Pullback attractors for three dimensional non-autonomous Navier-Stokes-Voigt equations

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Abstract

In this paper we consider a non-autonomous Navier-Stokes-Voigt model, to which a continuous process can be associated. We study the existence and relationship between minimal pullback attractors for this process in two different frameworks, namely, for the universe of fixed bounded sets, and also for another universe given by a tempered condition.

Since the model does not have a regularizing effect, to obtaining asymptotic compactness for the process is a more involved task. We prove this in a relatively simple way just by using an energy method. Our results simplify –and in some aspects generalize– some of those obtained previously for the autonomous and non-autonomous cases, since for example in Section 4, regularity is not required for the boundary of the domain and the force may take values in V'. Under additional suitable assumptions, regularity results for these families of attractors are also obtained, via bootstrapping arguments. Finally, we also conclude some results concerning the attraction in the $D(A)$ norm.

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1 Introduction and setting of the problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with, unless otherwise indicated, smooth enough (e.g., C^2) boundary $\partial\Omega$.

We consider the following problem for a system of non-autonomous Navier-Stokes-Voigt equations,

$$\begin{cases} \dfrac{\partial}{\partial t}\left(u - \alpha^2 \Delta u\right) - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) \text{ in } \Omega \times (\tau, \infty), \\ \operatorname{div} u = 0 \text{ in } \Omega \times (\tau, \infty), \\ u = 0 \text{ on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the unknown velocity field of the fluid and p is the unknown pressure, and we are given the kinematic viscosity $\nu > 0$, a length scale parameter $\alpha > 0$, characterizing the elasticity of the fluid (in the sense that the ratio α^2/ν describes the reaction time that is required for the fluid to respond to the applied force), an initial velocity field u_τ at the initial time $\tau \in \mathbb{R}$, and an external

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force term f, depending on time.

The Navier-Stokes-Voigt (NSV) model of viscoelastic incompresible fluid was introduced by Oskolkov in [20], gives an approximate description of the Kelvin-Voigt fluid (see [21, 14]), and recently was proposed as a regularization of the 3D-Navier-Stokes equation for the purpose of direct numerical simulations in [1].

The extra regularizing term $-\alpha^2 \Delta u_t$ changes the parabolic character of the equation, which makes it so that in 3D the problem is well-posed (forward and backward), but one does not observe any immediate smoothing of the solution, as expected in parabolic PDEs. Moreover, the generated semigroup is only asymptotically compact, similarly to damped hyperbolic systems.

One of the studied topics about the problem is the inviscid question in some different senses. It is also worth to observe that when $\nu = 0$, the inviscid equation that one recovers is the simplified Bardina subgrid scale model of turbulence. The relationship between the original and inviscid models was also addressed in [1]. On other hand, some questions on the inviscid regularization have been recently used for the study of a 2D surface quasi-geostrophic model (cf. [12]).

The long-time dynamics of the autonomous model was studied by Kalantarov [9] and Kalantarov and Titi [11]. Namely, the existence of global compact attractor was proved, and estimates on its fractal and Hausdorff dimensions, and upper bounds on the number of determining modes were given. Other related results are the Gévrey regularity of the global attractor (again for the autonomous model) when the force term is analytic of Gévrey type, and the establishment of similar statistical properties (and invariant measures) as the 3D-Navier-Stokes equations (cf. [10, 15, 22]).

On the other hand, the analysis of data in real applications indicates in general that the force term in applied fluid mechanic problems might not be autonomous, but non-autonomous. In this sense, there exist some different approaches in order to study the asymptotic behaviour, as that of uniform attractors (e.g., cf. [8] and [4, 26] and the references therein for earlier papers); trajectory attractors (again cf. [4] and the references therein); and more recently cocycle and pullback attractors (e.g., cf. [24, 25, 6, 13]) for both deterministic and random cases.

Our main goal in this paper is to obtain sufficient conditions such that the minimal pullback attractors for the process associated to (1.1) do exist. This can be done in different universes (see [18] for a comparison of these concepts, and [17, 19, 7] for some applications), namely, the classical one of fixed bounded sets, and more recently, in the framework of a general class of families, composed by time-dependent sets, given by a tempered growth condition at $-\infty$.

As commented before, the difference of this model in comparison with the *standard* 2D-Navier-Stokes (NS) model is that there exists a regularizing effect in the Navier-Stokes model (in 2D), while not here. For NS a continuous energy method can be applied thanks to the extra estimates that holds in higher norms (e.g., cf. [19]), which does not seem to hold for the Navier-Stokes-Voigt model. Some of the proofs in the previously cited references about NSV (e.g., cf. [11]) rely on splitting the problem in two, one with exponential decay, and the other with good asymptotic properties in the domain of a suitable fractional power of the Stokes operator. However, we will provide a simpler proof, which does not require the above-mentioned technicalities, but a sharp use of the energy equality, and the energy method used by Rosa in [23]. Moreover, it is worth to point out that our results in Section 4 do not use the regularity assumption on $\partial\Omega$ at all, and the force term may take values in V' instead of in L^2 as appears in [11].

We may also cite in this non-autonomous framework the paper [29], where the existence of uniform attractor for a Navier-Stokes-Voigt model is studied. However, there appears the same treatment with the fractional powers of the Stokes operator, and they require more regularity in the non-autonomous case that we need here.

As a second goal, we analyze some additional properties of the obtained attractor. Namely, extra regularity is deduced by using a bootstrapping argument, which now does rely on fractional powers of the Stokes operator, similarly as done in [11] for the autonomous case. Attraction in $D(A)$ norm is also proved by using the energy method as before and previous results on strong solutions.

The structure of the paper is the following. In Section 2 we recall some definitions of classical abstract functional spaces, useful for the establishment of our problem in an abstract setting. For completeness, we include the proof on existence and uniqueness of weak solution for problem (1.1), and a regularity property. Two continuous dependence results with respect to the initial datum, in the strong and weak senses, are also provided. In Section 3 we present a brief summary on abstract results in order to ensure the existence of minimal pullback attractor in a general universe. Moreover, we point out some relations between two possible families of attractors, each of them associated to the two cited universes, that of fixed bounded sets, and another one given by a tempered condition. This will be applied in Section 4. Namely, we prove there that the conditions in order to ensure the existence of pullback attractors in each suitable universe are fulfilled. To be more precise, both –pullback absorbing and pullback asymptotic compactness properties– are obtained from a rather general condition on the V' norm of f, square integrable in $(-\infty, 0)$ with an exponential weight. As a consequence the two announced families of pullback attractors, and relations among them, are obtained. In Section 5 regularity results for the obtained attractors will be deduced thanks to splitting the solution in sum of two for two different problems, using carefully a bootstrapping argument that involves fractional powers of the Stokes operator. Finally, in Section 6 the problem of attraction in $D(A)$ norm is studied, and indeed under suitable assumptions, all attractors are proved to coincide.

2 Existence and uniqueness of solution

In this section we analyze existence, uniqueness, and regularity properties of the solutions to problem (1.1). At least, part of these results may be found in [20], but for convenience of the reader, they are developed here. In order to proceed, we need initially to pose the problem in an abstract setting, recalling some definitions of functional spaces, operators, and some of their properties (for the details see [28]).

To start, we consider the usual spaces in the variational theory of Navier-Stokes equations: H, the closure of $\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}$ in $(L^2(\Omega))^3$ with norm $|\cdot|$, and inner product $(\cdot,\cdot)$, and V, the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with norm $\|\cdot\|$, and inner product $((\cdot,\cdot))$.

We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality $\langle V', V\rangle$. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v\rangle = (h, v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and compact.

Define the linear continuous operator $A : V \to V'$ as

$$\langle Au, v\rangle = ((u, v)) \quad \forall u, v \in V.$$

Let us denote $D(A) = \{u \in V : Au \in H\}$. Observe that by the regularity of $\partial\Omega$, one has that $D(A) = (H^2(\Omega))^3 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H). The set of eigenvalues of A, repeated according to their multiplicities, is an infinite sequence $\{\lambda_j\}_{j\geq 1}$, with $0 < \lambda_1 \leq \lambda_2 \leq \dots$, and $\lim_{j\to\infty} \lambda_j = \infty$. We will denote by $\{w_j\}_{j\geq 1} \subset D(A)$ the Hilbert basis of H of all the normalized eigenfunctions of the Stokes operator A ($Aw_j = \lambda_j w_j$, and $|w_j| = 1$).

For the fractional powers of A, we have the following inclusions with continuous injection (cf. [27, Ch.III, Lem.2.4.2, Lem.2.4.3])

$$D(A^{\beta}) \subset (L^{6/(3-4\beta)}(\Omega))^3, \quad \forall\, 0 \le \beta < 3/4, \tag{2.1}$$

$$D(A^{3/4}) \subset (L^p(\Omega))^3, \quad \forall\, 1 \le p < \infty, \tag{2.2}$$

and

$$D(A^{\beta}) \subset (L^{\infty}(\Omega))^3, \quad \forall\, 3/4 < \beta \le 1. \tag{2.3}$$

Let us define

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

for every functions $u, v, w : \Omega \to \mathbb{R}^3$ for which the right-hand side is well defined.

In particular, b has sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$, i.e., there exists a constant $C_1 > 0$ such that

$$|b(u,v,w)| \le C_1 \|u\| \|v\| \|w\| \quad \forall u, v, w \in V. \tag{2.4}$$

Important properties concerning b are that

$$b(u,v,w) = -b(u,w,v), \quad \forall u, v, w \in V, \tag{2.5}$$

$$b(u,v,v) = 0, \quad \forall u, v \in V, \tag{2.6}$$

and, using Agmon inequality (e.g., cf. [5]), we can assure that there exists a constant $C_2 > 0$ such that

$$|b(u,v,w)| \le C_2 |Au|^{1/2} \|u\|^{1/2} \|v\| |w|, \quad \forall u \in D(A), v \in V, w \in H. \tag{2.7}$$

For any $u \in V$, we will use $B(u)$ to denote the element of V' given by

$$\langle B(u), w \rangle = b(u,u,w), \quad \forall w \in V.$$

Thus, by (2.4),

$$\|B(u)\|_* \le C_1 \|u\|^2, \quad \forall u \in V, \tag{2.8}$$

and in particular, by (2.7) and the identification of H' with H, if $u \in D(A)$, then $B(u) \in H$, with

$$|B(u)| \le C_2 |Au|^{1/2} \|u\|^{3/2}, \quad \forall u \in D(A). \tag{2.9}$$

In fact, from (2.3), one also deduces that if $u \in D(A^{\beta})$ with $3/4 < \beta \le 1$, then $B(u) \in H$, and more exactly

$$|B(u)| \le C_{(\beta)} |A^{\beta} u| \|u\|, \quad \forall u \in D(A^{\beta}), \quad \forall\, 3/4 < \beta \le 1. \tag{2.10}$$

Analogously, if $0 \le \beta < 3/4$, from (2.1) one obtains that if $u \in D(A^{\beta}) \cap V$, $B(u) \in D(A^{\beta - 3/4})$, and more exactly

$$|A^{\beta - 3/4} B(u)| \le C_{(\beta)} |A^{\beta} u| \|u\|, \quad \forall u \in D(A^{\beta}) \cap V, \quad \forall\, 0 < \beta < 3/4. \tag{2.11}$$

Finally, in the case $\beta = 3/4$, from (2.2) one can see that if $u \in D(A^{3/4})$, then $B(u) \in D(A^{-\delta})$ for all $\delta > 0$, and more exactly

$$|A^{-\delta} B(u)| \le C_{(3/4,\delta)} |A^{3/4} u| \|u\|, \quad \forall u \in D(A^{3/4}), \quad \forall \delta > 0.$$

Before studying (1.1), we treat the autonomous equation $u + \alpha^2 Au = g$.

