Pullback attractors for a semilinear heat equation in a non-cylindrical domain

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Abstract

The existence and uniqueness of a variational solution satisfying energy equality is proved for a semilinear heat equation in a non-cylindrical domain with homogeneous Dirichlet boundary condition, under the assumption that the spatial domains are bounded and increase with time. In addition, the non-autonomous dynamical system generated by this class of solutions is shown to have a global pullback attractor.

Key words: semilinear heat equations, non-cylindrical domains, non-autonomous dynamical system, pullback attractor.
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1 Introduction

The theory of infinite dimensional dynamical systems and their attractors has been extensively developed over the past decades, especially for systems generated by parabolic partial differential equations. Both bounded and unbounded spatial domains have been considered as well as autonomous and non-autonomous attractors. During this same time period evolution equations

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on non-cylindrical domains have also been investigated, that is with a spatial domain which varies in time so its cartesian product with the time variable is a non-cylindrical set. Much of the progress here has been for nested spatial domains which expand in time. Moreover, the results focus mainly on formulation of the problems and existence and uniqueness theory. As far as we know, attractors of such systems have not yet been considered. This is not really surprising since such systems are intrinsically non-autonomous even if the equations themselves contain no time dependent terms and require the concept of a non-autonomous attractor, which has only been introduced in recent years.

In this paper we consider semilinear heat equations of the reaction-diffusion type on bounded spatial domains which are expanding in time. First we show how initial boundary value problems for these equations can be formulated as a variational problem with appropriate function spaces, and then we establish existence and uniqueness over a finite time interval of variational solutions satisfying an energy inequality. In the second part of the paper we show that the process or two-parameter semigroup generated by such solutions is dissipative under certain assumptions on the nonlinear term and thus has a non-autonomous pullback attractor, even when the external forcing term is independent of time.

2 Equations and notation

Let $\{\mathcal{O}_t\}_{t\in\mathbb{R}}$ be a family of nonempty bounded open subsets of $\mathbb{R}^N$ such that

$$s < t \quad \Rightarrow \quad \mathcal{O}_s \subset \mathcal{O}_t, \tag{1}$$

and

$$Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_t \times \{t\} \quad \text{is an open subset of } \mathbb{R}^{N+1} \text{ for any } T > \tau. \tag{2}$$

In addition, denote

$$Q_\tau := \bigcup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\}, \quad \forall\, \tau \in \mathbb{R},$$

$$\Sigma_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\mathcal{O}_t \times \{t\}, \qquad \Sigma_\tau := \bigcup_{t\in(\tau,+\infty)} \partial\mathcal{O}_t \times \{t\}, \quad \forall\, \tau < T.$$

We consider the following initial boundary value problem for a semilinear heat equation with homogeneous Dirichlet boundary condition,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in} \ \ Q_\tau, \\[2ex] u = 0 \ \ \text{on} \ \Sigma_\tau, \\[2ex] u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{3}$$

and, for each $T > \tau$, the auxiliary problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in} \ \ Q_{\tau,T}, \\[2ex] u = 0 \ \ \text{on} \ \Sigma_{\tau,T}, \\[2ex] u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{4}$$

where $\tau \in \mathbb{R}$, $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$ and $f : Q_\tau \to \mathbb{R}$ are given and $g \in C^1(\mathbb{R})$ is also a given function for which there exist nonnegative constants α_1, α_2, β and l, and $p \geq 2$ such that

$$-\beta + \alpha_1 |s|^p \leq g(s)s \leq \beta + \alpha_2 |s|^p \quad \forall\, s \in \mathbb{R} \tag{5}$$

and

$$g'(s) \geq -l \quad \forall\, s \in \mathbb{R}. \tag{6}$$

For later observe that, by (5), there then exist nonnegative constants $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, $\widetilde{\beta}$ such that

$$-\widetilde{\beta} + \widetilde{\alpha}_1 |s|^p \leq G(s) \leq \widetilde{\beta} + \widetilde{\alpha}_2 |s|^p \quad \forall\, s \in \mathbb{R}, \tag{7}$$

where

$$G(s) := \int_0^s g(r)\, \mathrm{d}r.$$

There are many papers devoted to the study of linear and nonlinear parabolic equations in non-cylindrical domains, most of which deal with the formulation of the problems and existence of solutions (see for example [2,13,16,17,19] and the bibliography therein). In contrast, the aim of the present paper is to study the asymptotic behavior of the solutions of (3), and more exactly to establish the existence of a global attractor for a class of solutions of (3). Since the open sets $\mathcal{O}_t$ change with time, the problem (3) is non-autonomous even when the external forcing f is independent of t. The appropriate concept of a non-autonomous attractor is provided by the theory of pullback attractor [3–7,9–12,14,15,20]. The main difficulty is to obtain an energy equality for the solutions of (3). For this we will adapt to the nonlinear parabolic problem (3) the method used in [1] to obtain the existence and uniqueness of a solution satisfying an energy equality for a linear Schrödinger-type equation in a non-cylindrical domain of the form (1). The new results here are thus the existence and uniqueness of solutions of (3) satisfying an energy equality and the existence of a pullback attractor for this class of solutions.

The structure of the paper is the following: after some preliminar results in Section 3, we proceed by a penalty method to solve approximated problems in Section 4. Then, Section 5 is devoted to the proof of existence of solution to the problems (3) and (4) satisfying an energy equality. A uniform estimate for the solutions is then obtained after an additional assumption in Section 6. This will lead to the proof of existence of attractor in an appropriate framework in Section 7.

3 Preliminary results

Define $H_r := L^2(\mathcal{O}_r)$ and $V_r := H_0^1(\mathcal{O}_r)$ for each $r \in \mathbb{R}$ and denote by $(\cdot,\cdot)_r$ and $|\cdot|_r$ the usual inner product and associated norm in H_r and by $((\cdot,\cdot))_r$ and $\|\cdot\|_r$ the usual gradient inner product and associated norm in V_r. For each $s < t$ consider V_s as a closed subspace of V_t with the functions belonging to V_s being trivially extended by zero. It follows from by (1) that $\{V_t\}_{t\in[\tau,T]}$ can be considered as a family of closed subspaces of V_T for each $T > \tau$ with

$$s < t \quad \Rightarrow \quad V_s \subset V_t. \tag{8}$$

Note that $(\cdot,\cdot)_r$ will also be used to denote the duality product between $L^{p/p-1}(\mathcal{O}_r)$ and $L^p(\mathcal{O}_r)$. In addition, H_r will be identified with its topological dual ${H_r}^*$ by means of the Riesz theorem and V_r will be considered as a subspace of ${H_r}^*$ with $v \in V_r$ identified with the element $f_v \in {H_r}^*$ defined by

$$f_v(h) = (v,h)_r, \quad h \in H_r.$$

The duality product between V_r^* and V_r will be denoted by $\langle\cdot,\cdot\rangle_r$.

Finally, for each $T > \tau$, denote

$$\mathcal{U}_{\tau,T} := \{\phi \in L^2(\tau,T;V_T) \cap L^p(\widetilde{Q}_{\tau,T}) : \phi' \in L^2(\tau,T;H_T),$$
$$\phi(\tau) = \phi(T) = 0,\ \ \phi(t) \in V_t \text{ a.e. in } (\tau,T)\},$$

where

$$\widetilde{Q}_{\tau,T} := \mathcal{O}_T \times (\tau,T),$$

and suppose that $u_\tau \in L^2(\mathcal{O}_\tau)$ and $f \in L^2(Q_{\tau,T})$, with trivial extensions by zero of u_τ and f being used where appropriate.

Definition 1 *A variational solution of* (4) *is a function* u *such that*

C1) $u \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$,

C2) for all $\phi \in \mathcal{U}_{\tau,T}$,

$$\int_\tau^T [-(u(t), \phi'(t))_T + ((u(t), \phi(t)))_T + (g(u(t)), \phi(t))_T] \,\mathrm{d}t = \int_\tau^T (f(t), \phi(t))_T \,\mathrm{d}t,$$

C3) $u(t) \in V_t$ *a.e. in* (τ, T),

C4) $\lim_{t\downarrow\tau} (t-\tau)^{-1} \int_\tau^t |u(r) - u_\tau|_T^2 \,\mathrm{d}r = 0.$

The form of condition C4) is due to the fact that it is not known a priori if u belongs to $C([\tau, T]; H_T)$ or to $C([\tau, T]; V_T^*)$. It implies that

$$\lim_{t\downarrow\tau} (t-\tau)^{-1} \int_\tau^t |u(r)|_T^2 \,\mathrm{d}r = |u_\tau|_T^2.$$

Note that a variational solution u of (4) satisfies the equation in (4) in the sense of distributions in $Q_{\tau,T}$, i.e.,

$$\int_{Q_{\tau,T}} (-u\partial_t\varphi + \nabla_x u \cdot \nabla_x \varphi + g(u)\varphi) \,\mathrm{d}x\mathrm{d}t = \int_{Q_{\tau,T}} f\varphi \,\mathrm{d}x\mathrm{d}t,$$

for all $\varphi \in C_0^\infty(Q_{\tau,T})$. Moreover, $u(\cdot, t) \in H_0^1(\mathcal{O}_t)$ a.e. $t \in (\tau, T)$.

Remark 2 *If* $T_2 > T_1 > \tau$ *and* u *is a variational solution of* (4) *with* $T = T_2$, *then the restriction of* u *to* $\widetilde{Q}_{\tau,T_1}$ *is a variational solution of* (4) *with* $T = T_1$.

Define

$$\widetilde{Q}_\tau := \bigcup_{T>\tau} \widetilde{Q}_{\tau,T}.$$

Definition 3 *A variational solution of* (3) *is a function* $u : \widetilde{Q}_\tau \to \mathbb{R}$ *such that for each* $T > \tau$ *its restriction to* $\widetilde{Q}_{\tau,T}$ *is a variational solution of* (4).

The following two results are similar to Lemmas 3.1 and 3.2 in [1].

Lemma 4 *Assume that* $v \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ *and there exist* $\xi \in L^2(\tau, T; V_T^*)$ *and* $\eta \in L^{p/p-1}(\widetilde{Q}_{\tau,T})$ *such that*

$$\int_\tau^T (v(t), \phi'(t))_T \,\mathrm{d}t = -\int_\tau^T \langle \xi(t), \phi(t) \rangle_T \,\mathrm{d}t - \int_\tau^T (\eta(t), \phi(t))_T \,\mathrm{d}t \qquad (9)$$

for every function $\phi \in \mathcal{U}_{\tau,T}$.

