Asymptotic behaviour of solutions for a three dimensional system of globally modified Navier-Stokes equations with a locally Lipschitz delay term*

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Abstract

Existence, uniqueness, and continuity properties of solution for a globally modified version of Navier-Stokes equations with finite delay terms within a locally Lipschitz operator are established. Moreover, we also analyze the stationary problem, and, under suitable assumptions, we prove that there exists a unique stationary solution, which is globally asymptotically exponentially stable.

Keywords: Globally Modified Navier-Stokes equations; finite delay.

1 Introduction

In this paper we consider the following system of globally modified Navier-Stokes equations (GMNSE) with delays on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, and with homogeneous Dirichlet boundary condition.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|)\left[(u\cdot\nabla)u\right] + \nabla p = g(t, u_t) \ \text{ in } (0,\infty)\times\Omega, \\ \nabla\cdot u \ = 0 \ \text{ in } (0,\infty)\times\Omega, \\ u = 0 \ \text{ on } (0,\infty)\times\partial\Omega, \\ u(s,x) = \phi(s,x), \ s\in[-h,0], \ x\in\Omega, \end{cases} \tag{1}$$

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where $N \in (0, \infty)$ is fixed, $F_N : [0, \infty) \to (0, 1]$ is defined by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}, \quad r \in [0, \infty),$$

$\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p is the pressure, ϕ is a given function (recent history of the velocity field) defined in interval $[-h, 0] \times \Omega$ with $h > 0$ a fixed value (memory effect), and g is an external force depending on t and u_t, where for each $t \geq 0$, we denote by u_t the function defined on $[-h, 0]$ by the relation $u_t(s) = u(t+s)$, $s \in [-h, 0]$.

The GMNSE (1), with or without delays, are indeed *global* modifications of the Navier-Stokes equations – the modifying factor $F_N(\|u\|)$ depends on the norm $\|u\| = \|\nabla u\|_{(L^2(\Omega))^{3\times 3}}$, which in turn depends on ∇u over the whole domain Ω and not just at or near the point $x \in \Omega$ under consideration. Essentially, it prevents large gradients dominating the dynamics and leading to explosions. It violates the basic laws of mechanics, but mathematically the GMNSE (1) are a well defined system of equations, just like the modified versions of the NSE of Leray and others with other mollifications of the nonlinear term, see the review paper [5]. It is worth mentioning that a global cut off function involving the $D(A^{1/4})$ norm for the two dimensional stochastic Navier-Stokes equations was used in [6], and a cut-off function similar to the one we use here was considered in [23].

The globally modified Navier-Stokes equations, in the case without delays, were introduced and studied in [1] (see also [2, 3, 4, 11, 12, 13] and the review paper [10]). Contrary to the original Navier-Stokes model, this modified model of the three-dimensional Navier-Stokes equations has some good properties as global existence, uniqueness, and regularity. These results are interesting in their own right, but also GMNSE are useful in obtaining new results about the three-dimensional Navier-Stokes equations, e.g., they were used in [1] to establish the existence of bounded entire weak solutions for them. Also, in [13], GMNSE were used to show that the attainability set of the weak solutions of the three-dimensional Navier-Stokes equations satisfying an energy inequality are weakly compact and weakly connected. For convergence results of solutions of GMNSE to solutions of the three-dimensional Navier-Stokes equations see [1, 16].

However, there are situations in which the model is better described if some terms containing delays appear in the equations (e.g. cf. [20] for a case concerning just the Navier-Stokes equations). These delays may appear, for instance, when one wants to control the system by applying a force which takes into account not only the present state but some history of the solutions.

In this paper we are interested in the case of a GMNSE model in which terms containing finite delays appear (see [17] for the case with infinite delays).

A particular case of problem (1) was studied in [4]. Our goal in this paper is to obtain existence and uniqueness of solution, and convergence to stationary solutions in a much more general situation, as is noticed in Remark 2 below. In particular, we only require on the delay term to be sublinear and locally Lipschitz.

The structure of the paper is as follows. Next section is devoted to some preliminaries and to establish the abstract framework for our problem. In Section 3 we prove existence and uniqueness of solution to (1), and a continuous dependence result. Finally, in Section 4, under suitable additional assumptions, we analyze the stationary problem and the convergence of solutions of the evolution problem toward the unique solution of the stationary one, which is proved to hold with an exponential decay.

2 Preliminaries

To set our problem in the abstract framework, we consider the following usual function spaces (see [14] and [18, 21, 22]).
H the closure of $\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0 \right\}$ in $(L^2(\Omega))^3$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$, where for $u, v \in (L^2(\Omega))^3$,

$$(u,v) = \sum_{j=1}^{3} \int_\Omega u_j(x) v_j(x) dx,$$

V the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with scalar product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$, where for $u, v \in (H_0^1(\Omega))^3$,

$$((u,v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

We will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot,\cdot \rangle$ for the duality pairing between V' and V. Finally, we will identify every $u \in H$ with the element $f_u \in V'$ given by

$$\langle f_u, v \rangle = (u,v) \quad \forall\, v \in V.$$

Then, it follows that $V \subset H \subset V'$, where the injections are dense and compact.

We consider the linear continuous operator $A : V \to V'$ defined by

$$\langle Au, v \rangle = ((u,v)) \quad \forall\, u, v \in V. \tag{2}$$

Denoting $D(A) = \{u \in V : \; Au \in H\}$, with inner product $(u,v)_{D(A)} = (Au, Av)$, then, by the regularity of $\partial\Omega$, $D(A) = (H^2(\Omega))^3 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H).

Let us denote

$$\lambda_1 = \inf_{v \in V \setminus \{0\}} \frac{\|v\|^2}{|v|^2} > 0,$$

the first eigenvalue of the Stokes operator.