From Lax-Milgram lemma, we know that for each $g \in V'$ there exists a unique $u_g \in V$ such that

$$u_g + \alpha^2 A u_g = g. \tag{2.12}$$

The mapping $\mathcal{C} : u \in V \mapsto u + \alpha^2 Au \in V'$ is linear and bijective, with $\mathcal{C}^{-1}g = u_g$. From (2.12), one has $|u_g|^2 + \alpha^2\|u_g\|^2 \leq \|g\|_*\|u_g\|$, and in particular, $\|u_g\| \leq \alpha^{-2}\|g\|_*$, i.e.,

$$\|\mathcal{C}^{-1}g\| \leq \alpha^{-2}\|g\|_*, \quad \forall g \in V'. \tag{2.13}$$

Observe that by the definition of $D(A)$, we also have that $\mathcal{C}^{-1}(H) = D(A)$, and reasoning as for the obtention of (2.13), we deduce that

$$\begin{aligned} |Au_g| &= \alpha^{-2}|g - u_g| \\ &\leq 2\alpha^{-2}|g|, \quad \forall g \in H. \end{aligned} \tag{2.14}$$

Assume that $u_\tau \in V$ and $f \in L^2_{loc}(\mathbb{R}; V')$.

Definition 1 *It is said that u is a weak solution to* (1.1) *if u belongs to $L^2(\tau, T; V)$ for all $T > \tau$, and satisfies*

$$\frac{d}{dt}(u(t) + \alpha^2 Au(t)) + \nu Au(t) + B(u(t)) = f(t), \quad \textit{in } \mathcal{D}'(\tau, \infty; V'), \tag{2.15}$$

and

$$u(\tau) = u_\tau. \tag{2.16}$$

Remark 2 *If $u \in L^2(\tau, T; V)$ for all $T > \tau$ and satisfies* (2.15)*, then the function v defined by*

$$v(t) = u(t) + \alpha^2 Au(t) \quad t > \tau, \tag{2.17}$$

belongs to $L^2(\tau, T; V')$ for all $T > \tau$, and by (2.8)*, $v' = \dfrac{dv}{dt} \in L^1(\tau, T; V')$ for all $T > \tau$.*

Consequently, $v \in C([\tau, \infty); V')$, and therefore, by (2.13)*, $u \in C([\tau, \infty); V)$. In particular,* (2.16) *has a sense.*

Moreover, again by (2.8) *and* (2.15)*, $v' \in L^2(\tau, T; V')$ for all $T > \tau$, and therefore, as $u' = \mathcal{C}^{-1}v'$, we deduce that $u' \in L^2(\tau, T; V)$ for all $T > \tau$.*

From these considerations, it is clear that u is a weak solution to (1.1) *if and only if $u \in C([\tau, \infty); V)$, $u' \in L^2(\tau, T; V)$ for all $T > \tau$, and*

$$u(t) + \alpha^2 Au(t) + \int_\tau^t (\nu Au(s) + B(u(s)))\, ds = u_\tau + \alpha^2 Au_\tau + \int_\tau^t f(s)\, ds \quad \textit{(equality in } V'\textit{)},$$

for all $t \geq \tau$.

We have the following energy equality for the solutions to (1.1).

Lemma 3 *If u is a weak solution to* (1.1)*, then*

$$\frac{1}{2}\frac{d}{dt}(|u(t)|^2 + \alpha^2\|u(t)\|^2) + \nu\|u(t)\|^2 = \langle f(t), u(t)\rangle, \quad \textit{a.e. } t > \tau. \tag{2.18}$$

Proof. We know from Remark 2 that $u \in W^{1,2}(\tau, T; V)$ and $v \in W^{1,2}(\tau, T; V')$ for all $T > \tau$, where v is given by (2.17). Thus,

$$\frac{d}{dt}\langle v(t), u(t)\rangle = \langle v'(t), u(t)\rangle + \langle v(t), u'(t)\rangle, \quad \text{a.e. } t > \tau. \tag{2.19}$$

But observing that $\mathcal{C}$ is self-adjoint, and using the fact that $v(t) = \mathcal{C}u(t)$ and $v'(t) = \mathcal{C}u'(t)$, we have $\langle v(t), u'(t)\rangle = \langle v'(t), u(t)\rangle$. Therefore, by (2.19), we have

$$\frac{d}{dt}\langle v(t), u(t)\rangle = 2\langle v'(t), u(t)\rangle, \quad \text{a.e. } t > \tau.$$

From this identity, taking into account (2.6) and (2.15), we obtain (2.18). ■

With respect to the existence and uniqueness of solution to (1.1), we have the following result.

Theorem 4 *Let $f \in L^2_{loc}(\mathbb{R}; V')$ be given. Then, for each $\tau \in \mathbb{R}$ and $u_\tau \in V$, there exists a unique weak solution $u = u(\cdot; \tau, u_\tau)$ of (1.1).*

Moreover, if $f \in L^2_{loc}(\mathbb{R}; H)$ and $u_\tau \in D(A)$, then the weak solution $u = u(\cdot; \tau, u_\tau)$ of (1.1) satisfies

$$u \in C([\tau, \infty); D(A)), \quad u' \in L^2(\tau, T; D(A)) \text{ for all } T > \tau, \tag{2.20}$$

and

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \alpha^2|Au(t)|^2) + \nu|Au(t)|^2 + (B(u(t)), Au(t)) = (f(t), Au(t)), \quad a.e.\ t > \tau. \tag{2.21}$$

Proof. We divide the proof in four steps, according to the claims of existence, uniqueness, regularity, and the energy equality (2.21).

Uniqueness.

Let $u^{(1)}$ and $u^{(2)}$ be two weak solutions to (1.1), corresponding to the same data f, τ and u_τ. Let us denote $\hat{u} = u^{(1)} - u^{(2)}$, and $\hat{v} = \hat{u} + \alpha^2 A\hat{u}$.

We have that $\hat{v} \in C([\tau, \infty); V')$, with

$$\hat{v}(t) = -\nu \int_\tau^t A\hat{u}(s)\, ds - \int_\tau^t (B(u^{(1)}(s)) - B(u^{(2)}(s)))\, ds \quad \text{for all } t \geq \tau. \tag{2.22}$$

Observe that by (2.4),

$$\begin{aligned} & \|B(u^{(1)}(s)) - B(u^{(2)}(s))\|_* \\ = & \sup_{w \in V, \|w\|=1} |b(u^{(1)}(s) - u^{(2)}(s), u^{(1)}(s), w) - b(u^{(2)}(s), u^{(2)}(s) - u^{(1)}(s), w)| \\ \leq & \ C_1(\|u^{(1)}(s)\| + \|u^{(2)}(s)\|)\|u^{(1)}(s) - u^{(2)}(s)\|. \end{aligned}$$

Thus, if we fix an arbitrary $T > \tau$, and denote $R_T = C_1 \max_{s\in[\tau,T]}(\|u^{(1)}(s)\| + \|u^{(2)}(s)\|)$, we have

$$\|B(u^{(1)}(s)) - B(u^{(2)}(s))\|_* \leq R_T\|u^{(1)}(s) - u^{(2)}(s)\| \quad \text{for all } s \in [\tau, T]. \tag{2.23}$$

Then, as $\|A\hat{u}(s)\|_* = \|\hat{u}(s)\|$, from (2.22) and (2.23) we deduce that

$$\|\hat{v}(t)\|_* \leq (\nu + R_T)\int_\tau^t \|\hat{u}(s)\|\, ds \quad \text{for all } t \in [\tau, T],$$

and therefore, by (2.13),

$$\|\hat{u}(t)\| \leq \alpha^{-2}(\nu + R_T)\int_\tau^t \|\hat{u}(s)\|\, ds \quad \text{for all } t \in [\tau, T].$$

From this inequality, by Gronwall's lemma, we deduce that $\|\hat{u}(t)\| = 0$ for all $t \in [\tau, T]$, and therefore, the uniqueness of weak solution to (1.1) holds.

Existence.

We can prove the existence of weak solution to (1.1) reasoning as in [1, pp. 844–846], but then with this method of proof, we do not know how to obtain the regularity result (2.20). We will proceed by using a Galerkin scheme.

Let $\{w_j\}_{j\geq1} \subset D(A)$ be the Hilbert basis of H formed by all the normalized eigenfunctions of the Stokes operator A introduced before.

Let $f \in L^2_{loc}(\mathbb{R}; V')$, $\tau \in \mathbb{R}$, and $u_\tau \in V$, be given.

For each integer $m \geq 1$, let define

$$u^m(t) = \sum_{j=1}^{m} \gamma_{m,j}(t) w_j,$$

where the coefficients $\gamma_{m,j}$ are required to satisfy the system

$$\frac{d}{dt}(u^m(t) + \alpha^2 A u^m(t), w_j) = -\langle \nu A u^m(t) + B(u^m(t)) - f(t), w_j \rangle, \quad \text{a.e. } t > \tau, \quad 1 \leq j \leq m, \tag{2.24}$$

and the initial condition

$$u^m(\tau) = P_m u_\tau,$$

where $P_m u_\tau = \sum_{j=1}^{m} (u_\tau, w_j) w_j$, is the orthogonal (in H and in V) projection of u_τ onto the space V_m =span$[w_1, \ldots, w_m]$.

The above system of ordinary differential equations fulfills the conditions of the Picard theorem for existence and uniqueness of local solution.

Next, we will deduce a priori estimates that assure that the solutions u^m do exist for all time $t \in [\tau, \infty)$.

Multiplying in (2.24) by $\gamma_{m,j}(t)$, summing from $j = 1$ to $j = m$, and taking into account (2.6), we obtain that a.e. $t > \tau$,

$$\begin{aligned} \frac{d}{dt}(|u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2) + 2\nu \|u^m(t)\|^2 &= 2\langle f(t), u^m(t) \rangle \\ &\leq \nu \|u^m(t)\|^2 + \nu^{-1} \|f(t)\|_*^2, \end{aligned}$$

and in particular,

$$|u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2 \leq |P_m u_\tau|^2 + \alpha^2 \|P_m u_\tau\|^2 + \nu^{-1} \int_\tau^t \|f(s)\|_*^2 \, ds,$$

for all $t \geq \tau$, and any $m \geq 1$.

Observe that as $u_\tau \in V$, one has that $|P_m u_\tau| \leq |u_\tau|$, $\|P_m u_\tau\| \leq \|u_\tau\|$, and $\lim_{m \to \infty} \|u_\tau - P_m u_\tau\| = 0$. Thus, the sequence $\{u^m\}_{m \geq 1}$ is bounded in $C([\tau, T]; V)$ for all $T > \tau$.

Now observe that by (2.24), $v^m = \mathcal{C} u^m$ satisfies

$$\frac{d}{dt}(v^m(t)) = \tilde{P}_m(-\nu A u^m(t) - B(u^m(t)) + f(t)), \quad \text{a.e. } t > \tau, \tag{2.25}$$

where

$$\langle \tilde{P}_m g, w \rangle = \langle g, P_m w \rangle \quad \forall g \in V', \ w \in V.$$

Consequently, as $\|\tilde{P}_m\|_{\mathcal{L}(V')} \leq 1$ for all $m \geq 1$, we deduce that the sequence $\{dv^m/dt\}_{m \geq 1}$ is bounded in $L^2(\tau, T; V')$ for all $T > \tau$, and therefore, taking into account that $du^m/dt = \mathcal{C}^{-1}(dv^m/dt)$, we have that the sequence $\{du^m/dt\}_{m \geq 1}$ is bounded in $L^2(\tau, T; V)$ for all $T > \tau$.

Thus, by the compactness of the injection of V into H and the Ascoli-Arzelà theorem, we deduce that there exist a subsequence $\{u^{m'}\}_{m' \geq 1} \subset \{u^m\}_{m \geq 1}$ and a function $u \in W^{1,2}(\tau, T; V)$ for all $T > \tau$, such that

$$\left\{ \begin{aligned} & u^{m'} \overset{*}{\rightharpoonup} u \text{ weakly-star in } L^\infty(\tau, T; V), \\ & u^{m'} \to u \text{ strongly in } C([\tau, T]; H), \\ & u^{m'} \to u \text{ a.e. in } \Omega \times (\tau, T), \\ & \frac{du^{m'}}{dt} \rightharpoonup \frac{du}{dt} \text{ weakly in } L^2(\tau, T; V), \\ & \frac{dv^{m'}}{dt} = \mathcal{C}\left(\frac{du^{m'}}{dt}\right) \rightharpoonup \mathcal{C}\left(\frac{du}{dt}\right) \text{ weakly in } L^2(\tau, T; V'), \end{aligned} \right. \tag{2.26}$$

for all $T > \tau$.