For each $0 < h < T - \tau$, define v_h by

$$v_h(t) := \begin{cases} h^{-1}(v(t+h) - v(t)) & a.e. \text{ in } (\tau, T-h); \\ 0 & a.e. \text{ in } (T-h, T). \end{cases}$$

Then

$$\lim_{h\downarrow 0} \int_\tau^T (v_h(t), w(t))_T \, \mathrm{d}t = \int_\tau^T \langle \xi(t), w(t) \rangle_T \, \mathrm{d}t + \int_\tau^T (\eta(t), w(t))_T \, \mathrm{d}t \tag{10}$$

for every function $w \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ such that $w(t) \in V_t$ a.e. in (τ, T).

Proof. Let be $0 < \varepsilon < (T - \tau)/2$ and $w \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ with $w(t) \in V_t$ a.e. in (τ, T) such that $w(t) = 0$ a.e. in $(\tau, \tau + \varepsilon) \cup (T - \varepsilon, T)$. For each $0 < h < \varepsilon$ define

$$\phi_h(t) := \begin{cases} h^{-1} \int_{t-h}^t w(s) \, \mathrm{d}s \text{ if } \tau + h \le t \le T, \\ 0 \qquad \text{otherwise.} \end{cases} \tag{11}$$

Since both V_T and $L^p(\mathcal{O}_T)$ are contained in $L^2(\mathcal{O}_T)$ with continuous injection, the integral appearing in (11) can be understood indifferently as in V_T or in $L^p(\mathcal{O}_T)$. Then, it is not difficult to see that $\phi_h \in \mathcal{U}_{\tau,T}$ (note that $\phi_h(t) \in V_t$ by (1)), the mapping $[\tau, T] \ni t \mapsto \phi_h(t) \in V_T \cap L^p(\mathcal{O}_T)$ is continuous, and $\phi_h'(t) = h^{-1}(w(t) - w(t-h))\chi_{(\tau+h,T)}(t)$. Moreover,

$$\|\phi_h\|_{L^2(\tau,T;V_T)} \le \|w\|_{L^2(\tau,T;V_T)}, \quad \|\phi_h\|_{L^p(\widetilde{Q}_{\tau,T})} \le \|w\|_{L^p(\widetilde{Q}_{\tau,T})}, \tag{12}$$

and $\lim_{h\downarrow 0} \phi_h = w$ in $L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ (see, e.g., Proposition 1.4.29 in [8]). Now

$$\begin{aligned}
&\int_\tau^T (v_h(t), w(t))_T \, \mathrm{d}t = \int_\tau^{T-h} (v_h(t), w(t))_T \, \mathrm{d}t \\
&= h^{-1} \left(\int_{\tau+h}^T (v(t), w(t-h))_T \, \mathrm{d}t - \int_\tau^{T-h} (v(t), w(t))_T \, \mathrm{d}t \right) \\
&= h^{-1} \int_{\tau+h}^T (v(t), w(t-h) - w(t))_T \, \mathrm{d}t \\
&= - \int_\tau^T (v(t), \phi_h'(t))_T \, \mathrm{d}t \\
&= \int_\tau^T \langle \xi(t), \phi_h(t) \rangle_T \, \mathrm{d}t + \int_\tau^T (\eta(t), \phi_h(t))_T \, \mathrm{d}t,
\end{aligned} \tag{13}$$

and thus (10) follows for this w on passing to the limit as $h \downarrow 0$.

Moreover, from (12) and (13), it holds

$$\left|\int_\tau^T (v_h(t), w(t))_T \,\mathrm{d}t\right| \leq \|\xi\|_{L^2(\tau,T;V_T^*)}\|w\|_{L^2(\tau,T;V_T)} + \|\eta\|_{L^{p/p-1}(\widetilde{Q}_{\tau,T})}\|w\|_{L^p(\widetilde{Q}_{\tau,T})} \tag{14}$$

for this particular w.

Now consider an arbitrary $w \in L^2(\tau,T;V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ with $w(t) \in V_t$ a.e. in (τ,T). A density argument will be used to prove (10) for this w. The sequence of functions $w_n(t) = w(t)\chi_{(\tau+1/n,T-1/n)}(t)$, for integers $n > 2/(T-\tau)$, satisfies

$$w_n \to w \quad \text{in } L^2(\tau,T;V_T) \cap L^p(\widetilde{Q}_{\tau,T}) \text{ as } n \to +\infty. \tag{15}$$

Then, by (14) applied to the function $w_n - w_m$,

$$\begin{aligned}&\left|\int_\tau^T (v_h(t), w_n(t) - w_m(t))_T \,\mathrm{d}t\right| \\ &\leq \|\xi\|_{L^2(\tau,T;V_T^*)}\|w_n - w_m\|_{L^2(\tau,T;V_T)} + \|\eta\|_{L^{p/p-1}(\widetilde{Q}_{\tau,T})}\|w_n - w_m\|_{L^p(\widetilde{Q}_{\tau,T})},\end{aligned}$$

for all $n, m > 2/(T-\tau)$. Hence, letting $m \to +\infty$ and taking into account (15), it follows that

$$\begin{aligned}&\left|\int_\tau^T (v_h(t), w_n(t) - w(t))_T \,\mathrm{d}t\right| \\ &\leq \|\xi\|_{L^2(\tau,T;V_T^*)}\|w_n - w\|_{L^2(\tau,T;V_T)} \\ &\qquad + \|\eta\|_{L^{p/p-1}(\widetilde{Q}_{\tau,T})}\|w_n - w\|_{L^p(\widetilde{Q}_{\tau,T})}, \quad \forall\, n > 2/(T-\tau).\end{aligned} \tag{16}$$

Now let $\varepsilon > 0$ be fixed, but otherwise arbitrary. By (15) and (16) there exists an $n_\varepsilon > 2/(T-\tau)$ such that

$$\begin{aligned}&\left|\int_\tau^T (v_h(t), w(t) - w_{n_\varepsilon}(t))_T \,\mathrm{d}t\right| \\ &\quad + \left|\int_\tau^T \langle \xi(t), w_{n_\varepsilon}(t) - w(t)\rangle_T \,\mathrm{d}t + \int_\tau^T (\eta(t), w_{n_\varepsilon}(t) - w(t))_T \,\mathrm{d}t\right| \leq \varepsilon\end{aligned}$$

for all $0 < h < T - \tau$. Then,

$$\left|\int_\tau^T (v_h(t), w(t))_T \,\mathrm{d}t - \int_\tau^T \langle \xi(t), w(t)\rangle_T \,\mathrm{d}t - \int_\tau^T (\eta(t), w(t))_T \,\mathrm{d}t\right|$$
$$\leq \varepsilon + \left|\int_\tau^T (v_h(t), w_{n_\varepsilon}(t))_T \,\mathrm{d}t - \int_\tau^T \langle \xi(t), w_{n_\varepsilon}(t)\rangle_T \,\mathrm{d}t - \int_\tau^T (\eta(t), w_{n_\varepsilon}(t))_T \,\mathrm{d}t\right|$$

for all $0 < h < T - \tau$ and, hence,

$$\limsup_{h\downarrow 0}\left|\int_\tau^T (v_h(t), w(t))_T \,\mathrm{d}t - \int_\tau^T \langle \xi(t), w(t)\rangle_T \,\mathrm{d}t - \int_\tau^T (\eta(t), w(t))_T \,\mathrm{d}t\right| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (10) then follows for this general w. ■

Remark 5 *If $\tau < T' < T$ and $\phi \in L^2(\tau, T'; V_T) \cap L^p(\mathcal{O}_T \times (\tau, T'))$, with $\phi' \in L^2(\tau, T'; H_T)$ satisfies $\phi(\tau) = \phi(T') = 0$ and $\phi(t) \in V_t$ a.e. in (τ, T'), then the trivial extension $\widetilde{\phi}$ of ϕ satisfies $\widetilde{\phi} \in \mathcal{U}_{\tau,T}$, with $(\widetilde{\phi})' = \widetilde{\phi}'$. Using the open sets $\widehat{\mathcal{O}}_t := \mathcal{O}_{t+T-T'}$, $\tau \leq t \leq T'$, it is easy to see that under the conditions of* Lemma 4*, one also has*

$$\lim_{h\downarrow 0}\int_\tau^{T'-h} (v_h(t), w(t))_T \,\mathrm{d}t = \int_\tau^{T'} \langle \xi(t), w(t)\rangle_T \,\mathrm{d}t + \int_\tau^{T'} (\eta(t), w(t))_T \,\mathrm{d}t,$$

for every $\tau < T' < T$ and every function $w \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ such that $w(t) \in V_t$ a.e. in (τ, T).

Lemma 6 *Let $v_i \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$, $i = 1, 2$, be two functions such that $v_i(t) \in V_t$ a.e. in (τ, T) for $i = 1, 2$. Assume that there exist $\xi_i \in L^2(\tau, T; V_T^*)$, $\eta_i \in L^{p/p-1}(\widetilde{Q}_{\tau,T})$, $i = 1, 2$, such that*

$$\int_\tau^T (v_i(t), \phi'(t))_T \,\mathrm{d}t = -\int_\tau^T \langle \xi_i(t), \phi(t)\rangle_T \,\mathrm{d}t - \int_\tau^T (\eta_i(t), \phi(t))_T \,\mathrm{d}t \quad i = 1, 2, \tag{17}$$

for every function $\phi \in \mathcal{U}_{\tau,T}$.

