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THREE DIMENSIONAL SYSTEM OF GLOBALLY MODIFIED NAVIER-STOKES EQUATIONS WITH INFINITE DELAYS

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Abstract. Existence and uniqueness of solution for a globally modified version of Navier-Stokes equations containing infinite delay terms are established. Moreover, we also analyze the stationary problem and, under suitable additional conditions, we obtain global exponential decay of the solutions of the evolutionary problem to the stationary solution.

Keywords: Globally Modified Navier-Stokes Equations; infinite delays.
Mathematics Subject Classifications (2000): 35K55, 35Q30, 34D23, 34K20

1. Introduction and statement of the problem.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary Γ, and let $N \in (0, +\infty)$ be fixed. Let us define $F_N : [0, +\infty) \to (0, 1]$ by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}, \quad r \in [0, +\infty),$$

and consider the following system of globally modified Navier-Stokes equations on Ω with homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N(\|u\|)\,[(u\cdot\nabla)u] + \nabla p = f(t) \quad \text{in} \ \ (\tau, T_*) \times \Omega, \\ \nabla \cdot u \ = 0 \quad \text{in} \ \ (\tau, T_*) \times \Omega, \\ u = 0 \ \ \text{on} \ (\tau, T_*) \times \Gamma, \\ u(\tau, x) = u^0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, u the velocity field of the fluid, p the pressure, $\tau \in \mathbb{R}$ an initial time, u^0 the initial velocity field, $f(t)$ a given external force field, and $T_* \in (\tau, +\infty]$ a given final time.

The system (1) is indeed a globally modified version of the Navier-Stokes system – the modifying factor $F_N(\|u\|)$ depends on the norm $\|u\|=\|\nabla u\|_{(L^2(\Omega))^{3\times 3}}$,

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which in turn depends on ∇u over the whole domain Ω and not just at or near the point $x \in \Omega$ under consideration. Essentially, it prevents large gradients dominating the dynamics and leading to explosions. It violates the basic laws of mechanics, but mathematically the system (1) is a well defined system of equations, just like the modified versions of the Navier-Stokes equations of Leray and others with other mollifications of the nonlinear term, see the review paper [8]. It is worth mentioning that a global cut off function involving the $D(A^{1/4})$ norm for the two dimensional stochastic Navier-Stokes equations is used in [9], and a cut-off function similar to the one we will use here was considered in [23].

The system (1) was introduced and studied in [1] (see also [2, 14, 15, 16, 3] and the review paper [13]). However, there are situations in which the model is better described if some terms containing delays appear in the equations. These delays may appear, for instance, when one wants to control the system by applying a force which takes into account not only the present state but the complete history of the solutions. Therefore, in this paper we are interested in the case in which terms containing infinite delays appear. We consider the following version:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N(\|u\|)\left[(u\cdot\nabla)u\right] + \nabla p = f(t) + g(t,u_t) \quad \text{in} \quad (\tau, T_*)\times\Omega, \\ \nabla\cdot u \ = 0 \quad \text{in} \quad (\tau,T_*)\times\Omega, \\ u = 0 \ \text{ on } (\tau,T_*)\times\Gamma, \\ u(\tau+s,x) = \phi(s,x), \ \ s\in(-\infty,0], x\in\Omega, \end{cases} \tag{2}$$

where g is another external force containing some hereditary characteristic and ϕ is a given function defined in the interval $(-\infty, 0]$.

Our goal is to establish the existence and uniqueness of solution for the above problem and to study its asymptotic behaviour (for a similar goal in Navier-Stokes models with finite delay cf. [5, 6, 7], and for ODEs and PDEs with unbounded delay terms cf. [4, 18]).

The structure of the paper is the following: in Section 2 we recall some spaces useful for the abstract framework and some properties and estimates related to the operators involved in the model. In Section 3 the existence and uniqueness of solution for the (evolutionary) problem is given. The stationary problem is treated in Section 4, where we also prove that under adequate assumptions, any solution of the evolutionary problem has an exponential decay toward the stationary solution.

2. **Preliminaries.**

To set our problem in the abstract framework, we consider the following usual abstract spaces (see [17] and [21, 22]):

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u \ = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$, where for $u, v \in (L^2(\Omega))^3$,

$$(u,v) = \sum_{j=1}^3 \int_\Omega u_j(x)v_j(x)\,dx,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with scalar product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$, where for $u, v \in (H_0^1(\Omega))^3$,

$$((u,v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality pairing between V' and V. Finally, we will identify every $u \in H$ with the element $f_u \in V'$ given by

$$\langle f_u, v\rangle = (u,v) \quad \forall\, v \in V.$$

It follows that $V \subset H \subset V'$, where the injections are dense and compact.

We consider the linear continuous operator $A : V \to V'$ defined by

$$\langle Au, v\rangle = ((u,v)) \quad \forall\, u, v \in V. \tag{3}$$

Denoting $D(A) = \{u \in V : \ Au \in H\}$, with inner product $(u,v)_{D(A)} = (Au, Av)$, then, by the regularity of Γ, $D(A) = (H^2(\Omega))^3 \cap V$, and $Au = -P\Delta u, \forall\, u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H).

Let us denote

$$\lambda_1 = \inf_{v \in V\setminus\{0\}} \frac{\|v\|^2}{|v|^2} > 0,$$

the first eigenvalue of the Stokes operator.

Now we define the trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j\, dx, \quad \forall\, u, v, w \in V,$$

and we denote

$$b_N(u,v,w) = F_N(\|v\|) b(u,v,w), \quad \forall\, u, v, w \in V.$$

The form b_N is linear in u and w, but it is nonlinear in v. Evidently we have $b_N(u,v,v) = 0$, for all $u, v \in V$. We will also make use of the following inequality (see [21] and [10])

$$|b(u,v,w)| \le 2^{-1}|u|^{1/4}\|u\|^{3/4}\|v\||w|^{1/4}\|w\|^{3/4}, \quad \forall\, u, v, w \in V. \tag{4}$$

In particular, this implies that there exists a constant $C_1 > 0$ only dependent on Ω (namely, $C_1 = (2\lambda_1^{1/4})^{-1}$) such that

$$|b(u,v,w)| \le C_1 \|u\|\|v\|\|w\|, \quad \forall\, u, v, w \in V.$$

Thus by the definition of F_N, if we denote

$$\langle B_N(u,v), w\rangle = b_N(u,v,w), \quad \forall\, u, v, w \in V,$$

we have

$$\|B_N(u,v)\|_* \le NC_1 \|u\|, \quad \forall\, u, v \in V. \tag{5}$$

We recall (see [21]) that there exists a constant $C_2 > 0$ depending only on Ω such that

$$|b(u,v,w)| \le C_2\|u\|^{1/2}|Au|^{1/2}\|v\||w|, \tag{6}$$

for all $u \in D(A), v \in V, w \in H$, and

$$|b(u,v,w)| \le C_2\|u\|\|v\||w|^{1/2}\|w\|^{1/2}, \tag{7}$$

for all $u, v, w \in V$. (See [20] for the proof of (7)).

Let X be a Banach space. The notation $B_X(a,r)$ will be used to denote the open ball of center a and radius r in the space X. Given a function $u:(-\infty,T_*)\to X$, for each $t<T_*$ we denote by u_t the function defined on $(-\infty,0)$ by the relation $u_t(s)=u(t+s), s\in(-\infty,0)$.

One possibility to deal with infinite delays, and which we will use here (cf. [18, 11, 12]), is to consider, for any $\gamma>0$, the space :

$$C_\gamma(H)=\left\{\varphi\in C((-\infty,0];H)\ :\ \exists\lim_{s\to-\infty}e^{\gamma s}\varphi(s)\in H\right\},$$

which is a Banach space with the norm

$$\|\varphi\|_\gamma:=\sup_{s\in(-\infty,0]}e^{\gamma s}|\varphi(s)|.$$

In order to state the problem in the correct framework, let us first establish some initial assumptions on some terms in the equation:

We will assume that $f\in L^2(\tau,T;(L^2(\Omega))^3)$ for all $T\in(\tau,T_*]\cap\mathbb{R}$.

