & 5/2 \\ 3/2 & 3/2 & 1 \end{pmatrix}$$

is in the dominance core and not in the preference core since $Y^{\{1,2\}} = (5/2, 3) \ngeqq (3, 1)$, the third element of the characteristic set $V(\{1, 2\})$.

Imputations in the core (any of them) will be acceptable if no coalition can argue against its allocated amount X^S. To this end, we use the following dominance concepts, where $\mathbb{R}^k_{\geqq}$ stands for $\{x \in \mathbb{R}^k : x \geqslant 0\}$.

Definition 3.3. Let us consider two matrices $X, Y \in \mathbb{R}^{k\times n}$ and a coalition $S \in N$.

1. Y dominates X through S according to $\nleq$, and we will denote $Y \overset{S}{\text{dom}}_{\nleq} X$, if:
   (a) $Y^S \nleq X^S$, $Y^S \neq X^S$,
   (b) $Y^S \in V(S) - \mathbb{R}^k_{\geqq}$.
2. Y dominates X through S according to $\geqq$, and we will denote $Y \overset{S}{\text{dom}}_{\geqslant} X$, if:
   (a) $Y^S \geqq X^S$, $Y^S \neq X^S$,
   (b) $Y^S \in V(S) - \mathbb{R}^k_{\geqq}$.

In scalar TU-games the set of non-dominated imputations has been widely considered (see Driessen, 1988 and the references therein). Nevertheless, in set-valued TU-games the concept which plays the important role is the NDA set. These sets are defined by:

1. $NDA(N, V; \nleq) = \{X \in I^*(N, V)\text{such that} \nexists S \subset N,\ Y \in I^*(N, V),\ Y \overset{S}{\text{dom}}_{\nleq} X\}$,
2. $NDA(N, V; \geqq) = \{X \in I^*(N, V)\text{such that} \nexists S \in N,\ Y \in I^*(N, V),\ Y \overset{S}{\text{dom}}_{\geqslant} X\}$.

Our following result proves that both core sets are sets of non-dominated allocations.

Theorem 3.1. *The core sets hold the following properties*:

1. $C(N, V; \geqq) = NDA(N, V; \nleq)$,
2. $C(N, V; \nleq) = NDA(N, V; \geqq)$.

Proof. We only prove 1. the proof of 2. being similar.

1. $\Rightarrow$ Suppose that $X \underset{S}{\in} C(N, V; \geqq)$ and that $X \notin NDA(N, V; \nleq)$. Then there exists $S \subset N$ and $Y \in I^*(N, V)$, such that $Y \mathrm{dom}_{\nleq} X$, that is, $Y^S \nleq X^S$, $Y^S \neq X^S$ and $Y^S \in V(S) - \mathbb{R}^k_{\geqq}$, but it is not possible because $X^S \geqq V(S)$.

$\Leftarrow$ Suppose that $X \in NDA(N, V; \nleq)$ and that $X \notin C(N, V; \geqq)$. Then, there exists $S \subset N$ and $v^S \in V(S)$, such that X^S is not better componentwise then v^S, that is, $v^S \nleq X^S$. Now let us construct an allocation, Y, of v^S as follows:

$$Y^i = \begin{cases} \frac{v^S}{|S|} & \forall i \in S, \\ \mathbf{0} & \forall i \notin S. \end{cases}$$

Allocation Y of $v^S \in V(S)$ dominates allocation X through coalition S according to $\nleq$ because $Y^S = v^S \nleq X^S$ and $Y^S \in V(S)$. Hence, it contradicts that $X \in NDA(N, V; \nleq)$. □

4. Existence theorems

Once, we have defined the two core concepts and their relationships it is important to give conditions that ensure non-emptiness of these cores.

4.1. Dominance core

For each scalarized vector $\lambda \in \Lambda$,

$$\Lambda = \left\{ \lambda \in \mathbb{R}^k, \lambda_j > 0, j = 1, \ldots, k \text{ such that } \sum_{j=1}^{k} \lambda_j = 1 \right\}$$

and any game $(N, V) \in G^V$, we define the scalar game $(N, v_\lambda) \in g^v$ as:

$$v_\lambda(\emptyset) = 0, \quad v_\lambda(S) = \max_{v^S \in V(S) - \mathbb{R}^k_{\geqq}} \lambda^t v^S, \ \forall S \subseteq N, \quad S \neq \emptyset. \tag{1}$$

Using the game defined in (1) we establish a sufficient condition for the non-emptiness of the *dominance core*.