Then, for every pair $\tau \leq s < t \leq T$ of Lebesgue points of the inner product function $(v_1, v_2)_T$ it holds

$$\begin{aligned}
&(v_1(t), v_2(t))_T - (v_1(s), v_2(s))_T \\
&= \int_s^t \langle \xi_1(r), v_2(r)\rangle_T \,\mathrm{d}r + \int_s^t \langle \xi_2(r), v_1(r)\rangle_T \,\mathrm{d}r \\
&\quad + \int_s^t (\eta_1(r), v_2(r))_T \,\mathrm{d}r + \int_s^t (\eta_2(r), v_1(r))_T \,\mathrm{d}r \\
&\quad + \lim_{h\downarrow 0} h^{-1}\int_s^{t-h} (v_1(r+h) - v_1(r), v_2(r+h) - v_2(r))_T \,\mathrm{d}r.
\end{aligned} \tag{18}$$

Proof. It is immediate that

$$
\begin{aligned}
&(v_1(r+h)-v_1(r),v_2(r))_T+(v_2(r+h)-v_2(r),v_1(r))_T\\
=&(v_1(r+h),v_2(r+h))_T-(v_1(r),v_2(r))_T\\
&-(v_1(r+h)-v_1(r),v_2(r+h)-v_2(r))_T.
\end{aligned}
\tag{19}
$$

Since for any $0<2h<t-s$

$$
\begin{aligned}
&\int_s^{t-h}(v_1(r+h),v_2(r+h))_T\,\mathrm{d}r-\int_s^{t-h}(v_1(r),v_2(r))_T\,\mathrm{d}r\\
=&\int_{s+h}^{t}(v_1(r),v_2(r))_T\,\mathrm{d}r-\int_s^{t-h}(v_1(r),v_2(r))_T\,\mathrm{d}r\\
=&\int_{t-h}^{t}(v_1(r),v_2(r))_T\,\mathrm{d}r-\int_s^{s+h}(v_1(r),v_2(r))_T\,\mathrm{d}r,
\end{aligned}
$$

it follows from (19) that

$$
\begin{aligned}
&h^{-1}\int_{t-h}^{t}(v_1(r),v_2(r))_T\,\mathrm{d}r-h^{-1}\int_s^{s+h}(v_1(r),v_2(r))_T\,\mathrm{d}r\\
=&h^{-1}\int_s^{t-h}(v_1(r+h)-v_1(r),v_2(r))_T\,\mathrm{d}r\\
&+h^{-1}\int_s^{t-h}(v_2(r+h)-v_2(r),v_1(r))_T\,\mathrm{d}r\\
&+h^{-1}\int_s^{t-h}(v_1(r+h)-v_1(r),v_2(r+h)-v_2(r))_T\,\mathrm{d}r.
\end{aligned}
\tag{20}
$$

Finally, (18) is an easy consequence of (20) and Remark 5. ■

Observe that if u is a variational solution of (4), then τ is a Lebesgue point of $|u|_T^2$ since C4) is satisfied. The next corollary gives an obvious consequence of (20).

Corollary 7 *If u is a variational solution of* (4)*, then for every Lebesgue point $t\in(\tau,T)$ of $|u|_T^2$ it holds*

$$
\begin{aligned}
&|u(t)|_T^2+2\int_\tau^t\|u(r)\|_T^2\,\mathrm{d}r+2\int_\tau^t(g(u(r)),u(r))_T\,\mathrm{d}r\\
=&|u_\tau|_T^2+2\int_\tau^t(f(r),u(r))_T\,\mathrm{d}r+\lim_{h\downarrow 0}h^{-1}\int_\tau^{t-h}|u(r+h)-u(r)|_T^2\,\mathrm{d}r.
\end{aligned}
\tag{21}
$$

Our aim is to obtain a variational solution u of (4) such that

$$
\begin{aligned}
&|u(t)|_T^2+2\int_\tau^t\|u(r)\|_T^2\,\mathrm{d}r+2\int_\tau^t(g(u(r)),u(r))_T\,\mathrm{d}r\\
=&|u_\tau|_T^2+2\int_\tau^t(f(r),u(r))_T\,\mathrm{d}r\quad\text{a.e. } t\in(\tau,T).
\end{aligned}
\tag{22}
$$

In this case we will say that u satisfies the energy equality a.e. in (τ, T). Analogously, if u is a variational solution of (3), we will say that u satisfies the energy equality a.e. in $(\tau, +\infty)$ if for each $T > \tau$ the restriction of u to $\widetilde{Q}_{\tau,T}$ satisfies the energy equality (22) a.e. in (τ, T).

Remark 8 *If u is a variational solution of* (4) *satisfying the energy equality a.e. in (τ, T), then $u \in L^\infty(\tau, T; H)$.*

For any function $v \in L^2(\tau, T; H_T)$ and any $t \in (\tau, T]$ define

$$\eta_{v,T}(t) := \limsup_{h \downarrow 0} h^{-1} \int_\tau^{t-h} |v(r+h) - v(r)|_T^2 \, \mathrm{d}r.$$

Remark 9 *$\eta_{v,T}$ is a nondecreasing function. Consequently, by* Corollary 7, *a variational solution u of* (4) *satisfies the energy equality a.e. in (τ, T) if and only if $\eta_{u,T}(t) = 0$ for all $t \in (\tau, T)$. In fact, using the continuity of the mapping*

$$t \in [\tau, T] \mapsto |u_\tau|_T^2 + 2 \int_\tau^t \Big[(f(r), u(r))_T - \|u(r)\|_T^2 - (g(u(r)), u(r))_T \Big] \, \mathrm{d}r \in \mathbb{R},$$

one can see that a variational solution u of (4) *satisfies the energy equality a.e. in (τ, T) if and only if $\eta_{u,T}(T) = 0$.*

The next lemma provides a sufficient condition for u to satisfy the energy equality a.e. in (τ, T).

Lemma 10 *Let u be a variational solution of* (4) *and suppose that there exists a sequence $\{t_n\} \subset (\tau, T)$ of Lebesgue points of $|u|_T^2$ such that $t_n \to T$ and*

$$\limsup_{n \uparrow \infty} |u(t_n)|_T^2 \le |u_\tau|_T^2 + 2 \int_\tau^T \Big[(f(r), u(r))_T - \|u(r)\|_T^2 - (g(u(r)), u(r))_T \Big] \, \mathrm{d}r. \tag{23}$$

Then, u satisfies the energy equality a.e. in (τ, T).

Proof. By Remark 9 it suffices to prove that $\eta_{u,T}(t) = 0$ for all $t \in (\tau, T)$. Since $t_n \to T$ and $\eta_{u,T}$ is nondecreasing, by Corollary 7

$$\begin{aligned}
&\eta_{u,T}(t) \le \limsup_{n \uparrow \infty} \eta_{u,T}(t_n) \\
=& \limsup_{n \uparrow \infty} \Big(|u(t_n)|_T^2 - |u_\tau|_T^2 - 2 \int_\tau^{t_n} \Big[(f(r), u(r))_T - \|u(r)\|_T^2 - (g(u(r)), u(r))_T \Big] \, \mathrm{d}r \Big) \\
\le& \limsup_{n \uparrow \infty} |u(t_n)|_T^2 - |u_\tau|_T^2 - 2 \int_\tau^{T} \Big[(f(r), u(r))_T - \|u(r)\|_T^2 - (g(u(r)), u(r))_T \Big] \, \mathrm{d}r \\
\le& 0
\end{aligned}$$

for any $t \in (\tau, T)$. ■

Proposition 11 *Let u, $\overline{u}$ be two variational solutions of* (4) *corresponding to the initial data u_τ, $\overline{u}_\tau \in L^2(\mathcal{O}_\tau)$, respectively, which satisfy the energy equality a.e. in (τ, T). Then,*

$$|u(t)-\overline{u}(t)|_T^2 + 2\int_\tau^t \|u(r)-\overline{u}(r)\|_T^2 \,\mathrm{d}r \le \mathrm{e}^{2l(t-\tau)}|u_\tau - \overline{u}_\tau|_T^2 \tag{24}$$

a.e. $t \in (\tau, T)$.

Proof. From the identity, $|u(t)-\overline{u}(t)|_T^2 = |u(t)|_T^2 + |\overline{u}(t)|_T^2 - 2(u(t), \overline{u}(t))_T$, Lemma 6, Corollary 7 and the energy equality a.e. it follows that

$$\eta_{u,T}(t) = \eta_{\overline{u},T}(t) = 0 \quad \text{for all } t \in (\tau, T), \tag{25}$$

and

$$\begin{aligned} &|u(t)-\overline{u}(t)|_T^2 + 2\int_\tau^t \|u(r)-\overline{u}(r)\|_T^2 \,\mathrm{d}r \\ &\qquad + 2\int_\tau^t (g(u(r)) - g(\overline{u}(r)), u(r)-\overline{u}(r))_T \,\mathrm{d}r \\ &= |u_\tau - \overline{u}_\tau|_T^2 - 2\lim_{h\downarrow 0} h^{-1}\int_\tau^{t-h} (u(r+h)-u(r), \overline{u}(r+h)-\overline{u}(r))_T \,\mathrm{d}r \end{aligned} \tag{26}$$

a.e. $t \in (\tau, T)$. Now

$$\begin{aligned} &\left| h^{-1}\int_\tau^{t-h} (u(r+h)-u(r), \overline{u}(r+h)-\overline{u}(r))_T \,\mathrm{d}r \right| \\ &\le \left(h^{-1}\int_\tau^{t-h} |u(r+h)-u(r)|_T^2 \,\mathrm{d}r \right)^{1/2} \left(h^{-1}\int_\tau^{t-h} |\overline{u}(r+h)-\overline{u}(r)|_T^2 \,\mathrm{d}r \right)^{1/2}, \end{aligned}$$

so by (25)

$$\lim_{h\downarrow 0} h^{-1}\int_\tau^{t-h} (u(r+h)-u(r), \overline{u}(r+h)-\overline{u}(r))_T \,\mathrm{d}r = 0 \quad \text{for all } t \in (\tau, T).$$

Using this and (6) in (26) one deduces

$$|u(t)-\overline{u}(t)|_T^2 + 2\int_\tau^t \|u(r)-\overline{u}(r)\|_T^2 \,\mathrm{d}r \le |u_\tau - \overline{u}_\tau|_T^2 + 2l\int_\tau^t |u(r)-\overline{u}(r)|_T^2 \,\mathrm{d}r,$$

a.e. $t \in (\tau, T)$. Finally, (24) then follows by an application of Gronwall's inequality. ■

An immediate consequence is the following uniqueness result.

Corollary 12 *For a given $u_\tau \in L^2(\mathcal{O}_\tau)$ there exists at most one variational solution of* (4) *satisfying the energy equality a.e. in (τ, T).*

4 Penalty method

The method of penalization due to J.L. Lions (see [17]) will now be used to prove existence and uniqueness of a solution to problem (4) satisfying the energy equality a.e. in (τ, T) and, as a consequence, the existence and uniqueness of a solution to problem (3) satisfying the energy equality a.e. in $(\tau, +\infty)$. To begin, fix $T > \tau$ and for each $t \in [\tau, T]$ denote by

$$V_t^\perp := \{v \in V_T : \ ((v, w))_T = 0 \quad \forall\, w \in V_t\}$$

the orthogonal subspace of V_t with respect the inner product in V_T and by $P(t) \in \mathcal{L}(V_T)$ the orthogonal projection operator from V_T onto $V_t^\perp$, which is defined as

$$P(t)v \in V_t^\perp, \quad v - P(t)v \in V_t,$$

for each $v \in V_T$. Finally, define $P(t) = P(T)$ for all $t > T$ and observe that $P(T)$ is the zero of $\mathcal{L}(V_T)$.