For the term g, in which the delay is present, we assume that $g:(\tau,T_*)\times C_\gamma(H)\to(L^2(\Omega))^3$ satisfies

($g1$) For any $\xi\in C_\gamma(H)$ the mapping $(\tau,T_*)\ni t\mapsto g(t,\xi)$ is measurable,

($g2$) $g(t,0)=0$ for all $t\in(\tau,T_*)$,

($g3$) there exists a constant $L_g>0$ such that for any $t\in(\tau,T_*)$ and all $\xi,\eta\in C_\gamma(H)$,

$$|g(t,\xi)-g(t,\eta)|\le L_g\|\xi-\eta\|_\gamma.$$

Remark 1. (i) Condition (g2) is not really a restriction, since otherwise, if $|g(\cdot,0)|\in L^2(\tau,T)$ for all $T\in(\tau,T_*]\cap\mathbb{R}$, we could redefine $\hat f(t)=f(t)+g(t,0)$ and $\hat g(t,\cdot)=g(t,\cdot)-g(t,0)$. In this way the problem is exactly the same and $\hat f$ and $\hat g$ satisfy the required assumptions.

(ii) Conditions ($g2$) and ($g3$) imply that

$$|g(t,\xi)|\le L_g\|\xi\|_\gamma,$$

so that $|g(\cdot,\xi)|\in L^\infty(\tau,T_*)$.

We will denote P_m the orthogonal projector of H onto the vector space generated by the first m eigenfunctions of the Stokes problem in Ω with homogeneous Dirichlet boundary conditions.

An example of operator satisfying assumptions ($g1$)-($g3$) is given here.

Example 1. We consider the operator $g:(\tau,T_*)\times C_\gamma(H)\to(L^2(\Omega))^3$ defined as follows:

$$g(t,\xi):=\int_{-\infty}^0 G(t,s,\xi(s))ds\quad\forall t\in(\tau,T_*),\forall\xi\in C_\gamma(H),$$

where the function $G:(\tau,T_*)\times(-\infty,0)\times\mathbb{R}^3\to\mathbb{R}^3$ satisfies the following assumptions:

(a) $G(t,s,0)=0$ for all $(t,s)\in(\tau,T_*)\times(-\infty,0)$.

(b) There exists a function $\kappa:(-\infty,0)\to(0,+\infty)$ such that

$$\begin{aligned}\|G(t,s,u)-G(t,s,v)\|_{\mathbb{R}^3}\ &\le\ \kappa(s)\|u-v\|_{\mathbb{R}^3}\\ &\qquad\forall u,v\in\mathbb{R}^3,\ \forall(t,s)\in(\tau,T_*),\times(-\infty,0),\end{aligned}$$

(c) and the function κ satisfies that $\kappa(\cdot)e^{-(\gamma+\epsilon)\cdot} \in L^2((-\infty,0))$ for some $\epsilon > 0$.

Namely, the operator g defines an element of $(L^2(\Omega))^3$ in the following way:

$$g(t,\xi)(x) = \int_{-\infty}^{0} G(t,s,\xi(s)(x))ds \quad \forall x \in \Omega.$$

We check now that g satisfies the assumption $(g3)$, and using (a) above, we obtain that it is well defined as a map with values in $(L^2(\Omega))^3$:

$$\begin{aligned}
& \int_\Omega \left(\int_{-\infty}^{0} \kappa(s) \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^3} ds \right)^2 dx \\
= & \int_\Omega \left(\int_{-\infty}^{0} \kappa(s) e^{-(\gamma+\epsilon)s} e^{(\gamma+\epsilon)s} \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^3} ds \right)^2 dx \\
\leq & \int_\Omega \left(\int_{-\infty}^{0} \kappa^2(s) e^{-2(\gamma+\epsilon)s} ds \right) \left(\int_{-\infty}^{0} e^{2(\gamma+\epsilon)s} \|\xi(s)(x) - \eta(s)(x)\|^2_{\mathbb{R}^3} ds \right) dx \\
= & \left(\int_{-\infty}^{0} \kappa^2(s) e^{-2(\gamma+\epsilon)s} ds \right) \int_\Omega \left(\int_{-\infty}^{0} e^{2(\gamma+\epsilon)s} \|\xi(s)(x) - \eta(s)(x)\|^2_{\mathbb{R}^3} ds \right) dx \\
= & C_\kappa \int_{-\infty}^{0} \int_\Omega e^{2(\gamma+\epsilon)s} \|\xi(s)(x) - \eta(s)(x)\|^2_{\mathbb{R}^3} dx ds \\
\leq & C_\kappa \left[\sup_{s \in (-\infty,0]} e^{2\gamma s} \int_\Omega \|\xi(s)(x) - \eta(s)(x)\|^2_{\mathbb{R}^3} dx \right] \int_{-\infty}^{0} e^{2\epsilon s} ds \\
= & C_\kappa \|\xi - \eta\|^2_\gamma \frac{1}{2\epsilon} \\
= & L_g \|\xi - \eta\|^2_\gamma,
\end{aligned}$$

where we have denoted $C_\kappa = \|\kappa(\cdot)e^{-(\gamma+\epsilon)\cdot}\|^2_{L^2(-\infty,0)}$ and $L_g = C_\kappa/(2\epsilon)$.

3. **Existence of solutions.** In this section we establish existence of solution for (2) by a compactness method using a Faedo-Galerkin scheme.

Definition 1. A weak solution of (2) is a function $u \in C((-\infty,T];H) \cap L^2(\tau,T;V)$ for all $T \in (\tau,T^*] \cap \mathbb{R}$, with $u_\tau = \phi$ and such that for all $v \in V$,

$$\frac{d}{dt}(u(t),v) + \nu((u(t),v)) + b_N(u(t),u(t),v) = (f(t),v) + (g(t,u_t),v),$$

in the sense of $\mathcal{D}'(\tau,T^*)$.

Remark 2. If u is a solution of (2) in the sense given above, then u satisfies an energy equality, namely:

$$\begin{aligned}
& |u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr \\
= & |u(s)|^2 + 2 \int_s^t \left[(f(r),u(r)) + (g(r,u_r),u(r)) \right] dr \quad \forall s,t \in [\tau,T_*] \cap \mathbb{R}.
\end{aligned}$$

First, we will prove the uniqueness of weak solutions for our model in a similar way as done in [20] for the model without delay. We will only include the detailed estimates which involve the delay term.

Theorem 1. *Under the above assumptions, there exists at most a weak solution u of (2).*

The proof is similar to, but a bit more complicated than for the 2D-Navier-Stokes equations and depends on the following

Lemma 1. *([20]) For every $u, v \in V$, and each $N > 0$,*

1. $0 \leq \|u\| F_N(\|u\|) \leq N$,
2. $|F_N(\|u\|) - F_N(\|v\|)| \leq \frac{1}{N} F_N(\|u\|) F_N(\|v\|) \|u - v\|$.