As in particular $H_0^1(\Omega) \subset L^4(\Omega)$ with continuous injection, for each $1 \leq i, j \leq 3$, the product $u_i^{m'} u_j^{m'}$ of the corresponding components of $u^{m'}$ is bounded in $L^\infty(\tau, T; L^2(\Omega))$, for all $T > \tau$, and by (2.26), $u_i^{m'} u_j^{m'} \to u_i u_j$ a.e. in $\Omega \times (\tau, T)$. So, by [16, Ch.1, Lem.1.3], we deduce that $u_i^{m'} u_j^{m'} \rightharpoonup u_i u_j$ weakly in $L^2(\Omega \times (\tau, T))$, for all $T > \tau$.

Therefore, taking into account (2.5), if $w \in L^2(\tau, T; V)$,

$$\begin{aligned} \int_\tau^T \langle B(u^{m'}(t)), w(t) \rangle \, dt &= -\int_\tau^T b(u^{m'}(t), w(t), u^{m'}(t)) \, dt \\ &= -\sum_{i,j=1}^3 \int_\tau^T \int_\Omega u_i^{m'}(x,t) u_j^{m'}(x,t) \frac{\partial w_j}{\partial x_i}(x,t) \, dx \, dt \\ &\to -\sum_{i,j=1}^3 \int_\tau^T \int_\Omega u_i(x,t) u_j(x,t) \frac{\partial w_j}{\partial x_i}(x,t) \, dx \, dt \\ &= \int_\tau^T \langle B(u(t)), w(t) \rangle \, dt. \end{aligned}$$

Hence, $B(u^{m'}) \rightharpoonup B(u)$ weakly in $L^2(\tau, T; V')$, for all $T > \tau$.

From all the convergences above, and (2.25), we can take limits and we obtain that u satisfies (2.15).

Observe that $u(\tau) = \lim_{m' \to \infty} u^{m'}(\tau) = \lim_{m' \to \infty} P_{m'} u_\tau = u_\tau$. Thus, u is the weak solution to (1.1).

Regularity.

Assume now that $u_\tau \in D(A)$ and $f \in L^2_{loc}(\mathbb{R}; H)$.

Multiplying in (2.24) by $\lambda_j \gamma_{m,j}(t)$, and summing from $j = 1$ to $j = m$, we obtain that a.e. $t > \tau$,

$$\frac{d}{dt}(\|u^m(t)\|^2 + \alpha^2 |Au^m(t)|^2) + 2\nu |Au^m(t)|^2 = -2 \left(B(u^m(t)), Au^m(t) \right) + 2 \left(f(t), Au^m(t) \right). \tag{2.27}$$

But by (2.9) and Young inequality,

$$2 | \left(B(u^m(t)), Au^m(t) \right) | \leq C_\nu \|u^m(t)\|^6 + \nu |Au^m(t)|^2,$$

where $C_\nu = 27 C_2^4 (16 \nu^3)^{-1}$.

Also,

$$2 | \left(f(t), Au^m(t) \right) | \leq \nu |Au^m(t)|^2 + \nu^{-1} |f(t)|^2.$$

Thus, observing that $|AP_m u_\tau| \leq |Au_\tau|$ and $\|P_m u_\tau\| \leq \|u_\tau\|$, from (2.27) we deduce in particular that

$$\alpha^2 |Au^m(t)|^2 \leq \|u_\tau\|^2 + \alpha^2 |Au_\tau|^2 + \nu^{-1} \int_\tau^t |f(s)|^2 \, ds + C_\nu (t - \tau) \sup_{s \in [\tau, t]} \|u^m(s)\|^6,$$

for all $t \geq \tau$, and any $m \geq 1$.

Consequently, as $\{u^m\}_{m \geq 1}$ is bounded in $C([\tau, T]; V)$, we have that $\{u^m\}_{m \geq 1}$ is bounded in $C([\tau, T]; D(A))$, for all $T > \tau$, and therefore, extracting a subsequence weakly-star convergent in $L^\infty(\tau, T; D(A))$, we see that $u \in L^\infty(\tau, T; D(A))$, for all $T > \tau$.

But then, $v = u + \alpha^2 Au \in L^\infty(\tau, T; H)$, with $v' = -\nu Au - B(u) + f \in L^2(\tau, T; H)$, for all $T > \tau$, and therefore, $v \in C([\tau, \infty); H)$.

Thus, $Au = \alpha^{-2}(v - u) \in C([\tau, \infty); H)$, i.e., $u \in C([\tau, \infty); D(A))$.

Moreover, as $v' \in L^2(\tau, T; H)$, by (2.14), then $u' = \mathcal{C}^{-1} v' \in L^2(\tau, T; D(A))$, for all $T > \tau$.

Identity (2.21)**.**

If $u_\tau \in D(A)$ and $f \in L^2_{loc}(\mathbb{R}; H)$, we have seen that $u \in W^{1,2}(\tau, T; D(A))$ and $v = \mathcal{C}u \in W^{1,2}(\tau, T; H)$, for all $T > \tau$. Then,

$$\frac{d}{dt}|v(t)|^2 = 2(v'(t), v(t)), \quad \text{a.e. } t > \tau,$$

and

$$\begin{aligned} \frac{d}{dt}(u(t), v(t)) &= 2(u(t), \mathcal{C}u'(t)) \\ &= 2(u(t), v'(t)), \quad \text{a.e. } t > \tau. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt}(Au(t), v(t)) &= \alpha^{-2}\frac{d}{dt}(v(t) - u(t), v(t)) \\ &= 2(v'(t), Au(t)), \quad \text{a.e. } t > \tau. \end{aligned}$$

From this equality, we have (2.21). ∎

Remark 5 *Observe that in the above proof, using the uniqueness of solution for the problem, for any $T > \tau$ the whole sequence of the Galerkin approximations satisfies that u^m converges to u in $C([\tau, T]; H)$, and actually, all convergences in* (2.26)*, except the third one, hold for the whole sequence. Analogously, one also deduces that for any $t \in [\tau, T]$, $u^m(t) \rightharpoonup u(t)$ weakly in V.*

Moreover, if $u_\tau \in D(A)$ and $f \in L^2_{loc}(\mathbb{R}; H)$, then in fact for any $T > \tau$ the sequence u^m converges to u in $C([\tau, T]; V)$, and weakly-star in $L^\infty(\tau, T; D(A))$, for any $t \in [\tau, T]$, $u^m(t) \rightharpoonup u(t)$ in $D(A)$, and the sequence du^m/dt converges to du/dt weakly in $L^2(\tau, T; D(A))$.

Now, we establish a result on the sequential weak continuity of the solutions to (1.1) with respect to the initial datum u_τ.

Theorem 6 *Let $f \in L^2_{loc}(\mathbb{R}; V')$ and $\tau < t$ be given. Consider a sequence $\{u_{\tau,n}\} \subset V$ weakly converging to u_τ in V. Then, the following convergences hold for the sequence of solutions $u(\cdot; \tau, u_{\tau,n})$ toward the solution $u(\cdot; \tau, u_\tau)$.*

$$u(\cdot; \tau, u_{\tau,n}) \stackrel{*}{\rightharpoonup} u(\cdot; \tau, u_\tau) \text{ weakly-star in } L^\infty(\tau, t; V),$$

$$u(\cdot; \tau, u_{\tau,n}) \to u(\cdot; \tau, u_\tau) \text{ strongly in } C([\tau, t]; H),$$

$$u(t; \tau, u_{\tau,n}) \rightharpoonup u(t; \tau, u_\tau) \text{ weakly in } V.$$

Moreover, if $f \in L^2_{loc}(\mathbb{R}; H)$ and the sequence $\{u_{\tau,n}\} \subset D(A)$ converges weakly to u_τ in $D(A)$, then, in fact,

$$u(\cdot; \tau, u_{\tau,n}) \stackrel{*}{\rightharpoonup} u(\cdot; \tau, u_\tau) \text{ weakly-star in } L^\infty(\tau, t; D(A)),$$

$$u(\cdot; \tau, u_{\tau,n}) \to u(\cdot; \tau, u_\tau) \text{ strongly in } C([\tau, t]; V),$$

$$u(t; \tau, u_{\tau,n}) \rightharpoonup u(t; \tau, u_\tau) \text{ weakly in } D(A).$$

Proof. The proof can be done analogously to that of Theorem 4, since the a priori estimates follow exactly the same. The fact that the whole sequence satisfies the above convergences is a consequence of the uniqueness of solution for the problem (cf. Remark 5). ∎

Remark 7 *Although the above result will be enough for our purposes, let us observe that the solution also depends continuously of the initial datum in the strong topology of* V*. Moreover, when* $f \in L^2_{loc}(\mathbb{R}; H)$*, the solution depends continuously of the initial datum in the strong topology of* $D(A)$*. Indeed, this can be proved similarly to the proof of uniqueness of weak solution to* (1.1)*, considering the difference of two solutions and using Gronwall's lemma.*

Remark 8 *Observe that actually in the existence and uniqueness part of Theorem 4 and also in the first part of Theorem 6 we do not need any regularity assumption on the boundary of the domain. This assumption is only required for the additional regularity results.*

3 Abstract results on minimal pullback attractors

In this section we recall some abstract results on pullback attractors theory. We present a resume of some results on the existence of minimal pullback attractors obtained in [7] (see also [18, 2, 3]).

Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t, \tau) \in \mathbb{R}^2 : \tau \leq t\}$.

A process U on X is a mapping $\mathbb{R}^2_d \times X \ni (t, \tau, x) \mapsto U(t, \tau)x \in X$ such that $U(\tau, \tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t, r)(U(r, \tau)x) = U(t, \tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$.

The process U is said to be continuous if for any pair $\tau \leq t$, the mapping $U(t, \tau) : X \to X$ is continuous. It is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $U(t, \tau)x_n \to y \in X$, then $U(t, \tau)x = y$. It is clear that every continuous process is closed.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

The process U is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.

We have the following result.

Proposition 9 *Let* U *be a closed process on* X*, and assume that is pullback* $\widehat{D}_0$*-asymptotically compact. Then, for any* $t \in \mathbb{R}$*, the set* $\Lambda_X(\widehat{D}_0, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)D_0(\tau)}^X$ *is a nonempty compact subset of* X *that satisfies:*

(a) $U(t, \tau)\Lambda_X(\widehat{D}_0, \tau) = \Lambda_X(\widehat{D}_0, t)$ *for all* $\tau \leq t$ *(invariance),*

(b) $\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t, \tau)D_0(\tau), \Lambda_X(\widehat{D}_0, t) = 0$ *for all* $t \in \mathbb{R}$ *(pullback attraction),*

where $\operatorname{dist}_X(\cdot, \cdot)$ *denotes the Hausdorff semi-distance in* X*.*

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 10 *A process* U *on* X *is said to be pullback* $\mathcal{D}-$*asymptotically compact if it is* $\widehat{D}-$*asymptotically compact for any* $\widehat{D} \in \mathcal{D}$*.*

It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *is pullback* $\mathcal{D}-$*absorbing for the process* U *on* X *if for any* $t \in \mathbb{R}$ *and any* $\widehat{D} \in \mathcal{D}$*, there exists a* $\tau_0(t, \widehat{D}) \leq t$ *such that*

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \textit{for all } \tau \leq \tau_0(t, \widehat{D}).$$

We have the following result (cf. [7]) on existence of minimal pullback attractors.