We will now approximate $P(t)$ by operators which are more regular in time. Consider the family $p(t; \cdot, \cdot)$ of symmetric bilinear forms on V_T defined by

$$p(t; v, w) := ((P(t)v, w))_T \quad \forall\, v, w \in V_T, \quad \forall\, t \geq \tau.$$

In view of (8), it can be proved (see [1]) that the mapping $[\tau, +\infty) \ni t \mapsto \mathrm{p}(t; v, w) \in \mathbb{R}$ is measurable for all v, $w \in V_T$. Moreover, $|p(t; v, w)| \leq \|v\|_T \|w\|_T$. For each integer $k \geq 1$ and each $t \geq \tau$ define

$$p_k(t; v, w) := k \int_0^{1/k} p(t + r; v, w)\, \mathrm{d}r \quad \forall\, v, w \in V_T, \quad \forall\, t \geq \tau,$$

and denote by $P_k(t) \in \mathcal{L}(V_T)$ the associated operator defined by

$$((P_k(t)v, w))_T := p_k(t; v, w) \quad \forall\, v, w \in V, \quad \forall\, t \geq \tau. \tag{27}$$

The following lemma can be proved (see [1]) on account of (8).

Lemma 13 *For any integers $1 \leq h \leq k$, any $t \geq \tau$, and every $v, w \in V_T$*

$$p_k(t; v, w) = p_k(t; w, v), \tag{28}$$

$$0 \leq p_h(t; v, v) \leq p_k(t; v, v) \leq p(t; v, v) = \|P(t)v\|_T^2 \leq \|v\|_T^2, \tag{29}$$

$$p_k'(t; v, v) := \frac{\mathrm{d}}{\mathrm{d}t} p_k(t; v, v) = k(p(t + 1/k; v, v) - p(t; v, v)) \leq 0, \tag{30}$$

$$((P_k(t)v, z))_T = 0 \quad \forall\, z \in V_t. \tag{31}$$

Moreover, for every sequence $v_k \in L^2(\tau, T; V_T)$ weakly convergent to v in $L^2(\tau, T; V_T)$

$$\liminf_{k \to +\infty} \int_\tau^T p_k(t; v_k(t), v_k(t))\, \mathrm{d}t \geq \int_\tau^T p(t; v(t), v(t))\, \mathrm{d}t. \tag{32}$$

Let $J : V_T \to V_T^*$ be the Riesz isomorphism defined by

$$\langle Jv, w\rangle_T := ((v, w))_T \quad \forall\, v, w \in V_T,$$

and for each integer $k \geq 1$ and each $t \in [\tau, T]$ denote

$$A_k(t) := -\Delta + kJP_k(t). \tag{33}$$

Obviously, $A_k(t) \in \mathcal{L}(V_T, V_T^*)$, $t \in [\tau, T]$, is a family of symmetric linear operators such that the mapping $t \in [\tau, T] \mapsto A(t) \in \mathcal{L}(V_T, V_T^*)$ is measurable and bounded, and satisfies

$$\langle A_k(t)v, v\rangle_T \geq \|v\|_T^2 \quad \forall\, v \in V_T \quad \forall\, t \in [\tau, T]. \tag{34}$$

Let $u_\tau \in H_T$ be given and for each $k \geq 1$ consider the problem

$$\begin{aligned}(u_k(t), v)_T &+ \int_\tau^t \langle A_k(r)u_k(r), v\rangle_T \,\mathrm{d}r + \int_\tau^t (g(u_k(r)), v)_T \,\mathrm{d}r \\ = (u_\tau, v)_T &+ \int_\tau^t (f(r), v)_T \,\mathrm{d}r \quad \forall\, t \in [\tau, T], \quad \forall\, v \in V_T \cap L^p(\mathcal{O}_T).\end{aligned} \tag{35}$$

Theorem 14 *Suppose that* (1), (2), (5) *and* (6) *hold. Then, for each* $k \geq 1$, $f \in L^2(\tau, T; H_T)$ *and* $u_\tau \in H_T$ *there exists a unique solution* $u_k \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ *of* (35). *Moreover,* $u_k \in C([\tau, T]; H_T)$. *In addition, if* $u_\tau \in V_T \cap L^p(\mathcal{O}_T)$, *then* u_k *also satisfies*

$$u_k \in L^\infty(\tau, T; V_T) \cap L^\infty(\tau, T; L^p(\mathcal{O}_T)), \quad u_k' \in L^2(\tau, T; H_T), \tag{36}$$

and

$$\begin{aligned}&\int_\tau^T |u_k'(t)|_T^2 \,\mathrm{d}t + \|u_k\|^2_{L^\infty(\tau,T;V_T)} \\ &\qquad + k\int_\tau^T ((P_k(t)u_k(t), u_k(t)))_T \,\mathrm{d}t + 2\widetilde{\alpha}_1 \|u_k\|^p_{L^\infty(\tau,T;L^p(\mathcal{O}_T))} \\ &\leq (3 + T - \tau)\Big[\|u_\tau\|_T^2 + k((P_k(\tau)u_\tau, u_\tau))_T \\ &\qquad + 2\widetilde{\alpha}_2 \|u_\tau\|^p_{L^p(\mathcal{O}_T)} + 4\widetilde{\beta}|\mathcal{O}_T| + \int_\tau^T |f(r)|_T^2 \,\mathrm{d}r\Big],\end{aligned} \tag{37}$$

where $\widetilde{\alpha}_2$ *and* $\widetilde{\beta}$ *are given in* (7).

Proof. It is well known (see [17]) by the monotonicity of the involved operators, that problem (35) has a unique solution $u_k \in L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ and that $u_k \in C([\tau, T]; H_T)$. This solution can be obtained by the Galerkin method. More exactly, one can take an orthonormal Hilbert basis $\{e_j\}$ of H_T formed by elements of $V_T \cap L^p(\mathcal{O}_T)$ such that the vector space generated by $\{e_j\}$ is dense in V_T and in $L^p(\mathcal{O}_T)$. Then, one takes a sequence u_{τ_m} converging to u_τ in H_T, with u_{τ_m} in the vector space spanned by the m first e_j. For each integer $m \geq 1$ one considers the approximation $u_{k_m}(t) = \sum_{j=1}^m \gamma_{k_m,j}(t) e_j$, defined as the unique solution of

$$\begin{aligned}(u_{k_m}(t), e_j) &+ \int_\tau^t \langle A_k(r) u_{k_m}(r), e_j\rangle_T \,\mathrm{d}r + \int_\tau^t (g(u_{k_m}(r)), e_j)_T \,\mathrm{d}r \\ &= (u_{\tau_m}, e_j)_T + \int_\tau^t (f(r), e_j)_T \,\mathrm{d}r, \quad \forall\, t \in [\tau, T], \quad \forall 1 \leq j \leq m. \qquad (38)\end{aligned}$$

Moreover, the solution u_{k_m} satisfies the energy equality

$$\begin{aligned}|u_{k_m}(t)|_T^2 &+ 2\int_\tau^t \langle A_k(r) u_{k_m}(r), u_{k_m}(r)\rangle_T \,\mathrm{d}r + 2\int_\tau^t (g(u_{k_m}(r)), u_{k_m}(r))_T \,\mathrm{d}r \\ &= |u_{\tau_m}|_T^2 + 2\int_\tau^t (f(r), u_{k_m}(r))_T \,\mathrm{d}r \quad \forall\, t \in [\tau, T]\end{aligned}$$

and it follows by (5) and (34) that

$$\begin{aligned}|u_{k_m}(t)|_T^2 &+ 2\int_\tau^t \|u_{k_m}(r)\|_T^2 \,\mathrm{d}r + 2\alpha_1 \int_\tau^t \|u_{k_m}(r)\|^p_{L^p(\mathcal{O}_T)} \,\mathrm{d}r \\ &\leq |u_{\tau_m}|_T^2 + \int_\tau^t |f(r)|_T^2 \,\mathrm{d}r + \int_\tau^t |u_{k_m}(r)|_T^2 \,\mathrm{d}r + 2\beta(t-\tau)|\mathcal{O}_T| \qquad (39)\end{aligned}$$

for all $t \in [\tau, T]$. Then, from (39) and the Gronwall inequality it follows that the sequence $\{u_{k_m}\}$ is bounded in $L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ and in $L^\infty(\tau, T; H_T)$. Finally, it is well known that one can also prove that the sequence $\{u_{k_m}\}$ converges weakly in $L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ and weak-star in $L^\infty(\tau, T; H_T)$ to the unique solution u_k of (35).