Now we define the trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall\, u, v, w \in V,$$

and we denote

$$b_N(u,v,w) = F_N(\|v\|) b(u,v,w) \quad \forall\, u, v, w \in V.$$

The form b_N is linear in u and w, but it is nonlinear in v. Evidently we have $b_N(u,v,v)=0$ for all $u,v\in V$. We will also make use of the following inequality (see [21] and [8]).

$$|b(u,v,w)|\leq 2^{-1}|u|^{1/4}\|u\|^{3/4}\|v\||w|^{1/4}\|w\|^{3/4}\quad\forall\, u,v,w\in V. \tag{3}$$

In particular, this implies that there exists a constant $C_1>0$ only dependent on Ω (namely, $C_1=(2\lambda_1^{1/4})^{-1}$) such that

$$|b(u,v,w)|\leq C_1\,\|u\|\|v\|\|w\|\quad\forall\, u,v,w\in V.$$

Thus by the definition of F_N, if we denote

$$\langle B_N(u,v),w\rangle=b_N(u,v,w)\quad\forall\, u,v,w\in V,$$

we have

$$\|B_N(u,v)\|_*\leq NC_1\,\|u\|\quad\forall\, u,v\in V. \tag{4}$$

We recall (see [21]) that there exists a constant $C_2>0$ depending only on Ω such that

$$|b(u,v,w)|\leq C_2\|u\|^{1/2}|Au|^{1/2}\|v\||w|\quad\forall\, u\in D(A),v\in V,w\in H, \tag{5}$$
$$|b(u,v,w)|\leq C_2\|u\|\|v\||w|^{1/2}\|w\|^{1/2}\quad\forall\, u,v,w\in V. \tag{6}$$

(See [19] for the proof of (6)).
We will also use the the following result.

Lemma 1 *([19]) For any $u,v\in V$, and each $N>0$,*

1. $0\leq\|u\|F_N(\|u\|)\leq N$,
2. $|F_N(\|u\|)-F_N(\|v\|)|\leq\frac{1}{N}F_N(\|u\|)F_N(\|v\|)\|u-v\|$.

We denote $C_H=C([-h,0];H)$, with norm $|\phi|_{C_H}=\sup_{s\in[-h,0]}|\phi(s)|$, and $\mathbb{R}_+=(0,\infty)$.

For the term g, we assume that $g:\mathbb{R}_+\times C_H\to(L^2(\Omega))^3$ satisfies

g1) For any $\xi\in C_H$ the mapping $\mathbb{R}_+\ni t\mapsto g(t,\xi)\in(L^2(\Omega))^3$ is measurable,

g2) there exists a nondecreasing function $L_g:\mathbb{R}_+\to\mathbb{R}_+$, such that for all $R\geq 0$, if $|\xi|_{C_H},|\eta|_{C_H}\leq R$, then

$$|g(t,\xi)-g(t,\eta)|\leq L_g(R)\,|\xi-\eta|_{C_H}\quad\forall\, t\in\mathbb{R}_+,$$

g3) there exists a constant $C_g>0$ and a nonnegative function $f\in L^1(0,T)$ for all $T>0$, such that for any $\xi\in C_H$,

$$|g(t,\xi)|^2\leq C_g\,|\xi|^2_{C_H}+f(t)\quad\forall\, t\in\mathbb{R}_+.$$

Finally, we suppose that $\phi\in C_H$.

Remark 2 (cf. [7]) *Consider a globally Lipschitz function* $G : H \to (L^2(\Omega))^3$*, with Lipschitz constant* $L_G > 0$*, and a measurable function* $\rho : \mathbb{R} \to [0, h]$*.*

Then, it is not difficult to check that the operator $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$*, defined by*

$$\mathbb{R} \times C_H \ni (t, \xi) \mapsto g(t, \xi) := G(\xi(-\rho(t)))$$

satisfies the assumptions g1)–g3) given above.

Observe that the only assumption on ρ *is that it is measurable, in contrast with the condition* $\rho \in C^1$*, with derivative* $\rho'(t) \le \rho_* < 1$ *appearing in [4].*

The above example can be generalized in several senses. The most immediate generalization is to take into account more than one delay term in the problem, and to take G *depending on time. Namely, consider* m *measurable functions* $\rho_i : \mathbb{R} \to [0, h]$ *for* $i = 1$ *to* m*, a measurable mapping* $G : \mathbb{R}_+ \times H^m \to (L^2(\Omega))^3$ *such that* $G(t, \cdot)$ *is locally Lipschitz (in the sense given above) and sublinear in* H^m *uniformly with respect to time. Then, consider* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ *given by* $g(t, \xi) := G(t, \xi(-\rho_1(t)), \ldots, \xi(-\rho_m(t)))$*. This operator* g *also satisfies conditions g1)–g3).*

3 Existence of solutions

In this section we establish existence of weak solution to (1).

Definition 3 *A weak solution to (1) is a function* $u \in C([-h, T]; H) \cap L^2(0, T; V)$ *for all* $T > 0$*, with* $u(t) = \phi(t)$ *for all* $t \in [-h, 0]$ *and such that for all* $v \in V$*,*

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b_N(u(t), u(t), v) = (g(t, u_t), v), \quad \text{in } \mathcal{D}'(0, \infty).$$

Remark 4 *If* u *is a weak solution to (1), then* u *satisfies the energy equality*

$$|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr = |u(s)|^2 + 2 \int_s^t (g(r, u_r), u(r)) dr \quad \forall\, 0 \le s \le t < \infty.$$

Theorem 5 *Suppose that* $g : \mathbb{R}_+ \times C_H \to (L^2(\Omega))^3$ *satisfying the assumptions g1)–g3), and* $\phi \in C_H$ *are given. Then, there exists a unique weak solution* $u = u(\cdot; \phi)$ *to (1), which in fact is a strong solution in the sense that*

$$u \in C((0, T]; V) \cap L^2(\varepsilon, T; D(A)) \quad \forall\, 0 < \varepsilon < T.$$

Moreover, if $\phi(0) \in V$*, then* u *satisfies*

$$u \in C([0, T]; V) \cap L^2(0, T; D(A)) \quad \forall\, T > 0.$$

Proof. The proof of uniqueness is similar to that in [15, Theorem 1], and we omit it.

We split the proof of existence in three steps.

Step 1: A Galerkin scheme. First a priori estimates. Let us consider $\{v_j\} \subset V$ the orthonormal basis of H of all the eigenfunctions of the Stokes operator. Denote $V_m = \text{span}[v_1, \ldots, v_m]$ and consider the projector $P_m u = \sum_{j=1}^m (u, v_j) v_j$.

Define also

$$u^m(t) = \sum_{j=1}^{m} \alpha_{m,j}(t) v_j$$

where the upper script m will be used instead of (m) for short since no confusion is possible with powers of u, and where the coefficients $\alpha_{m,j}$ are required to satisfy the following system in the sense of $\mathcal{D}'(0,\infty)$.

$$\frac{d}{dt}(u^m(t), v_j) + \nu((u^m(t), v_j)) + b_N(u^m(t), u^m(t), v_j) = (g(t, u_t^m), v_j), \quad 1 \le j \le m, \tag{7}$$

and the initial condition $u^m(s) = P_m\phi(s)$ for $s \in [-h, 0]$.