Theorem 4.1. *The core $C(N, V; \nleq)$ of the game $(N, V) \in G^V$ is non-empty if there exists $\hat{\lambda} \in \Lambda$ such that the scalar game $(N, v_{\hat{\lambda}}) \in g^v$ is balanced and it satisfies $v_{\hat{\lambda}}(N) \neq 0$.*

Proof. Let it $\hat{\lambda}$ be a weight in Λ such that the scalar game $(N, v_{\hat{\lambda}}) \in g^v$, defined in (1), is balanced and verify $v_{\hat{\lambda}}(N) \neq 0$. Consider $z^S \in (V(S) - \mathbb{R}^k_{\geqq})$ such that $\hat{\lambda}^t z^S = v_{\hat{\lambda}}(S)\ \forall S \subseteq N$. Notice that $z^S \in V(S)$, otherwise it is possible to find another vector $v^S \in V(S)$ such that $z^S \leqq v^S$, $z^S \neq v^S$, and then $\hat{\lambda}^t z^S < \hat{\lambda}^t v^S$. By Bondareva and Shapley theorem (see Bondareva, 1963) there exists an allocation $x \in C(N, v_{\hat{\lambda}})$.

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Now consider the matrix $X \in \mathbb{R}^{k \times n}$ whose columns are:

$$X^i = \frac{x^i}{v_{\hat{\lambda}}(N)} z^N \quad \forall i \in N.$$

Since $v_{\hat{\lambda}}(N) \neq 0$, we prove that $X \in C(N, V; \not\leq)$. Indeed,

$$X^N = \sum_{i=1}^{n} \frac{x^i}{v_{\hat{\lambda}}(N)} z^N = z^N$$

and then $X \in I^*(N, V)$. Assume that $X \notin C(N, V; \not\leq)$. Then, there exists a coalition $S \subset N$ and a vector $w^S \in V(S)$ such that $X^S \leqslant w^S$, $X^S \neq w^S$, that is, $\hat{\lambda}^t X^S < \hat{\lambda}^t w^S$. Then:

$$\max_{v^S \in V(S) - \mathbb{R}^k_{\geqq}} \hat{\lambda}^t v^S \geqslant \hat{\lambda}^t w^S > \hat{\lambda}^t X^S = \sum_{i \in S} \hat{\lambda}^t X^i = \frac{\sum_{i \in S} x^i}{v_{\hat{\lambda}}(N)} \hat{\lambda}^t z^N = x^S \geqslant v_{\hat{\lambda}}(S) = \max_{v^S \in V(S) - \mathbb{R}^k_{\geqq}} \hat{\lambda}^t v^S.$$

This is a contradiction. □

This results is useful in finding elements in the dominance core of different set-valued games.

4.1.1. Multiobjective linear programming games

The set-valued characteristic function is usually defined through the set of non-dominated values of a multiobjective programming problem. A particular case of these games are the Multiobjective Linear Production Games. These games are characterized because the objective functions of the multiobjective program are linear. In this situation we can obtain an allocation of the dominance core for any $z = Cx \in V(N)$. Indeed, given $z^* = Cx^* \in V(N)$, it is well-known that there exists a vector of weights $\hat{\lambda} \in \mathbb{R}^k$, $\hat{\lambda} > 0$, such that x^* is the solution of the scalar problem:

$$[P_N(\hat{\lambda})] \begin{array}{ll} \max & \hat{\lambda}^t C x \\ \text{s.t.:} & x \in F(P_N). \end{array}$$

Let u^* be an optimal solution of the dual problem of $[P_N(\hat{\lambda})]$. The matrix $X^* = (X^1, X^2, \ldots, X^n)$ whose columns are $X^i = (u^* b^i / \hat{\lambda}^t z^*) z^*$ belongs to the dominance core. This follows from Theorem 4.1. Notice that X^* is an allocation of z^*.

We note in passing that the choice of $z \in V(N)$ can be done taking a weighting vector $\lambda > 0$. Procedures guiding the agents to the choice of weighting vectors are described in Marmol et al. (2002) and the references therein.