Suppose now that $u_\tau \in V_T \cap L^p(\mathcal{O}_T)$. Multiply by $\gamma'_{k_m,j}(t)$ the equation

$$(u'_{k_m}(t), e_j)_T + \langle A_k(t) u_{k_m}(t), e_j\rangle_T + (g(u_{k_m}(t)), e_j)_T = (f(t), e_j)_T, \quad \text{a.e. } t \in (\tau, T),$$

and sum from $j = 1$ to m to obtain

$$\begin{aligned}|u'_{k_m}(t)|_T^2 &+ ((u_{k_m}(t), u'_{k_m}(t)))_T + k((P_k(t) u_{k_m}(t), u'_{k_m}(t)))_T + (g(u_{k_m}(t)), u'_{k_m}(t))_T \\ &= (f(t), u'_{k_m}(t))_T, \quad \text{a.e. } t \in (\tau, T).\end{aligned}$$

Hence

$$
\begin{aligned}
&|u'_{k_m}(t)|_T^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|u_{k_m}(t)\|_T^2 + 2k((P_k(t)u_{k_m}(t), u'_{k_m}(t)))_T + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}_T} G(u_{k_m}(x,t))\,\mathrm{d}x \\
&\le |f(t)|_T^2, \quad \text{a.e. } t \in (\tau, T).
\end{aligned} \tag{40}
$$

Now, observe that

$$
\begin{aligned}
((P_k(t)u_{k_m}(t), u'_{k_m}(t)))_T &= k\int_0^{1/k} ((P(t+r)u_{k_m}(t), u'_{k_m}(t)))_T\,\mathrm{d}r \\
&= k\int_t^{t+1/k} ((P(r)u_{k_m}(t), u'_{k_m}(t)))_T\,\mathrm{d}r,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_t^{t+1/k} ((P(r)u_{k_m}(t), u_{k_m}(t)))_T\,\mathrm{d}r \\
&= 2\int_t^{t+1/k} ((P(r)u_{k_m}(t), u'_{k_m}(t)))_T\,\mathrm{d}r \\
&\qquad + ((P(t+1/k)u_{k_m}(t), u_{k_m}(t)))_T - ((P(t)u_{k_m}(t), u_{k_m}(t)))_T \\
&\le 2\int_t^{t+1/k} ((P(r)u_{k_m}(t), u'_{k_m}(t)))_T\,\mathrm{d}r,
\end{aligned}
$$

where (30) has been used in the last inequality. Thus,

$$
((P_k(t)u_{k_m}(t), u'_{k_m}(t)))_T \;\ge\; \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T,
$$

and hence, by (40),

$$
\begin{aligned}
&|u'_{k_m}(t)|_T^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|u_{k_m}(t)\|_T^2 + k\frac{\mathrm{d}}{\mathrm{d}t}((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \\
&\qquad + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}_T} G(u_{k_m}(x,t))\,\mathrm{d}x \\
&\le |f(t)|_T^2 \quad \text{a.e. } t \in (\tau, T).
\end{aligned} \tag{41}
$$

This last estimate and (7) give

$$
\begin{aligned}
&\int_\tau^t |u'_{k_m}(r)|_T^2 \,\mathrm{d}r + \|u_{k_m}(t)\|_T^2 \\
&\qquad + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\widetilde{\alpha}_1 \|u_{k_m}(t)\|^p_{L^p(\mathcal{O}_T)} \\
&\leq \|u_{\tau_m}\|_T^2 + k((P_k(\tau)u_{\tau_m}, u_{\tau_m}))_T + 2\widetilde{\alpha}_2 \|u_{\tau_m}\|^p_{L^p(\mathcal{O}_T)} \\
&\qquad + 4\widetilde{\beta}|\mathcal{O}_T| + \int_\tau^t |f(r)|_T^2 \,\mathrm{d}r,
\end{aligned} \tag{42}
$$

for all $t \in [\tau, T]$.

Since $u_\tau \in V_T \cap L^p(\mathcal{O}_T)$, there is a sequence u_{τ_m} converging to u_τ in $V_T \cap L^p(\mathcal{O}_T)$ with u_{τ_m} in the vector space spanned by the m first e_j. With this choice of initial values one easily deduces from (42) that the sequence $\{u_{k_m}\}$ is bounded in $L^\infty(\tau, T; V_T) \cap L^\infty(\tau, T; L^p(\mathcal{O}_T))$ and that the sequence $\{u'_{k_m}\}$ is bounded in $L^2(\tau, T; H_T)$, and therefore that u_k satisfies (36). Moreover, by the uniqueness of u_k, the complete sequence u_{k_m} converges weakly-star to u_k in $L^\infty(\tau, T; V_T) \cap L^\infty(\tau, T; L^p(\mathcal{O}_T))$ and u'_{k_m} converges weakly to u'_k in $L^2(\tau, T; H_T)$ as $m \to +\infty$. Then it follows from (42) and the weak and weak-star lower semicontinuity of the norms that

$$
\begin{aligned}
&\int_\tau^T |u'_k(t)|_T^2 \,\mathrm{d}t + \|u_k\|^2_{L^\infty(\tau,T;V_T)} + 2\widetilde{\alpha}_1 \|u_k\|^p_{L^\infty(\tau,T;L^p(\mathcal{O}_T))} \\
&\leq 3\|u_\tau\|_T^2 + 3k((P_k(\tau)u_\tau, u_\tau))_T + 6\widetilde{\alpha}_2 \|u_\tau\|^p_{L^p(\mathcal{O}_T)} \\
&\quad + 12\widetilde{\beta}|\mathcal{O}_T| + 3\int_\tau^T |f(r)|_T^2 \,\mathrm{d}r.
\end{aligned} \tag{43}
$$

On the other hand, by (42) again,

$$
\begin{aligned}
&k\int_\tau^T ((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T \,\mathrm{d}t \\
&\leq (T-\tau)\Big[\|u_{\tau_m}\|_T^2 + k((P_k(\tau)u_{\tau_m}, u_{\tau_m}))_T + 2\widetilde{\alpha}_2 \|u_{\tau_m}\|^p_{L^p(\mathcal{O}_T)} \\
&\qquad\qquad + 4\widetilde{\beta}|\mathcal{O}_T| + \int_\tau^t |f(r)|_T^2 \,\mathrm{d}r\Big].
\end{aligned}
$$

Hence, since a functional $\Phi : L^2(\tau, T; V_T) \to \mathbb{R}$ defined by

$$
\Phi(v) = \int_\tau^T ((P_k(t)v(t), v(t)))_T \,\mathrm{d}t, \quad v \in L^2(\tau, T; V_T),
$$

is continuous and convex, it follows that

$$
\begin{aligned}
&k\int_\tau^T ((P_k(t)u_k(t),u_k(t)))_T\,\mathrm{d}t\\
&\le k\liminf_{m\to+\infty}\int_\tau^T ((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T\,\mathrm{d}t\\
&\le (T-\tau)\Bigg[\|u_\tau\|_T^2+k((P_k(\tau)u_\tau,u_\tau))_T+2\widetilde{\alpha}_2\|u_\tau\|^p_{L^p(\mathcal{O}_T)}\\
&\qquad\qquad +4\widetilde{\beta}|\mathcal{O}_T|+\int_\tau^t |f(r)|_T^2\,\mathrm{d}r\Bigg]. \qquad (44)
\end{aligned}
$$

The inequalities (43) and (44) then give (37). ■

5 Variational solution of (4) satisfying the energy equality

The purpose of this section is to establish the existence of variational solutions satisfying the energy inequality.

Theorem 15 *Suppose that* (1), (2), (5) *and* (6) *hold. Then for each* $f \in L^2(\tau,T;H_T)$ *and* $u_\tau \in L^2(\mathcal{O}_\tau)$ *there exists a unique variational solution* u *of* (4) *satisfying the energy equality a.e. in* (τ,T)*. In addition,* $u\in C([\tau,T];H_T)$ *and satisfies the energy equality* (22) *for all* $t\in[\tau,T]$*. Moreover, if* $u_\tau \in V_\tau\cap L^p(\mathcal{O}_\tau)$*, then* u *also satisfies*

$$
u\in L^\infty(\tau,T;V_T)\cap L^\infty(\tau,T;L^p(\mathcal{O}_T)),\quad u'\in L^2(\tau,T;H_T).
$$

Proof. First suppose that $u_\tau\in V_\tau\cap L^p(\mathcal{O}_\tau)$. Then, by (31),

$$
((P_k(\tau)u_\tau,u_\tau))=0\quad \forall\, k\ge 1.
$$

Hence, from estimate (37), there exists a subsequence of the sequence $\{u_k\}$ of solutions of (35), which for simplicity will continue to be denoted by $\{u_k\}$, and a function u such that

$$
u_k \rightharpoonup u \quad \text{weak-star in } L^\infty(\tau,T;V_T)\cap L^\infty(\tau,T;L^p(\mathcal{O}_T)), \qquad (45)
$$

and

$$
u_k' \rightharpoonup u' \quad \text{weakly in } L^2(\tau,T;H_T).
$$

Hence, $u\in C([\tau,T];H_T)$.

Observe that $u_k\rightharpoonup u$ weakly in $L^2(\tau,T;V_T)$, and consequently, by Lemma 13 and (37), that

$$\int_\tau^T \|P(t)u(t)\|_T^2 \,\mathrm{d}t \le \liminf_{k\to+\infty} \int_\tau^T ((P_k(t)u_k(t), u_k(t)))\,\mathrm{d}t$$
$$\le \liminf_{k\to+\infty} \frac{C}{k} = 0,$$

where

$$C = (3+T-\tau)\left[\|u_\tau\|_\tau^2 + 2\widetilde{\alpha}_2\|u_\tau\|^p_{L^p(\mathcal{O}_\tau)} + 4\widetilde{\beta}|\mathcal{O}_T| + \int_\tau^T |f(r)|_T^2\,\mathrm{d}r\right], \quad (46)$$

from which it follows that $P(t)u(t) = 0$ a.e. in (τ, T), i.e.,

$$u(t) \in V_t \quad \text{a.e. in } (\tau, T). \quad (47)$$

On the other hand, (37) and the equality

$$u_k(t) - u_k(s) = \int_s^t u_k'(r)\,\mathrm{d}r \quad \forall\, s,t \in [\tau, T], \quad \forall\, k \ge 1,$$

give

$$|u_k(t) - u_k(s)|_T \le C^{1/2}|t-s|^{1/2} \quad \forall\, s,t \in [\tau,T], \quad \forall\, k \ge 1, \quad (48)$$

with C defined by (46). It allows follows by (37) that $\|u_k(t)\|_T^2 \le C$ for all $t \in [\tau, T]$ and each $k \ge 1$. Since the injection of V_T into H_T is compact, the set $\{v \in V_T : \ \|v\|_T^2 \le C\}$ is compact in H_T. By (48) and the Ascoli-Arzelà Theorem there thus exists a subsequence that will still be denoted $\{u_k\}$ such that

$$u_k \to u \quad \text{in } C([\tau,T]; H_T) \text{ as } k \to +\infty. \quad (49)$$

Hence, in particular, $u_k \to u$ in $L^2(\tau, T; H_T)$, and, extracting a subsequence if necessary, $u_k(x,t) \to u(x,t)$ a.e. in $\mathcal{O}_T \times (\tau, T)$, so

$$g(u_k(x,t)) \to g(u(x,t)) \quad \text{a.e. in } \mathcal{O}_T \times (\tau, T). \quad (50)$$

From (50) and the fact that by (5) and (37) the sequence $g(u_k)$ is bounded in $L^{p/p-1}(\widetilde{Q}_{\tau,T})$, it follows by an application of Lemma 1.3, Chapter 1, in [17] that

$$g(u_k) \rightharpoonup g(u) \quad \text{weakly in } L^{p/p-1}(\widetilde{Q}_{\tau,T}). \quad (51)$$

Observe by (31) and (35) that

$$\int_\tau^T [-(u_k(t), \phi'(t))_T + ((u_k(t), \phi(t)))_T + (g(u_k(t)), \phi(t))_T]\,\mathrm{d}t$$
$$= \int_\tau^T (f(t), \phi(t))_T\,\mathrm{d}t \quad (52)$$

for any $\phi \in \mathcal{U}_{\tau,T}$. In view of (45), (49) and (51), it is possible to take the limit as $k \to +\infty$ in (52) and to conclude that u is a variational solution of (4).