In principle, the above system of ordinary functional differential equations has a unique local solution defined in $[0, t_m)$, with $0 < t_m \le \infty$ (see [9]).

We will obtain a priori estimates that guarantee that the solutions do exist for any time $t \in [0, \infty)$.

Let us fix $0 < T < t_m$. Multiplying (7) by u^m, we obtain

$$\begin{aligned} \frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 &= 2(g(t, u_t^m), u^m(t)) \\ &\le 2|g(t, u_t^m)||u^m(t)|, \quad \text{a.e. } t \in (0, T), \end{aligned} \tag{8}$$

and therefore, using Young inequality, and taking into account g3), we can deduce that

$$\frac{d}{dt}|u^m(t)|^2 \le (2\nu\lambda_1)^{-1}(C_g|u_t^m|_{C_H}^2 + f(t)), \quad \text{a.e. } t \in (0, T). \tag{9}$$

Hence, integrating between 0 and t, we deduce

$$|u^m(t)|^2 \le |\phi(0)|^2 + (2\nu\lambda_1)^{-1} \int_0^t \left(C_g|u_s^m|_{C_H}^2 + f(s)\right) ds \quad \forall\, t \in [0, T].$$

From this,

$$|u_t^m|_{C_H}^2 \le |\phi|_{C_H}^2 + (2\nu\lambda_1)^{-1} \int_0^t \left(C_g|u_s^m|_{C_H}^2 + f(s)\right) ds \quad \forall\, t \in [0, T],$$

and therefore, by Gronwall lemma we have

$$|u_t^m|_{C_H}^2 \le e^{C_g(2\nu\lambda_1)^{-1}t} \left(|\phi|_{C_H}^2 + (2\nu\lambda_1)^{-1} \int_0^t f(s) ds \right) \quad \forall\, t \in [0, T]. \tag{10}$$

Thus, we obtain that for any $T \in (0, t_m)$ there exists a constant $C = C(T, |\phi|_{C_H})$, depending on some constants of the problem (namely λ_1, ν, C_g and f), and on T and $|\phi|_{C_H}$, such that

$$|u_t^m|_{C_H}^2 \le C(T, |\phi|_{C_H}) \quad \forall\, t \in [0, T], \quad m \ge 1. \tag{11}$$

In particular, this implies that $t_m = \infty$ for all m, and taking also into account that $u^m(s) = P_m\phi(s)$ for $s \in [-h, 0]$,

$$\{u^m\} \text{ is bounded in } L^\infty(-h, T; H) \quad \forall\, T > 0. \tag{12}$$

Moreover, it also follows from (8), g3) and (11) that

$$\{u^m\} \text{ is bounded in } L^2(0,T;V) \quad \forall T>0. \tag{13}$$

Now, observe that (7) is equivalent to

$$\frac{du^m}{dt} = -\nu A u^m - \widetilde{P}_m B_N(u^m,u^m) + \widetilde{P}_m g(t,u_t^m), \tag{14}$$

where $\widetilde{P}_m : V' \to V'$ is given by $\langle \widetilde{P}_m k, w\rangle = \langle k, P_m w\rangle$ for any $k \in V'$ and $w \in V$.

From (4), g3), and (12)–(14), by the choice of the basis one also deduces that

$$\{(u^m)'\} \text{ is bounded in } L^2(0,T;V') \quad \forall T>0. \tag{15}$$

So, this implies the existence of a function $u \in L^\infty(-h,T;H) \cap L^2(0,T;V)$, with $u' \in L^2(0,T;V')$, for all $T>0$, and a subsequence of $\{u^m\}$ which converges weak-star to u in $L^\infty(-h,T;H)$ and weakly to u in $L^2(0,T;V)$, with $\{(u^m)'\}$ converging weakly to u' in $L^2(0,T;V')$ for all $T>0$. Observe that in particular $u \in C([0,\infty);H)$.

By the Aubin-Lions compactness result (cf. [14, Chapter 1, Theorem 5.1]), one can then deduce that a subsequence in fact converges strongly to u in $L^2(0,T;H)$ and a.e. in $(0,T)$ with values in H and a.e. in $(0,T)\times\Omega$ for all $T>0$.

Step 2: Some more a priori estimates. We need to find a stronger estimate to ensure that $\|u^m(t)\| \to \|u(t)\|$ or at least $F_N(\|u^m(t)\|) \to F_N(\|u(t)\|)$ for a.e. $t>0$, which is needed to pass to the limit in the nonlinear term $B_N(u^m,u^m)$.

Taking the inner product of the Galerkin functional differential system (7) with Au^m, we obtain

$$\frac{d}{dt}\|u^m(t)\|^2 + 2\nu|Au^m(t)|^2 + 2b_N(u^m(t),u^m(t),Au^m(t)) = 2(g(t,u_t^m),Au^m(t)) \tag{16}$$

a.e. $t>0$.

By (5), Lemma 1 and Young inequality, it follows

$$\begin{aligned} |b_N(u^m(t),u^m(t),Au^m(t))| &\le \frac{N}{\|u^m(t)\|} C_2\|u^m(t)\|^{3/2}|Au^m(t)|^{3/2} \\ &\le \frac{\nu}{4}|Au^m(t)|^2 + C_N\|u^m(t)\|^2, \quad a.e.\, t>0, \end{aligned}$$

with $C_N = \dfrac{27(NC_2)^4}{4\nu^3}$, and we also have that

$$|(g(t,u_t^m),Au^m(t))| \le \frac{\nu}{4}|Au^m(t)|^2 + \frac{1}{\nu}|g(t,u_t^m)|^2, \quad a.e.\, t>0.$$

So, (16) simplifies to

$$\frac{d}{dt}\|u^m(t)\|^2 + \nu|Au^m(t)|^2 \le \frac{2}{\nu}|g(t,u_t^m)|^2 + 2C_N\|u^m(t)\|^2, \quad a.e.\, t>0.$$

Integrating between s and t with $0 \leq s \leq t \leq T$ we obtain that

$$
\begin{aligned}
& \|u^m(t)\|^2 + \nu \int_s^t |Au^m(r)|^2 dr \qquad (17) \\
\leq \quad & \|u^m(s)\|^2 + \frac{2}{\nu} \int_0^T |g(r, u_r^m)|^2 dr + 2C_N \int_0^T \|u^m(r)\|^2 dr \quad \forall\, 0 \leq s \leq t \leq T.
\end{aligned}
$$