Theorem 11 *Consider a closed process* $U : \mathbb{R}^2_d \times X \to X$*, a universe* $\mathcal{D}$ *in* $\mathcal{P}(X)$*, and a family* $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *which is pullback* $\mathcal{D}-$*absorbing for* U*, and assume also that* U *is pullback* $\widehat{D}_0-$*asymptotically compact.*

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ *defined by* $\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda_X(\widehat{D}, t)}^X$*, has the following properties:*

(a) for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda_X(\widehat{D}_0, t)$,

(b) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}-$attracting, i.e., $\lim_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ for all $\widehat{D} \in \mathcal{D}$, and any $t \in \mathbb{R}$,

(c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e., $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \leq t$,

(d) if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda_X(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, $\lim_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), C(t)) = 0$, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark 12 *Under the assumptions of Theorem 11, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}-$attractor for the process U.*

If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

We will denote $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result.

Corollary 13 *Under the assumptions of Theorem 11, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F^X$, then both attractors, $\mathcal{A}_{\mathcal{D}_F^X}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \quad \text{for all } t \in \mathbb{R}.$$

Remark 14 *It can be proved (cf. [18]) that, under the assumptions of the preceding corollary, if for some $T \in \mathbb{R}$, the set $\cup_{t\leq T} D_0(t)$ is a bounded subset of X, then*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \text{for all } t \leq T.$$

4 Existence of minimal pullback attractors in V norm

Now, by the previous results, we are able to define a process U on V associated to (1.1), and under suitable assumptions on f, we can obtain the existence of minimal pullback attractors. As pointed out in the Introduction, in the results of this section we do not require any regularity assumption on $\partial\Omega$, and the force term may take values in V' instead of in L^2 as appears in [11].

Proposition 15 *Assume that $f \in L^2_{loc}(\mathbb{R}; V')$ is given. Then, the bi-parametric family of maps $U(t,\tau) : V \to V$, with $\tau \leq t$, given by*

$$U(t,\tau)u_\tau = u(t;\tau,u_\tau), \tag{4.1}$$

where $u = u(\cdot;\tau,u_\tau)$ is the unique weak solution to (1.1), defines a closed process on V.

Proof. It is a consequence of Theorem 4 and Theorem 6. ■

Remark 16 *Observe that, by Remark 7, U is in fact a continuous process on V.*

For the obtention of a pullback absorbing family for the process U, we have the following result.

Lemma 17 *Assume that* $f \in L^2_{loc}(\mathbb{R}; V')$ *and* $u_\tau \in V$. *Then, for any*

$$0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}, \tag{4.2}$$

the solution $u = u(\cdot; \tau, u_\tau)$ *of* (1.1) *satisfies*

$$\begin{aligned} &\|u(t)\|^2 + \varepsilon\alpha^{-2} \int_\tau^t e^{\sigma(s-t)} \|u(s)\|^2 \, ds \\ \leq \quad & (1 + \alpha^{-2}\lambda_1^{-1}) e^{\sigma(\tau - t)} \|u_\tau\|^2 + \alpha^{-2}\varepsilon^{-1} \int_\tau^t e^{\sigma(s-t)} \|f(s)\|_*^2 \, ds \end{aligned} \tag{4.3}$$

for all $t \geq \tau$, *where*

$$\varepsilon = \nu - \frac{\sigma}{2}(\lambda_1^{-1} + \alpha^2). \tag{4.4}$$

Proof. By (2.18), for all $\varepsilon > 0$,

$$\begin{aligned} \frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) &= \sigma e^{\sigma t}|u(t)|^2 + \alpha^2 \sigma e^{\sigma t}\|u(t)\|^2 - 2\nu e^{\sigma t}\|u(t)\|^2 + 2e^{\sigma t}\langle f(t), u(t)\rangle \\ &\leq \{\sigma(\lambda_1^{-1} + \alpha^2) - 2\nu + \varepsilon\} e^{\sigma t}\|u(t)\|^2 + \varepsilon^{-1} e^{\sigma t}\|f(t)\|_*^2, \end{aligned}$$

a.e. $t > \tau$.

Thus, if σ satisfies (4.2), then ε given by (4.4) is positive, and for this ε we have

$$\frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) + \varepsilon e^{\sigma t}\|u(t)\|^2 \leq \varepsilon^{-1} e^{\sigma t}\|f(t)\|_*^2,$$

a.e. $t > \tau$.

From this inequality we obtain (4.3). ∎

Taking into account the estimate (4.3), we define the following universe.

Definition 18 *For any* $\sigma > 0$, *we will denote by* $\mathcal{D}_\sigma^V$ *the class of all families of nonempty subsets* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V)$ *such that* $\lim_{\tau \to -\infty}(e^{\sigma\tau} \sup_{v \in D(\tau)} \|v\|^2) = 0$.

Accordingly to the notation introduced in the previous section, $\mathcal{D}_F^V$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of V.

Remark 19 *Observe that for any* $\sigma > 0$, $\mathcal{D}_F^V \subset \mathcal{D}_\sigma^V$ *and that both are inclusion-closed.*

As an evident consequence of Lemma 17, we have the following result.

Corollary 20 *Assume that* $f \in L^2_{loc}(\mathbb{R}; V')$ *satisfies that*

$$\int_{-\infty}^0 e^{\sigma s}\|f(s)\|_*^2 \, ds < \infty, \tag{4.5}$$

for some $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$.

Then, the family $\widehat{D}_\sigma = \{D_\sigma(t) : t \in \mathbb{R}\}$ *defined by*

$$D_\sigma(t) = \overline{B}_V(0, R_\sigma^{1/2}(t)), \tag{4.6}$$

the closed ball in V *of center zero and radius* $R_\sigma^{1/2}(t)$, *where*

$$R_\sigma(t) = 1 + \alpha^{-2}\varepsilon^{-1}e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}\|f(s)\|_*^2 \, ds, \tag{4.7}$$

with ε *given by* (4.4), *is pullback* $\mathcal{D}_\sigma^V-$*absorbing for the process* $U : \mathbb{R}_d^2 \times V \to V$ *given by* (4.1) *(and therefore* $\mathcal{D}_F^V-$*absorbing too), and* $\widehat{D}_\sigma \in \mathcal{D}_\sigma^V$.

In order to prove that the process U is pullback $\widehat{D}_\sigma-$asymptotically compact, we will apply an energy method used by Rosa (cf. [23], see also [17]), which does not require any additional estimate on the solutions in higher norms in contrast with the *energy continuous method* (e.g., cf. [19]), or the method used in [11] with the fractional powers of the operator A. Our proof here relies on a sharp use of the differential equality that lead to the existence of an absorbing family, the use of weak limits in V in a diagonal argument, and the fact that the process is sequentially weakly continuous.

Lemma 21 *Assume that $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies (4.5). Then, the process U defined by (4.1) is pullback $\widehat{D}_\sigma-$ asymptotically compact, where $\widehat{D}_\sigma = \{D_\sigma(t) : t \in \mathbb{R}\}$ is defined in Corollary 20.*

Proof. Let $t \in \mathbb{R}$, and $\tau_n \to -\infty$ with $\tau_n \leq t$ and $u_{\tau_n} \in D_\sigma(\tau_n)$ for all n, be given. We must prove that the sequence $\{U(t, \tau_n)u_{\tau_n}\}$ is relatively compact in V. By Corollary 20, for each integer $k \geq 0$, there exists $\tau_{\widehat{D}_\sigma}(k) \leq t - k$ such that $U(t - k, \tau)D_\sigma(\tau) \subset D_\sigma(t - k)$ for all $\tau \leq \tau_{\widehat{D}_\sigma}(k)$. Recall that each $D_\sigma(t)$, defined in (4.6), is a bounded set of V. From this and a diagonal argument, we can extract a subsequence $\{u_{\tau_{n'}}\} \subset \{u_{\tau_n}\}$ such that

$$U(t - k, \tau_{n'})u_{\tau'_n} \rightharpoonup w_k \quad \text{weakly in } V, \ \forall k \geq 0, \tag{4.8}$$

where $w_k \in D_\sigma(t - k)$.

Now, applying Theorem 6 on each fixed interval $[t - k, t]$ we obtain that

$$\begin{aligned} w_0 &= V - \text{weak} \lim_{n' \to \infty} U(t, \tau_{n'})u_{\tau_{n'}} \\ &= V - \text{weak} \lim_{n' \to \infty} U(t, t - k)U(t - k, \tau_{n'})u_{\tau_{n'}} \\ &= U(t, t - k) \left[V - \text{weak} \lim_{n' \to \infty} U(t - k, \tau_{n'})u_{\tau_{n'}} \right] \\ &= U(t, t - k)w_k. \end{aligned}$$

In particular, observe that $\|w_0\| \leq \liminf_{n' \to \infty} \|U(t, \tau_{n'})u_{\tau_{n'}}\|$. We will prove now that it also holds that

$$\limsup_{n' \to \infty} \|U(t, \tau_{n'})u_{\tau_{n'}}\| \leq \|w_0\|, \tag{4.9}$$

which combined with (4.8) for $k = 0$, will imply the convergence in the strong topology of V, and the asymptotic compactness.

Observe that, as we already used in Lemma 17, for any pair (τ, u_τ) with $u_\tau \in V$, the solution $u(\cdot; \tau, u_\tau)$, for short denoted $u(\cdot)$, satisfies the differential equality

$$\begin{aligned} & \frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) \\ = \ & \sigma e^{\sigma t}|u(t)|^2 + \alpha^2 \sigma e^{\sigma t}\|u(t)\|^2 - 2\nu e^{\sigma t}\|u(t)\|^2 + 2e^{\sigma t}\langle f(t), u(t)\rangle, \ a.e. \, t > \tau. \end{aligned} \tag{4.10}$$

Since we have chosen σ satisfying (4.2), observe that $[\cdot]$, with $[v]^2 = (2\nu - \alpha^2\sigma)\|v\|^2 - \sigma|v|^2$, defines an equivalent norm to $\|\cdot\|$ in V.

We integrate the above expression in the interval $[t - k, t]$ for the solutions $U(\cdot, \tau_{n'})u_{\tau_{n'}}$ with $\tau_{n'} \leq t - k$, which yields

$$\begin{aligned} & |U(t, \tau_{n'})u_{\tau_{n'}}|^2 + \alpha^2\|U(t, \tau_{n'})u_{\tau_{n'}}\|^2 \\ = \ & |U(t, t - k)U(t - k, \tau_{n'})u_{\tau_{n'}}|^2 + \alpha^2\|U(t, t - k)U(t - k, \tau_{n'})u_{\tau_{n'}}\|^2 \\ = \ & e^{-\sigma k}\left(|U(t - k, \tau_{n'})u_{\tau_{n'}}|^2 + \alpha^2\|U(t - k, \tau_{n'})u_{\tau_{n'}}\|^2\right) \\ & +2\int_{t-k}^t e^{\sigma(s-t)}\langle f(s), U(s, t - k)U(t - k, \tau_{n'})u_{\tau_{n'}}\rangle ds \\ & -\int_{t-k}^t e^{\sigma(s-t)}[U(s, t - k)U(t - k, \tau_{n'})u_{\tau_{n'}}]^2 ds. \end{aligned} \tag{4.11}$$

On other hand, by (4.8) and Theorem 6, we deduce that

$$U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}} \rightharpoonup U(\cdot,t-k)w_k \quad \text{weakly in } L^2(t-k,t;V).$$

From this, as $e^{\sigma(\cdot-t)}f(\cdot)\in L^2(t-k,t;V')$, it yields

$$\begin{aligned}&\lim_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s),U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\rangle ds\\ =\;&\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s),U(s,t-k)w_k\rangle ds. \qquad (4.12)\end{aligned}$$

Since $\int_{t-k}^{t} e^{\sigma(s-t)}[v(s)]^2ds$ defines an equivalent norm in $L^2(t-k,t;V)$, we also deduce from above that

$$\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)w_k]^2ds \le \liminf_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}]^2ds. \qquad (4.13)$$

From (4.11)–(4.13), taking into account (4.8) with $k=0$, the compactness of the injection of V into H, and (4.6), we conclude that

$$\begin{aligned}&|w_0|^2+\alpha^2\limsup_{n'\to\infty}\|U(t,\tau_{n'})u_{\tau_{n'}}\|^2\\ \le\;& e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)R_\sigma(t-k)+2\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s),U(s,t-k)w_k\rangle ds\\ &-\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)w_k]^2ds.\end{aligned}$$

Now, taking into account that $w_0=U(t,t-k)w_k$, integrating again in (4.10), we obtain

$$\begin{aligned}|w_0|^2+\alpha^2\|w_0\|^2 \;=\;& e^{-\sigma k}(|w_k|^2+\alpha^2\|w_k\|^2)+2\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s),U(s,t-k)w_k\rangle ds\\ &-\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)w_k]^2ds.\end{aligned}$$

Comparing the above two expressions, we conclude that in particular

$$|w_0|^2+\alpha^2\limsup_{n'\to\infty}\|U(t,\tau_{n'})u_{\tau_{n'}}\|^2 \le e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)R_\sigma(t-k)+|w_0|^2+\alpha^2\|w_0\|^2.$$

But from (4.7) and (4.5), we have that $\lim_{k\to\infty} e^{-\sigma k}R_\sigma(t-k)=0$, so (4.9) holds, and the proof is finished. ■

As a consequence of the above results, we obtain the existence of minimal pullback attractors for the process $U:\mathbb{R}^2_d\times V\to V$ defined by (4.1).