Example 2.1 (*continued*). Let us take $\hat{\lambda} = (0.8, 0.2)$. The problem $P_N(\hat{\lambda})$ is:

$$\begin{array}{ll} \max & 1.9x_1 + 3.4x_2 \\ \text{s.a.:} & x_1 + 8x_2 \leqslant 43;\; 7x_1 + 4x_2 \leqslant 41;\; 7x_1 + 8x_2 \leqslant 57;\; x_1, x_2 \geqslant 0. \end{array}$$

An optimal solution of $P_N(\hat{\lambda})$ is $x_1 = (2.\hat{3}, 5.08\hat{3})$ with objective value $z_1 = (25, 8.58\hat{3})$. An the optimal solution of the dual of $P_N(\hat{\lambda})$ is $u^* = (0.179167, 0, 0.245833)$. The allocation in the *dominance core* obtained by the above method, is:

$$X^* = \begin{pmatrix} 6.567 & 9.938 & 8.495 \\ 2.254 & 3.412 & 2.917 \end{pmatrix}.$$

4.1.2. Multiobjective continuous single facility location games

For this class of games we can provide a method to construct allocations in the dominance core. The approach consists of applying Theorem 4.1 transforming the multiobjective game into a scalar continuous single facility location game. Conditions for the non-emptiness of the corresponding core set are given in Puerto et al. (2001).

Example 2.2 (*continued*)**.** We apply Theorem 4.1 with $\lambda = 1/2$. Thus, we obtain the corresponding scalar game whose characteristic function $v_\lambda(S) = K + L(S)$ where $L(S)$ is given by:

S	{1},{2},{3}	{1,2}	{1,3}	{2,3}	N
$L(S)$	0	1/4	$3/4 - \varepsilon$	1/4	3/4

According to Puerto et al. (2001) the egalitarian allocation $(K/3 + 1/4, K/3 + 1/4, K/3 + 1/4)$ belongs to the core of this scalar game. Therefore, the egalitarian allocation of the vector

$$\begin{pmatrix} K + 5/4 - \varepsilon \\ K + 3/4 - \varepsilon \end{pmatrix},$$

that corresponds to the non-dominated value in $V(N)$ for $x = 2$, belongs to the dominance core.

4.1.3. Multiobjective minimum cost spanning tree games

We can provide a method to obtain allocations in the dominance core. A way to deal with this problem is using topological orders in $\mathbb{R}^k$. As was shown in Ehrgott (2000), every Pareto optimal spanning tree of a graph is a conventional mcst using the appropriate topological order. Restricting to topological orders induced by an increasing linear utility function, the mcst obtained from the weighted graph is a Pareto optimal tree.

In order to find a condition that permits to divide among the players a total cost $z^N \in V(N)$ accordingly with a given strictly increasing linear utility function, u, we will define the following scalar game (N, v_u):

$$v_u(\emptyset) = 0, \quad v_u(S) = \min_{z^S \in V(S)} u(z^S), \quad \forall S \subseteq N, S \neq \emptyset.$$

Using any allocation in the core of the game (N, v_u), we can construct dominance core allocations for some $z^N \in V(N)$.

Let $x = (x^1, \ldots, x^n)$ be the Bird's allocation of the game (N, v_u) (see Bird, 1976). This vector allows us to give a proportional allocation of $z^N \in V(N)$ defined by:

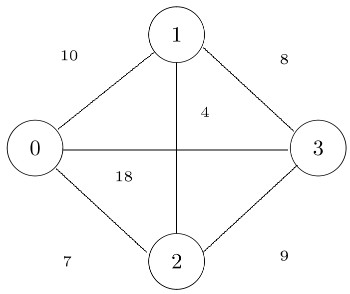
$$X = (X^1, \ldots, X^n), \quad \text{where } X^i = \frac{x^i}{u(z^N)} z^N \; \forall i \in N.$$

This allocation belongs to the dominance core by Theorem 4.1.