Proof of Theorem 1: Let u, v be two weak solutions with the same initial conditions and set $w = v - u$. Then, using the energy equality, we obtain

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + \langle \mathcal{NL}(u,v), w\rangle = (g(t, v_t) - g(t, u_t), w), \quad t \in (\tau, T_*),$$

where we have set $\langle \mathcal{NL}(u,v), w\rangle = F_N(\|u\|) b(u,u,w) - F_N(\|v\|) b(v,v,w)$. From the properties of the trilinear form b it easily follows that

$$\begin{aligned}\langle \mathcal{NL}(u,v), w\rangle &= F_N(\|u\|) b(w,u,w) + (F_N(\|u\|) - F_N(\|v\|)) b(v,u,w) \\ &\quad + F_N(\|v\|) b(v,w,w).\end{aligned}$$

Now using Lemma 1, formula (7) and Young's inequality (see [20] for the details) there exists a constant $C_3 > 0$, which depends on C_2 and ν, such that,

$$|\langle \mathcal{NL}(u,v), w\rangle| \leq \nu\|w\|^2 + C_3 N^4 |w|^2.$$

Consequently, taking (g3) into account, we obtain

$$\frac{d}{dt}|w(t)|^2 \leq 2C_3 N^4 |w(t)|^2 + 2L_g \|w_t\|_\gamma |w(t)|, \quad t \in (\tau, T_*).$$

Observe that $w(s) = 0$ if $s \leq \tau$. Therefore, for $t \in (\tau, T_*)$:

$$\|w_t\|_\gamma = \sup_{\theta \leq 0} e^{\gamma\theta} |w(t+\theta)| \leq \sup_{\theta \in [\tau - t, 0]} |w(t+\theta)|.$$

Thus we obtain

$$\begin{aligned}|w(t)|^2 &\leq 2C_3 N^4 \int_\tau^t |w(s)|^2 ds + 2L_g \int_\tau^t \sup_{r \in [\tau, s]} |w(r)||w(s)| ds \\ &\leq (2C_3 N^4 + 2L_g) \int_\tau^t \sup_{r \in [\tau, s]} |w(r)|^2 ds,\end{aligned}$$

for any $t \in [\tau, T_*)$.

Now we deduce that

$$\sup_{r \in [\tau, t]} |w(r)|^2 \leq (2C_3 N^4 + 2L_g) \int_\tau^t \sup_{r \in [\tau, s]} |w(r)|^2 ds,$$

for any $t \in [\tau, T_*)$, whence the Gronwall lemma finishes the proof. □

Our main result is the following

Theorem 2. *Suppose that $f \in L^2(\tau, T; (L^2(\Omega))^3)$ for all $T \in (\tau, T_*] \cap \mathbb{R}$, $g : (\tau, T_*) \times C_\gamma(H) \to (L^2(\Omega))^3$ satisfying the assumptions (g1)–(g3), and $\phi \in C_\gamma(H)$ are given, and that $2\gamma > \nu\lambda_1$. Then, there exists a unique weak solution u of (2), which in fact is a strong solution in the sense that*

$$u \in C((\tau, T]; V) \cap L^2(\tau + \varepsilon, T; D(A)),$$

for all $0 < \varepsilon < T_* - \tau$ *and any* $T \in (\tau + \varepsilon, T_*] \cap \mathbb{R}$.

Moreover, if $\phi(0) \in V$, *then* u *satisfies*

$$u \in C([\tau, T]; V) \cap L^2(\tau, T; D(A)),$$

for all $T \in (\tau, T_*] \cap \mathbb{R}$.

Proof. We split the proof in several steps.

Step 1: A Galerkin scheme. Let us consider $\{v_j\} \subset V$, the orthonormal basis of H of all the eigenfunctions of the Stokes operator. Denote $V_m = \text{span}[v_1, \ldots, v_m]$ and consider the projector $P_m u = \sum_{j=1}^{m}(u, v_j)v_j$.

Define also

$$u^m(t) = \sum_{j=1}^{m} \alpha_{m,j}(t)v_j$$

where the upper script m will be used instead of (m) for short since no confusion is possible with powers of u, and where the coefficients $\alpha_{m,j}$ are required to satisfy the following system:

$$\begin{aligned} & \frac{d}{dt}(u^m(t), v_j) + \nu((u^m(t), v_j)) + b_N(u^m(t), u^m(t), v_j) \\ = \quad & (f(t), v_j) + (g(t, u_t^m), v_j), \quad 1 \le j \le m, \end{aligned} \tag{8}$$

where the equations are understood in the sense of $\mathcal{D}'(\tau, T_*)$, and the initial conditions is $u^m(\tau + s) = P_m\phi(s)$ for $s \in (-\infty, 0]$.

The above system of ordinary functional differential equations with infinite delay fulfills the conditions for existence and uniqueness of local solution of [12, Th.1.1, p.36].

Next, we will deduce a priori estimates that assure that the solutions do exist for all time $t \in [\tau, T_*] \cap \mathbb{R}$.

Step 2: A priori estimates. Multiplying (8) by u^m we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u^m(t)|^2 + \frac{\nu\lambda_1}{2}|u^m(t)|^2 + \frac{\nu}{2}\|u^m(t)\|^2 \; & \le \; (f(t), u^m(t)) + (g(t, u_t^m), u^m(t)) \\ & \le \; |f(t)||u^m(t)| + L_g\|u_t^m\|_\gamma |u^m(t)| \\ & \le \; \frac{\nu}{4}\|u^m(t)\|^2 + \frac{|f(t)|^2}{\nu\lambda_1} + L_g\|u_t^m\|_\gamma^2. \end{aligned}$$

Hence

$$\begin{aligned} & |u^m(t)|^2 + \frac{\nu}{2}\int_\tau^t e^{-\nu\lambda_1(t-s)}\|u^m(s)\|^2 ds \\ \le \quad & e^{-\nu\lambda_1(t-\tau)}|u(\tau)|^2 + 2\int_\tau^t e^{-\nu\lambda_1(t-s)}\left(\frac{|f(s)|^2}{\nu\lambda_1} + L_g\|u_s^m\|_\gamma^2\right) ds. \end{aligned} \tag{9}$$

Further

$$\begin{aligned} \|u_t^m\|_\gamma^2 \; \le \; \max\Bigg\{ & \sup_{\theta\in(-\infty,\tau-t]} e^{2\gamma\theta}|\phi(\theta + t - \tau)|^2, \sup_{\theta\in[\tau-t,0]} \Bigg[e^{2\gamma\theta-\nu\lambda_1(t-\tau+\theta)}|u(\tau)|^2 \\ & + 2e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-\nu\lambda_1(t+\theta-s)}\left(\frac{|f(s)|^2}{\nu\lambda_1} + L_g\|u_s^m\|_\gamma^2\right) ds\Bigg]\Bigg\}. \end{aligned}$$

On the one hand

$$\begin{aligned}\sup_{\theta\in(-\infty,\tau-t]} e^{\gamma\theta}|\phi(\theta+t-\tau)| &= \sup_{\theta\le 0} e^{\gamma(\theta-(t-\tau))}|\phi(\theta)| \\ &= e^{-\gamma(t-\tau)}\|\phi\|_\gamma.\end{aligned}$$

On the other hand, as we are assuming that $2\gamma > \nu\lambda_1$,

$$\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta-\nu\lambda_1(t-\tau+\theta)}|u(\tau)|^2 \le e^{-\nu\lambda_1(t-\tau)}|u(\tau)|^2$$

and

$$\begin{aligned}&\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-\nu\lambda_1(t+\theta-s)}\left(\frac{|f(s)|^2}{\nu\lambda_1}+L_g\|u_s^m\|_\gamma^2\right)ds \\ \le\ & \int_\tau^t e^{-\nu\lambda_1(t-s)}\left(\frac{|f(s)|^2}{\nu\lambda_1}+L_g\|u_s^m\|_\gamma^2\right)ds.\end{aligned}$$

Collecting these inequalities we deduce

$$\|u_t^m\|_\gamma^2 \le e^{-\nu\lambda_1(t-\tau)}\|\phi\|_\gamma^2 + 2\int_\tau^t e^{-\nu\lambda_1(t-s)}\left(\frac{|f(t)|^2}{\nu\lambda_1}+L_g\|u_s^m\|_\gamma^2\right)ds.$$

By the Gronwall lemma we have

$$\|u_t^m\|_\gamma^2 \le e^{-(\nu\lambda_1-2L_g)(t-\tau)}\|\phi\|_\gamma^2 + \frac{2}{\nu\lambda_1}\int_\tau^t e^{-(\lambda_1\nu-2L_g)(t-s)}|f(s)|^2 ds.$$

