Reaction-diffusion equations with non-autonomous force in H^{-1} and delays under measurability conditions on the driving delay term

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Abstract

In this paper we analyze the existence of solutions for a reaction-diffusion problem with hereditary effects and a time-dependent force term with values in H^{-1}. The main novelty is that the delay term may be driven by a function under very minimal assumptions, namely, just measurability. This is due to the fact that we only deal with a phase-space of functions continuous in time, allowing this general setting, which might be more useful when less regularity is known in the hereditary mechanism.

After that, we obtain uniform estimates and asymptotic compactness properties (via an energy method) that allow us to ensure the existence of pullback attractors for the associated process to the problem. Actually, we obtain two different families of minimal pullback attractors, namely, those of fixed bounded sets but also for a class of time-dependent families (universe) given by a tempered condition. Finally, from comparison results, we establish relations among them, and under suitable additional assumptions we conclude that these families of attractors are in fact the same object.

Keywords: reaction-diffusion equations; delay; pullback attractors.

1 Introduction and statement of the problem

Reaction-diffusion equations have been intensively developed during the last decades because of their applications in Chemistry, Biology, etcetera. Even just in the last few years it is possible to find many related results and features of such models in the literature, as epidemic systems, cellular neural networks, or problems within random environments (e.g. cf. [5, 15, 22, 25] and the references therein among many others).

Apart of the analysis in finite-time intervals, it has also been deeply studied the asymptotic behaviour of solutions for such kind of problems. Jointly with stability questions, the theory of attractors for the associated

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dynamical systems has also experimented significant improvements. Namely, examples of such advances are the internal structure of the global attractor for Chaffee-Infante equations (for autonomous systems), finite-dimensionality reduction, and trajectory and pullback attractors for non-autonomous dynamical systems (e.g. cf. some results in the monographs [21, 13, 6], the recent study [1], or the papers [12, 20, 26, 18, 23, 19, 24, 17, 2, 16, 3, 4, 14]).

Our aim in this paper is to present an improvement on the conditions for dealing with time-dependent delayed reaction-diffusion problems. Namely, we may consider a reaction-diffusion problem only under measurability conditions on the driving delay terms appearing in the equation and we are able to establish existence results and to study the long-time behaviour of such solutions as well. Before continuing with the description of our results, let us state our problem properly.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We consider the following non-autonomous reaction-diffusion problem with delay effects and homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u) + g(t, u_t) + k(t) & \text{in } \Omega \times (\tau, \infty), \\ u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau + s) = \phi(x, s) & x \in \Omega,\ s \in [-h, 0], \end{cases} \tag{1}$$

where $\tau \in \mathbb{R}$, $f \in C(\mathbb{R})$, g is an operator acting on the solution containing some hereditary characteristics (assumptions on g are given below), the time-dependent force term is $k \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$, and $\phi \in C([-h, 0]; L^2(\Omega))$ is the initial datum, being $h > 0$ the length of the delay effect, and where for each $t \geq \tau$, we denote by u_t the function defined in $[-h, 0]$ by $u_t(s) = u(t + s)$, $s \in [-h, 0]$.

Before describing the assumptions on the function f and the operator g, we introduce some notation that will be used in the paper.

We will denote by $(\cdot, \cdot)$ and $|\cdot|$ the scalar product and norm in $L^2(\Omega)$ respectively; the norm in $L^p(\Omega)$ will be written as $\|\cdot\|_{L^p(\Omega)}$; in $H^1_0(\Omega)$ we will use as (equivalent) scalar product $((\cdot, \cdot)) = (\nabla \cdot, \nabla \cdot)$, with corresponding norm $\|\cdot\|$. We will denote by C_{L^2} the Banach space $C([-h, 0]; L^2(\Omega))$, equipped with the sup-norm. For an element $u \in C_{L^2}$, its norm will be written as $|u|_{C_{L^2}} = \max_{t \in [-h, 0]} |u(t)|$. The duality between $L^p(\Omega)$ and its dual $L^q(\Omega)$, where $q = p/(p-1)$ (when $1 < p < \infty$), will be also denoted by $(\cdot, \cdot)$; and the duality between $H^1_0(\Omega)$ and its dual $H^{-1}(\Omega)$ will be written as $\langle \cdot, \cdot \rangle$. Finally, we will use $\|\cdot\|_*$ for the norm in $H^{-1}(\Omega)$.

Concerning the function f and the operator g, we make the following assumptions.

There exist positive constants α_1, α_2, κ, l, and $p > 2$ such that

$$-\kappa - \alpha_1 |s|^p \leq f(s)s \leq \kappa - \alpha_2 |s|^p \quad \forall\, s \in \mathbb{R}, \tag{2}$$

$$(f(s) - f(r))(s - r) \leq l(s - r)^2 \quad \forall\, r, s \in \mathbb{R}. \tag{3}$$

From (2) we deduce that there exists $c > 0$ such that

$$|f(s)| \leq c(|s|^{p-1} + 1) \quad \forall\, s \in \mathbb{R}. \tag{4}$$

It is well-known that any polynomial of odd degree $f(s) = \sum_{j=0}^{2m+1} c_j s^j$ with $c_{2m+1} < 0$ satisfies the above assumptions (a typical example is the function $f(s) = -s^3 + \lambda s$, that in reaction-diffusion leads to Chaffee-

Infante equations, and more in particular, when $\lambda = 1$, gives rise to the double-potential, for instance, in phase-field models).

Since it will be used below, let us denote

$$\mathcal{F}(s) := \int_0^s f(r)dr.$$

From (2), there exist positive constants $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, and $\widetilde{\kappa}$ such that

$$-\widetilde{\kappa} - \widetilde{\alpha}_1|s|^p \leq \mathcal{F}(s) \leq \widetilde{\kappa} - \widetilde{\alpha}_2|s|^p \quad \forall\, s \in \mathbb{R}. \tag{5}$$

For the operator g we will assume that it is well-defined as $g : \mathbb{R} \times C_{L^2} \to L^2(\Omega)$, and it satisfies

(I) for all $\xi \in C_{L^2}$, the function $\mathbb{R} \ni t \mapsto g(t, \xi) \in L^2(\Omega)$ is measurable,

(II) $g(t, 0) = 0$ for all $t \in \mathbb{R}$,

(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$ and ξ, $\eta \in C_{L^2}$, it holds

$$|g(t, \xi) - g(t, \eta)| \leq L_g|\xi - \eta|_{C_{L^2}}.$$

Observe that the assumption (II) does not represent any restriction, since we may consider new functions $\tilde{g}$ and $\tilde{k}$ defined for each $t \in \mathbb{R}$ and $\xi \in C_{L^2}$ by $\tilde{g}(t, \xi) = g(t, \xi) - g(t, 0)$ and $\tilde{k}(t) = k(t) + g(t, 0)$. Now, $\tilde{g}$ satisfies (I)–(III) provided that g satisfied (I) and (III); and $\tilde{k} \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ if $g(\cdot, 0) \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$.