Theorem 22 *Assume that $f\in L^2_{loc}(\mathbb{R};V')$ satisfies (4.5). Then, there exist the minimal pullback $\mathcal{D}^V_F$-attractor $\mathcal{A}_{\mathcal{D}^V_F}=\{\mathcal{A}_{\mathcal{D}^V_F}(t):t\in\mathbb{R}\}$ and the minimal pullback $\mathcal{D}^V_\sigma$-attractor $\mathcal{A}_{\mathcal{D}^V_\sigma}=\{\mathcal{A}_{\mathcal{D}^V_\sigma}(t):t\in\mathbb{R}\}$, for the process U defined by (4.1), $\mathcal{A}_{\mathcal{D}^V_\sigma}$ belongs to $\mathcal{D}^V_\sigma$, and the following relation holds,*

$$\mathcal{A}_{\mathcal{D}^V_F}(t)\subset\mathcal{A}_{\mathcal{D}^V_\sigma}(t)\subset\overline{B}_V(0,R_\sigma^{1/2}(t)) \quad \forall t\in\mathbb{R}, \qquad (4.14)$$

where R_σ is given by (4.7).

Finally, if f satisfies the stronger requirement

$$\sup_{r\le 0}\left(e^{-\sigma r}\int_{-\infty}^{r} e^{\sigma s}\|f(s)\|_*^2ds\right)<\infty, \qquad (4.15)$$

then

$$\mathcal{A}_{\mathcal{D}^V_F}(t)=\mathcal{A}_{\mathcal{D}^V_\sigma}(t) \quad \forall t\in\mathbb{R}. \qquad (4.16)$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_F^V}$ and $\mathcal{A}_{\mathcal{D}_\sigma^V}$ is a direct consequence of Theorem 11, Corollary 13, Proposition 15, Corollary 20, and Lemma 21.

The inclusions in (4.14) are a consequence of Theorem 11 and Corollary 13.

Finally, the equality (4.16) is a consequence of Remark 14, and the fact that (4.15) is equivalent to have that $\sup_{t\leq T} R_\sigma(t)$ is bounded for any $T \in \mathbb{R}$. ■

Remark 23 *Observe that, as it can be easily proved, in general, if $g \in L^2_{loc}(\mathbb{R}; X)$, with X a Banach space with norm $\|\cdot\|_X$, the three following conditions are equivalent:*

(1) $\sup_{r\leq 0}\left(e^{-\sigma r}\int_{-\infty}^{r} e^{\sigma s}\|g(s)\|_X^2 ds\right) < \infty$, *for some* $\sigma > 0$.

(2) $\sup_{r\leq 0}\int_{r-1}^{r} \|g(s)\|_X^2 ds < \infty$.

(3) $\sup_{r\leq 0}\left(e^{-\hat{\sigma} r}\int_{-\infty}^{r} e^{\hat{\sigma} s}\|g(s)\|_X^2 ds\right) < \infty$, *for all* $\hat{\sigma} > 0$.

Remark 24 *Observe that if $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies (4.5), then it also satisfies*

$$\int_{-\infty}^{0} e^{\hat{\sigma} s}\|f(s)\|_*^2 ds < \infty, \quad \textit{for all } \hat{\sigma} \in (\sigma, 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}).$$

So, there exists the corresponding minimal pullback $\mathcal{D}^V_{\hat{\sigma}}$- attractor, $\mathcal{A}_{\mathcal{D}^V_{\hat{\sigma}}}$.

In fact, since $\mathcal{D}^V_\sigma \subset \mathcal{D}^V_{\hat{\sigma}}$, for any such $\hat{\sigma}$, then

$$\mathcal{A}_{\mathcal{D}^V_\sigma}(t) \subset \mathcal{A}_{\mathcal{D}^V_{\hat{\sigma}}}(t) \quad \textit{for all } t \in \mathbb{R}, \textit{ and any } \sigma < \hat{\sigma} < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}.$$

Moreover, if (4.15) holds, then we conclude by (4.16) and Remark 23 that

$$\mathcal{A}_{\mathcal{D}^V_F}(t) = \mathcal{A}_{\mathcal{D}^V_\sigma}(t) = \mathcal{A}_{\mathcal{D}^V_{\hat{\sigma}}}(t) \quad \textit{for all } t \in \mathbb{R}, \textit{ and any } 0 < \hat{\sigma} < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}.$$

Thus, $\mathcal{A}_{\mathcal{D}^V_F}$ is the minimal pullback $\mathcal{D}^V_{max}$- attractor, where

$$\mathcal{D}^V_{max} = \bigcup_{0<\hat{\sigma}<2\nu(\lambda_1^{-1}+\alpha^2)^{-1}} \mathcal{D}^V_{\hat{\sigma}}.$$

5 Regularity of the pullback attractors

The main goal of this paragraph is to provide some extra regularity for the attractors obtained in the previous section. This will be obtained by a bootstrapping argument, and making the most of a representation of the solutions to the problem splitting it in two parts, the linear part with an exponential decay, and the nonlinear part with good enough estimates. In order to achieve these results, we will use the fractional powers of the Stokes operator, introduced in Section 2.

Observe that for every $\tau \in \mathbb{R}$, any $u_\tau \in V$, and $f \in L^2_{loc}(\mathbb{R}; V')$, by Theorem 4, there exists a unique weak solution u to problem (1.1). Moreover, let us point out that the following representation of the solution holds:

$$u(t) = U(t,\tau)u_\tau = Y(t,\tau)u_\tau + Z(t,\tau)u_\tau \quad \forall\, t \geq \tau,$$

where $y = Y(\cdot,\tau)u_\tau$ and $z = Z(\cdot,\tau)u_\tau$, are solutions of

$$\begin{cases} y \in C([\tau,\infty); V), \\ \dfrac{d}{dt}(y(t) + \alpha^2 Ay(t)) + \nu Ay(t) = 0, \quad \text{in } \mathcal{D}'(\tau,\infty; V'), \\ y(\tau) = u_\tau, \end{cases} \tag{5.1}$$

and

$$\begin{cases} z \in C([\tau, \infty); V), \\ \dfrac{d}{dt}(z(t) + \alpha^2 Az(t)) + \nu Az(t) = f(t) - B(u(t)), \quad \text{in } \mathcal{D}'(\tau, \infty; V'), \\ z(\tau) = 0, \end{cases} \tag{5.2}$$

respectively.

The existence and uniqueness of weak solution to (5.1) and to (5.2) can be obtained reasoning as in the proof of Theorem 4.

For the problem (5.1) we have the following result.

Lemma 25 *For any $\tau \in \mathbb{R}$, $u_\tau \in V$ and σ fulfilling the assumption* (4.2)*, the solution $y = Y(\cdot, \tau)u_\tau$ of* (5.1) *satisfies*

$$\|Y(t,\tau)u_\tau\|^2 \le (1 + \alpha^{-2}\lambda_1^{-1})e^{\sigma(\tau - t)}\|u_\tau\|^2 \quad \textit{for all } t \ge \tau. \tag{5.3}$$

Proof. It is analogous to the proof of (4.3), and we omit it. ■

For the study of the problem (5.2), we will make use of the following lemma.

Lemma 26 *Assume that $g \in L^2_{loc}(\mathbb{R}; D(A^{-\beta}))$ with $0 \le \beta \le 1/2$. Then, for each $\tau \in \mathbb{R}$ and σ satisfying the assumption* (4.2)*, the unique solution z of the problem*

$$\begin{cases} z \in C([\tau, \infty); V), \\ \dfrac{d}{dt}(z(t) + \alpha^2 Az(t)) + \nu Az(t) = g(t), \quad \textit{in } \mathcal{D}'(\tau, \infty; V'), \\ z(\tau) = 0, \end{cases} \tag{5.4}$$

satisfies

$$z \in C([\tau, \infty); D(A^{1-\beta})), \tag{5.5}$$

and

$$|A^{1-\beta}z(t)|^2 \le \alpha^{-2}\varepsilon^{-1} \int_\tau^t e^{\sigma(s-t)}|A^{-\beta}g(s)|^2\, ds \quad \textit{for all } t \ge \tau, \tag{5.6}$$

where ε is given by (4.4)*.*

Proof. We give a formal proof, the rigorous one should be made using the Galerkin approximations constructed with the basis $\{w_j\}_{j\ge1}$ of eigenfunctions of the Stokes operator A.

Multiplying in (5.4) by $A^{1-2\beta}z(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|A^{(1-2\beta)/2}z(t)|^2 + \alpha^2|A^{1-\beta}z(t)|^2\right) + \nu|A^{1-\beta}z(t)|^2 = (A^{-\beta}g(t), A^{1-\beta}z(t)), \ \textit{a.e.}\, t > \tau.$$

Thus,

$$\begin{aligned} &\frac{d}{dt}\left\{e^{\sigma t}\left(|A^{(1-2\beta)/2}z(t)|^2 + \alpha^2|A^{1-\beta}z(t)|^2\right)\right\} + 2\nu e^{\sigma t}|A^{1-\beta}z(t)|^2 \\ = \ &\sigma e^{\sigma t}\left(|A^{(1-2\beta)/2}z(t)|^2 + \alpha^2|A^{1-\beta}z(t)|^2\right) + 2e^{\sigma t}(A^{-\beta}g(t), A^{1-\beta}z(t)), \ \textit{a.e.}\, t > \tau. \end{aligned} \tag{5.7}$$

Now, using that $2e^{\sigma t}|(A^{-\beta}g(t), A^{1-\beta}z(t))| \le \varepsilon e^{\sigma t}|A^{1-\beta}z(t)|^2 + \varepsilon^{-1}e^{\sigma t}|A^{-\beta}g(t)|^2$, and

$$\begin{aligned} |A^{1-\beta}z(t)|^2 &= |A^{1/2}(A^{(1-2\beta)/2}z(t))|^2 \\ &\ge \lambda_1|A^{(1-2\beta)/2}z(t)|^2, \end{aligned}$$

from (5.7) and the fact that $z(\tau) = 0$, we obtain (5.6).

Now, from (5.6) we have that $v = z + \alpha^2 Az$ and its derivative v' belong to $L^2(\tau, T; D(A^{-\beta}))$ for any $T > \tau$. So, it holds that $v \in C([\tau, \infty); D(A^{-\beta}))$, whence using the mapping $\mathcal{C}$, (5.5) follows. ■

Now we can prove the following regularity result for the pullback attractors in V norm.