Then we obtain the following estimates: for any $R>0$ and $T\in(\tau,T_*]\cap\mathbb{R}$, there exists a constant $C=C(\tau,T,R)$, depending on some constants of the problem (namely, λ_1, ν, L_g and f), and on τ, T and R, such that

$$\|u_t^m\|_\gamma^2 \le C(\tau,T,R) \quad \forall t\in[\tau,T], \quad \forall\,\|\phi\|_\gamma \le R, \quad \forall m\ge 1. \tag{10}$$

In particular, this implies that

$$\{u^m\} \text{ is bounded in } L^\infty(\tau,T;H) \quad \forall\, T\in(\tau,T_*]\cap\mathbb{R}. \tag{11}$$

Now, it follows from (9) and (10) that

$$\begin{aligned}&\frac{\nu}{2}e^{-\nu\lambda_1(T-\tau)}\int_\tau^T \|u^m(s)\|^2 ds \\ \le\ & \frac{\nu}{2}\int_\tau^T e^{-\nu\lambda_1(T-s)}\|u^m(s)\|^2 ds \\ \le\ & |u(\tau)|^2 + 2\int_\tau^T e^{-\nu\lambda_1(T-s)}\left(\frac{|f(s)|^2}{\nu\lambda_1}+L_g\|u_s^m\|_\gamma^2\right)ds \\ \le\ & R^2 + 2\int_\tau^T e^{-\nu\lambda_1(T-s)}\left(\frac{|f(s)|^2}{\nu\lambda_1}+L_gC(\tau,T,R)\right)ds,\end{aligned}$$

so that we conclude the existence of another constant (relabelled the same) $C(\tau,T,R)$ such that

$$\|u^m\|_{L^2(\tau,T;V)}^2 \le C(\tau,T,R) \quad \forall\,\|\phi\|_\gamma\le R \quad \forall m\ge 1, \quad \forall\, T\in(\tau,T_*]\cap\mathbb{R}. \tag{12}$$

Now, observe that (8) is equivalent to

$$\frac{du^m}{dt} = -\nu Au^m - P_mB_N(u^m,u^m) + P_mf(t) + P_mg(t,u_t^m). \tag{13}$$

From (5), (11), (12) and (13), by the choice of the basis one also deduces that

$$\|(u^m)'\|^2_{L^2(\tau,T;V')} \le C(\tau,T,R) \quad \forall \|\phi\|_\gamma \le R \; \forall m \ge 1, \; \forall T \in (\tau, T_*] \cap \mathbb{R}. \tag{14}$$

So, this implies the existence of a

$$u \in L^\infty(\tau,T;H) \cap L^2(\tau,T;V) \text{ with } u' \in L^2(\tau,T;V'), \quad \forall T \in (\tau,T_*] \cap \mathbb{R},$$

and a subsequence of $\{u^m\}$ which converges weak-star to u in $L^\infty(\tau,T;H)$, weakly to u in $L^2(\tau,T;V)$, with $\{(u^m)'\}$ converging weakly to u' in $L^2(\tau,T;V')$ for all $T \in (\tau,T_*] \cap \mathbb{R}$.

Observe in particular that $u \in C([\tau,T];H)$ for all $T \in (\tau,T_*] \cap \mathbb{R}$.

By a compactness result (cf. [17, Ch.1,Th.5.1]), one can then deduce that a subsequence in fact converges strongly to u in $L^2(\tau,T;H)$ and a.e. in (τ,T) with values in H and a.e. in $(\tau,T) \times \Omega$ for all $T \in (\tau,T_*] \cap \mathbb{R}$.

Step 3: Some more a priori estimates. The estimates obtained above are not enough to pass to the limit and deduce that u is a solution of (2). Namely, we have two main difficulties. On the one hand, we need to pass to the limit in $g(u^m)$, this will be done in Step 4, proving that actually $u_t^m \to u_t$ in $C_\gamma(H)$. On other hand, the weak convergence in $L^2(\tau,T;V)$ is not enough to ensure that

$$\|u^m(t)\| \to \|u(t)\|$$

or at least

$$F_N(\|u^m(t)\|) \to F_N(\|u(t)\|) \quad \text{for a.a. } t,$$

which is needed to manage the nonlinear term $B_N(u^m,u^m)$.

In order to sort out this last trouble, we need to find a stronger estimate. We proceed now with that. Take the inner product of the Galerkin ODE (8) with Au^m and obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^m(t)\|^2 + \nu|Au^m(t)|^2 + b_N(u^m(t),u^m(t),Au^m(t)) \\ = \;& (f(t),Au^m(t)) + (g(t,u_t^m),Au^m(t)). \end{aligned} \tag{15}$$

Obviously,

$$(f(t),Au^m(t)) \le |f(t)||Au^m(t)| \le \frac{\nu}{8}|Au^m(t)| + \frac{2}{\nu}|f(t)|^2$$

and

$$|(g(t,u_t^m),Au^m(t))| \le \frac{\nu}{8}|Au^m(t)|^2 + \frac{2}{\nu}|g(t,u_t^m)|^2.$$

By (6), Lemma 1 and Young's inequality, it follows

$$\begin{aligned} |b_N(u^m(t),u^m(t),Au^m(t))| &\le \frac{N}{\|u^m(t)\|} C_2 \|u^m(t)\|^{3/2}|Au^m(t)|^{3/2} \\ &= NC_2\|u^m(t)\|^{1/2}|Au^m(t)|^{3/2} \\ &\le \frac{\nu}{4}|Au^m(t)|^2 + C_N\|u^m(t)\|^2, \end{aligned}$$

with $C_N = \dfrac{27(NC_2)^4}{4\nu^3}$.

Thus (15) simplifies to

$$\frac{d}{dt}\|u^m(t)\|^2 + \nu|Au^m(t)|^2 \le \frac{4}{\nu}|f(t)|^2 + \frac{4}{\nu}|g(t,u_t^m)|^2 + 2C_N\|u^m(t)\|^2. \tag{16}$$

Integrating between s and t with $\tau \le s \le t \le T$ and for all $T \in (\tau, T_*] \cap \mathbb{R}$, we deduce that

$$\begin{aligned} & \|u^m(t)\|^2 + \nu \int_s^t |Au^m(r)|^2 dr \\ \le \quad & \|u^m(s)\|^2 + \frac{4}{\nu} \int_\tau^T |f(r)|^2 + |g(r, u_r^m)|^2 dr \\ & +2C_N \int_\tau^T \|u^m(r)\|^2 dr, \quad \forall \tau \le s \le t \le T, \ \forall T \in (\tau, T_*] \cap \mathbb{R}. \end{aligned}$$

Now, integrating once more between τ and t we obtain

$$\begin{aligned} (t-\tau)\|u^m(t)\|^2 \ \le \ & \int_\tau^T \|u^m(s)\|^2 ds + \frac{4(T-\tau)}{\nu} \int_\tau^T |f(r)|^2 + |g(r, u_r^m)|^2 dr \\ & +2C_N (T-\tau) \int_\tau^T \|u^m(r)\|^2 dr, \ \forall \tau \le s \le t \le T, \ \forall T \in (\tau, T_*] \cap \mathbb{R}. \end{aligned}$$

The important fact is that the right hand side of the above expression is bounded by some constant $C_{\tau,T}$, which is independent of m.