In order to show that u satisfies the energy equality in (τ, T), observe from the energy equality for u_k that

$$
\begin{aligned}
&|u_k(T)|_T^2 + 2\int_\tau^T \|u_k(r)\|_T^2 \,\mathrm{d}r + 2\int_\tau^T (g(u_k(r)), u_k(r))_T \,\mathrm{d}r \\
\leq\, &|u_\tau|_T^2 + 2\int_\tau^T (f(r), u_k(r))_T \,\mathrm{d}r.
\end{aligned} \tag{53}
$$

Now

$$
\begin{aligned}
&\int_\tau^T (g(u_k(r)), u_k(r))_T \,\mathrm{d}r \\
=&\int_\tau^T (g(u_k(r)) - g(u(r)), u_k(r) - u(r))_T \,\mathrm{d}r + \int_\tau^T (g(u_k(r)), u(r))_T \,\mathrm{d}r \\
&\qquad + \int_\tau^T (g(u(r)), u_k(r))_T \,\mathrm{d}r - \int_\tau^T (g(u(r)), u(r))_T \,\mathrm{d}r.
\end{aligned}
$$

Thus, by (6),

$$
\begin{aligned}
&\int_\tau^T (g(u_k(r)), u_k(r))_T \,\mathrm{d}r \\
\geq &-l\int_\tau^T |u_k(r) - u(r)|_T^2 \,\mathrm{d}r + \int_\tau^T (g(u_k(r)), u(r))_T \,\mathrm{d}r \\
&\qquad + \int_\tau^T (g(u(r)), u_k(r))_T \,\mathrm{d}r - \int_\tau^T (g(u(r)), u(r))_T \,\mathrm{d}r.
\end{aligned}
$$

This inequality and (53) give

$$
\begin{aligned}
&|u_k(T)|_T^2 + 2\int_\tau^T \|u_k(r)\|_T^2 \,\mathrm{d}r \\
\leq\, &|u_\tau|_T^2 + 2l\int_\tau^T |u_k(r) - u(r)|_T^2 \,\mathrm{d}r \\
&-2\int_\tau^T (g(u_k(r)), u(r))_T \,\mathrm{d}r - 2\int_\tau^T (g(u(r)), u_k(r))_T \,\mathrm{d}r \\
&+2\int_\tau^T (g(u(r)), u(r))_T \,\mathrm{d}r + 2\int_\tau^T (f(r), u_k(r))_T \,\mathrm{d}r.
\end{aligned} \tag{54}
$$

By the lower semicontinuity of the norm, (49) and (51), it follows from $u_k \rightharpoonup u$ weakly in $L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T})$ and (54) that

$$
|u(T)|_T^2 \leq |u_\tau|_T^2 + 2\int_\tau^T \Big[(f(r), u(r))_T - \|u(r)\|_T^2 - (g(u(r)), u(r))_T\Big] \,\mathrm{d}r.
$$

By Lemma 10 and the fact that $u \in C([\tau, T]; H_T)$ one concludes that u in fact satisfies the energy equality for all $t \in [\tau, T]$.

Finally, suppose that $u_\tau \in L^2(\mathcal{O}_\tau)$ and consider a sequence $\{u_{\tau_n}\} \subset V_\tau \cap L^p(\mathcal{O}_\tau)$ such that $u_{\tau_n} \to u_\tau$ in $L^2(\mathcal{O}_\tau)$ as $n \to +\infty$. For each n let $u_n \in C([\tau, T]; H_T)$ be the unique variational solution of (4) satisfying the energy equality in $[\tau, T]$ with initial value u_{τ_n}. Thus, for each $n \geq 1$,

$$
\begin{aligned}
&|u_n(t)|_T^2 + 2\int_\tau^t \|u_n(r)\|_T^2 \,\mathrm{d}r + 2\int_\tau^t (g(u_n(r)), u_n(r))_T \,\mathrm{d}r \\
&= |u_{\tau_n}|_T^2 + 2\int_\tau^t (f(r), u_n(r))_T \,\mathrm{d}r, \qquad (55)
\end{aligned}
$$

for all $t \in [\tau, T]$. From this equality and (5), it is then standard to prove that

$$\{u_n\} \text{ is bounded in } L^\infty(\tau, T; H_T) \cap L^2(\tau, T; V_T) \cap L^p(\widetilde{Q}_{\tau,T}). \qquad (56)$$

On the other hand, reasoning as in the proof of Proposition 11,

$$
\begin{aligned}
&|u_n(t) - u_m(t)|_T^2 + 2\int_\tau^t \|u_n(r) - u_m(r)\|_T^2 \,\mathrm{d}r \\
&\leq |u_{\tau_n} - u_{\tau_m}|_T^2 + 2l\int_\tau^t |u_n(r) - u_m(r)|_T^2 \,\mathrm{d}r
\end{aligned}
$$

for all $t \in [\tau, T]$ and any $n, m \geq 1$. Using Gronwall's lemma, one concludes from this inequality that

$$\{u_n\} \text{ is a Cauchy sequence in } L^2(\tau, T; V_T) \cap C([\tau, T]; H_T). \qquad (57)$$

From (56) and (57) it follows that $u_n \to u$ in $L^2(\tau, T; V_T) \cap C([\tau, T]; H_T)$ as $n \to +\infty$, with $u \in L^p(\widetilde{Q}_{\tau,T})$. Then, reasoning as before, there is a convergent subsequence $g(u_n) \rightharpoonup g(u)$ weakly in $L^{p/p-1}(\widetilde{Q}_{\tau,T})$ as $n \to +\infty$. One can thus pass to the limit in the equation satisfied by the u_n and conclude that u is a variational solution of (4). Finally, as each u_n satisfies the energy equality in $[\tau, T]$, applying Lemma 10 it is easy to see that u also satisfies the energy equality in $[\tau, T]$. ■

6 A uniform estimate in V_T for the solution of (4)

A uniform estimate in V_T will be established now for the solutions of (4) satisfying the energy equality under an additional assumption on f. The proof requires the following lemma.

Lemma 16 (*cf.* [18]) *Let $X \subset Y$ be Banach spaces such that X is reflexive and the injection of X in Y is compact. Suppose that $\{v_n\}$ is a bounded* *sequence in* $L^\infty(t_0,T;X)$ *such that* $v_n \rightharpoonup v$ *weakly in* $L^p(t_0,T;X)$ *for some* $p\in[1,+\infty)$ *and* $v\in C^0([t_0,T];Y)$. *Then,* $v(t)\in X$ *for all* $t\in[t_0,T]$ *and*

$$\|v(t)\|_X \le \liminf_{n\to+\infty}\|v_n\|_{L^\infty(t_0,T;X)}, \quad \forall\, t\in[t_0,T]. \tag{58}$$

Proposition 17 *Assume that* (1), (2), (5) *and* (6) *hold. In addition, suppose that* $T>\tau+1$ *and that* $f\in L^2_{loc}(\mathbb{R}^{N+1})$ *satisfies*

$$C_{f,T} := \sup_{t\le T}\int_{t-1}^{t}|f(r)|_T^2\,\mathrm{d}r < +\infty. \tag{59}$$

Then, for any $u_\tau\in L^2(\mathcal{O}_\tau)$ *the corresponding solution* u *of* (4) *satisfying the energy equality in* (τ,T) *also satisfies*

$$u(t)\in V_T \quad \forall\,\tau+1\le t\le T, \tag{60}$$

and

$$\begin{aligned}\|u(t)\|_T^2 \le{}& \alpha_3|u_\tau|_\tau^2 e^{\lambda_{1,T}(\tau-t+2)} + [4\widetilde{\beta}+2\alpha_3\beta(1+\lambda_{1,T}^{-1})]|\mathcal{O}_T|\\ &+\left(1+2\alpha_3\lambda_{1,T}^{-1}(1-e^{-\lambda_{1,T}})^{-1}\right)C_{f,T},\end{aligned} \tag{61}$$

for all $\tau+1\le t\le T$, *where* $\alpha_3 := (1+\widetilde{\alpha}_2\alpha_1^{-1})$, *and* $\lambda_{1,T}$ *is the first eigenvalue for the operator* $-\Delta$ *in* $\mathcal{O}_T$ *with homogeneous Dirichlet boundary condition.*

Proof. Suppose first that $u_\tau\in V_\tau\cap L^p(\mathcal{O}_\tau)$ and denote

$$y_{k_m}(t) := \|u_{k_m}(t)\|_T^2 + k((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T + 2\int_{\mathcal{O}_T}G(u_{k_m}(x,t))\,\mathrm{d}x + 2\widetilde{\beta}|\mathcal{O}_T|,$$

where the u_{k_m} are the Galerkin approximations of u_k defined by (38). Then, by (41) and (7),

$$y_{k_m}(t)\ge 0, \quad \text{and} \quad y'_{k_m}(t)\le|f(t)|_T^2 \quad \text{a.e. } t\in(\tau,T). \tag{62}$$

On the other hand, from the energy equality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_{k_m}(t)|_T^2 + \|u_{k_m}(t)\|_T^2\\ &\qquad +k((P_k(t)u_{k_m}(t),u_{k_m}(t)))_T + (g(u_{k_m}(t)),u_{k_m}(t))_T\\ &= (f(t),u_{k_m}(t))_T, \quad \text{a.e. } t\in(\tau,T),\end{aligned}$$

it follows that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|u_{k_m}(t)|_T^2 + \|u_{k_m}(t)\|_T^2 \\
&\qquad +2k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\alpha_1 \|u_{k_m}(t)\|^p_{L^p(\mathcal{O}_T)} \\
&\le 2\beta|\mathcal{O}_T| + \lambda_{1,T}^{-1}|f(t)|_T^2, \quad \text{a.e. } t \in (\tau, T),
\end{aligned} \tag{63}
$$

and, in particular, that

$$
\frac{\mathrm{d}}{\mathrm{d}t}|u_{k_m}(t)|_T^2 + \lambda_{1,T}|u_{k_m}(t)|_T^2 \le 2\beta|\mathcal{O}_T| + \lambda_{1,T}^{-1}|f(t)|_T^2, \quad \text{a.e. } t \in (\tau, T).
$$