Theorem 27 *Assume that* $f \in L^2_{loc}(\mathbb{R}; D(A^{-\beta}))$ *for some* $0 \le \beta \le 1/2$, *and that*

$$\sup_{r\le 0}\int_{r-1}^{r} \|f(s)\|_*^2 ds < \infty. \tag{5.8}$$

Then:

(1) If f *also satisfies*

$$\int_{-\infty}^{0} e^{\sigma s}|A^{-\beta} f(s)|^2 ds < \infty, \tag{5.9}$$

for some $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$, *and*

$$\begin{cases} \sup_{r\le 0}\int_{r-1}^{r} |A^{-1/4-\beta} f(s)|^2 ds < \infty, & \text{if } 0 < \beta < 1/4, \\ \sup_{r\le 0}\int_{r-1}^{r} |A^{-\delta} f(s)|^2 ds < \infty & \text{for some } 0 < \delta < 1/4, \ \text{if } \beta = 0, \end{cases} \tag{5.10}$$

then the pullback attractor $\mathcal{A}_{\mathcal{D}^V_{max}} = \mathcal{A}_{\mathcal{D}^V_F}$ *fulfills that*

$$\bigcup_{t_1\le t\le t_2} \mathcal{A}_{\mathcal{D}^V_{max}}(t) = \bigcup_{t_1\le t\le t_2} \mathcal{A}_{\mathcal{D}^V_F}(t) \text{ is a bounded subset of } D(A^{1-\beta}), \text{ for any } t_1 < t_2. \tag{5.11}$$

(2) If f *also satisfies*

$$\sup_{r\le 0}\int_{r-1}^{r} |A^{-\beta} f(s)|^2 ds < \infty, \tag{5.12}$$

then for any $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$,

$$\bigcup_{t\le t_2} \mathcal{A}_{\mathcal{D}^V_\sigma}(t) = \bigcup_{t\le t_2} \mathcal{A}_{\mathcal{D}^V_F}(t) \text{ is a bounded subset of } D(A^{1-\beta}), \text{ for any } t_2 \in \mathbb{R}. \tag{5.13}$$

Proof. Let us fix $t \in \mathbb{R}$ and $v \in \mathcal{A}_{\mathcal{D}^V_\sigma}(t) = \mathcal{A}_{\mathcal{D}^V_F}(t)$. By (4.14), (5.8) and Remark 23, we see that

$$\bigcup_{r\le t} \mathcal{A}_{\mathcal{D}^V_\sigma}(r) \subset \overline{B}_V(0, \widetilde{R}_\sigma^{1/2}(t)), \tag{5.14}$$

where $\widetilde{R}_\sigma(t) = 1 + \alpha^{-2}\varepsilon^{-1}\sup_{r\le t}(e^{-\sigma r}\int_{-\infty}^{r} e^{\sigma s}\|f(s)\|_*^2 ds)$, with ε given by (4.4).

Let $\{\tau_n\}_{n\ge 1} \subset (-\infty, t]$ be a sequence with $\tau_n \to -\infty$ as $n \to \infty$. By the invariance of $\mathcal{A}_{\mathcal{D}^V_\sigma}$, for each $n \ge 1$ there exists $u_{\tau_n} \in \mathcal{A}_{\mathcal{D}^V_\sigma}(\tau_n)$ such that $v = U(t,\tau_n)u_{\tau_n}$, and therefore,

$$v = Y(t,\tau_n)u_{\tau_n} + Z(t,\tau_n)u_{\tau_n}.$$

From (5.3) and (5.14) we deduce that $\|Y(t,\tau_n)u_{\tau_n}\| \to 0$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \|Z(t,\tau_n)u_{\tau_n} - v\| = 0. \tag{5.15}$$

Let us denote $u_n(r) = U(r,\tau_n)u_{\tau_n}$ for $r \ge \tau_n$ and $n \ge 1$. By (5.14) and the invariance of $\mathcal{A}_{\mathcal{D}^V_\sigma}$,

$$u_n(r) \in \mathcal{A}_{\mathcal{D}^V_\sigma}(r) \subset \overline{B}_V(0, \widetilde{R}_\sigma^{1/2}(t)), \quad \forall \tau_n \le r \le t, \quad \forall n \ge 1. \tag{5.16}$$

Now we distinguish three cases.
Case 1. If $1/4 \le \beta \le 1/2$.

In this case, from (2.11), the continuous injection of V in $D(A^{3/4-\beta})$ and (5.16), we deduce that

$$\begin{aligned} |A^{-\beta}B(u_n(r))| &\leq C_{(3/4-\beta)}|A^{3/4-\beta}u_n(r)|\|u_n(r)\| \\ &\leq \widetilde{C}_{(3/4-\beta)}\|u_n(r)\|^2 \\ &\leq \widetilde{C}_{(3/4-\beta)}\widetilde{R}_\sigma(t), \quad \forall \tau_n \leq r \leq t, \quad \forall n \geq 1. \end{aligned}$$

Thus, if we assume (5.9), from Lemma 26 we obtain that

$$|A^{1-\beta}Z(t,\tau_n)u_{\tau_n}|^2 \leq M_{\sigma,\beta}(t), \tag{5.17}$$

where

$$M_{\sigma,\beta}(t) = 2\alpha^{-2}\varepsilon^{-1}\left(\int_{-\infty}^{t} e^{\sigma(s-t)}|A^{-\beta}f(s)|^2 ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^2_\sigma(t)\right).$$

From (5.15), (5.17) and the weak lower semi-continuity of the norm, we deduce that v belongs to $\overline{B}_{D(A^{1-\beta})}(0, M^{1/2}_{\sigma,\beta}(t))$, and therefore (5.11) holds.

Moreover, if f satisfies (5.12), then (5.13) holds, and more exactly,

$$\bigcup_{t\leq t_2}\mathcal{A}_{\mathcal{D}^V_\sigma}(t) \subset \overline{B}_{D(A^{1-\beta})}(0, \widetilde{M}^{1/2}_{\sigma,\beta}(t_2)), \text{ for all } t_2 \in \mathbb{R}, \tag{5.18}$$

where

$$\widetilde{M}_{\sigma,\beta}(t_2) = 2\alpha^{-2}\varepsilon^{-1}\left(\sup_{t\leq t_2}\int_{-\infty}^{t} e^{\sigma(s-t)}|A^{-\beta}f(s)|^2 ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^2_\sigma(t_2)\right).$$

Case 2. If $0 < \beta < 1/4$.

In this case, if f satisfies (5.10), as $1/4 < 1/4+\beta < 1/2$, from (5.18) we have that

$$\bigcup_{r\leq t}\mathcal{A}_{\mathcal{D}^V_\sigma}(r) \subset \overline{B}_{D(A^{3/4-\beta})}(0, \widetilde{M}^{1/2}_{\sigma,1/4+\beta}(t)).$$

Thus, by (2.11) and (5.16), we obtain that

$$\begin{aligned} |A^{-\beta}B(u_n(r))| &\leq C_{(3/4-\beta)}|A^{3/4-\beta}u_n(r)|\|u_n(r)\| \\ &\leq C_{(3/4-\beta)}\widetilde{M}^{1/2}_{\sigma,1/4+\beta}(t)\widetilde{R}^{1/2}_\sigma(t), \quad \forall \tau_n \leq r \leq t, \quad \forall n \geq 1. \end{aligned}$$

Thus, if we assume (5.9), from Lemma 26 we deduce that

$$|A^{1-\beta}Z(t,\tau_n)u_{\tau_n}|^2 \leq R_{\sigma,\beta}(t), \tag{5.19}$$

where

$$R_{\sigma,\beta}(t) = 2\alpha^{-2}\varepsilon^{-1}\left(\int_{-\infty}^{t} e^{\sigma(s-t)}|A^{-\beta}f(s)|^2 ds + \sigma^{-1}C^2_{(3/4-\beta)}\widetilde{M}_{\sigma,1/4+\beta}(t)\widetilde{R}_\sigma(t)\right).$$

Again, from (5.15), (5.19) and the weak lower semi-continuity of the norm, we deduce that v belongs to $\overline{B}_{D(A^{1-\beta})}(0, R^{1/2}_{\sigma,\beta}(t))$, and therefore (5.11) holds.

Moreover, if f satisfies (5.12), then (5.13) holds, and more exactly,

$$\bigcup_{t\leq t_2}\mathcal{A}_{\mathcal{D}^V_\sigma}(t) \subset \overline{B}_{D(A^{1-\beta})}(0, \widetilde{R}^{1/2}_{\sigma,\beta}(t_2)), \text{ for all } t_2 \in \mathbb{R}, \tag{5.20}$$

where

$$\widetilde{R}_{\sigma,\beta}(t_2) = 2\alpha^{-2}\varepsilon^{-1}\left(\sup_{t\leq t_2}\int_{-\infty}^{t} e^{\sigma(s-t)}|A^{-\beta}f(s)|^2 ds + \sigma^{-1}C^2_{(3/4-\beta)}\widetilde{M}_{\sigma,1/4+\beta}(t_2)\widetilde{R}_\sigma(t_2)\right).$$

Case 3. If $\beta = 0$.

In this case, if f satisfies (5.10), as $0 < \delta < 1/4$, from (5.20) we see that

$$\bigcup_{r \le t} \mathcal{A}_{\mathcal{D}_\sigma^V}(r) \subset \overline{B}_{D(A^{1-\delta})}(0, \widetilde{R}^{1/2}_{\sigma,\delta}(t)).$$

So, by (2.10) and (5.16), we deduce that

$$\begin{aligned} |B(u_n(r))| &\le C_{(1-\delta)} |A^{1-\delta} u_n(r)| \|u_n(r)\| \\ &\le C_{(1-\delta)} \widetilde{R}^{1/2}_{\sigma,\delta}(t) \widetilde{R}^{1/2}_{\sigma}(t), \quad \forall \tau_n \le r \le t, \quad \forall n \ge 1. \end{aligned}$$

Thus, if we assume (5.9), from Lemma 26 we deduce that

$$|AZ(t,\tau_n)u_{\tau_n}|^2 \le R_{\sigma,\delta,0}(t), \tag{5.21}$$

where

$$R_{\sigma,\delta,0}(t) = 2\alpha^{-2}\varepsilon^{-1} \left(\int_{-\infty}^t e^{\sigma(s-t)} |f(s)|^2 ds + \sigma^{-1} C^2_{(1-\delta)} \widetilde{R}_{\sigma,\delta}(t) \widetilde{R}_{\sigma}(t) \right).$$

Again, from (5.15), (5.21) and the weak lower semi-continuity of the norm, we deduce that

$$v \in \overline{B}_{D(A)}(0, R^{1/2}_{\sigma,\delta,0}(t)),$$

and therefore (5.11) holds.

Moreover, if f satisfies (5.12), then (5.13) holds, and more exactly,

$$\bigcup_{t \le t_2} \mathcal{A}_{\mathcal{D}_\sigma^V}(t) \subset \overline{B}_{D(A)}(0, \widetilde{R}^{1/2}_{\sigma,\delta,0}(t_2)), \text{ for all } t_2 \in \mathbb{R},$$

where

$$\widetilde{R}_{\sigma,\delta,0}(t_2) = 2\alpha^{-2}\varepsilon^{-1} \left(\sup_{t \le t_2} \int_{-\infty}^t e^{\sigma(s-t)} |f(s)|^2 ds + \sigma^{-1} C^2_{(1-\delta)} \widetilde{R}_{\sigma,\delta}(t_2) \widetilde{R}_{\sigma}(t_2) \right).$$

■

6 Attraction in $D(A)$ norm

By the previous results, when $f \in L^2_{loc}(\mathbb{R}; H)$, the restriction to $D(A)$ of the process U defined by (4.1) is a process on $D(A)$. Now, we will prove that under suitable assumptions on f, we can obtain the existence of minimal pullback attractors for U on $D(A)$.