When comparing with similar non-autonomous delayed problems in the literature (for instance, with variable time delay), one may check that the assumptions on the delay operator in order to deal with phase-spaces square integrable in time include more restrictions than our hypotheses, as C^1 regularity for the function driving the delay time and control of square norms in time of the delay operators by the original arguments in an augmented time interval. We do not require such additional conditions, since our phase-space is composed by functions continuous in time, which in this framework does not really seem a restriction.

Indeed, we emphasize that the assumptions on g (introduced in [8]) and the fact that we are dealing just with C_{L^2} terms allow us to include driving delay terms under the only assumption of measurability. Namely, suppose that $\rho : \mathbb{R} \to [0, h]$ measurable and $G : L^2(\Omega) \to L^2(\Omega)$ globally Lipschitz with $G(0) = 0$ are given. Then, $g : \mathbb{R} \times C_{L^2} \to L^2(\Omega)$ defined by $\mathbb{R} \times C_{L^2} \ni (t, \xi) \mapsto g(t, \xi) := G(\xi(-\rho(t)))$ satisfies the assumptions (I)–(III) given above. Moreover, examples with multiple delays can also be included (cf. [8, Remark 2.1]).

The structure of the paper is the following. In Section 2 we establish existence and uniqueness of solution for the problem under the above conditions. Continuity with respect to initial data of the solution operator is also studied. Then, in Section 3, after a very brief survey of some abstract results of pullback attractors theory, we consider the natural (non-autonomous) dynamical system associated to the problem through the previous result and we analyze conditions in order to obtain pullback attractors for it. An energy method that relies strongly on the continuity of the solutions is involved. Actually, we obtain two different families of minimal pullback attractors, namely, those of fixed bounded sets but also for a class of time-dependent families (universe) given by a tempered condition. Finally, from comparison results, we establish relations among them, and under suitable additional assumptions we conclude that these families of attractors are in fact the same object.

2 Existence and uniqueness of solution

In this paragraph we analyze the existence and uniqueness of solution to problem (1).

Definition 1 *A weak solution to (1) is a function* $u \in C([\tau - h, \infty); L^2(\Omega))$ *such that* $u \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ *for all* $T > \tau$, *with* $u(t) = \phi(t-\tau)$ *for all* $t \in [\tau - h, \tau]$, *and such that for all* $v \in H_0^1(\Omega) \cap L^p(\Omega)$, *it satisfies*

$$\frac{d}{dt}(u(t), v) + ((u(t), v)) = (f(u(t)), v) + (g(t, u_t), v) + \langle k(t), v\rangle \quad \text{in } \mathcal{D}'(\tau, \infty).$$

Remark 2 *Observe that if* u *is a weak solution to (1), then it satisfies the energy equality*

$$|u(t)|^2 + 2\int_s^t \|u(r)\|^2 dr = |u(s)|^2 + 2\int_s^t [(f(u(r)), u(r)) + (g(r, u_r), u(r)) + \langle k(r), u(r)\rangle] dr$$

for all $\tau \leq s \leq t$.

The main result of this section is the following

Theorem 3 *Consider* f *and* g *satisfying (2) and (3), and (I)–(III) respectively,* $k \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$, $\tau \in \mathbb{R}$, *and* $\phi \in C_{L^2}$ *given. Then, there exists a unique weak solution* $u(\cdot) = u(\cdot; \tau, \phi)$ *to (1).*

Proof. Uniqueness. Suppose that u and v are two weak solutions to (1), and denote $w = u - v$, which solves the equation

$$\frac{\partial w}{\partial t} - \Delta w = f(u) - f(v) + g(t, u_t) - g(t, v_t)$$

with $w_\tau = 0$ as initial datum, and homogeneous Dirichlet boundary condition.

Then, from the energy equality (cf. Remark 2) we have

$$|w(t)|^2 + 2\int_\tau^t \|w(s)\|^2 ds = 2\int_\tau^t [(f(u(s)) - f(v(s)), w(s)) + (g(s, u_s) - g(s, v_s), w(s))] ds$$

for all $t \geq \tau$.

Using (3), (III), and Cauchy-Schwarz inequality, we deduce that

$$|w(t)|^2 + 2\int_\tau^t \|w(s)\|^2 ds \leq 2l \int_\tau^t |w(s)|^2 ds + 2L_g \int_\tau^t |w_s|^2_{C_{L^2}} ds$$

for all $t \geq \tau$.

In particular, and since $w_\tau = 0$, we conclude that

$$|w_t|^2_{C_{L^2}} \leq 2(l + L_g) \int_\tau^t |w_s|^2_{C_{L^2}} ds \quad \forall\, t \geq \tau,$$

whence uniqueness follows by Gronwall's inequality.

Existence. We split the proof in three steps, proving an intermediate result. Consider $\{w_j\}_{j\geq 1} \subset H_0^1(\Omega) \cap L^p(\Omega)$ a Hilbert basis of $L^2(\Omega)$ such that $\text{span}\{w_j\}_{j\geq 1}$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$.

Claim. Consider $k \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, $n \in \mathbb{N}$ fixed, and $\phi \in C([-h, 0]; \text{span}\{w_j\}_{j=1}^n)$ (in particular, observe that $\phi(0) \in H_0^1(\Omega) \cap L^p(\Omega)$). Then, there exists a (unique) weak solution $\tilde{u}$ to (1).

The uniqueness has been proved in the previous argument. In order to establish the existence, we state the following approximate system for any $m \geq n$. We seek $\tilde{u}^m(t, x) = \sum_{j=1}^m \gamma_{mj}(t) w_j(x)$ (observe that we use an upper script to avoid misleading with the delay terminology, but no brackets are used since no possible confusion arises in this sense) that solves

$$\begin{cases} \dfrac{d}{dt}(\tilde{u}^m(t), w_j) + ((\tilde{u}^m(t), w_j)) = (f(\tilde{u}^m(t)), w_j) + (g(t, \tilde{u}^m_t), w_j) + (k(t), w_j), \ a.e.\, t > \tau, \ 1 \le j \le m, \\ \tilde{u}^m_\tau = \phi. \end{cases}$$

It is well-known that the above finite-dimensional delayed system is well-posed (e.g. cf. [9]), at least locally. We will provide a priori estimates that show that these solutions are well-defined in every interval $[\tau - h, T]$ for any $T > \tau$.