Now, integrating with respect to s between 0 and t, in particular we obtain

$$
t\|u^m(t)\|^2 \leq \int_0^T \|u^m(s)\|^2 ds + \frac{2T}{\nu} \int_0^T |g(r, u_r^m)|^2 dr + 2C_N T \int_0^T \|u^m(r)\|^2 dr \quad \forall\, 0 \leq t \leq T.
$$

The above two inequalities, (11), (13) and g3) imply that

$$
\{u^m\} \text{ is bounded in } L^\infty(\varepsilon, T; V) \cap L^2(\varepsilon, T; D(A)) \quad \forall\, T > \varepsilon > 0. \qquad (18)
$$

Moreover, observe that if $\phi(0) \in V$, then, since $\|u^m(0)\| = \|P_m \phi(0)\| \leq \|\phi(0)\|$, from (17), (11), (13) and g3) one deduces directly that

$$
\{u^m\} \text{ is bounded in } L^\infty(0, T; V) \cap L^2(0, T; D(A)) \quad \forall\, T > 0. \qquad (19)
$$

Assuming again only that $\phi(0) \in H$, as $D(A) \subset V \subset H$ with compact injection, by [14, Chapter 1, Theorem 5.1], from Step 1 and (18), using a sequence of positive values $\varepsilon_n \downarrow 0$ and a diagonal argument, we deduce that

$$
u \in L^\infty(-h, T; H) \cap L^2(0, T; V) \cap L^\infty(\varepsilon, T; V) \cap L^2(\varepsilon, T; D(A))
$$

for any $T > \varepsilon > 0$, and for a subsequence of $\{u^m\}$, that we relabel the same, we have

$$
\begin{cases}
u^m \rightharpoonup u & \text{weak in } L^2(0, T; V), \\
u^m \overset{*}{\rightharpoonup} u & \text{weak-star in } L^\infty(0, T; H), \\
u^m \to u & \text{a.e. in } (0, T) \times \Omega, \\
u^m \to u & \text{strong in } L^2(\varepsilon, T; V), \\
u^m \rightharpoonup u & \text{weak in } L^2(\varepsilon, T; D(A)), \\
u^m \overset{*}{\rightharpoonup} u & \text{weak-star in } L^\infty(\varepsilon, T; V),
\end{cases} \qquad (20)
$$

for any $T > \varepsilon > 0$.

As long as u^m converges to u in $L^2(\varepsilon, T; V)$ for all $T > \varepsilon > 0$, we may assume, eventually extracting a subsequence, that $\|u^m(t)\| \to \|u(t)\|$ a.e. in $(0, \infty)$, and therefore

$$
F_N(\|u^m(t)\|) \to F(\|u(t)\|), \quad \text{a.e. in } (0, \infty). \qquad (21)
$$

Step 3: Convergence in C_H and existence of solution.

We will prove that for a subsequence (relabelled the same) it holds that $u_t^m \to u_t$ in C_H, for all $t \in [0, \infty)$. To see this, it is enough to prove:

$$
P_m \phi \to \phi \quad \text{in } C_H, \qquad (22)
$$

$$
u^m \to u \quad \text{in } C([0, T]; H) \quad \forall\, T > 0. \qquad (23)
$$

Indeed, for the delay initial datum $\phi \in C_H$, if (22) is not true, there would exist $\varepsilon > 0$ and a subsequence, that we relabel the same, such that

$$|P_m\phi(\theta_m) - \phi(\theta_m)| > \varepsilon \quad \forall\, m \geq 1. \tag{24}$$

Then, there exists $\theta \in [-h, 0]$ such that a subsequence $\{\theta_m\}$ (relabelled the same) satisfies $\theta_m \to \theta$. Now, observe that we have by the triangular inequality

$$|P_m\phi(\theta_m) - \phi(\theta)| \leq |P_m\phi(\theta_m) - P_m\phi(\theta)| + |P_m\phi(\theta) - \phi(\theta)|.$$

Clearly, since $\phi(\theta)$ is fixed, the second addend in the right-hand side goes to zero as $m \to \infty$. Moreover, since P_m is a projection from H onto V_m, it is non-expansive. Therefore, the first addend in the right-hand side satisfies $|P_m\phi(\theta_m) - P_m\phi(\theta)| \leq |\phi(\theta_m) - \phi(\theta)|$, which also converges to zero as $m \to \infty$ by the continuity of ϕ.

So we conclude that $P_m\phi(\theta_m) \to \phi(\theta)$ in H as $m \to \infty$. But this is a contradiction with (24). So, (22) holds.

Now, in order to check (23), we will use an energy method based upon the estimates obtained in Step 1. This method essentially relies on the convergence in almost every time of an interval of a sequence of continuous and non-increasing functions toward another continuous and non-increasing function. This implies the convergence in the whole time interval, which will lead to an absurd in a contradiction argument analogous to the above.

Consider a fixed (but arbitrary) value $T > 0$.
Due to the strong convergence of $\{u^m\}$ to u in $L^2(0, T; H)$, we deduce that a subsequence (still denoted the same) $u^m(t) \to u(t)$ in H a.e. $t \in (0, T)$.

Since we have the equality in V'

$$u^m(t) - u^m(s) = \int_s^t (u^m)'(r)dr \quad \forall\, 0 \leq s \leq t \leq T,$$

from (15) we deduce that $\{u^m\}$ is equi-continuous on $[0, T]$ with values in V'. Therefore, by the Ascoli-Arzelà theorem we have that a subsequence (relabelled the same) satisfies

$$u^m \to u \quad \text{in } C([0, T]; V'). \tag{25}$$

Again from (12) we obtain that for any sequence $\{t_m\} \subset [0, T]$ with $t_m \to t$, one has

$$u^m(t_m) \rightharpoonup u(t) \quad \text{weakly in } H, \tag{26}$$

where we have used (25) in order to identify the weak limit.

Now, we will prove (23) by a contradiction argument. If it would not be so, then, taking into account that $u \in C([0, T]; H)$, there would exist $\varepsilon > 0$, a value $t_0 \in [0, T]$ and subsequences (still denoted the same) $\{u^m\}$ and $\{t_m\} \subset [0, T]$ with $\lim_{m\to\infty} t_m = t_0$ such that

$$|u^m(t_m) - u(t_0)| \geq \varepsilon \quad \forall\, m. \tag{27}$$

In order to prove that this is absurd, we will use an energy method.