Thus, these two inequalities imply (first the last one, and then using the previous one) that for any $\varepsilon \in (0, T-\tau)$, and for any $t \in [\tau+\varepsilon, T)$ with $T \in (\tau+\varepsilon, T_*] \cap \mathbb{R}$ one obtains additional estimates for $\{u^m\}$. Namely,

$$\{u^m\} \text{ is bounded in } L^\infty(\tau+\varepsilon, T; V) \cap L^2(\tau+\varepsilon, T; D(A)), \ \forall T \in (\tau+\varepsilon, T_*] \cap \mathbb{R}. \quad (17)$$

Moreover, observe that if $\phi(0) \in V$, then, thanks to the fact that $\|u^m(\tau)\| = \|P_m \phi(0)\| \le \|\phi(0)\|$, from (16), (11) and (12), one deduces directly that

$$\{u^m\} \quad \text{is bounded in } L^\infty(\tau, T; V) \cap L^2(\tau, T; D(A)), \quad \forall T \in (\tau, T_*] \cap \mathbb{R}. \quad (18)$$

We extract some consequences from above assuming again that $\phi(0) \in H$. As $D(A) \subset V \subset H$ with compact injection, by [17, Th.5.1,Ch.1], from the convergences obtained in Step 2, (17), and using a sequence of positive values $\varepsilon_n \downarrow 0$ and a diagonal argument, we deduce that there exists an element

$$u \in L^\infty(-\infty, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau+\varepsilon, T; V) \cap L^2(\tau+\varepsilon, T; D(A))$$

for all $T \in (\tau, T_*] \cap \mathbb{R}$, and any $\varepsilon > 0$, any $T > \tau + \varepsilon > \tau$, and a subsequence of $\{u^m\}$, that we will also denote by $\{u^m\}$, such that they satisfy

$$\begin{cases} u^m \rightharpoonup u & \text{weak in } \ L^2(\tau, T; V), \\ u^m \overset{*}{\rightharpoonup} u & \text{weak-star in } \ L^\infty(\tau, T; H), \\ u^m \to u & \text{a.e. in } \ (\tau, T) \times \Omega, \\ u^m \to u & \text{strong in } \ L^2(\tau+\varepsilon, T; V), \\ u^m \rightharpoonup u & \text{weak in } \ L^2(\tau+\varepsilon, T; D(A)), \\ u^m \overset{*}{\rightharpoonup} u & \text{weak-star in } \ L^\infty(\tau+\varepsilon, T; V), \end{cases}$$

for all $T \in (\tau, T_*] \cap \mathbb{R}$, and any $T > \tau + \varepsilon > \tau$.

Also, as u^m converges to u in $L^2(\tau+\varepsilon, T; V)$ for all $T > \tau + \varepsilon > \tau$, we can assume, eventually extracting a subsequence, that

$$\|u^m(t)\| \to \|u(t)\| \quad \text{a.e. in } (\tau, T_*),$$

and therefore

$$F_N(\|u^m(t)\|) \to F(\|u(t)\|) \quad \text{a.e. in } (\tau, T_*). \quad (19)$$

Step 4: Convergence in $C_\gamma(H)$ and existence of solution. We will prove now that actually the convergences obtained for $\{u^m\}$ to u in the above steps can be improved. Indeed, we will see that

$$u_t^m \to u_t \text{ in } C_\gamma(H),\ \forall t \in (-\infty, T_*] \cap \mathbb{R}.$$

It is not difficult to check that this holds if we prove the following:

$$P_m\phi \to \phi \quad \text{in } C_\gamma(H), \tag{20}$$

$$u^m \to u \quad \text{in } C([\tau,T];H)\ \forall T \in (\tau, T_*] \cap \mathbb{R}. \tag{21}$$

A similar result was proved in [18], however there are some differences between both arguments, so for the sake of completeness, we reproduce the proof here.

Step 4.1: Approximation in $C_\gamma(H)$ of the initial datum. We check now the convergence claimed in (20). For the delay initial datum $\phi \in C_\gamma(H)$, we have used the projections in the Galerkin scheme in Step 1.

Indeed, if not, there would exist $\varepsilon > 0$ and a subsequence, that we relabel the same, such that

$$e^{\gamma\theta_m}|P_m\phi(\theta_m) - \phi(\theta_m)| > \varepsilon. \tag{22}$$

One can assume that $\theta_m \to -\infty$, otherwise if $\theta_m \to \theta$, then $P_m\phi(\theta_m) \to \phi(\theta)$, since $|P_m\phi(\theta_m) - \phi(\theta)| \le |P_m\phi(\theta_m) - P_m\phi(\theta)| + |P_m\phi(\theta) - \phi(\theta)| \to 0$ as $m \to +\infty$. But with $\theta_m \to -\infty$ as $m \to +\infty$, if we denote $x = \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)$, we obtain that

$$\begin{aligned} & e^{\gamma\theta_m}|P_m\phi(\theta_m) - \phi(\theta_m)| \\ = \;& |P_m(e^{\gamma\theta_m}\phi(\theta_m)) - e^{\gamma\theta_m}\phi(\theta_m)| \\ \le \;& |P_m(e^{\gamma\theta_m}\phi(\theta_m)) - P_m x| + |P_m x - x| + |x - e^{\gamma\theta_m}\phi(\theta_m)| \to 0. \end{aligned}$$

This is a contradiction with (22), so (20) holds.

Step 4.2: convergence of u^m to u in $C([\tau,T];H)$ for all $T \in (\tau, T_*] \cap \mathbb{R}$. Hereon consider a fixed value (but otherwise arbitrary) $T \in (\tau, T_*] \cap \mathbb{R}$.

From the strong convergence of $\{u^m\}$ to u in $L^2(\tau,T;H)$ obtained at the end of Step 2, we deduce that

$$u^m(t) \to u(t) \quad \text{in } H \text{ a.e. } t \in (\tau, T_*] \cap \mathbb{R}.$$

Since

$$u^m(t) - u^m(s) = \int_s^t (u^m)'(r)dr \quad \text{in } V',\ \forall s,t \in [\tau,T],$$

from (14) we have that $\{u^m\}$ is equi-continuous on $[\tau,T]$ with values in V'. By the compactness of the injection of H into V', from (11) and the equi-continuity in V', by the Ascoli-Arzelà theorem we have that

$$u^m \to u \quad \text{in } C([\tau,T];V'). \tag{23}$$

Again from (11) we obtain that for any sequence $\{t_m\} \subset [\tau,T]$, with $t_m \to t$, one has

$$u^m(t_m) \rightharpoonup u(t) \quad \text{weakly in } H, \tag{24}$$

where we have used (23) in order to identify which is the weak limit.

Now, we are ready to prove (21) by a contradiction argument. If it would not be so, then, taking into account that $u \in C([\tau,T];H)$, there would exist $\varepsilon > 0$, a value $t_0 \in [\tau, T]$ and subsequences (relabelled the same) $\{u^m\}$ and $\{t_m\} \subset [\tau, T]$ with $\lim_{m\to+\infty} t_m = t_0$ such that

$$|u^m(t_m) - u(t_0)| \geq \varepsilon \quad \forall m. \tag{25}$$

To prove that this is absurd, we will use an energy method.

Observe that the following energy inequality holds for all u^m:

$$\begin{aligned} &\frac{1}{2}|u^m(t)|^2 + \frac{\nu}{2}\int_s^t \|u^m(r)\|^2 dr \\ \leq\ &\int_s^t (f(r), u^m(r))dr + \frac{1}{2}|u^m(s)|^2 + C_4(t-s), \quad \forall s, t \in [\tau, T], \end{aligned} \tag{26}$$

where $C_4 = \frac{D}{2\nu\lambda_1}$ and D corresponds to the upper bound

$$\int_s^t |g(r, u_r^m)|^2 dr \leq D(t-s), \quad \text{for all } \tau \leq s \leq t \leq T,$$

by $(g2)$, $(g3)$ and (11).

On the other hand, from (11), $(g2)$ and $(g3)$, we deduce the existence of $\xi_g \in L^2(\tau, T; (L^2(\Omega))^3)$, which we cannot identify by the moment, such that $\{g(u^m)\}$ converges weakly to ξ_g in $L^2(\tau, T; (L^2(\Omega))^3)$.

Then, taking into account (11), (12) and (19), reasoning as in [1] for the case without delays (see also [17] for the case of the Navier-Stokes system), we can pass to the limit in equation (13) and deduce that u is solution of

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + \langle B_N(u, u), v\rangle = (f(t), v) + (\xi_g(t), v), \ \forall v \in V. \tag{27}$$

Therefore, it satisfies the energy equality

$$\begin{aligned} &|u(t)|^2 + 2\int_s^t \nu\|u(r)\|^2 dr \\ =\ &|u(s)|^2 + 2\int_s^t (f(r), u(r)) + (\xi_g(r), u(r))dr, \quad \forall s, t \in [\tau, T]. \end{aligned}$$

Of course, for the weak limit ξ_g we have the estimate

$$\begin{aligned} \int_s^t |\xi_g(r)|^2 dr &\leq \liminf_{m\to+\infty} \int_s^t |g(r, u_r^m)|^2 dr \\ &\leq D(t-s), \quad \forall \tau \leq s \leq t \leq T. \end{aligned}$$

So, we have that u also satisfies the inequality (26) with the same constant C_4.