Example 2.3 (*continued*)**.** Suppose that the strictly increasing linear utility function, u, used to compare the worth of the coalitions consists of giving triple importance to the second criterion, that is, the utility of vector a is $u(a) = a_1 + 3a_2$. Then, the scalar game (N, v_u) is:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v_u(S)$	10	7	6	11	18	16	19

In this case, $v_u(N) = u((4, 5)^t)$, the mcst for the weighted graph is the Pareto-optimal tree associated to $z^N = (4, 5)^t$ and (N, v_u) is the mcst-game associated to the weighted graph.



Therefore Bird's cost allocation $x = (4, 7, 8)$ is in the core of (N, v_u). Then the proportional allocation

$$X = \begin{pmatrix} \frac{16}{19} & \frac{28}{19} & \frac{32}{19} \\ \frac{20}{19} & \frac{35}{19} & \frac{40}{19} \end{pmatrix} \in C(N, V, \ngeqq).$$

4.2. Preference core

This section is devoted to characterize the non-emptiness of the preference core. Associated with a coalition S in the game $(N, V) \in G^V$ we consider k different scalar problems:

$$[P_S(j)] \begin{array}{ll} \max & v_j^S \\ \text{s.t.:} & v^S \in V(S) - \mathbb{R}^k_{\geqq}, \end{array}$$

where v_j^S, $j = 1, 2, \ldots, k$, is the jth component of vector v^S. The reader may notice that for cost games the corresponding problems $[P_S(j)]$ would be minimization problems.

Let us denote by $z^*(S, j)$ the value associated with an optimal solution of problem $[P_S(j)]$ and by $z^*(S)$ the k-dimensional vector $z^*(S) = (z^*(S, 1), z^*(S, 2), \ldots, z^*(S, k))$.

Notice that for a fixed coalition S if an allocation X of the set-valued TU-game, $(N, V) \in G^V$, satisfies $X^S \geqq V(S)$ then $X^S \geqq z^*(S)$ and conversely.

For each $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_k) \in V(N)$, we introduce $(N, v_j^{\hat{z}})$, the scalar j-component game, $j = 1, 2, \ldots, k$, defined as follows:

$$v_j^{\hat{z}}(\emptyset) = 0, \quad v_j^{\hat{z}}(S) = z^*(S, j) \quad \forall S \subset N \text{ and } v_j^{\hat{z}}(N) = \hat{z}_j. \tag{2}$$

A necessary and sufficient condition for the non-emptiness of the preference core is given in the next result.

Theorem 4.2. *The preference core is non-empty if and only if there exists at least one $\hat{z} \in V(N)$ such that all the scalar j-component games $(N, v_j^{\hat{z}})$ are balanced.*

Proof. If every scalar j-component game $(N, v_j^{\hat{z}})$ is balanced, consider any allocation, X_j, in the core of $(N, v_j^{\hat{z}})$, $j = 1, 2, \ldots, k$. Then, the $k \times n$-matrix X whose rows are X_j, $j = 1, 2, \ldots, k$, is an allocation associated with $\hat{z}$. Moreover, for each $S \subset N$, $X^S \geqq z^*(S)$ and $X^S \geqq V(S)$.

Conversely, let X be an allocation in the preference core such that $X^N = \hat{z} \in V(N)$. Then $X^S \geqq V(S)$, $\forall S \subset N$ and $X^S \geqslant z^*(S)$, $\forall S \subset N$. Therefore, X_j is an allocation in the core $(N, v_j^{\hat{z}})$. $\square$

We can also give a similar but refined sufficient condition. Let $\bar{z}$ be a k-dimensional vector not necessarily in $V(N)$ and consider the scalar game $(N, v_j^{\bar{z}})$ as defined above.

Corollary 4.1. *If* $(N, v_j^{\bar{z}})$ *is balanced for any* $j = 1, 2, \ldots, k$ *and there exists* $\hat{z} \in V(N)$ *such that* $\hat{z} \geqq \bar{z}$, *then there exist allocations associated with* $\hat{z}$ *in the preference core.*