By (9) and (11), we have that

$$\begin{aligned} |u^m(t)|^2 \quad &\leq \quad |u^m(s)|^2 + (2\nu\lambda_1)^{-1} C_g C(T, |\phi|_{C_H})(t - s) \\ &\quad + (2\nu\lambda_1)^{-1} \int_s^t f(r)dr \quad \forall\, 0 \leq s \leq t \leq T. \end{aligned} \tag{28}$$

On the other hand, from (11) and $g3$) we deduce the existence of $\xi_g \in L^2(0,T;(L^2(\Omega))^3)$, such that (a subsequence of) $\{g(\cdot, u^m_\cdot)\}$ converges weakly to ξ_g in $L^2(0,T;(L^2(\Omega))^3)$.

Then, from (20) and (21), reasoning as in [1] for the case without delays, we can pass to the limit in equation (14) and deduce that u is solution of

$$\frac{d}{dt}(u(t),v) + \nu((u(t),v)) + \langle B_N(u,u),v\rangle = (\xi_g(t),v) \quad \forall\, v \in V \tag{29}$$

in the sense of $\mathcal{D}'(0,\infty)$.

Therefore, by the energy equality and Young inequality,

$$|u(t)|^2 \le |u(s)|^2 + (2\nu\lambda_1)^{-1}\int_s^t |\xi_g(r)|^2 dr \quad \forall\, 0 \le s \le t \le T.$$

Now, observe that for the weak limit ξ_g we have the estimate

$$\begin{aligned}\int_s^t |\xi_g(r)|^2 dr &\le \liminf_{m\to\infty}\int_s^t |g(r,u^m_r)|^2 dr \\ &\le C_g C(T,|\phi|_{C_H})(t-s) + \int_s^t f(r)dr \quad \forall\, 0 \le s \le t \le T.\end{aligned}$$

So, u also satisfies the inequality (28).

Now, consider the continuous functions $J_m,\ J : [0,T] \to \mathbb{R}$ defined by

$$J_m(t) = |u^m(t)|^2 - \frac{1}{2\nu\lambda_1}\int_0^t f(r)dr - \frac{C_g C(T,|\phi|_{C_H})}{2\nu\lambda_1}t,$$

$$J(t) = |u(t)|^2 - \frac{1}{2\nu\lambda_1}\int_0^t f(r)dr - \frac{C_g C(T,|\phi|_{C_H})}{2\nu\lambda_1}t.$$

From (28) for $\{u^m\}$ and u, it is clear that J_m and J are non-increasing functions. Moreover, by the convergence of u^m to u a.e. in time with values in H, it holds that

$$J_m(t) \to J(t), \quad \text{a.e. } t \in [0,T]. \tag{30}$$

Now we will prove that

$$u^m(t_m) \to u(t_0) \quad \text{in } H, \tag{31}$$

which contradicts (27).

Firstly, recall that from (26) we have that

$$|u(t_0)| \le \liminf_{m\to\infty} |u^m(t_m)|.$$

Therefore, if we show that

$$\limsup_{m\to\infty} |u^m(t_m)| \le |u(t_0)|, \tag{32}$$

we conclude the convergence of the norms, which jointly with (26) implies (31).

Now, observe that the case $t_0 = 0$ follows directly from (28) with $s = 0$ and the definition of $u^m(0) = P_m\phi(0)$. So, we may assume that $t_0 > 0$. This is important, since we will approach this value t_0 from the left by a sequence $\{\tilde{t}_k\}$, i.e. $\lim_{k\to\infty} \tilde{t}_k \nearrow t_0$, being $\{\tilde{t}_k\}$ values where (30) holds. Since $u(\cdot)$ is continuous at t_0, for an arbitrary value $\epsilon > 0$ there is k_ϵ such that

$$|J(\tilde{t}_k) - J(t_0)| < \epsilon/2 \quad \forall\, k \geq k_\epsilon.$$

On other hand, taking $m \geq m(k_\epsilon)$ such that $t_m > \tilde{t}_{k_\epsilon}$, as J_m is non-increasing and for all $\tilde{t}_k$ the convergence (30) holds, one has that

$$J_m(t_m) - J(t_0) \leq |J_m(\tilde{t}_{k_\epsilon}) - J(\tilde{t}_{k_\epsilon})| + |J(\tilde{t}_{k_\epsilon}) - J(t_0)|,$$

and obviously, taking $m \geq m'(k_\epsilon)$, it is possible to obtain $|J_m(\tilde{t}_{k_\epsilon}) - J(\tilde{t}_{k_\epsilon})| < \epsilon/2$. We also have that

$$\int_0^{t_m} f(r)dr \to \int_0^{t_0} f(r)dr \quad \text{as } m \to \infty.$$

So, since $\epsilon > 0$ was arbitrary, we conclude that (32) holds. Thus, (31) and finally (23) are also true, as we wanted to check.

Now, we are ready to pass to the limit in the equations satisfied by the $\{u^m\}$ and to complete the information obtained in (29).

Assume initially that $\phi(0) \in H$. The first clear consequence from the convergence proved above, since g satisfies g2), is that

$$g(\cdot, u^m_\cdot) \to g(\cdot, u_\cdot) \quad \text{in } L^2(0,T;(L^2(\Omega))^3) \quad \forall\, T > 0.$$

Thus, we can identify $\xi_g(t) = g(t, u_t)$ in (29). Therefore u is a solution to (1).

Finally, if $\phi(0) \in V$, from (19) and analogous arguments to those given above we conclude that $u \in C([0,T];V) \cap L^2(0,T;D(A))$ for all $T > 0$. ■

Proposition 6 *Under the assumptions of Theorem 5, the solution to (1) is continuous with respect to the initial condition ϕ. More exactly, there exists a constant $C_3 > 0$, only dependent on ν and the constant C_2 appearing in (6), such that u^i, for $i = 1, 2$, the corresponding solutions to initial data $\phi^i \in C_H$, $i = 1, 2$, satisfy*

$$|u^1_t - u^2_t|^2_{C_H} \leq |\phi^1 - \phi^2|^2_{C_H} e^{(2C_3 N^4 + 2L_g(R_{T,\phi_1,\phi_2}))t}, \tag{33}$$

for all $t \in [0,T]$, where $R_{T,\phi_1,\phi_2} \geq 0$ is given by

$$R^2_{T,\phi_1,\phi_2} = \left(\max(|\phi^1|^2_{C_H}, |\phi^2|^2_{C_H}) + (2\nu\lambda_1)^{-1} \int_0^T f(s)ds \right) e^{C_g(2\nu\lambda_1)^{-1}T}.$$

Proof. For short in what follows, let us introduce the operator $\mathcal{NL} : V \times V \to V'$ given by

$$\langle \mathcal{NL}(u,v), w\rangle := F_N(\|u\|)b(u,u,w) - F_N(\|v\|)b(v,v,w) \quad \forall u, v, w \in V.$$

If we denote $w = u^1 - u^2$, from the energy equality we obtain

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 + \langle \mathcal{NL}(u^1(t), u^2(t)), w(t)\rangle = (g(t,u^1_t) - g(t,u^2_t), w(t)) \tag{34}$$

a.e. $t \in (0,T)$.