Step 1: First a priori estimates. Multiplying each equation in the above system by $\gamma_{mj}(t)$ respectively and summing from $j = 1$ to m, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\tilde{u}^m(t)|^2 + \|\tilde{u}^m(t)\|^2 &= (f(\tilde{u}^m(t)), \tilde{u}^m(t)) + (g(t, \tilde{u}^m_t), \tilde{u}^m(t)) + (k(t), \tilde{u}^m(t)) \\ &\le \kappa|\Omega| - \alpha_2\|\tilde{u}^m(t)\|^p_{L^p(\Omega)} + L_g|\tilde{u}^m_t|_{C_{L^2}}|\tilde{u}^m(t)| + \frac{1}{2}\|k(t)\|^2_* + \frac{1}{2}\|\tilde{u}^m(t)\|^2, \ a.e.\, t > \tau, \end{aligned}$$

where we have used (2), (II), (III), and Cauchy-Schwarz and Young inequalities.

Therefore, we arrive at

$$\frac{d}{dt}|\tilde{u}^m(t)|^2 + \|\tilde{u}^m(t)\|^2 + 2\alpha_2\|\tilde{u}^m(t)\|^p_{L^p(\Omega)} \le 2\kappa|\Omega| + 2L_g|\tilde{u}^m_t|^2_{C_{L^2}} + \|k(t)\|^2_*, \quad a.e.\, t > \tau.$$

Integrating in $[\tau, t]$ we obtain

$$\begin{aligned} & |\tilde{u}^m(t)|^2 + \int_\tau^t \|\tilde{u}^m(s)\|^2 ds + 2\alpha_2 \int_\tau^t \|\tilde{u}^m(s)\|^p_{L^p(\Omega)} ds \\ \le\ & |\tilde{u}^m(\tau)|^2 + 2L_g \int_\tau^t |\tilde{u}^m_s|^2_{C_{L^2}} ds + \int_\tau^t \|k(s)\|^2_* ds + 2\kappa|\Omega|(t - \tau) \end{aligned} \tag{6}$$

for all $t \ge \tau$.

In particular, putting $t + \theta$ instead of t, with $\theta \in [-h, 0]$, we deduce that

$$|\tilde{u}^m_t|^2_{C_{L^2}} \le |\phi|^2_{C_{L^2}} + 2L_g \int_\tau^t |\tilde{u}^m_s|^2_{C_{L^2}} ds + \int_\tau^t \|k(s)\|^2_* ds + 2\kappa|\Omega|(t - \tau) \quad \forall\, t \ge \tau,$$

and by Gronwall's inequality, it yields

$$|\tilde{u}^m_t|^2_{C_{L^2}} \le \left(|\phi|^2_{C_{L^2}} + \int_\tau^t \|k(s)\|^2_* ds + 2\kappa|\Omega|(t - \tau) \right) e^{2L_g(t-\tau)} \quad \forall\, t \ge \tau, \ m \ge n. \tag{7}$$

So, from this and (6), we obtain that

$$\{\tilde{u}^m\}_{m \ge n} \text{ is bounded in } L^2(\tau, T; H^1_0(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau, T]; L^2(\Omega))$$

for all $T > \tau$.

Now, using (4), we have that $\{f(\tilde{u}^m)\}_{m \ge n}$ is bounded in $L^q(\tau, T; L^q(\Omega))$ for all $T > \tau$.

So, there exist functions $\tilde{u} \in L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega))$ and $\widetilde{\chi} \in L^q(\tau, T; L^q(\Omega))$ for all $T > \tau$, and a subsequence (relabelled the same) such that

$$\begin{cases} \tilde{u}^m \overset{*}{\rightharpoonup} \tilde{u} & \text{weakly-star in } L^\infty(\tau, T; L^2(\Omega)), \\ \tilde{u}^m \rightharpoonup \tilde{u} & \text{weakly in } L^p(\tau, T; L^p(\Omega)), \\ \tilde{u}^m \rightharpoonup \tilde{u} & \text{weakly in } L^2(\tau, T; H^1_0(\Omega)), \\ f(\tilde{u}^m) \rightharpoonup \widetilde{\chi} & \text{weakly in } L^q(\tau, T; L^q(\Omega)) \end{cases} \tag{8}$$

for all $T > \tau$.

Step 2: Uniform estimates for the time-derivatives. Now, we proceed to obtain a second set of estimates that will provide the convergence of a subsequence in C_{L^2}.

Multiplying each equation of the approximate system by $\gamma'_{mj}(t)$ and summing from $j = 1$ to m, we obtain

$$\begin{aligned} |(\tilde{u}^m)'(t)|^2 + \frac{1}{2}\frac{d}{dt}\|\tilde{u}^m(t)\|^2 &= (f(\tilde{u}^m(t)), (\tilde{u}^m)'(t)) + (g(t, \tilde{u}^m_t) + k(t), (\tilde{u}^m)'(t)) \\ &\leq \frac{d}{dt}\int_\Omega \mathcal{F}(\tilde{u}^m(t,x))dx + |g(t,\tilde{u}^m_t)|^2 + |k(t)|^2 + \frac{1}{2}|(\tilde{u}^m)'(t)|^2, \ a.e.\, t > \tau. \end{aligned}$$

Integrating between τ and t, from (II), (III), and by (5), we have

$$\begin{aligned} &\int_\tau^t |(\tilde{u}^m)'(s)|^2 ds + \|\tilde{u}^m(t)\|^2 + 2\widetilde{\alpha}_2\|\tilde{u}^m(t)\|^p_{L^p(\Omega)} \\ \leq\ & 4\tilde{\kappa}|\Omega| + 2\widetilde{\alpha}_1\|\tilde{u}^m(\tau)\|^p_{L^p(\Omega)} + \|\tilde{u}^m(\tau)\|^2 + 2L_g^2\int_\tau^t |\tilde{u}^m_s|^2_{C_{L^2}} ds + 2\int_\tau^t |k(s)|^2 ds \end{aligned}$$

for all $t \geq \tau$ and any $m \geq n$.