Now, consider the functions J_m, $J : [\tau, T] \to \mathbb{R}$ defined by

$$J_m(t) = \frac{1}{2}|u^m(t)|^2 - \int_\tau^t (f, u^m(r))dr - C_4 t,$$

$$J(t) = \frac{1}{2}|u(t)|^2 - \int_\tau^t (f, u(r))dr - C_4 t.$$

From the inequalities fulfilled by $\{u^m\}$ and u, it is clear that J_m and J are non-increasing (and continuous) functions. Moreover, by the convergence of u^m to u a.e. in time with values in H, and weakly in $L^2(\tau, T; H)$, it holds that

$$J_m(t) \to J(t) \text{ a.e. } t \in [\tau, T]. \tag{28}$$

Now we will prove that

$$u^m(t_m) \to u(t_0) \quad \text{in } H, \tag{29}$$

which contradicts (25). Firstly, recall from (24) that

$$u^m(t_m) \rightharpoonup u(t_0) \quad \text{weakly in } H. \tag{30}$$

So, we have that

$$|u(t_0)| \leq \liminf_{m\to+\infty} |u^m(t_m)|.$$

Therefore, if we show that

$$\limsup_{m\to+\infty} |u^m(t_m)| \leq |u(t_0)|, \tag{31}$$

we obtain that $\lim_{m\to+\infty} |u^m(t_m)| = |u(t_0)|$, which jointly with (30) imply (29).

Now, observe that the case $t_0 = \tau$ follows directly from (26) with $s = \tau$ and the definition of $u^m(\tau) = P_m\phi(0)$. So, we may assume that $t_0 > \tau$. This is important, since we will approach this value t_0 from the left by a sequence $\{\tilde{t}_k\}$, i.e. $\lim_{k\to+\infty} \tilde{t}_k \nearrow t_0$, being $\{\tilde{t}_k\}$ values where (28) holds. Since $u(\cdot)$ is continuous at t_0, there is k_ε such that

$$|J(\tilde{t}_k) - J(t_0)| < \varepsilon/2, \quad \forall k \geq k_\varepsilon.$$

On other hand, taking $m \geq m(k_\varepsilon)$ such that $t_m > \tilde{t}_{k_\varepsilon}$, as J_m is non-increasing and for all $\tilde{t}_k$ the convergence (28) holds, one has that

$$J_m(t_m) - J(t_0) \leq |J_m(\tilde{t}_{k_\varepsilon}) - J(\tilde{t}_{k_\varepsilon})| + |J(\tilde{t}_{k_\varepsilon}) - J(t_0)|,$$

and obviously, taking $m \geq m'(k_\varepsilon)$, it is possible to obtain $|J_m(\tilde{t}_{k_\varepsilon}) - J(\tilde{t}_{k_\varepsilon})| < \varepsilon/2$. It can also be deduced from Step 2 that

$$\int_\tau^{t_m} (f, u^m(r))dr \to \int_\tau^{t_0} (f, u(r))dr,$$

so we conclude that (31) holds. Thus, (29) and finally (21) are also true, as we wanted to check.

Now, we are ready to pass to the limit in the equations satisfied by the $\{u^m\}$ and to complete the information obtained in (27).

Assume initially that $\phi(0) \in H$. The first clear consequence from the convergence proved above, since g satisfies (g3), is that

$$g(\cdot, u^m_\cdot) \to g(\cdot, u_\cdot) \quad \text{in } L^2(\tau, T; (L^2(\Omega))^3), \quad \forall T \in (\tau, T_*] \cap \mathbb{R}.$$

Thus, we can identify $\xi_g(t) = g(t, u_t)$ in (27). Therefore u is a solution of (2).

Finally, if $\phi(0) \in V$, from (18) and analogous arguments to those given above we conclude that $u \in C([\tau, T]; V) \cap L^2(\tau, T; D(A))$, for all $T \in (\tau, T_*] \cap \mathbb{R}$. □

Proposition 1 (Continuity of solutions with respect to initial data)**.** *Under the assumptions of Theorem 2, the solutions obtained for (2) are continuous with respect to the initial condition ϕ, and more exactly, there exists a constant $C_3 > 0$, only dependent on ν and the constant C_2 appearing in (7), such that if u^i, for $i = 1, 2$, are the corresponding solutions to initial data $\phi^i \in C_\gamma(H)$, $i = 1, 2$, the following estimate holds:*

$$\max_{r\in[\tau,t]} |u^1(r) - u^2(r)|^2 \leq \left(|\phi^1(0) - \phi^2(0)|^2 + \frac{L_g}{2\gamma}\|\phi^1 - \phi^2\|_\gamma^2\right) \times e^{(3L_g + 2C_3N^4)(t-\tau)}, \tag{32}$$

for all $t \in [\tau, T_*] \cap \mathbb{R}$.

Proof. If we proceed as in the Proof of Theorem 1 we have that there exists a constant $C_3 > 0$ which depends on C_2 and ν, such that ,

$$\frac{d}{dt}|u^1(t) - u^2(t)|^2 \leq 2C_3N^4|u^1(t) - u^2(t)|^2 + 2L_g\|u_t^1 - u_t^2\|_\gamma|u^1(t) - u^2(t)|, \quad (33)$$

for all $t \in [\tau, T_*] \cap \mathbb{R}$. As for $s \in [\tau, t]$ one has

$$\begin{aligned}
\|u_s^1 - u_s^2\|_\gamma &= \sup_{\theta \leq 0} e^{\gamma\theta}|u^1(s+\theta) - u^2(s+\theta)| \\
&= \max\Bigg\{ \sup_{\theta \in (-\infty, \tau - s]} e^{\gamma\theta}|\phi^1(s+\theta-\tau) - \phi^2(s+\theta-\tau)|, \\
&\qquad \sup_{\theta \in [\tau - s, 0]} e^{\gamma\theta}|u^1(s+\theta) - u^2(s+\theta)| \Bigg\} \\
&\leq \max\left\{ e^{\gamma(\tau - s)}\|\phi^1 - \phi^2\|_\gamma, \max_{\theta \in [\tau, s]}|u^1(\theta) - u^2(\theta)| \right\},
\end{aligned}$$

we conclude from (33) that for all $t \in [\tau, T_*] \cap \mathbb{R}$,

$$\begin{aligned}
|u^1(t) - u^2(t)|^2 \leq\ & |u^1(\tau) - u^2(\tau)|^2 + 2L_g\|\phi^1 - \phi^2\|_\gamma \int_\tau^t e^{\gamma(\tau - s)}|u^1(s) - u^2(s)|ds \\
& + 2L_g \int_\tau^t |u^1(s) - u^2(s)| \max_{\theta \in [\tau, s]}|u^1(\theta) - u^2(\theta)|ds \\
& + 2C_3N^4 \int_\tau^t |u^1(s) - u^2(s)|^2 ds.
\end{aligned}$$

If we now substitute t by $r \in [\tau, t]$ and consider the maximum when varying this r, from the above we can conclude that

$$\begin{aligned}
\max_{r \in [\tau, t]}|u^1(r) - u^2(r)|^2 \leq\ & |u^1(\tau) - u^2(\tau)|^2 + \frac{L_g}{2\gamma}\|\phi^1 - \phi^2\|_\gamma^2 \\
& + \int_\tau^t \left(3L_g + 2C_3N^4\right) \max_{r \in [\tau, s]}|u^1(r) - u^2(r)|^2 ds.
\end{aligned}$$

Hence, by the Gronwall lemma we obtain (32). □

4. **Stationary solutions and their stability.** In this section we are interested in proving that the problem (2), with some obvious restrictions, admits stationary solutions, and that under additional assumptions, in fact the stationary solution is unique and is globally asymptotically exponentially stable.