From the properties of the trilinear form b, it easily follows that

$$\begin{aligned}\left\langle \mathcal{NL}(u^1(t),u^2(t)),w(t)\right\rangle &= F_N(\|u^1(t)\|)b(w(t),u^1(t),w(t)) \\ &\quad +(F_N(\|u^1(t)\|)-F_N(\|u^2(t)\|))b(u^2(t),u^1(t),w(t)).\end{aligned}$$

Then, using Lemma 1, formula (6) and Young inequality (see [19] for the details), we deduce that there exists a constant $C_3>0$, which depends on C_2 and ν, such that,

$$\left\langle \mathcal{NL}(u^1(t),u^2(t)),w(t)\right\rangle \leq \nu\|w(t)\|^2 + C_3N^4|w(t)|^2. \tag{35}$$

On the other hand, reasoning as for the obtention of (10), we deduce that

$$|u_t^i|_{C_H}^2 \leq \left(|\phi^i|_{C_H}^2 + (2\nu\lambda_1)^{-1}\int_0^T f(s)ds\right)e^{C_g(2\nu\lambda_1)^{-1}T}$$

for all $t\in[0,T]$.

From this inequality we obtain, by g2), a particular (local) Lipschitz constant for g, which can be used, jointly with (35), in (34), whence

$$\begin{aligned}\frac{d}{dt}|w(t)|^2 &\leq 2C_3N^4|w(t)|^2 + 2L_g(R_{T,\phi_1,\phi_2})|w_t|_{C_H}|w(t)| \\ &\leq (2C_3N^4+2L_g(R_{T,\phi_1,\phi_2}))|w_t|_{C_H}^2\end{aligned}$$

for all $t\in[0,T]$.

Thus,

$$|w_t|_{C_H}^2 \leq |\phi^1-\phi^2|_{C_H}^2 + (2C_3N^4+2L_g(R_{T,\phi_1,\phi_2}))\int_0^t |w_s|_{C_H}^2 ds$$

for all $t\in[0,T]$, and therefore, by Gronwall lemma, we conclude (33). ■

4 Stationary solutions and their stability

In this section we will prove that under additional assumptions, the problem (1) admits a unique stationary solution that is globally asymptotically exponentially stable.

From now on we assume that $g:\mathbb{R}_+\times C_H\to L^2(\Omega)^3$ satisfies g1)–g3) with $f(t)=|f|\geq 0$ for all $t\geq 0$, a constant function. We also suppose that g is autonomous, in the sense that there exists a function $g_0: H\to H$ such that

g4) $g(t,w)=g_0(w)$ for all $(t,w)\in[0,\infty)\times H$,

where, with a slight abuse of notation, we identify every element $w\in H$ with the constant function in C_H that is equals to w for any time $t\in[-h,0]$.

We consider the equation

$$\frac{du}{dt}+\nu Au + B_N(u,u) = g(t,u_t) \quad t>0, \tag{36}$$

where A is the operator given by (2). A stationary solution to (36) will be an element $u^*\in V$ such that

$$\nu((u^*,v)) + b_N(u^*,u^*,v) = (g_0(u^*),v) \quad \forall\, v\in V. \tag{37}$$

The proof of the following result is analogous to [15, Theorem 3], and we omit it.

Theorem 7 *Under the above assumptions and notation, if $\lambda_1\nu > C_g^{1/2}$, then:*

(a) The problem (36) admits at least one stationary solution u^*, which indeed belongs to $D(A)$. Moreover, any such stationary solution satisfies the estimate

$$(\nu\lambda_1 - C_g^{1/2})\|u^*\| \le \lambda_1^{1/2}|f|^{1/2}. \tag{38}$$

(b) If the following condition holds,

$$\min\left\{N\lambda_1^{-1/4}, \frac{\lambda_1^{1/4}|f|^{1/2}}{\nu\lambda_1 - C_g^{1/2}}\right\} < \nu - \lambda_1^{-1}L_g(R_g), \tag{39}$$

where

$$R_g = \frac{|f|^{1/2}}{\nu\lambda_1 - C_g^{1/2}}, \tag{40}$$

then, the stationary solution of (36) is unique.

Theorem 8 *Assume that g1)–g4) hold with f a positive constant, $\nu\lambda_1 > C_g^{1/2}$, and that (39) is satisfied. Let u^* be the unique stationary solution of (36). Then,*

a) If $L_g(R) = L_g$ is independent of R, there exist two constants $\lambda > 0$ and $C_\lambda > 0$ such that for any $\phi \in C_H$,

$$|u(t;\phi) - u^*|^2 \le C_\lambda|\phi - u^*|_{C_H}^2 e^{-\lambda t} \quad \forall\, t \ge 2h. \tag{41}$$

b) If $L_g(R)$ is a continuous function of R, there exists a $\mu \in (0, 2\nu\lambda_1)$ such that $\mu(2\nu\lambda_1 - \mu)e^{-\mu h} > C_g$, and

$$\min\left\{N\lambda_1^{-1/4}, \frac{\lambda_1^{1/4}|f|^{1/2}}{\nu\lambda_1 - C_g^{1/2}}\right\} < \nu - \lambda_1^{-1}L_g(\widetilde{R}_g), \tag{42}$$

where $\widetilde{R}_g$ is the positive number given by

$$\widetilde{R}_g^2 = \max\{e^{\mu h}(2\nu\lambda_1 - \mu)^{-1}(\mu - (2\nu\lambda_1 - \mu)^{-1}e^{\mu h}C_g)^{-1}|f|, R_g^2\},$$

with R_g defined by (40), then there exists a constant $\lambda > 0$ such that for each $\phi \in C_H$ there exist $T_\phi \ge 2h$ and $C_{\lambda,\phi} > 0$ such that

$$|u(t;\phi) - u^*|^2 \le C_{\lambda,\phi}|\phi - u^*|_{C_H}^2 e^{-\lambda t} \quad \forall\, t \ge T_\phi. \tag{43}$$

Proof. For short denote $u(t) = u(t;\phi)$. Let us also denote $w(t) = u(t) - u^*$. Considering equations (36) for $u(t)$ and (37) for u^*, one has

$$\frac{d}{dt}(w(t), v) + \nu((w(t), v)) + b_N(u(t), u(t), v) - b_N(u^*, u^*, v) = (g(t, u_t) - g(t, u^*), v)$$

in $\mathcal{D}'(0,\infty)$ for any $v \in V$.