Multiplying this inequality by $e^{\lambda_{1,T}t}$ and integrating gives

$$
\begin{aligned}
|u_{k_m}(t)|_T^2 \le{}& |u_{\tau_m}|_T^2 e^{\lambda_{1,T}(\tau - t)} + 2\beta\lambda_{1,T}^{-1}|\mathcal{O}_T| \\
&+\lambda_{1,T}^{-1}e^{-\lambda_{1,T}t}\int_\tau^t e^{\lambda_{1,T}r}|f(r)|_T^2\,\mathrm{d}r, \quad \forall\, t \in [\tau, T].
\end{aligned} \tag{64}
$$

Then, integrating (63) from t to $t+1$ and using (64), gives

$$
\begin{aligned}
&\int_t^{t+1}\|u_{k_m}(s)\|_T^2\,\mathrm{d}s + 2k\int_t^{t+1}((P_k(s)u_{k_m}(s), u_{k_m}(s)))_T\,\mathrm{d}s \\
&\qquad +2\alpha_1\int_t^{t+1}\|u_{k_m}(s)\|^p_{L^p(\mathcal{O}_T)}\,\mathrm{d}s \\
&\le |u_{\tau_m}|_T^2 e^{\lambda_{1,T}(\tau-t)} + 2\beta(1+\lambda_{1,T}^{-1})|\mathcal{O}_T| + \lambda_{1,T}^{-1}\int_t^{t+1}|f(r)|_T^2\,\mathrm{d}r \\
&\qquad +\lambda_{1,T}^{-1}e^{-\lambda_{1,T}t}\int_\tau^t e^{\lambda_{1,T}r}|f(r)|_T^2\,\mathrm{d}r,
\end{aligned} \tag{65}
$$

for all $\tau \le t \le T-1$.

Observing that, as can be easily deduced,

$$
\int_\tau^t e^{\lambda_{1,T}r}|f(r)|_T^2\,\mathrm{d}r \le C_{f,T}e^{\lambda_{1,T}t}(1-e^{-\lambda_{1,T}})^{-1},
$$

the inequality (65) becomes

$$
\begin{aligned}
&\int_t^{t+1}\|u_{k_m}(s)\|_T^2\,\mathrm{d}s + 2k\int_t^{t+1}((P_k(s)u_{k_m}(s), u_{k_m}(s)))_T\,\mathrm{d}s \\
&\qquad +2\alpha_1\int_t^{t+1}\|u_{k_m}(s)\|^p_{L^p(\mathcal{O}_T)}\,\mathrm{d}s \\
&\le |u_{\tau_m}|_T^2 e^{\lambda_{1,T}(\tau-t)} + 2\beta(1+\lambda_{1,T}^{-1})|\mathcal{O}_T| + 2\lambda_{1,T}^{-1}C_{f,T}(1-e^{-\lambda_{1,T}})^{-1},
\end{aligned} \tag{66}
$$

for all $\tau \le t \le T-1$.

Thus, by (7) and the definition of y_{k_m},

$$y_{k_m}(t) \leq \|u_{k_m}(t)\|_T^2 + k((P_k(t)u_{k_m}(t), u_{k_m}(t)))_T + 2\widetilde{\alpha}_2\|u_{k_m}(t)\|_{L^p(\mathcal{O}_T)}^p + 4\widetilde{\beta}|\mathcal{O}_T|$$

for $t \in (\tau, T)$. From this and (66), it follows that

$$\begin{aligned}\int_t^{t+1} y_{k_m}(s)\,\mathrm{d}s \leq \alpha_3|u_{\tau_m}|_T^2 e^{\lambda_{1,T}(\tau-t)} + [4\widetilde{\beta} + 2\alpha_3\beta(1+\lambda_{1,T}^{-1})]|\mathcal{O}_T| \\ +2\alpha_3\lambda_{1,T}^{-1}C_{f,T}(1-e^{-\lambda_{1,T}})^{-1},\end{aligned} \tag{67}$$

for all $\tau \leq t \leq T-1$. Now, by (62),

$$y_{k_m}(t+1) \leq y_{k_m}(s) + \int_t^{t+1} |f(r)|_T^2\,\mathrm{d}r \quad \forall\, \tau \leq t \leq s \leq t+1 \leq T,$$

so integrating between t and $t+1$ one has

$$y_{k_m}(t+1) \leq \int_t^{t+1} y_{k_m}(s)\,\mathrm{d}s + C_{f,T} \quad \forall\, \tau \leq t \leq T-1.$$

It then follows from this inequality and (67) that

$$\begin{aligned}\|u_{k_m}(t)\|_T^2 \leq \alpha_3|u_{\tau_m}|_T^2 e^{\lambda_{1,T}(\tau-t+1)} + [4\widetilde{\beta} + 2\alpha_3\beta(1+\lambda_{1,T}^{-1})]|\mathcal{O}_T| \\ + \left(1+2\alpha_3\lambda_{1,T}^{-1}(1-e^{-\lambda_{1,T}})^{-1}\right)C_{f,T},\end{aligned} \tag{68}$$

for all $\tau+1 \leq t \leq T$.

Now it is known that $u_{k_m} \rightharpoonup u_k$ in $L^2(\tau, T; V_T)$ as $m \to +\infty$ and, in particular, that

$$u_{k_m} \rightharpoonup u_k \quad \text{in } L^2(t, t+1; V_T) \text{ as } m \to +\infty, \text{ for all } \tau \leq t \leq T-1.$$

Hence, by (68) and Lemma 16, it follows that for any $k \geq 1$

$$u_k(t) \in V_T \quad \forall\, \tau+1 \leq t \leq T,$$

and

$$\begin{aligned}\|u_k(t)\|_T^2 \leq \alpha_3|u_\tau|_\tau^2 e^{\lambda_{1,T}(\tau-t+2)} + \left[4\widetilde{\beta} + 2\alpha_3\beta(1+\lambda_{1,T}^{-1})\right]|\mathcal{O}_T| \\ + \left(1+2\alpha_3\lambda_{1,T}^{-1}(1-e^{-\lambda_{1,T}})^{-1}\right)C_{f,T},\end{aligned}$$

for all $\tau+1 \leq t \leq T$.

Finally, since $u_k \rightharpoonup u$ in $L^2(\tau, T; V_T)$ as $k \to +\infty$ and $u \in C([\tau, T]; H_T)$, with u being the solution of (4) with the initial value $u_\tau \in V_\tau \cap L^p(\mathcal{O}_\tau)$, one can use the same argument to show that u satisfies (60) and (61).

It is easy to see, again with the same arguments as above, that (60) and (61) also hold for the solution of (4) corresponding to any $u_\tau \in L^2(\mathcal{O}_\tau)$. ■

7 Pullback attractors for asymptotically compact non-autonomous dynamical systems

Some basic ideas and results from the abstract theory of non-autonomous dynamical systems that are needed to study the existence of global attractor for (3) will now be sketched. This will be done in terms of the process formulation of a non-autonomous dynamical system rather than the cocycle formalism (see [6,12]) since the former is more appropriate in the present context. The results in this section are modifications of those in [5].

Consider a process (also called a two-parameter semigroup) U on a family of metric spaces $\{(X_t, d_t);\ t \in \mathbb{R}\}$, i.e., a family $\{U(t,\tau);\ -\infty < \tau \le t < +\infty\}$ of continuous mappings $U(t,\tau) : X_\tau \to X_t$ such that $U(\tau,\tau)x = x$ for all $x \in X_\tau$ and

$$U(t,\tau) = U(t,r)U(r,\tau) \quad \text{for all} \ \ \tau \le r \le t. \tag{69}$$

In addition, suppose $\mathcal{D}$ is a nonempty class of parameterized sets of the form $\widehat{D} = \{D(t);\ D(t) \subset X_t,\ D(t) \ne \emptyset,\ t \in \mathbb{R}\}$.

Definition 18 *The process $U(\cdot,\cdot)$ is said to be pullback $\mathcal{D}$-asymptotically compact if the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X_t for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, and any sequences $\{\tau_n\}$ and $\{x_n\}$ with $\tau_n \to -\infty$, and $x_n \in D(\tau_n)$.*

Definition 19 *A family $\widehat{B} \in \mathcal{D}$ is said to be pullback $\mathcal{D}$-absorbing for the process $U(\cdot,\cdot)$ if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \le t$ such that*

$$U(t,\tau)D(\tau) \subset B(t) \quad \textit{for all } \tau \le \tau_0(t,\widehat{D}).$$

Remark 20 *Note that if $\widehat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $U(\cdot,\cdot)$, and $B(t)$ is a compact subset of X_t for any $t \in \mathbb{R}$, then the process $U(\cdot,\cdot)$ is pullback $\mathcal{D}$-asymptotically compact.*

For each $t \in \mathbb{R}$ let $\mathrm{dist}_t(D_1, D_2)$ be the Hausdorff semi-distance between nonempty subsets D_1 and D_2 of X_t, which is defined as

$$\mathrm{dist}_t(D_1, D_2) = \sup_{x\in D_1} \inf_{y \in D_2} d_t(x,y) \quad \text{for} \quad D_1, D_2 \subset X_t.$$

Definition 21 *The family $\widehat{A} = \{A(t);\ A(t) \subset X_t,\ A(t) \ne \emptyset,\ t \in \mathbb{R}\}$ is said to be a pullback $\mathcal{D}$-attractor for $U(\cdot,\cdot)$ if*

(1) $A(t)$ is a compact subset of X_t for all $t \in \mathbb{R}$,

(2) $\widehat{A}$ *is pullback $\mathcal{D}$-attracting, i.e.,*

$$\lim_{\tau\to-\infty} \operatorname{dist}_t(U(t,\tau)D(\tau), A(t)) = 0 \quad \textit{for all } \widehat{D}\in\mathcal{D} \textit{ and all } \ t\in\mathbb{R},$$