Since $\tilde{u}^m_\tau = \phi$ for all $m \geq n$ and in particular $\tilde{u}^m(\tau) = \phi(0) \in H^1_0(\Omega) \cap L^p(\Omega)$, by (7), we deduce that

$$\{\tilde{u}^m\}_{m\geq n} \text{ is bounded in } L^\infty(\tau, T; H^1_0(\Omega) \cap L^p(\Omega))$$

for all $T > \tau$ and

$$\{(\tilde{u}^m)'\}_{m\geq n} \text{ is bounded in } L^2(\tau, T; L^2(\Omega)) \tag{9}$$

for all $T > \tau$. Then, we improve the regularity of $\tilde{u}$ obtained in Step 1. Actually, $\tilde{u} \in L^\infty(\tau, T; H^1_0(\Omega) \cap L^p(\Omega))$ with $\tilde{u}' \in L^2(\tau, T; L^2(\Omega))$ for all $T > \tau$.

Fixing (an arbitrary value) $T > \tau$, since

$$|\tilde{u}^m(t_2) - \tilde{u}^m(t_1)|^2 = \left|\int_{t_1}^{t_2} (\tilde{u}^m)'(s)ds\right|^2 \leq \|(\tilde{u}^m)'\|^2_{L^2(\tau,T;L^2(\Omega))}|t_2 - t_1| \quad \forall\, t_1, t_2 \in [\tau, T],$$

from (9), the compactness of the embedding of $H^1_0(\Omega)$ into $L^2(\Omega)$, by the Ascoli-Arzelà Theorem, and taking into account the initial data for all the sequence, we deduce that there exists a subsequence (relabelled the same) such that

$$\tilde{u}^m \to \tilde{u} \text{ in } C([\tau - h, T]; L^2(\Omega)) \tag{10}$$

for all $T > \tau$ and a.e. in $\Omega \times (\tau, \infty)$.

Since $f \in C(\mathbb{R})$, we conclude that $f(\tilde{u}^m) \to f(\tilde{u})$ a.e. in $\Omega \times (\tau, \infty)$. So, from (8) and [10, Lemma 1.3, page 12] we may identify $\widetilde{\chi} = f(\tilde{u})$.

Thus, from (8) and (10) we may pass to the limit in the equations satisfied by $\{\tilde{u}^m\}$ and, thanks to the fact that span$\{w_j\}_{j\geq 1}$ is dense in $H^1_0(\Omega) \cap L^p(\Omega)$, we conclude that $\tilde{u}$ is a weak solution to (1). The claim is proved.

Step 3: Proof of the general statement by density. For each $n \in \mathbb{N}$, define $\phi^n(s) = \sum_{j=1}^n (\phi(s), w_j)w_j$, $s \in [-h, 0]$. [Due to the fact that $\{w_j\}_{j\geq 1}$ is a Hilbert basis of $L^2(\Omega)$, it is not difficult to check by contradiction that $\phi^n \to \phi$ in C_{L^2}.]

Let also consider a sequence $\{k^n\} \subset L^2_{loc}(\mathbb{R}; L^2(\Omega))$ converging to k in $L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$.

Denote by u^n the corresponding solution to (1) with k replaced by k^n and initial data $u^n_\tau = \phi^n$ (according to the claim proved in steps 1 and 2).

From the energy equality for each u^n, we have

$$|u^n(t)|^2 + 2\int_\tau^t \|u^n(s)\|^2 ds = |u^n(\tau)|^2 + 2\int_\tau^t (f(u^n(s)), u^n(s))ds + 2\int_\tau^t (g(s, u^n_s) + k^n(s), u^n(s))ds \quad \forall\, t \geq \tau.$$

Reasoning analogously to Step 1, we conclude that

$$\{u^n\} \text{ is bounded in } L^2(\tau, T; H^1_0(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau - h, T]; L^2(\Omega))$$

for all $T > \tau$.

Now, using (4), we have that $\{f(u^n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$ for all $T > \tau$.

So, there exist functions $u \in L^\infty(\tau - h, T; L^2(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega))$ and $\chi \in L^q(\tau, T; L^q(\Omega))$ for all $T > \tau$, and a subsequence (relabelled the same) such that

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau - h, T; L^2(\Omega)), \\ u^n \rightharpoonup u & \text{weakly in } L^p(\tau, T; L^p(\Omega)), \\ u^n \rightharpoonup u & \text{weakly in } L^2(\tau, T; H^1_0(\Omega)), \\ f(u^n) \rightharpoonup \chi & \text{weakly in } L^q(\tau, T; L^q(\Omega)) \end{cases} \tag{11}$$

for all $T > \tau$.

Actually, we may improve some of the above convergences. Reasoning as in the proof of the uniqueness of solution, i.e. taking into account the difference of the equations satisfied by u^n and u^m, and writing the energy equality for $u^n - u^m$, we have

$$\begin{aligned} &|u^m(t) - u^n(t)|^2 + \int_\tau^t \|u^m(s) - u^n(s)\|^2 ds \\ \leq\ &|u^m(\tau) - u^n(\tau)|^2 + 2l\int_\tau^t |u^m(s) - u^n(s)|^2 ds + 2L_g\int_\tau^t |u^m_s - u^n_s|^2_{C_{L^2}} ds + \int_\tau^t \|k^m(s) - k^n(s)\|^2_* ds \end{aligned} \tag{12}$$

for all $t \geq \tau$, and so, in particular,

$$|u^m_t - u^n_t|^2_{C_{L^2}} \leq |\phi^m - \phi^n|^2_{C_{L^2}} + 2(l + L_g)\int_\tau^t |u^m_s - u^n_s|^2_{C_{L^2}} ds + \int_\tau^t \|k^m(s) - k^n(s)\|^2_* ds \quad \forall\, t \geq \tau.$$

From Gronwall's inequality and (12), we have

$$\{u^n\} \text{ is a Cauchy sequence in } L^2(\tau, T; H^1_0(\Omega)) \cap C([\tau - h, T]; L^2(\Omega))$$

for all $T > \tau$.

So, up to a subsequence (still relabelled the same), we have $u^n \to u$ a.e. in $\Omega \times (\tau, \infty)$.

Thus, as before, from (11) and [10, Lemma 1.3, page 12] we may identify $\chi = f(u)$; and from (11) we may pass to the limit in the equations satisfied by $\{u^n\}$ and we conclude that u is a weak solution to (1). ■

Although the following regularity result is not important for the goal of this paper (but of a forthcoming one), it is worth to present it here since is a simple generalization of the intermediate claim proved above.