From the energy equality, we obtain for any $\lambda > 0$,

$$\begin{aligned}\frac{d}{dt}(e^{\lambda t}|w(t)|^2) \ \le\ & e^{\lambda t}\left(\lambda|w(t)|^2 - 2\nu\|w(t)\|^2 + 2|b_N(u(t), u(t), w(t)) - b_N(u^*, u^*, w(t))|\right.\\ &\left. +2|g(t, u_t) - g(t, u^*)| \cdot |w(t)|\right), \quad a.e.\, t > 0.\end{aligned} \tag{44}$$

Now observe that

$$\begin{aligned}
& |b_N(u(t),u(t),w(t)) - b_N(u^*,u^*,w(t))| \\
\leq\ & |b_N(u(t),u(t),w(t)) - b_N(u(t),u^*,w(t))| + |b_N(w(t),u^*,w(t))| \\
=\ & |F_N(\|u(t)\|)b(u(t),u(t),w(t)) - F_N(\|u^*\|)b(u(t),u^*,w(t))| \\
& + |F_N(\|u^*\|)b(w(t),u^*,w(t))|,
\end{aligned}$$

and therefore, taking into account that

$$b(u(t),u(t),w(t)) = b(u(t),u^*,w(t)) = -b(u(t),w(t),u^*),$$

from Lemma 1 and inequality (3), we obtain

$$\begin{aligned}
& |b_N(u(t),u(t),w(t)) - b_N(u^*,u^*,w(t))| \\
\leq\ & N^{-1}F_N(\|u(t)\|)F_N(\|u^*\|)\|w(t)\|2^{-1}\lambda_1^{-1/4}\|u(t)\|\|w(t)\|\|u^*\| \\
& + F_N(\|u^*\|)2^{-1}\lambda_1^{-1/4}\|w(t)\|\|u^*\|\|w(t)\|.
\end{aligned}$$

Thus, observing that $F_N(\|u(t)\|)\|u(t)\| \leq N$, we see that

$$|b_N(u(t),u(t),w(t)) - b_N(u^*,u^*,w(t))| \leq \lambda_1^{-1/4}F_N(\|u^*\|)\|u^*\|\|w(t)\|^2,$$

and therefore, as $F_N(\|u^*\|)\|u^*\| \leq N$, and also, $F_N(\|u^*\|) \leq 1$ and $\|u^*\|$ satisfies (38), we deduce that

$$|b_N(u(t),u(t),w(t)) - b_N(u^*,u^*,w(t))| \leq \sigma\|u(t)-u^*\|^2, \quad t>0, \tag{45}$$

where

$$\sigma = \min\left\{N\lambda_1^{-1/4}, \frac{\lambda_1^{1/4}|f|^{1/2}}{\nu\lambda_1 - C_g^{1/2}}\right\}.$$

Case a) Assume that g is globally Lipschitz, i.e., $L_g(R) = L_g$ is independent of R.

In this case, from (44), (45) and Young inequality, we conclude that

$$\frac{d}{dt}(e^{\lambda t}|w(t)|^2) \leq e^{\lambda t}(-2\nu + \lambda\lambda_1^{-1} + 2\sigma + \lambda_1^{-1}L_g)\|w(t)\|^2 + L_g e^{\lambda t}|w_t|^2_{C_H},$$

a.e. $t>0$, for any $\lambda > 0$.

Therefore, integrating from $2h$ to t, and observing that

$$\sup_{r\in[s-h,s]}|w(r)|^2 \leq \lambda_1^{-1}\sup_{r\in[s-h,s]}\|w(r)\|^2 \quad \forall\, s \geq 2h,$$

we have

$$\begin{aligned}
e^{\lambda t}|w(t)|^2 \ \leq\ & e^{2\lambda h}|w(2h)|^2 \\
& + \big(-2\nu+\lambda\lambda_1^{-1}+2\sigma+2\lambda_1^{-1}L_g\big)\int_{2h}^t e^{\lambda s}\sup_{r\in[s-h,s]}\|w(r)\|^2 ds
\end{aligned}$$

for all $t \geq 2h$ and any $\lambda > 0$.

Thus, if we assume (39), and we take $\lambda = 2(\nu - \sigma)\lambda_1 - 2L_g > 0$, we obtain

$$e^{\lambda t}|w(t)|^2 \leq e^{2\lambda h}|w(2h)|^2 \quad \text{for all } t \geq 2h.$$

Hence, by Proposition 6, we deduce (41) with $C_\lambda = e^{2(\lambda+2L_g+2C_3N^4)h}$.

Case b) In this case, we claim that for any $\mu \in (0, 2\nu\lambda_1)$ we have that

$$\begin{aligned} |u_t|^2_{C_H} \quad &\leq \quad e^{\mu h}\{|\phi|^2_{C_H} e^{((2\nu\lambda_1-\mu)^{-1}e^{\mu h}C_g-\mu)t} \\ &\quad +(2\nu\lambda_1 - \mu)^{-1}(\mu - (2\nu\lambda_1 - \mu)^{-1}e^{\mu h}C_g)^{-1}|f|\} \end{aligned} \tag{46}$$

for all $t \geq 0$.

Indeed, from the energy equality and Young inequality, for any $\mu \in (0, 2\nu\lambda_1)$ it holds

$$\begin{aligned} \frac{d}{dt}(e^{\mu t}|u(t)|^2) \quad &\leq \quad e^{\mu t}(\mu - 2\nu\lambda_1)|u(t)|^2 + 2e^{\mu t}|g(t, u_t)||u(t)| \\ &\leq \quad e^{\mu t}\,(2\nu\lambda_1 - \mu)^{-1}\,|g(t, u_t)|^2 \end{aligned}$$

a.e. $t > 0$, and therefore, by g3),

$$\frac{d}{dt}(e^{\mu t}|u(t)|^2) \leq (2\nu\lambda_1 - \mu)^{-1}\,e^{\mu t}\left(C_g|u_t|^2_{C_H} + |f|\right), \quad \text{a.e. } t > 0.$$

Therefore, it is easy to deduce that

$$e^{\mu t}|u_t|^2_{C_H} \quad \leq \quad e^{\mu h}|\phi|^2_{C_H} + (2\nu\lambda_1 - \mu)^{-1}\,e^{\mu h}\int_0^t e^{\mu s}\left(|f| + C_g|u_s|^2_{C_H}\right)ds$$

for all $t \geq 0$, and therefore, by Gronwall lemma, we deduce (46).