The restrictions we must impose to give sense to a stationary solution are that $f \in (L^2(\Omega))^3$ and g and are now autonomous, i.e. without dependence on time, and we must clarify how g acts over a fixed element of H. This is done with a slight abuse of notation in the following sense: we consider $g(w)$ as $g(\tilde{w})$, where $\tilde{w} \in C_\gamma(H)$ is the element that has the only value w for all time $t \leq 0$ [observe that $\tilde{w}$ is a well defined element of $C_\gamma(H)$ and that $\|\tilde{w}\|_\gamma = |w|$; so we will continue denoting directly w instead of $\tilde{w}$ since no confusion arises]. Of course, as an immediate consequence of the assumptions for g, it follows that

$$|g(x_1) - g(x_2)| \leq L_g|x_1 - x_2|, \quad \forall x_1, x_2 \in H.$$

So, consider the following equation,

$$\frac{du}{dt} + \nu Au + B_N(u,u) = f + g(u_t) \quad \forall t \geq 0, \tag{34}$$

where A is the operator given by (3). By a stationary solution to (34) we mean an element $u^* \in V$ such that

$$\nu((u^*,v)) + b_N(u^*,u^*,v) = (f,v) + (g(u^*),v) \quad \forall v \in V. \tag{35}$$

Theorem 3. *Under the above assumptions and notation, if $\lambda_1\nu > L_g$, then:*

(a) *The problem (34) admits at least one stationary solution u^*, which indeed belongs to $D(A)$. Moreover, any such stationary solution satisfies the estimate*

$$(\nu - \lambda_1^{-1}L_g)\|u^*\| \leq \lambda_1^{-1/2}|f|. \tag{36}$$

(b) *If the following condition holds,*

$$\min\{N\lambda_1^{-1/4}, |f|^{1/2}\lambda_1^{-3/8}\} < \nu - \lambda_1^{-1}L_g, \tag{37}$$

then the stationary solution of (34) is unique.

Proof. **Existence** The estimate (36) can be obtained taking into account that in particular any stationary solution u^*, if it exists, should verify

$$\nu\langle Au^*, u^*\rangle = (f,u^*) + (g(u^*),u^*),$$

and therefore

$$\nu\|u^*\|^2 \leq |f||u^*| + L_g|u^*|^2 \leq \lambda_1^{-1/2}|f|\|u^*\| + L_g\lambda_1^{-1}\|u^*\|^2.$$

For the existence, let us consider $\{v_j\} \subset V$, the orthonormal basis of H of all the eigenfunctions of the Stokes operator. For each integer $m \geq 1$, let us denote V_m =span$[v_1,\dots,v_m]$, with the inner product $((\cdot,\cdot))$ and norm $\|\cdot\|$. Define the operators $R_m : V_m \to V_m$, $m \geq 1$, by

$$((R_m u, v)) = \nu\langle Au, v\rangle + \langle B_N(u,u), v\rangle - (f,v) - (g(u),v), \ \forall u,v \in V_m. \tag{38}$$

Since the right hand side is a continuous linear map from V_m to $\mathbb{R}$, by the Riesz theorem, each $R_m u \in V_m$ is well defined. We check now that R_m is continuous. Indeed, taking into account the assumptions for g, Lemma 1 and the properties of b, we have

$$\begin{aligned}
& ((R_m u - R_m\widetilde{u}, v)) \\
= \ & \nu\langle Au - A\widetilde{u}, v\rangle + \langle B_N(u,u) - B_N(\widetilde{u},\widetilde{u}), v\rangle - (g(u) - g(\widetilde{u}), v) \\
\leq \ & \nu\|u-\widetilde{u}\|\,\|v\| + F_N(\|u\|)b(u-\widetilde{u},u,v) + (F_N(\|u\|) - F_N(\|\widetilde{u}\|))b(\widetilde{u},u,v) \\
& + F_N(\|\widetilde{u}\|)b(\widetilde{u},u-\widetilde{u},v) + L_g\lambda_1^{-1}\|u-\widetilde{u}\|\,\|v\| \\
\leq \ & (\nu + 3NC_1 + L_g\lambda_1^{-1})\|u-\widetilde{u}\|\,\|v\|,
\end{aligned} \tag{39}$$

for all $u, u', v \in V_m$. So,

$$\|R_m u - R_m\widetilde{u}\| \leq \left(\nu + 3NC_1 + L_g\lambda_1^{-1}\right)\|u-\widetilde{u}\|,$$

for all u, u'.

On the other hand, for all $u \in V_m$,

$$\begin{aligned}
((R_m u, u)) &= \nu\langle Au, u\rangle - (f,u) - (g(u),u) \\
&\geq \nu\|u\|^2 - \lambda_1^{-1/2}|f|\|u\| - \lambda_1^{-1}L_g\|u\|^2.
\end{aligned}$$

Thus, if we take

$$\beta = \frac{\lambda_1^{-1/2}|f|}{\nu - L_g\lambda_1^{-1}},$$

we obtain $((R_m u, u)) \geq 0 \ \forall u \in V_m$ such that $\|u\| = \beta$.

Consequently by a corollary of the Brouwer's fixed point theorem (see [17, p.53]), for each $m \geq 1$ there exist $u_m \in V_m$ such that $R_m(u_m) = 0$, with $\|u_m\| \leq \beta$.

Observe moreover that $Au_m \in V_m$, and therefore

$$\begin{aligned} \nu|Au_m|^2 &= -\langle B_N(u_m, u_m), Au_m\rangle + (f, Au_m) + (g(u_m), Au_m) \\ &\leq \frac{\nu}{2}|Au_m|^2 - \langle B_N(u_m, u_m), Au_m\rangle + \frac{|f|^2}{\nu} + \frac{L_g^2\beta^2}{\nu\lambda_1}. \end{aligned} \tag{40}$$

Moreover, by (6) and Young's inequality,

$$\begin{aligned} |\langle B_N(u_m, u_m), Au_m\rangle| &\leq \frac{\nu}{4}|Au_m|^2 + C_N\|u_m\|^2 \\ &\leq \frac{\nu}{4}|Au_m|^2 + C_N\beta^2, \end{aligned} \tag{41}$$

with $C_N = \dfrac{27(NC_2)^4}{4\nu^3}$.

From (40) and (41), we deduce that the sequence $\{u_m\}$ is bounded in $D(A)$, and consequently, by the compact injection of $D(A)$ in V, we can extract a subsequence $\{u_{m'}\} \subset \{u_m\}$, that converges weakly in $D(A)$ and strongly in V to an element $u^* \in D(A)$. It is now standard to take limits in (38) and to obtain that u^* is an stationary solution.

In order to prove the final regularity remark, it is enough to take into account that every stationary solution u_* to (34) is also a solution to (2), but with initial data $\phi(t) = u_*$ for $t \in (-\infty, 0]$, and forcing term $f + g(u_*)$. Thus, we can apply Theorem 2.