Corollary 4 *Consider f and g satisfying (2) and (3), and (I)–(III) respectively, $k \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, $\phi \in C_{L^2}$ with $\phi(0) \in H^1_0(\Omega) \cap L^p(\Omega)$, and suppose that there exist a family $\{w_j\}_{j\geq 1} \subset H^1_0(\Omega) \cap L^p(\Omega)$ dense in $H^1_0(\Omega) \cap L^p(\Omega)$ and $\{\phi^n\} \subset C_{L^2}$ with each $\phi^n \in C([-h, 0]; span\{w_i\}_{i=1}^n)$, $\phi^n \to \phi$ in C_{L^2} and $\phi^n(0) \to \phi(0)$ in $H^1_0(\Omega) \cap L^p(\Omega)$. Then, there exists a unique weak solution u to (1), which also satisfies $u \in L^\infty(\tau, T; H^1_0(\Omega) \cap L^p(\Omega))$ and $u' \in L^2(\tau, T; L^2(\Omega))$ for all $T > \tau$.*

Proof. It is analogous to steps 1 and 2 in the proof of Theorem 3, with the natural modification in the initial data in the Galerkin scheme, taking into account that, without loss of generality, we may assume that the functions $\{w_j\}_{j\geq 1}$ are linearly independent. ■

Remark 5 *Given a family $\{w_j\}_{j\geq 1} \subset H^1_0(\Omega) \cap L^p(\Omega)$ dense in $H^1_0(\Omega) \cap L^p(\Omega)$ (and therefore dense in $L^2(\Omega)$), it is always possible to assume that $\{w_j\}_{j\geq 1}$ is a Hilbert basis in $L^2(\Omega)$ –by a Gram-Schmidt ortho-normalization process–, and then $P_n\phi := \sum_{j=1}^n (\phi, w_j)w_j \to \phi$ in C_{L^2}. However, it might not occur at the same time that $P_n\phi(0) \to \phi(0)$ in $H^1_0(\Omega) \cap L^p(\Omega)$, which motivates this requirement in the statement given above.*

Nevertheless, it is really the case in some situations, as for instance if $N \leq 2p/(p-2)$ (without extra conditions on Ω) or with $\Omega \subset \mathbb{R}^N$ a bounded C^k domain with $k \geq 2$ and $k \geq N(p-2)/(2p)$, since then, a special basis formed by eigenfunctions of $-\Delta$ with homogeneous Dirichlet boundary conditions satisfies all the requirements (namely in the first case $H^1_0(\Omega)$ is continuously embedded in $L^p(\Omega)$; and in the second case $span\{w_j\}_{j\geq 1}$ is dense in $H^k(\Omega) \cap H^1_0(\Omega)$, which is continuously embedded in $L^p(\Omega)$).

Concerning the solutions to (1), we have the following result, which shows continuity with respect to initial data.

Proposition 6 *Consider f and g satisfying (2) and (3), and (I)–(III) respectively, and $k \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ given. Then, for any $\tau \in \mathbb{R}$, and ϕ, $\psi \in C_{L^2}$, the solutions $u = u(\cdot; \tau, \phi)$ and $v = v(\cdot; \tau, \psi)$ to (1) with respective initial data ϕ and ψ satisfy*

$$|u_t - v_t|^2_{C_{L^2}} \leq |\phi - \psi|^2_{C_{L^2}} e^{2(l+L_g)(t-\tau)} \quad \forall\, t \geq \tau. \tag{13}$$

Proof. Denote by $w = u - v$, which solves the problem $\partial w/\partial t - \Delta w = f(u) - f(v) + g(t, u_t) - g(t, v_t)$, with $w_\tau = \phi - \psi$ and homogeneous Dirichlet boundary condition. Then, from the energy equality we have

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \|w(t)\|^2 = (f(u(t)) - f(v(t)), w(t)) + (g(t, u_t) - g(t, v_t), w(t)), \quad a.e.\ t > \tau.$$

Using (3) and (III), and integrating in time, we may deduce that

$$|w(t)|^2 + 2\int_\tau^t \|w(s)\|^2 ds \leq |w(\tau)|^2 + 2(l + L_g)\int_\tau^t |w_s|^2_{C_{L^2}} ds \quad \forall\, t \geq \tau,$$

whence in particular

$$|w_t|^2_{C_{L^2}} \leq |w_\tau|^2_{C_{L^2}} + 2(l + L_g)\int_\tau^t |w_s|^2_{C_{L^2}} ds \quad \forall\, t \geq \tau.$$

Now, by applying Gronwall's inequality, it yields to (13). ■

3 Pullback attractors

The goal of this section is to ensure, under certain conditions, the existence of pullback attractors for a suitable dynamical system associated to problem (1). In order to proceed, let us first introduce some definitions and abstract results that will be applied later.

Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \le t\}$.

A process U on X is a mapping $\mathbb{R}^2_d \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ such that $U(\tau,\tau)x = x$ for any $(\tau,x) \in \mathbb{R} \times X$, and $U(t,r)(U(r,\tau)x) = U(t,\tau)x$ for any $\tau \le r \le t$ and all $x \in X$.

Definition 7 *A process U on X is said to be continuous if for any $\tau \le t$, the map $U(t,\tau) : X \to X$ is continuous.*

The process U is said to be closed if for any $\tau \le t$, and any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.

If a process is continuous, then it is closed. Therefore, it is more general (and can be more useful for applications) to establish a theory within the concept of closed process.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 8 We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Denote

$$\Lambda_X(\widehat{D}_0, t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau) D_0(\tau)}^X \qquad \forall\, t \in \mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Given two subsets of X, $\mathcal{O}_1$ and $\mathcal{O}_2$, we denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between them, defined as

$$\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x \in \mathcal{O}_1} \inf_{y \in \mathcal{O}_2} d_X(x,y).$$

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 9 *A process U on X is called pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.*

It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \le t$ such that $U(t,\tau)D(\tau) \subset D_0(t)$ for all $\tau \le \tau_0(t,\widehat{D})$.

With the above definitions, we may establish the following result (cf. [7, Theorem 3.11]).

Theorem 10 *Consider a closed process $U : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}$-absorbing for U, and assume also that U is pullback $\widehat{D}_0$-asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by $\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda_X(\widehat{D}, t)}^X$, satisfies* *(a) for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda_X(\widehat{D}_0, t)$,*

(b) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e., $\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ for all $\widehat{D} \in \mathcal{D}$, and any $t \in \mathbb{R}$,

(c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e., $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \leq t$,

(d) if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda_X(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, $\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), C(t)) = 0$, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark 11 *Under the assumptions of Theorem 10, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}$-attractor for the process U.*

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

We will denote by $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result (cf. [7, 11]).

Corollary 12 *Under the assumptions of Theorem 10, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F(X)$, then both attractors, $\mathcal{A}_{\mathcal{D}_F(X)}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and $\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$.*

Moreover, if for some $T \in \mathbb{R}$, the set $\cup_{t \leq T} D_0(t)$ is a bounded subset of X, then $\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \leq T$.