(3) $\widehat{A}$ *is invariant, i.e.,*

$$U(t,\tau)A(\tau) = A(t) \quad \textit{for } -\infty < \tau \le t < +\infty.$$

Remark 22 *Observe that* Definition 21 *does not guarantee the uniqueness of pullback $\mathcal{D}$-attractors (see* [4] *for a discussion on this point). In order to ensure uniqueness one needs to impose additional conditions as, for instance, that the attractor belongs to the same family $\mathcal{D}$ or enjoys some kind of minimality. These assumptions have not been included in the definition above since they do not always hold. However, as it will be seen in* Theorem 23*, it is possible under very general hypotheses to ensure the existence of a global pullback $\mathcal{D}$-attractor which is minimal in an appropriate sense. Actually, in* Theorem 24 *both conditions -minimality and inclusion of the attractor in the attracted family- hold.*

Theorem 23 *Suppose that the process $U(\cdot,\cdot)$ is pullback $\mathcal{D}$-asymptotically compact and that $\widehat{B}\in\mathcal{D}$ is a family of pullback $\mathcal{D}$-absorbing sets for $U(\cdot,\cdot)$.*

Then, the family $\widehat{A}=\{A(t);\ t\in\mathbb{R}\}$ defined by $A(t) := \Lambda(\widehat{B},t)$, $t\in\mathbb{R}$, where for each $\widehat{D}\in\mathcal{D}$ and $t\in\mathbb{R}$

$$\Lambda(\widehat{D},t) := \bigcap_{s\le t}\left(\overline{\bigcup_{\tau\le s} U(t,\tau)D(\tau)}^{X_t}\right) \qquad (\textit{closure in } X_t)$$

is a pullback $\mathcal{D}$-attractor for $U(\cdot,\cdot)$, which in addition satisfies

$$A(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D},t)}^{X_t}, \quad \forall t\in\mathbb{R}.$$

Furthermore, $\widehat{A}$ is minimal in the sense that if $\widehat{C}=\{C(t);\ t\in\mathbb{R}\}$ is a family of nonempty sets such that $C(t)$ is a closed subset of X_t and

$$\lim_{\tau\to-\infty} \operatorname{dist}_t(U(t,\tau)B(\tau), C(t)) = 0$$

for any $t\in\mathbb{R}$, then $A(t)\subset C(t)$ for all $t\in\mathbb{R}$.

Proof. (Sketch) The proof follows from the properties of the pullback omega limit sets $\Lambda(\widehat{D},t)$. Indeed, it follows that $\Lambda(\widehat{D},t)$ is a compact nonempty set in X_t for all $\widehat{D}\in\mathcal{D}$ and all $t\in\mathbb{R}$. Moreover, the family $\{\Lambda(\widehat{D},t), t\in\mathbb{R}\}$ pullback attracts $\widehat{D}$, is invariant in the sense of Definition 21, and satisfies that $\Lambda(\widehat{B},t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}}\Lambda(\widehat{D},t)}$, with the closure taken in X_t. The minimality property follows immediately (see [6] for more details). ■

The aim in this subsection is to establish the existence of a global pullback attractor for the problem (3).

Suppose that $f \in L^2_{loc}(\mathbb{R}^{N+1})$. Then, according to Theorem 15 and Remark 2, for each $\tau \in \mathbb{R}$ and $u_\tau \in H_\tau$ there exists a unique variational solution $u(\cdot;\tau,u_\tau)$ of (3) satisfying the energy equality a.e. in (τ,T) for all $T > \tau$. Moreover, $u(\cdot;\tau,u_\tau) \in C([\tau,T];H_T)$ and, in fact, satisfies the energy equality for all $t \in [\tau,T]$ and for any $T > \tau$.

Define

$$U(t,\tau)u_\tau := u(t;\tau,u_\tau), \quad -\infty < \tau \le t < +\infty, \quad u_\tau \in H_\tau, \tag{70}$$

It is evident that $U(\tau,\tau)u_\tau = u_\tau$, and by the uniqueness of variational solution to problem (3) satisfying the energy equality a.e. in (τ,T) for all $T > \tau$, it is not difficult to see that the family of mappings $\{U(t,\tau);\ -\infty < \tau \le t < +\infty\}$ satisfy (69). In addition, by Proposition 11 and the fact that $u(\cdot;\tau,u_\tau) \in C([\tau,T];H_T)$ and actually satisfies the energy equality for all $t \in [\tau,T]$, for any $T > \tau$, it follows that for all $\tau \le t$ the mapping $U(t,\tau) : H_\tau \to H_t$ is continuous. Hence the family of mappings $\{U(t,\tau);\ -\infty < \tau \le t < +\infty\}$ defined by (70) is a process $U(\cdot,\cdot)$ for the family of Hilbert spaces $\{H_t;\ t \in \mathbb{R}\}$.

Let $\mathcal{R}_{\lambda_1}$ be the set of all functions $r : \mathbb{R} \to (0,+\infty)$ such that

$$\lim_{t\to-\infty} e^{t\lambda_{1,t}} r^2(t) = 0, \tag{71}$$

where $\lambda_{1,t}$ is the first eigenvalue for the operator $-\Delta$ in $\mathcal{O}_t$ with homogeneous Dirichlet boundary condition (see Proposition 17) and denote by $\mathcal{D}_{\lambda_1}$ the class of all families $\widehat{D} = \{D(t);\ D(t) \subset H_t,\ D(t) \neq \emptyset,\ t \in \mathbb{R}\}$ such that $D(t) \subset \overline{B}(0, r_{\widehat{D}}(t))$ for some $r_{\widehat{D}} \in \mathcal{R}_{\lambda_1}$, where $\overline{B}(0, r_{\widehat{D}}(t))$ is the closed ball in H_t centered at zero with radius $r_{\widehat{D}}(t)$.

Theorem 24 *Suppose that the assumptions in* Theorem 15 *hold and that* $f \in L^2_{loc}(\mathbb{R}^{N+1})$ *and satisfies* (59). *Then, there exists a unique global pullback* $\mathcal{D}_{\lambda_1}$-*attractor belonging to* $\mathcal{D}_{\lambda_1}$ *for the process* U *defined by* (70).

Proof. For each $t \in \mathbb{R}$ define $R_{\lambda_1}(t)$ as the positive constant given by

$$(R_{\lambda_1}(t))^2 = 1 + [4\widetilde{\beta} + 2\alpha_3\beta(1 + \lambda_{1,t}^{-1})]|\mathcal{O}_t| + \left(1 + 2\alpha_3\lambda_{1,t}^{-1}(1 - e^{-\lambda_{1,t}})^{-1}\right) C_{f,t},$$

and consider the family of closed balls $\widehat{B}_{\lambda_1} = \{B_{\lambda_1}(t);\ t \in \mathbb{R}\}$ defined by

$$B_{\lambda_1}(t) = \{v \in V_t;\ \|v\|_t \le R_{\lambda_1}(t)\}, \quad t \in \mathbb{R}.$$

Taking into account (1) and the variational characterization of $\lambda_{1,t}$, one sees that $\lambda_{1,t}$ is a non-increasing function of t. It is then not difficult to check that $\widehat{B}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ and that by (61) $\widehat{B}_{\lambda_1}$ is pullback $\mathcal{D}_{\lambda_1}$-absorbing for the process $U(\cdot,\cdot)$. Moreover, by the compactness of the injection of V_t into H_t, it is clear that $B_{\lambda_1}(t)$ is a compact subset of H_t for any $t \in \mathbb{R}$. The asserted result then follows from Theorem 23 and the fact that the universe $\mathcal{D}_{\lambda_1}$ is inclusion closed. ■

Remark 25 *Denote* $\mathcal{O}_\infty = \cup_{t\in\mathbb{R}} \mathcal{O}_t$ *and let* $\mathcal{B}_\infty$ *be the class of bounded subsets of* $L^2(\mathcal{O}_\infty)$. *For each* $B \in \mathcal{B}_\infty$ *consider the family* $\widehat{D}_B = \{D_B(t);\ t \in \mathbb{R}\}$, *where*

$$D_B(t) := \{v_{|\mathcal{O}_t};\ v \in B\} \quad \forall\, t \in \mathbb{R}.$$

It is easy to see that $\widehat{D}_B \in \mathcal{D}_{\lambda_1}$ *and hence that the global pullback* $\mathcal{D}_{\lambda_1}$*-attractor* $\widehat{A}_{\lambda_1} \in \mathcal{D}_{\lambda_1}$ *satisfies, in particular,*

$$\lim_{\tau\to-\infty} \sup_{u_\tau\in B} \inf_{v\in A_{\lambda_1}(t)} |u(t;\tau,u_\tau) - v|_{L^2(\mathcal{O}_t)} = 0, \quad \text{for any } B \in \mathcal{B}_\infty.$$

On the other hand, if $\mathcal{O}_\infty$ *is bounded and* $\sup_{T\in\mathbb{R}} C_{f,T} < +\infty$, *then* $R_{\lambda_1}(t)$ *defined in the proof of* Theorem 24 *remains uniformly bounded and one can verify that the uniform forward attractor in the sense of Chepyzhov and Vishik (see* [10]*) exists, i.e., a compact subset* A_∞ *of* $L^2(\mathcal{O}_\infty)$ *such that, amongst other properties,*

$$\lim_{t\to+\infty} \sup_{\tau\in\mathbb{R}} \mathrm{dist}_{L^2(\mathcal{O}_\infty)}(U(t+\tau,\tau)B, A_\infty) = 0 \quad \textit{for all } B \in \mathcal{B}_\infty,$$

or equivalently (uniform pullback attraction)

$$\lim_{s\to+\infty} \sup_{t\in\mathbb{R}} \mathrm{dist}_{L^2(\mathcal{O}_\infty)}(U(t,t-s)B, A_\infty) = 0 \quad \textit{for all } B \in \mathcal{B}_\infty.$$

Moreover, in this case,

$$A_{\lambda_1}(t) \subset A_\infty \quad \forall\, t \in \mathbb{R}.$$

Conclusion

We have proved the existence and uniqueness of solution satisfying an energy equality to a semilinear heat equation in a non-cylindrical domain. The result is obtained under the assumption (1) of spatial domains which are expanding in time.

This problem is intrinsically non-autonomous, and the previous result has been used to study the asymptotic behaviour of the solutions, namely to establish a result on existence of pullback attractor.

It would be interesting to reproduce similar results for other nonlinear PDEs on non-cylindrical domains and/or without the cited restriction (1) on the spatial domains, and more exactly to obtain solutions still satisfying an energy equality. To our knowledge these are open problems.

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