By (42) and the continuity of L_g, there exists an $\varepsilon > 0$ such that

$$\sigma = \min\left\{N\lambda_1^{-1/4}, \frac{\lambda_1^{1/4}|f|^{1/2}}{\nu\lambda_1 - C_g^{1/2}}\right\} < \nu - \lambda_1^{-1}L_g(\widetilde{R}_g + \varepsilon),$$

and a $T_\phi \geq 2h$ such that

$$|u_t|_{C_H} \leq \widetilde{R}_g + \varepsilon \quad \forall\, t \geq T_\phi.$$

Now, reasoning as in the case a), we can prove that if we take

$$\lambda = 2(\nu - \sigma)\lambda_1 - 2L_g(\widetilde{R}_g + \varepsilon) > 0,$$

we obtain

$$e^{\lambda t}|w(t)|^2 \leq e^{\lambda T_\phi}|w(T_\phi)|^2 \quad \forall\, t \geq T_\phi.$$

Thus, reasoning as in the proof of Proposition 6, we deduce (43) with

$$C_{\lambda,\phi} = \exp\{(\lambda + 2L_g(\widetilde{R}_g + \varepsilon) + 2C_3N^4)T_\phi\}.$$

■

Remark 9 *The case $f = 0$ can also be treated in the above result. Namely, statement a) follows without changes; however, for statement b), we must perform slight changes. Indeed, observe that if $f = 0$, then $\widetilde{R}_g = 0$ and $L_g(0)$ does not have sense since L_g is defined in $(0, \infty)$. We may circumvent this in two ways, either with L_g extended to $[0, \infty)$ and continuous, either with the more general requirement (at light of the above proof) that $L_g(\widetilde{R}_g)$ in (42) is replaced by $\lim_{t \to \widetilde{R}_g^+} L_g(t)$, which must also be asked to exists.*

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References

[1] T. Caraballo, P. E. Kloeden, and J. Real, Unique strong solutions and V-attractors of a three dimensional system of Globally Modified Navier-Stokes equations, *Adv. Nonlinear Stud.* **6** (2006), 411–436.

[2] T. Caraballo, P. E. Kloeden, and J. Real, Addendum to the paper "Unique strong solutions and V-attractors of a three dimensional system of Globally Modified Navier-Stokes equations", Advanced Nonlinear Studies 6 (2006), 411–436, *Adv. Nonlinear Stud.* **10** (2010), 245–247.

[3] T. Caraballo, P. E. Kloeden, and J. Real, Invariant measures and statistical solutions of the globally modified Navier-Stokes equations, *Discrete Contin. Dyn. Syst. Ser. B* **10** (2008), 761–781.

[4] T. Caraballo, J. Real, and A. M. Márquez-Durán, Three dimensional system of globally modified Navier-Stokes equations with delay, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **20** (2010), 2869–2883.

[5] P. Constantin, Near identity transformations for the Navier-Stokes equations, in "*Handbook of Mathematical Fluid Dynamics*", Vol. II, 117–141, North-Holland, Amsterdam, 2003.

[6] F. Flandoli and B. Maslowski, Ergodicity of the 2-D Navier-Stokes Equation under random perturbations, *Comm. Math. Phys.* **172** (1995), 119–141.

[7] J. García-Luengo, P. Marín-Rubio, and J. Real, Pullback attractors for 2D Navier-Stokes equations with delays and their regularity, *Adv. Nonlinear Stud.* To appear.

[8] M. J. Garrido-Atienza and P. Marín-Rubio, Navier-Stokes equations with delays on unbounded domains, *Nonlinear Anal.* **64** (2006), 1100–1118.

[9] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.

[10] P. E. Kloeden, T. Caraballo, J. A. Langa, J. Real, and J. Valero, The three dimensional globally modified Navier-Stokes equations, in *"Advances in Nonlinear Analysis: Theory Methods and Applications"*, Math. Probl. Eng. Aerosp. Sci. **3** 11–22, Cambridge Scientific Publishers, Cambridge, 2009.

[11] P. E. Kloeden, J. A. Langa, and J. Real, Pullback V-attractors of the 3-dimensional globally modified Navier-Stokes equations, *Commun. Pure Appl. Anal.* **6** (2007), 937–955.

[12] P. E. Kloeden, P. Marín-Rubio, and J. Real, Equivalence of Invariant measures and Stationary Statistical solutions for the autonomous globally modified Navier-Stokes equations, *Commun. Pure Appl. Anal.* **8** (2009), 785–802.

[13] P. E. Kloeden and J. Valero, The weak connectedness of the attainability set of weak solutions of the three-dimensional Navier-Stokes equations, *Proc. R. Soc. Lond. Ser. A* **463** (2007), 1491–1508.

[14] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[15] P. Marín-Rubio, A. M. Márquez-Durán, and J. Real, Three dimensional system of globally modified Navier-Stokes equations with infinite delays, *Discrete Contin. Dyn. Syst. Ser. B* **14** (2010), 655–673.

[16] P. Marín-Rubio, A. M. Márquez-Durán, and J. Real, On the convergence of solutions of Globally Modified Navier-Stokes equations with delays to solutions of Navier-Stokes equations with delays, *Adv. Nonlinear Stud.* **11** (2011), 917–927.

[17] P. Marín-Rubio, A. M. Márquez-Durán, and J. Real, Pullback attractors for globally modified Navier-Stokes equations with infinite delays, *Discrete Contin. Dyn. Syst.* **31** (2011), 779–796.

[18] J. C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge University Press, Cambridge, 2001.

[19] M. Romito, The uniqueness of weak solutions of the globally modified Navier-Stokes equations, *Adv. Nonlinear Stud.* **9** (2009), 425–427.

[20] T. Taniguchi, The exponential behavior of Navier-Stokes equations with time delay external force, *Discrete Contin. Dyn. Syst.* **12** (2005), 997–1018.

[21] R. Temam, *Navier-Stokes equations, Theory and Numerical Analysis*, Second edition, North Holland, Amsterdam, 1979.

[22] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, Second edition, SIAM, Philadelphia, 1995.

[23] Z. Yoshida and Y. Giga, A nonlinear semigroup approach to the Navier-Stokes system, *Comm. Partial Differential Equations* **9** (1984), 215–230.