After the above briefly recall of basic concepts on non-autonomous dynamical systems and pullback attractors, we apply them to problem (1).

Proposition 13 *Consider f and g satisfying (2) and (3), and (I)–(III) respectively, and $k \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ given. Then, the bi-parametric family of maps $U(t,\tau) : C_{L^2} \to C_{L^2}$, with $\tau \leq t$, given by*

$$U(t,\tau)\phi = u_t, \tag{14}$$

where $u = u(\cdot;\tau,\phi)$ is the unique weak solution to (1), defines a continuous process on C_{L^2}.

Proof. The well-possedness of U follows from Theorem 3, and the continuity of this process is a consequence of Proposition 6. ■

The following result will be useful to have asymptotic estimates for the process defined above. To state it, let us denote

$$0 < \lambda_1 = \min_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{\|v\|^2}{|v|^2}.$$

Lemma 14 *Under the assumptions of Proposition 13, there exist positive constants* δ_1 *and* δ_2 *such that for any* $\tau \in \mathbb{R}$ *and* $\phi \in C_{L^2}$*, the solution* $u(\cdot;\tau,\phi)$ *to (1) satisfies*

$$|u_t|^2_{C_{L^2}} \leq e^{\lambda_1 h-(\lambda_1-\delta_1^{-1}L_g^2 e^{\lambda_1 h})(t-\tau)}|u_\tau|^2_{C_{L^2}} + e^{\lambda_1 h}\int_\tau^t e^{-(\lambda_1-\delta_1^{-1}L_g^2e^{\lambda_1 h})(t-s)}(2\kappa|\Omega| + \delta_2^{-1}\|k(s)\|_*^2)ds, \quad (15)$$

$$\int_\tau^t (\|u(s)\|^2 + 2\alpha_2\|u(s)\|^p_{L^p(\Omega)})ds \leq |u(\tau)|^2 + \int_\tau^t (2\kappa|\Omega| + \delta_1^{-1}L_g^2|u_s|^2_{C_{L^2}} + \delta_2^{-1}\|k(s)\|_*^2)ds \quad (16)$$

for all $t \geq \tau$.

Proof. Analogously as done in the proof of existence in Theorem 3, we have that the energy equality for the solution u, jointly with assumptions (2), (II), (III), and a sharper use of Young inequality, leads to

$$\frac{d}{dt}|u(t)|^2 + \|u(t)\|^2 + 2\alpha_2\|u(t)\|^p_{L^p(\Omega)} \leq 2\kappa|\Omega| + \delta_1^{-1}L_g^2|u_t|^2_{C_{L^2}} + \delta_2^{-1}\|k(t)\|_*^2, \quad a.e.\ t > \tau, \quad (17)$$

where we have chosen $0 < \delta_1$, δ_2 such that $\delta_2 + \delta_1\lambda_1^{-1} = 1$.

In particular, introducing the exponential $e^{\lambda_1 t}$, and integrating, we deduce that

$$e^{\lambda_1 t}|u(t)|^2 \leq e^{\lambda_1\tau}|u(\tau)|^2 + \int_\tau^t e^{\lambda_1 s}(2\kappa|\Omega| + \delta_1^{-1}L_g^2|u_s|^2_{C_{L^2}} + \delta_2^{-1}\|k(s)\|_*^2)ds \quad \forall\, t \geq \tau.$$

Now, this yields

$$e^{\lambda_1 t}|u_t|^2_{C_{L^2}} \leq e^{\lambda_1(h+\tau)}|u_\tau|^2_{C_{L^2}} + e^{\lambda_1 h}\int_\tau^t e^{\lambda_1 s}(2\kappa|\Omega| + \delta_1^{-1}L_g^2|u_s|^2_{C_{L^2}} + \delta_2^{-1}\|k(s)\|_*^2)ds \quad \forall\, t \geq \tau.$$

Therefore, (15) follows from Gronwall's inequality. Finally, (16) is a direct consequence of (17). ■

Here on we will assume that

$$0 < \lambda_1 - L_g e^{\lambda_1 h/2}, \quad (18)$$

$$\int_{-\infty}^0 e^{(\lambda_1 - L_g e^{\lambda_1 h/2})s}\|k(s)\|_*^2 ds < \infty. \quad (19)$$

Observe that if we assume that $k \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$, then (19) is equivalent to have

$$\int_{-\infty}^t e^{(\lambda_1 - L_g e^{\lambda_1 h/2})s}\|k(s)\|_*^2 ds < \infty \quad \forall\, t \in \mathbb{R}.$$

Remark 15 *It is clear that a combination of the estimate (15) for a particular choice of* δ_1 *and* δ_2 *and assumption (18) will play an essential role in the study of asymptotic estimates. Nevertheless, it should be pointed out that other dissipativity conditions (again by using a generalized Young inequality) would be possible. These conditions would depend on the delay parameters, h and* L_g*, and on* λ_1*, and its benefit is related to each particular situation but not general. Therefore, we keep in the sequel with* $\lambda_1 - L_g e^{\lambda_1 h/2}$ *just for the sake of clarity in the exposition.*

Definition 16 *For any* $\sigma > 0$*, we will denote by* $\mathcal{D}_\sigma(C_{L^2})$ *the class of all families of nonempty subsets* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_{L^2})$ *such that*

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{v\in D(\tau)}|v|^2_{C_{L^2}}\right) = 0.$$

As in the abstract setting introduced above, we will denote by $\mathcal{D}_F(C_{L^2})$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded set of C_{L^2}.

Remark 17 *Observe that for any $\sigma > 0$, $\mathcal{D}_F(C_{L^2}) \subset \mathcal{D}_\sigma(C_{L^2})$ and that $\mathcal{D}_\sigma(C_{L^2})$ is inclusion-closed.*

For short, we will denote from now on

$$\hat{\sigma} = \lambda_1 - L_g e^{\lambda_1 h/2}.$$

Corollary 18 *Under the assumptions of Proposition 13, if moreover (18) and (19) are satisfied, then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ with $D_0(t) = \overline{B}_{C_{L^2}}(0, \rho(t))$, the closed ball in C_{L^2} of center zero and radius $\rho(t)$, where*

$$\rho^2(t) = 1 + e^{\lambda_1 h}\hat{\sigma}^{-1}\left(2\kappa|\Omega| + \lambda_1 \int_{-\infty}^{t} e^{-\hat{\sigma}(t-s)}\|k(s)\|_*^2 ds\right), \tag{20}$$

is pullback $\mathcal{D}_{\hat{\sigma}}(C_{L^2})$-absorbing for the process U defined by (14). Moreover, $\widehat{D}_0 \in \mathcal{D}_{\hat{\sigma}}(C_{L^2})$.