Uniqueness Let us suppose that u^* and $\widetilde{u}^*$ are two stationary solutions of (34). Then,

$$\begin{aligned} &\nu\langle Au^* - A\widetilde{u}^*, v\rangle + \langle B_N(u^*, u^*) - B_N(\widetilde{u}^*, \widetilde{u}^*), v\rangle \\ = \ &(g(u^*) - g(\widetilde{u}^*), v), \ \forall v \in V. \end{aligned} \tag{42}$$

Taking $v = u^* - \widetilde{u}^*$ and proceeding as in (39) we obtain from (42)

$$\begin{aligned} \nu\|u^* - \widetilde{u}^*\|^2 \leq \ &|F_N(\|u^*\|)b(u^* - \widetilde{u}^*, u^*, u^* - \widetilde{u}^*)| \\ &+|(F_N(\|u^*\|) - F_N(\|\widetilde{u}^*\|))b(\widetilde{u}^*, u^*, u^* - \widetilde{u}^*)| \\ &+\lambda_1^{-1}L_g\|u^* - \widetilde{u}^*\|^2. \end{aligned} \tag{43}$$

From this inequality, taking into account (4), that $b(\widetilde{u}^*, u^*, u^* - \widetilde{u}^*) = 0$, and the fact that $F_N(\|u^*\|) \leq 1$, we obtain

$$\nu\|u^* - \widetilde{u}^*\|^2 \leq \lambda_1^{-1/4}\|u^*\|\,\|u^* - \widetilde{u}^*\|^2 + \lambda_1^{-1}L_g\|u^* - \widetilde{u}^*\|^2,$$

and therefore, by the estimate (36),

$$\nu\|u^* - \widetilde{u}^*\|^2 \leq \left(\lambda_1^{-1/4}\frac{\lambda_1^{-1/2}|f|}{\nu - \lambda_1^{-1}L_g} + \lambda_1^{-1}L_g\right)\|u^* - \widetilde{u}^*\|^2. \tag{44}$$

On the other hand, if in (43) we use (4), and Lemma 1, i.e. $\|u^*\|F_N(\|u^*\|) \le N$ and $\|\widetilde{u}^*\|F_N(\|\widetilde{u}^*\|) \le N$, we obtain

$$\nu \|u^* - \widetilde{u}^*\|^2 \le N\lambda_1^{-1/4} \|u^* - \widetilde{u}^*\|^2 + \lambda_1^{-1} L_g \|u^* - \widetilde{u}^*\|^2 . \tag{45}$$

From (44) and (45) we deduce that if (37) holds, then $u^* = \widetilde{u}^*$. □

Theorem 4. *Assume that the assumptions in Theorem 2 with f and g independent of time and (37) hold. Then there exists a value $0 < \lambda < 2\gamma$ such that for the solution $u(\cdot, 0, \phi)$ of (2) with $\tau = 0$, $T_* = +\infty$ and $\phi \in C_\gamma(H)$, the following estimates hold for all $t \ge 0$:*

$$|u(t,0,\phi) - u^*|^2 \le e^{-\lambda t}\left(|\phi(0) - u^*|^2 + \frac{L_g}{2\gamma - \lambda}\|\phi - u^*\|_\gamma^2\right), \tag{46}$$

$$\begin{aligned} &\|u_t(\cdot,0,\phi) - u^*\|_\gamma^2 \\ \le \quad &\max\left\{e^{-2\gamma t}\|\phi - u^*\|_\gamma^2, e^{-\lambda t}\left(|\phi(0) - u^*|^2 + \frac{L_g}{2\gamma - \lambda}\|\phi - u^*\|_\gamma^2\right)\right\}, \end{aligned} \tag{47}$$

where u^* is the unique stationary solution of (34) given by Theorem 3.

Proof. For short denote $u(t) = u(\cdot, 0, \phi)$. Let us also denote $w(t) = u(t) - u^*$. Considering equations (34) for $u(t)$ and (35) for u^*, one has

$$\frac{d}{dt}(w(t), v) + \nu((w(t), v)) + b_N(u(t), u(t), v) - b_N(u^*, u^*, v) = (g(u_t) - g(u^*), v),$$

for $t > 0$, for any $v \in V$.

From energy equality and the Lipschitz condition on g, and introducing an exponential term $e^{\lambda t}$ with a positive value λ to be fixed later on, we obtain

$$\begin{aligned} \frac{d}{dt}(e^{\lambda t}|w(t)|^2) \le e^{\lambda t} \big(&\lambda|w(t)|^2 - 2\nu\|w(t)\|^2 \\ &+2|b_N(u(t), u(t), w(t)) - b_N(u^*, u^*, w(t))| + 2L_g\|w_t\|_\gamma |w(t)|\big), \end{aligned}$$

for $t > 0$.

Reasoning as for (44) and (45), we have

$$|b_N(u(t), u(t), w) - b_N(u^*, u^*, w)| \le \mu\|u(t) - u^*\|^2,$$

where

$$\mu := \min\left\{N\lambda_1^{-1/4}, \frac{\lambda_1^{-3/4}|f|}{\nu - \lambda_1^{-1}L_g}\right\}.$$

Hence, using a Young inequality with $\delta > 0$ to be fixed later on, we conclude that

$$\frac{d}{dt}(e^{\lambda t}|w(t)|^2) \le e^{\lambda t}(-2\nu + \lambda\lambda_1^{-1} + 2\mu + \delta\lambda_1^{-1}L_g)\|w(t)\|^2 + \frac{L_g}{\delta}e^{\lambda t}\|w_t\|_\gamma^2.$$

Therefore, integrating from 0 to t, we have

$$\begin{aligned} e^{\lambda t}|w(t)|^2 \quad \le \quad &|w(0)|^2 + \frac{L_g}{\delta}\int_0^t e^{\lambda s}\|w_s\|_\gamma^2 ds \\ &+(-2\nu + \lambda\lambda_1^{-1} + 2\mu + \delta\lambda_1^{-1}L_g)\int_0^t e^{\lambda s}\|w(s)\|^2 ds. \end{aligned} \tag{48}$$

In order to control the term $\int_0^t e^{\lambda s}\|w_s\|_\gamma^2 ds$, we proceed as follows.

$$\begin{aligned}&\int_0^t e^{\lambda s}\sup_{\theta\le 0} e^{2\gamma\theta}|w(s+\theta)|^2 ds\\ =\;&\int_0^t e^{\lambda s}\max\{\sup_{\theta\le -s} e^{2\gamma\theta}|w(s+\theta)|^2, \sup_{\theta\in[-s,0]} e^{2\gamma\theta}|w(s+\theta)|^2\}ds\\ =\;&\int_0^t \max\{e^{-(2\gamma-\lambda)s}\|\phi-u^*\|_\gamma^2, \sup_{\theta\in[-s,0]} e^{(2\gamma-\lambda)\theta}e^{\lambda(s+\theta)}|w(s+\theta)|^2\}ds.\end{aligned}$$

So, if $\lambda\le 2\gamma$, using the above equality in (48), we obtain

$$\begin{aligned}e^{\lambda t}|w(t)|^2 \;\le\; & |w(0)|^2+\frac{L_g}{\delta}\|\phi-u^*\|_\gamma^2\int_0^t e^{(\lambda-2\gamma)s}ds\\ &+\big(-2\nu+\lambda\lambda_1^{-1}+2\mu+\delta\lambda_1^{-1}L_g+L_g(\lambda_1\delta)^{-1}\big)\int_0^t \max_{r\in[0,s]} e^{\lambda r}\|w(r)\|^2 ds.\end{aligned}$$

Observe that the (optimal) choice of $\delta=1$ makes that $\delta\lambda_1^{-1}L_g+L_g(\lambda_1\delta)^{-1}$ is minimal and the coefficient of the last integral is negative with a suitable choice of $\lambda\in(0,2\gamma)$ by (37). So, we can omit this term and deduce that

$$e^{\lambda t}|w(t)|^2\le |w(0)|^2+\frac{L_g}{2\gamma-\lambda}(1-e^{(\lambda-2\gamma)t})\|\phi-u^*\|_\gamma^2,$$

whence (46) follows.

Finally, (47) can be deduced in the following way:

$$\begin{aligned}\|w_t\|_\gamma^2 &= \sup_{\theta\le 0} e^{2\gamma\theta}|w(t+\theta)|^2\\ &= \max\{\sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}|\phi(t+\theta)-u^*|^2, \max_{\theta\in[-t,0]} e^{2\gamma\theta}|w(t+\theta)|^2\}\\ &= \max\{e^{-2\gamma t}\|\phi-u^*\|_\gamma^2, \max_{\theta\in[-t,0]} e^{2\gamma\theta}|w(t+\theta)|^2\},\end{aligned}$$

and the second term can be estimated using (46) and that $e^{(2\gamma-\lambda)\theta}\le 1$. □

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