Proposition 28 *Assume that $f \in L^2_{loc}(\mathbb{R}; H)$ is given. Then, the restriction to $D(A)$ of the bi-parametric family of maps $U(t,\tau)$, with $\tau \le t$, given by (4.1), is a closed process on $D(A)$.*

Proof. It is a consequence of Theorem 4 and Theorem 6. ■

Remark 29 *Observe that, by Remark 7, U restricted to $D(A)$ is in fact a continuous process on $D(A)$.*

For the obtention of a pullback absorbing family for the process U restricted to $D(A)$, we first have the following result.

Lemma 30 *Assume that $f \in L^2_{loc}(\mathbb{R}; H)$ satisfies (5.8). Then, for any $\tau \in \mathbb{R}$, $u_\tau \in D(A)$,*

$$0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}, \quad \textit{and} \quad 0 < \underline{\sigma} < \sigma/3, \tag{6.1}$$

the solution $u = u(\cdot;\tau,u_\tau)$ of (1.1) satisfies

$$\begin{aligned}\|u(t)\|^2+\alpha^2|Au(t)|^2 &\le e^{\sigma(\tau-t)}(\|u_\tau\|^2+\alpha^2|Au_\tau|^2)+2\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}|f(s)|^2ds\\ &\quad +4C_\varepsilon C_{\underline{\sigma}}^3(\sigma-3\underline{\sigma})^{-1}\left(e^{-3\underline{\sigma}(t-\tau)}\|u_\tau\|^6+M_{t,\underline{\sigma}}^3\right)\end{aligned} \tag{6.2}$$

for all $t\ge\tau$, where $\varepsilon>0$ is given by (4.4),

$$C_\varepsilon = 27C_2^4(2\varepsilon^3)^{-1}, \tag{6.3}$$

$$C_{\underline{\sigma}} = \alpha^{-2}\max\left\{(\alpha^2+\lambda_1^{-1}),\left(\nu-\frac{\underline{\sigma}}{2}(\lambda_1^{-1}+\alpha^2)\right)^{-1}\right\}, \tag{6.4}$$

and

$$M_{t,\underline{\sigma}} = \sup_{r\le t}\int_{-\infty}^r e^{\underline{\sigma}(s-r)}\|f(s)\|_*^2ds. \tag{6.5}$$

Proof. Let $\tau\in\mathbb{R}$, $u_\tau\in D(A)$, σ and $\underline{\sigma}$ satisfying (6.1) be fixed. From Lemma 17 we deduce in particular that $u=u(\cdot;\tau,u_\tau)$ satisfies

$$\|u(s)\|^2\le C_{\underline{\sigma}}\left(e^{\underline{\sigma}(\tau-s)}\|u_\tau\|^2+M_{t,\underline{\sigma}}\right),\quad \forall\tau\le s\le t. \tag{6.6}$$

On the other hand, by (2.21),

$$\begin{aligned}&\frac{d}{dt}(e^{\sigma t}\|u(t)\|^2+\alpha^2e^{\sigma t}|Au(t)|^2)+2\nu e^{\sigma t}|Au(t)|^2+2e^{\sigma t}(B(u(t)),Au(t))\\ =\;&\sigma e^{\sigma t}\|u(t)\|^2+\alpha^2\sigma e^{\sigma t}|Au(t)|^2+2e^{\sigma t}(f(t),Au(t)),\quad \text{a.e. } t>\tau.\end{aligned}$$

Thus, taking into account that $\|u(t)\|^2\le\lambda_1^{-1}|Au(t)|^2$,

$$\begin{aligned}2|\,(B(u(t)),Au(t))\,| &\le 2C_2\|u(t)\|^{3/2}|Au(t)|^{3/2}\\ &\le C_\varepsilon\|u(t)\|^6+\frac{\varepsilon}{2}|Au(t)|^2,\end{aligned}$$

and

$$2|\,(f(t),Au(t))\,|\le\frac{\varepsilon}{2}|Au(t)|^2+\frac{2}{\varepsilon}|f(t)|^2,$$

we deduce that

$$\begin{aligned}\|u(t)\|^2+\alpha^2|Au(t)|^2 &\le e^{\sigma(\tau-t)}(\|u_\tau\|^2+\alpha^2|Au_\tau|^2)+2\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}|f(s)|^2ds\\ &\quad +C_\varepsilon\int_\tau^t e^{\sigma(s-t)}\|u(s)\|^6ds\end{aligned}$$

for all $t\ge\tau$.

From this inequality and (6.6), we easily obtain (6.2). ■

Definition 31 *For any σ, $\underline{\sigma}>0$, we will consider the universe $\mathcal{D}_\sigma^{D(A)}\cap\mathcal{D}_{\underline{\sigma}}^V$ formed by the class of all families of nonempty subsets $\widehat{D}=\{D(t):t\in\mathbb{R}\}\subset\mathcal{P}(D(A))$ such that*

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{v\in D(\tau)}|Av|^2\right)=\lim_{\tau\to-\infty}\left(e^{\underline{\sigma}\tau}\sup_{v\in D(\tau)}\|v\|^2\right)=0.$$

Accordingly to the notation introduced in Section 3, $\mathcal{D}_F^{D(A)}$ will denote the class of families $\widehat{D}=\{D(t)=D:t\in\mathbb{R}\}$ with D a fixed nonempty bounded subset of $D(A)$.

Remark 32 *Observe that for any* σ, $\underline{\sigma} > 0$, $\mathcal{D}_F^{D(A)} \subset \mathcal{D}_\sigma^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^V$ *and that both universes are inclusion-closed.*

As a consequence of Lemma 30, we have the following result.

Corollary 33 *Assume that* $f \in L^2_{loc}(\mathbb{R}; H)$ *satisfies* (5.8) *and*

$$\int_{-\infty}^{0} e^{\sigma s} |f(s)|^2 \, ds < \infty, \tag{6.7}$$

for some $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$.

Then, for any $0 < \underline{\sigma} < \sigma/3$, *the family* $\widehat{D}_{\sigma,\underline{\sigma}} = \{D_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\}$ *defined by*

$$D_{\sigma,\underline{\sigma}}(t) = \overline{B}_{D(A)}(0, R^{1/2}_{\sigma,\underline{\sigma}}(t)), \tag{6.8}$$

the closed ball in $D(A)$ *of center zero and radius* $R^{1/2}_{\sigma,\underline{\sigma}}(t)$, *where*

$$R_{\sigma,\underline{\sigma}}(t) = \alpha^{-2} \left(1 + 2\varepsilon^{-1} \int_{-\infty}^{t} e^{\sigma(s-t)} |f(s)|^2 ds + 4 C_\varepsilon C_{\underline{\sigma}}^3 (\sigma - 3\underline{\sigma})^{-1} M^3_{t,\underline{\sigma}} \right), \tag{6.9}$$

with ε, C_ε, $C_{\underline{\sigma}}$ *and* $M_{t,\underline{\sigma}}$, *given by* (4.4), (6.3), (6.4) *and* (6.5), *respectively, is pullback* $\mathcal{D}_\sigma^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^V-$*absorbing for the restriction to* $D(A)$ *of the process* U *given by* (4.1) *(and therefore* $\mathcal{D}_F^{D(A)}-$*absorbing too).*

Now, we prove that the process U is pullback $\mathcal{D}_\sigma^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^V-$asymptotically compact. We will apply, with obvious necessary changes, the same energy method used in the proof of Lemma 21.

Lemma 34 *Assume that* $f \in L^2_{loc}(\mathbb{R}; H)$ *satisfies* (5.8) *and* (6.7) *for some* $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$. *Then, for any* $0 < \underline{\sigma} < \sigma/3$, *the restriction to* $D(A)$ *of the process* U *defined by* (4.1) *is pullback* $\widehat{D}-$*asymptotically compact for any* $\widehat{D} \in \mathcal{D}_\sigma^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^V$.

Proof. Let us fix $0 < \underline{\sigma} < \sigma/3$. Let $\widehat{D} \in \mathcal{D}_\sigma^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^V$, $t \in \mathbb{R}$, $\tau_n \to -\infty$ with $\tau_n \leq t$ and $u_{\tau_n} \in D(\tau_n)$ for all n, be given. We must prove that the sequence $\{U(t, \tau_n) u_{\tau_n}\}$ is relatively compact in $D(A)$. By Corollary 33, for each integer $k \geq 0$, there exists $\tau_{\widehat{D}}(k) \leq t - k$ such that

$$U(t-k, \tau) D(\tau) \subset D_{\sigma,\underline{\sigma}}(t-k), \quad \forall \tau \leq \tau_{\widehat{D}}(k). \tag{6.10}$$

From this and a diagonal argument, we can extract a subsequence $\{u_{\tau_{n'}}\} \subset \{u_{\tau_n}\}$ such that

$$U(t-k, \tau_{n'}) u_{\tau'_n} \rightharpoonup w_k \quad \text{weakly in } D(A), \ \forall k \geq 0, \tag{6.11}$$

where $w_k \in D_{\sigma,\underline{\sigma}}(t-k)$.

Now, applying Theorem 6 on each fixed interval $[t-k, t]$ we obtain that

$$\begin{aligned} w_0 &= D(A) - \text{weak} \lim_{n' \to \infty} U(t, \tau_{n'}) u_{\tau_{n'}} \\ &= D(A) - \text{weak} \lim_{n' \to \infty} U(t, t-k) U(t-k, \tau_{n'}) u_{\tau_{n'}} \\ &= U(t, t-k) \left[D(A) - \text{weak} \lim_{n' \to \infty} U(t-k, \tau_{n'}) u_{\tau_{n'}} \right] \\ &= U(t, t-k) w_k. \end{aligned}$$

In particular, observe that $|Aw_0| \leq \liminf_{n' \to \infty} |AU(t, \tau_{n'}) u_{\tau_{n'}}|$. We will prove now that it also holds that

$$\limsup_{n' \to \infty} |AU(t, \tau_{n'}) u_{\tau_{n'}}| \leq |Aw_0|, \tag{6.12}$$

which combined with (6.11) for $k = 0$, will imply the convergence in the strong topology of $D(A)$, and the asymptotic compactness.

Observe that, as we already used in Lemma 30, for any pair (τ, u_τ) with $u_\tau \in D(A)$, the solution $u(\cdot; \tau, u_\tau)$, for short denoted $u(\cdot)$, satisfies the differential equality (2.21).

Since $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$, we observe that $[[\cdot]]$, with $[[v]]^2 = (2\nu - \alpha^2\sigma)|Av|^2 - \sigma\|v\|^2$, defines an equivalent norm to $|\cdot|_{D(A)}$ in $D(A)$.