Proof. It is an immediate application of Lemma 14, taking $\delta_1 = L_g e^{\lambda_1 h/2} > 0$ and $\delta_2 = 1 - \delta_1\lambda_1^{-1} > 0$. ■

The final key in order to establish the existence of minimal pullback attractors for the process U is the following

Proposition 19 *Under the assumptions of Corollary 18, the process U defined by (14) is pullback $\mathcal{D}_{\hat{\sigma}}(C_{L^2})$-asymptotically compact.*

Proof. Consider fixed $t_0 \in \mathbb{R}$, $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_{\hat{\sigma}}(C_{L^2})$, $\{\tau_n\} \subset \mathbb{R}$ with $\tau_n < t_0$ for all $n \geq 1$, and $\lim_n \tau_n = -\infty$, and $\{\varphi^n\} \subset C_{L^2}$ with $\varphi^n \in D(\tau_n)$ for all n.

We have to check that the set $\{u_{t_0}^n\} \subset C_{L^2}$ is relatively compact, where $u^n = u(\cdot; \tau_n, \varphi^n)$ is the weak solution to (1) with $u_{\tau_n}^n = \varphi^n$.

Fix a value $T > h$. From Corollary 18 we have that there exists $n_0 = n_0(t_0, T)$ such that $\tau_n < t_0 - T$ for all $n \geq n_0$, and

$$|u_t^n|_{C_{L^2}}^2 \leq R(t_0, T) \quad \forall\, t \in [t_0 - T, t_0],\, n \geq n_0, \tag{21}$$

where

$$R(t_0, T) = 1 + e^{\lambda_1 h}\hat{\sigma}^{-1}\left(2\kappa|\Omega| + \lambda_1 e^{\hat{\sigma}T}\int_{-\infty}^{t_0} e^{-\hat{\sigma}(t_0-s)}\|k(s)\|_*^2 ds\right).$$

In particular, $|u^n(t)|^2 \leq R(t_0, T)$ for all $t \in [t_0 - T, t_0]$ and $n \geq n_0$. If we denote $y^n(t) = u^n(t + t_0 - T)$ for all $t \in [0, T]$, we have that $\{y^n\}_{n \geq n_0}$ is bounded in $L^\infty(0, T; L^2(\Omega))$.

Moreover, for each $n \geq n_0$, y^n is a weak solution on $[0, T]$ to an analogous problem to (1), namely with k and g replaced respectively by

$$\tilde{k}(t) = k(t + t_0 - T) \quad \text{and} \quad \tilde{g}(t, \cdot) = g(t + t_0 - T, \cdot), \quad t \in (0, T),$$

and with $y_0^n = u_{t_0 - T}^n$ and $y_T^n = u_{t_0}^n$.

We also have from (21) that $|y_0^n|_{C_{L^2}}^2 \leq R(t_0, T)$ for all $n \geq n_0$, whence, using (16) with values δ_1 and δ_2 as in the proof of Corollary 18, it yields

$$\|y^n\|_{L^2(0,T;H_0^1(\Omega))}^2 + 2\alpha_2\|y^n\|_{L^p(0,T;L^p(\Omega))}^p \leq K(t_0, T) \quad \forall\, n \geq n_0,$$

where

$$K(t_0,T) = R(t_0,T) + 2\kappa|\Omega|T + L_g e^{-\lambda_1 h/2} R(t_0,T)T + \lambda_1 \hat{\sigma}^{-1} \int_{t_0-T}^{t_0} \|k(s)\|_*^2 ds.$$

Therefore, $\{y^n\}_{n\geq n_0}$ is bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$ (in particular, also in $L^2(0,T;H_0^1(\Omega) \cap L^p(\Omega))$). Then, $\{f(y^n)\}_{n\geq n_0}$ is bounded in $L^q(0,T;L^q(\Omega))$ and $\{(y^n)'\}_{n\geq n_0}$ is bounded in $L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega))$ (in particular, in $L^q(0,T;H^{-1}(\Omega)+L^q(\Omega))$), thanks to (21), (II), (III), and the fact that $-\Delta$ defines an isometric isomorphism from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

So, from compactness results (e.g. cf. [10, 6]) we conclude that there exist $y \in L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega)) \cap C([0,T];L^2(\Omega))$ with $y' \in L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega))$, and $\chi \in L^q(0,T;L^q(\Omega))$ such that a subsequence of $\{y^n\}_{n\geq n_0}$ (relabelled the same) converges to y weakly-star in $L^\infty(0,T;L^2(\Omega))$, weakly in $L^2(0,T;H_0^1(\Omega))$ and in $L^p(0,T;L^p(\Omega))$, strongly in $L^2(0,T;L^2(\Omega))$ and a.e. in $\Omega\times(0,T)$, with $(y^n)'$ converging to y' weakly in $L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega))$, and with $f(y^n)$ converging to χ weakly in $L^q(0,T;L^q(\Omega))$. In particular, from above and by [10, Lemma 1.3, page 12], we may identify $\chi = f(y)$.

From the boundedness of $\{y^n\}_{n\geq n_0}$ and $\{(y^n)'\}_{n\geq n_0}$ in $L^\infty(0,T;L^2(\Omega))$ and $L^q(0,T;H^{-1}(\Omega)+L^q(\Omega))$ respectively, and the compactness of the injection of $L^2(\Omega)$ in $H^{-1}(\Omega)$, and therefore in $H^{-1}(\Omega)+L^q(\Omega)$, we deduce (by applying Ascoli-Arzelà Theorem) that for any sequence $\{s_n\} \subset [0,T]$ with $s_n \to s_*$, one has

$$y^n(s_n) \rightharpoonup y(s_*) \quad \text{weakly in } L^2(\Omega). \tag{22}$$

On the other hand, from (II), (III), and (21) we have

$$\int_s^t |\tilde{g}(r,y_r^n)|^2 dr \leq C(t-s) \quad \forall\, 0 \leq s \leq t \leq T,\ n \geq n_0, \tag{23}$$

where $C>0$ is independent of n. This implies (up to another subsequence) that there exists $\xi \in L^2(0,T;L^2(\Omega))$ such that

$$\tilde{g}(\cdot, y_\cdot^n) \rightharpoonup \xi \quad \text{weakly in } L^2(0,T;L^2(\Omega)),$$

and therefore,

$$\int_s^t |\xi(r)|^2 dr \leq C(t-s) \quad \forall\, 0 \leq s \leq t \leq T. \tag{24}$$

Thus, from all the above convergences we may conclude that y is the unique weak solution to the equation

$$\frac{\partial u}{\partial t} - \Delta u = f(u) + \xi(t) + \tilde{k}(t)$$

fulfilled with homogeneous Dirichlet boundary condition and $u(0) = y(0)$.