Example 4.1. Consider the following bi-objective linear production game with three players in which the matrix that represents the two objectives is

$$C = \begin{pmatrix} 2.5 & 5 \\ 3 & 2 \end{pmatrix}$$

the technological matrix is

$$A = \begin{pmatrix} 2 & 9 \\ 6 & 4 \\ 8 & 9 \end{pmatrix}$$

and the resource vectors for the players are:

$$b^1 = (400, 5, 35)^t,$$

$$b^2 = (15, 400, 35)^t,$$

$$b^3 = (15, 5, 500)^t.$$

In this case all the vectors in $V(N)$ can be allocated within the preference core. Let us consider the vector $z = (192, 155.2)$ that is a vector less or equal than all the vectors in $V(N)$. It is easy to prove that the game (N, v_1^z) defined as:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v_1^z(S)$	6.3	13	6.25	38.9	12.5	37.5	192

and the game (N, v_2^z) is defined as:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v_2^z(S)$	2.5	13	2.5	26.3	5	45	155.2

are balanced. Therefore, since $z = (192, 155.2) \leqslant \hat{z}\ \forall \hat{z} \in V(N)$ we can obtain allocations in the preference core for all vectors in $V(N)$, using Corollary 4.1.

In order to obtain an allocation, for instance, of vector $\hat{z} = (192, 205) \in V(N)$, we search for vectors in the core of the corresponding component games.

Vector $X_1 = (60, 60, 72)$ is in the core of the game $(N, v_1^{\hat{z}})$:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v_1^z(S)$	6.3	13	6.25	38.9	12.5	37.5	192

Vector $X_2 = (70, 70, 65)$ is in the core of the game $(N, v_1^{\hat{z}})$:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v_2^{\hat{z}}(S)$	2.5	13	2.5	26.3	5	45	205

Therefore, the matrix

$$Y = \begin{pmatrix} 60 & 60 & 72 \\ 70 & 70 & 65 \end{pmatrix}$$

is an allocation of the vector $(192, 205)$ in the preference core.

Although the example above shows that every $z \in V(N)$ can be allocated within the preference core, there are also cases where this is not possible.

Example 2.1 (*continued*)**.** Consider the two scalar 1,2-component games defined in (2):

The scalar 1-component game is:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v(S)$	6.5	9	8.25	16.75	15	17.38	$v_1^{\hat{z}}$

The scalar 2-component game is:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v(S)$	2.786	2.25	4	5.357	6.964	6.269	$v_2^{\hat{z}}$

It is easy to see that the first scalar component games is not balanced for any $\hat{z} \in V(N)$. Therefore the preference core in this game is empty by Theorem 4.2.

Example 2.2 (*continued*). Let us fix the setup cost $K = 3$. Consider the two component games obtained for the non-dominated value $V(N)$ with $x = 2$:

$$\begin{pmatrix} 17/4 - \varepsilon \\ 15/4 - \varepsilon \end{pmatrix}.$$

The scalar 1-component game is:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v(S)$	3	3	3	$\frac{13}{4} + \varepsilon$	$\frac{7}{2}$	$\frac{13}{4}$	$\frac{17}{4} - \varepsilon$

The scalar 2-component game is:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	N
$v(S)$	3	3	3	$\frac{13}{4} - \varepsilon$	$\frac{7}{2}$	$\frac{13}{4} - \varepsilon$	$\frac{15}{4} - \varepsilon$

The reader can check that the scalar component games are balanced and the allocation

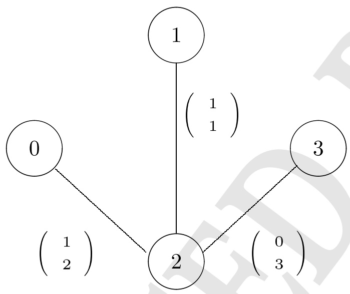
$$\begin{pmatrix} \frac{17}{12} & \frac{17}{12}-\varepsilon & \frac{17}{12} \\ 1 & \frac{7}{4}-\varepsilon & 1 \end{pmatrix}$$

belongs to the preference core.

Example 2.3 (*continued*). In this example, we can allocate $(2,6)^t \in V(N)$ by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

that is in the preference core. This allocation has been obtained applying Bird's rule to the Pareto-minimum tree given in the following figure.



It is worth noting that there are classes of OR games for which the preference core is always non-empty. This is the case of the so called *Multiobjective maintenance games* (see Borm et al., 2001 for the definition of the scalar game). These games consist of a multiobjective minimum cost spanning tree game where the underline graph G is a tree. In this case any proportional allocation rule, as for instance Bird's rule, always belongs to the preference core.

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