We integrate (2.21) in the interval $[t-k, t]$ for the solutions $U(\cdot, \tau_{n'})u_{\tau_{n'}}$, which yields

$$\begin{aligned}
& \|U(t,\tau_{n'})u_{\tau_{n'}}\|^2 + \alpha^2|AU(t,\tau_{n'})u_{\tau_{n'}}|^2 \\
= \ & \|U(t,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\|^2 + \alpha^2|AU(t,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}|^2 \\
= \ & e^{-\sigma k}\left(\|U(t-k,\tau_{n'})u_{\tau_{n'}}\|^2 + \alpha^2|AU(t-k,\tau_{n'})u_{\tau_{n'}}|^2\right) \\
& +2\int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}})ds \\
& -2\int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}), AU(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}})ds \\
& -\int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}]]^2 ds. \qquad (6.13)
\end{aligned}$$

From (6.11) and Theorem 6, we have that

$$U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}} \to U(\cdot,t-k)w_k \quad \text{strongly in } C([t-k,t];V),$$

and also

$$U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}} \rightharpoonup U(\cdot,t-k)w_k \quad \text{weakly in } L^2(t-k,t;D(A)).$$

Then, it is not difficult to see that

$$\begin{aligned}
& \lim_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}), AU(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}})ds \\
= \ & \int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)w_k), AU(s,t-k)w_k)ds. \qquad (6.14)
\end{aligned}$$

Also, as $e^{\sigma(\cdot-t)}f(\cdot) \in L^2(t-k,t;H)$, it yields

$$\begin{aligned}
& \lim_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}})ds \\
= \ & \int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)w_k)ds. \qquad (6.15)
\end{aligned}$$

Finally, as $\int_{t-k}^{t} e^{\sigma(s-t)}[[v(s)]]^2 ds$ defines an equivalent norm in $L^2(t-k,t;D(A))$, we also deduce from above that

$$\int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)w_k]]^2 ds \le \liminf_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}]]^2 ds. \qquad (6.16)$$

From (6.10), (6.11) with $k = 0$, the compactness of the injection of $D(A)$ into V, and (6.13), (6.14)–(6.16), we conclude that

$$\begin{aligned}
& \|w_0\|^2 + \alpha^2 \limsup_{n'\to\infty}|AU(t,\tau_{n'})u_{\tau_{n'}}|^2 \\
\le \ & e^{-\sigma k}(\lambda_1^{-1} + \alpha^2)R_{\sigma,\underline{\sigma}}(t-k) + 2\int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)w_k)ds \\
& -2\int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)w_k), AU(s,t-k)w_k)ds - \int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)w_k]]^2 ds.
\end{aligned}$$

Now, taking into account that $w_0 = U(t, t-k)w_k$, integrating again in (2.21), we obtain

$$\begin{aligned}\|w_0\|^2 + \alpha^2 |Aw_0|^2 &= e^{-\sigma k}(\|w_k\|^2 + \alpha^2 |Aw_k|^2) + 2\int_{t-k}^{t} e^{\sigma(s-t)}(f(s), AU(s,t-k)w_k)ds \\ &\quad -2\int_{t-k}^{t} e^{\sigma(s-t)}(B(U(s,t-k)w_k), AU(s,t-k)w_k)ds \\ &\quad -\int_{t-k}^{t} e^{\sigma(s-t)}[[U(s,t-k)w_k]]^2 ds.\end{aligned}$$

Comparing the above two expressions, we conclude that

$$\begin{aligned}&\|w_0\|^2 + \alpha^2 \limsup_{n'\to\infty} |AU(t,\tau_{n'})u_{\tau_{n'}}|^2 \\ \le\ & e^{-\sigma k}(\lambda_1^{-1} + \alpha^2) R_{\sigma,\underline{\sigma}}(t-k) + \|w_0\|^2 + \alpha^2 |Aw_0|^2 - e^{-\sigma k}\left(\|w_k\|^2 + \alpha^2 |Aw_k|^2\right).\end{aligned}$$

But from (6.9), we have that $\lim_{k\to\infty} e^{-\sigma k} R_{\sigma,\underline{\sigma}}(t-k) = 0$, so (6.12) holds. ■

In general, the pullback absorbing family $\widehat{D}_{\sigma,\underline{\sigma}}$ defined by (6.8) does not belong to $\mathcal{D}_{\sigma}^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^{V}$, and we do not know if U is pullback $\widehat{D}_{\sigma,\underline{\sigma}}$−asymptotically compact. Thus, we cannot apply Theorem 11 to the family $\widehat{D}_{\sigma,\underline{\sigma}}$. Nevertheless we can prove the following result.

Theorem 35 *Assume that $f \in L^2_{loc}(\mathbb{R}; H)$ satisfies* (5.8) *and* (6.7) *for some $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$. Then, for any $0 < \underline{\sigma} < \sigma/3$, the family of sets*

$$X_{\sigma,\underline{\sigma}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}_{\sigma}^{D(A)}\cap\mathcal{D}_{\underline{\sigma}}^{V}} \Lambda_{D(A)}(\widehat{D}, t)}^{D(A)} \qquad t \in \mathbb{R}, \tag{6.17}$$

has the following properties:

- *(a) $\lim_{\tau\to-\infty} \mathrm{dist}_{D(A)}(U(t,\tau)D(\tau), X_{\sigma,\underline{\sigma}}(t)) = 0$ for all $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_{\sigma}^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^{V}$ (pullback attraction).*
- *(b) It is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A))$ is a family of closed subsets of $D(A)$ such that $\lim_{\tau\to-\infty} \mathrm{dist}_{D(A)}(U(t,\tau)D(\tau), C(t)) = 0$ for all $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_{\sigma}^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^{V}$, then $X_{\sigma,\underline{\sigma}}(t) \subset C(t)$ for all $t \in \mathbb{R}$.*
- *(c) $U(t,\tau)X_{\sigma,\underline{\sigma}}(\tau) = X_{\sigma,\underline{\sigma}}(t)$ for all $\tau \le t$ (invariance).*

Proof. The assertion (a) is an easy consequence of Proposition 9 and Lemma 34.

For the proof of (b), assume that $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A))$ is a family of closed subsets of $D(A)$ such that $\lim_{\tau\to-\infty} \mathrm{dist}_{D(A)}(U(t,\tau)D(\tau), C(t)) = 0$ for all $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_{\sigma}^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^{V}$. Now, let us fix $t \in \mathbb{R}$. In this case, it is easy to see that, for any $u \in \Lambda_{D(A)}(\widehat{D}, t)$, with $\widehat{D} \in \mathcal{D}_{\sigma}^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^{V}$, one has that $\mathrm{dist}_{D(A)}(x, C(t)) = 0$. Thus, as $C(t)$ is closed in $D(A)$, we deduce that $\Lambda_{D(A)}(\widehat{D}, t) \subset C(t)$, and therefore, $X_{\sigma,\underline{\sigma}}(t) \subset C(t)$.

Finally, let $\tau \le t$ be fixed. In order to prove (c) we observe that by Proposition 9, we also have that

$$U(t,\tau)\Lambda_{D(A)}(\widehat{D}, \tau) = \Lambda_{D(A)}(\widehat{D}, t) \quad \text{for any } \widehat{D} \in \mathcal{D}_{\sigma}^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^{V}. \tag{6.18}$$

If $y \in X_{\sigma,\underline{\sigma}}(t)$, there exist two sequences $\{\widehat{D}_n\} \subset \mathcal{D}_{\sigma}^{D(A)} \cap \mathcal{D}_{\underline{\sigma}}^{V}$ and $\{y_n\} \subset D(A)$, such that $y_n \in \Lambda_{D(A)}(\widehat{D}_n, t)$ and $y = D(A) - \lim_{n\to\infty} y_n$. But by (6.18), $y_n = U(t,\tau)x_n$, with $x_n \in \Lambda_{D(A)}(\widehat{D}_n, \tau) \subset$

$X_{\sigma,\underline{\sigma}}(\tau)$. By Corollary 33, we can also deduce that $X_{\sigma,\underline{\sigma}}(\tau) \subset \overline{B}_{D(A)}(0, R^{1/2}_{\sigma,\underline{\sigma}}(\tau))$, and therefore, by the compactness of the injection of $D(A)$ into V, $X_{\sigma,\underline{\sigma}}(\tau)$ is a compact subset of V. Thus, there exists a subsequence $\{x_{n'}\} \subset \{x_n\}$ such that $x_{n'} \to x \in X_{\sigma,\underline{\sigma}}(\tau)$ in V. But then, as U is a closed process on V, $y = U(t,\tau)x$, and this proves that $X_{\sigma,\underline{\sigma}}(t) \subset U(t,\tau)X_{\sigma,\underline{\sigma}}(\tau)$. The reverse inclusion can be proved analogously. ■

Under the additional assumption

$$\sup_{r\le 0} \int_{r-1}^{r} |f(s)|^2\, ds < \infty, \tag{6.19}$$

the pullback absorbing family $\widehat{D}_{\sigma,\underline{\sigma}}$ defined by (6.8) does belong to $\mathcal{D}^{D(A)}_{\sigma} \cap \mathcal{D}^{V}_{\underline{\sigma}}$, whence we can apply Theorem 11, and actually we have the following result.

Theorem 36 *Assume that $f \in L^2_{loc}(\mathbb{R}; H)$ satisfies (6.19). Then, for any $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$ and $0 < \underline{\sigma} < \sigma/3$, we have that:*

(a) The family of sets $X_{\sigma,\underline{\sigma}}(t)$ defined by (6.17) is the minimal pullback $\mathcal{D}^{D(A)}_{\sigma} \cap \mathcal{D}^{V}_{\underline{\sigma}}$-attractor, and in fact is a family of compact subsets of $D(A)$.

(b) $X_{\sigma,\underline{\sigma}}(t) = \mathcal{A}_{\mathcal{D}^V_F}(t)$ for all $t \in \mathbb{R}$.

(c) Indeed, $\mathcal{A}_{\mathcal{D}^V_F}$ is the unique family of closed subsets for the norm of $D(A)$ in any universe of the form $\mathcal{D}^{D(A)}_{\sigma} \cap \mathcal{D}^{V}_{\underline{\sigma}}$ that is invariant for U and attracts any $\widehat{D} \in \mathcal{D}^{D(A)}_{\sigma} \cap \mathcal{D}^{V}_{\underline{\sigma}}$ in the pullback sense.

Proof. Let us fix $0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1}$ and $0 < \underline{\sigma} < \sigma/3$.

Observe that under the above assumptions on f, the family $\widehat{D}_{\sigma,\underline{\sigma}} = \{D_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\}$ defined by (6.8)–(6.9) belongs to $\mathcal{D}^{D(A)}_{\sigma} \cap \mathcal{D}^{V}_{\underline{\sigma}}$.

Therefore, the assertion (a) is a direct consequence of Theorem 11, Proposition 28, Corollary 33, and Lemma 34.

Now, let us fix $t \in \mathbb{R}$. It is evident that by (6.19),

$$X_{\sigma,\underline{\sigma}}(t) \subset \overline{\bigcup_{\widehat{D}\in\mathcal{D}^V_{\underline{\sigma}}} \Lambda_{D(A)}(\widehat{D}, t)}^{D(A)} \subset \overline{\bigcup_{\widehat{D}\in\mathcal{D}^V_{\underline{\sigma}}} \Lambda_{D(A)}(\widehat{D}, t)}^{V} = \mathcal{A}_{\mathcal{D}^V_{\underline{\sigma}}}(t) = \mathcal{A}_{\mathcal{D}^V_F}(t).$$

On the other hand, again by (6.19), from Theorem 27 we have that $\bigcup_{r\le t} \mathcal{A}_{\mathcal{D}^V_F}(r)$ is a bounded subset of $D(A)$, and therefore,

$$\mathrm{dist}_{D(A)}(U(t,\tau) \bigcup_{r\le t} \mathcal{A}_{\mathcal{D}^V_F}(r), X_{\sigma,\underline{\sigma}}(t)) \le \mathrm{dist}_{D(A)}(U(t,\tau) \bigcup_{r\le t} \mathcal{A}_{\mathcal{D}^V_F}(r), \Lambda_{D(A)}(\bigcup_{r\le t} \mathcal{A}_{\mathcal{D}^V_F}(r), t)).$$

From this inequality, Proposition 9, Lemma 34, and the invariance of $\mathcal{A}_{\mathcal{D}^V_F}$, we deduce that

$$\mathrm{dist}_{D(A)}(\mathcal{A}_{\mathcal{D}^V_F}(t), X_{\sigma,\underline{\sigma}}(t)) = 0,$$

and therefore $\mathcal{A}_{\mathcal{D}^V_F}(t) \subset X_{\sigma,\underline{\sigma}}(t)$. Thus, (b) is proved.

Finally, (c) is a direct consequence of Remark 12. ■

Remark 37 *Observe that in particular, if $f \in L^2_{loc}(\mathbb{R}; H)$ satisfies (6.19), by Remark 14 the minimal attractor $\mathcal{A}_{\mathcal{D}^{D(A)}_F}$ does exist, and it also coincides with the family $\mathcal{A}_{\mathcal{D}^V_F}$. Moreover, this last family attracts in the pullback sense in the norm of $D(A)$ to all the families of the universe*

$$\mathcal{D}^{D(A),V}_{max} = \bigcup_{\substack{0<\sigma<2\nu(\lambda_1^{-1}+\alpha^2)^{-1} \\ 0<\underline{\sigma}<\sigma/3}} \mathcal{D}^{D(A)}_{\sigma} \cap \mathcal{D}^{V}_{\underline{\sigma}}.$$

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