From the energy equality, (2), (23), (24), and Young inequality, we obtain that

$$\frac{1}{2}|z(t)|^2 \leq \frac{1}{2}|z(s)|^2 + \int_s^t \langle \tilde{k}(r), z(r)\rangle dr + \left(\frac{C}{4\lambda_1} + \kappa|\Omega|\right)(t-s) \quad \forall\, 0 \leq s \leq t \leq T,$$

where $z = y^n$ or $z = y$. Then, the maps J_n and $J : [0,T] \to \mathbb{R}$ defined by

$$\begin{aligned} J_n(t) &= \frac{1}{2}|y^n(t)|^2 - \int_0^t \langle \tilde{k}(r), y^n(r)\rangle dr - \left(\frac{C}{4\lambda_1} + \kappa|\Omega|\right) t, \\ J(t) &= \frac{1}{2}|y(t)|^2 - \int_0^t \langle \tilde{k}(r), y(r)\rangle dr - \left(\frac{C}{4\lambda_1} + \kappa|\Omega|\right) t, \end{aligned}$$

are non-increasing, continuous, and satisfy

$$J_n(t) \to J(t) \quad a.e.\, t \in (0,T). \tag{25}$$

We may use this to prove that $y^n \to y$ in $C([\delta,T];L^2(\Omega))$ for any $0<\delta<T$. Indeed, if this is not true, there exist $0<\delta_*<T$, $\varepsilon_*>0$, and subsequences $\{y^m\}\subset\{y^n\}_{n\geq n_0}$ and $\{t_m\}\subset[\delta_*,T]$ with $t_m\to t_*\in[\delta_*,T]$ such that

$$|y^m(t_m)-y(t_*)|\geq\varepsilon_* \quad \forall\, m. \tag{26}$$

Fix $\varepsilon>0$. On the one hand, from (25) and since J is continuous and non-increasing, there exists $0<t_\varepsilon<t_*$ such that

$$\lim_m J_m(t_\varepsilon)=J(t_\varepsilon) \qquad \text{and} \qquad 0\leq J(t_\varepsilon)-J(t_*)\leq\varepsilon.$$

On the other hand, as $t_m\to t_*$, there exists m_ε such that $t_\varepsilon<t_m$ for all $m\geq m_\varepsilon$. Then, from above we have

$$\begin{aligned} J_m(t_m)-J(t_*) &\leq J_m(t_\varepsilon)-J(t_*) \\ &\leq |J_m(t_\varepsilon)-J(t_\varepsilon)|+|J(t_\varepsilon)-J(t_*)| \\ &\leq |J_m(t_\varepsilon)-J(t_\varepsilon)|+\varepsilon \end{aligned}$$

for all $m\geq m_\varepsilon$, which implies that $\limsup_m J_m(t_m)\leq J(t_*)+\varepsilon$. But, since $\varepsilon>0$ is arbitrary, we conclude that

$$\limsup_m J_m(t_m)\leq J(t_*).$$

As long as for $t_m\to t_*$ we have

$$\int_0^{t_m}\langle\tilde{k}(r),y^m(r)\rangle dr\to\int_0^{t_*}\langle\tilde{k}(r),y(r)\rangle dr,$$

we deduce that $\limsup_m|y^m(t_m)|\leq|y(t_*)|$. From this inequality and (22), we have that $y^m(t_m)\to y(t_*)$ strongly in $L^2(\Omega)$, which is a contradiction with (26).

Thus, we have that $y^n\to y$ in $C([\delta,T];L^2(\Omega))$ for any $0<\delta<T$. As we chose $T>h$, we conclude in particular that $u^n_{t_0}\to y_T$ in C_{L^2}. ■

We may now establish the main result of the paper.

Theorem 20 *Consider f and g satisfying (2) and (3), and (I)–(III) respectively, and $k\in L^2_{loc}(\mathbb{R};H^{-1}(\Omega))$ given. Suppose moreover that conditions (18) and (19) are satisfied. Then, there exist the minimal pullback $\mathcal{D}_F(C_{L^2})$-attractor $\mathcal{A}_{\mathcal{D}_F(C_{L^2})}$ and the minimal pullback $\mathcal{D}_{\hat{\sigma}}(C_{L^2})$-attractor $\mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(C_{L^2})}$ for the process U defined by (14). The family $\mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(C_{L^2})}$ belongs to $\mathcal{D}_{\hat{\sigma}}(C_{L^2})$, and it holds that*

$$\mathcal{A}_{\mathcal{D}_F(C_{L^2})}(t)\subset\mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(C_{L^2})}(t)\subset\overline{B}_{C_{L^2}}(0,\rho(t)) \quad \forall\, t\in\mathbb{R}. \tag{27}$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(C_{L^2})}$ is a consequence of Theorem 10, since U is continuous (cf. Proposition 13) and therefore closed, the existence of a pullback absorbing family was given by Corollary 18, and in Proposition 19 we have proved the pullback $\mathcal{D}_{\hat{\sigma}}(C_{L^2})$-asymptotic compactness.

By Remark 17 and Corollary 12, the case of fixed bounded sets follows immediately. Then, we also deduce the first inclusion in (27).

Finally, Theorem 10 also implies the last inclusion in (27) and the fact that $\mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(C_{L^2})} \in \mathcal{D}_{\hat{\sigma}}(C_{L^2})$, since the sufficient conditions in Remark 11 hold. Namely, $\mathcal{D}_{\hat{\sigma}}(C_{L^2})$ is inclusion-closed, by construction $D_0(t)$ is closed in C_{L^2} for all $t \in \mathbb{R}$, and $\widehat{D}_0 \in \mathcal{D}_{\hat{\sigma}}(C_{L^2})$ (cf. Corollary 18). ■

Remark 21 *If, additionally, it holds that*

$$\sup_{r \leq 0} \int_{-\infty}^{r} e^{-\hat{\sigma}(r-s)} \|k(s)\|_*^2 ds < \infty,$$

then $\rho(\cdot)$, given by (20), is uniformly bounded. We may apply then Corollary 12 to deduce that

$$\mathcal{A}_{\mathcal{D}_F(C_{L^2})}(t) = \mathcal{A}_{\mathcal{D}_{\hat{\sigma}}(C_{L^2})}(t) \quad \forall\, t \in \mathbb{R}.$$

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