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GLOBAL ATTRACTOR AND OMEGA-LIMIT SETS STRUCTURE FOR A PHASE-FIELD MODEL OF THERMAL ALLOYS

DEDICATED TO PROF. JOSÉ REAL ON HIS 60^{TH} BIRTHDAY

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Abstract. In this paper, the existence of weak solutions is established for a phase-field model of thermal alloys supplemented with Dirichlet boundary conditions. After that, the existence of global attractors for the associated multi-valued dynamical systems is proved, and relationship among these sets is established. Finally, we provide a more detailed description of the asymptotic behaviour of solutions via the omega-limit sets. Namely, we obtain a characterization –through the natural stationary system associated to the model– of the elements belonging to the omega-limit sets under suitable assumptions.

1. Introduction

In recent years the phase-field methodology has achieved considerable importance in the modelling and numerical simulation of a range of phase transitions and complex growth structures like dendrites occurring during solidification (see, for instance, [24]). There exists a wide literature devoted to phase-field modelling of various phase transition phenomena, from the classical phase-field system of Caginalp [7] to problems with constraints leading to variational inequalities (cf. [15] and the references therein). Phase-field models have also been used to describe the evolution in time of diffusive phase interfaces in solid materials, in which martensitic phase transformations driven by configurational forces take place. In such models, partial differential equations of linear elasticity were coupled with a degenerate parabolic equation of second order for the phase-field (cf. [2]). In addition, we remark that the method has been extended to multi-phase systems by introducing a multi-phase approach, see [13] for an isothermal phase-field model of ternary two-phase diffusion couples.

In this paper we study an initial boundary value problem of a model describing a binary mixture with thermal properties and a phase transition. Phase-field models for binary alloys were proposed in [9, 8] and its derivation is based on thermodynamic principles. These models include as a special case the phase-field equations for a pure material introduced by Caginalp [7].

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With several simplifications and idealization, it is possible to obtain the "simplest" phase-field-alloy model which retains the key characteristics of an alloy. It involves the following highly nonlinear parabolic system of three partial differential equations with three independent variables: phase-field, solute concentration, and temperature

$$\begin{cases} \alpha\varepsilon^2\phi_t - \varepsilon^2\Delta\phi = \dfrac{1}{2}(\phi - \phi^3) + \beta(\theta - \theta_A c - \theta_B(1-c)) \quad \text{in} \quad Q, \\ C_V\theta_t + \dfrac{l}{2}\phi_t = \nabla \cdot [K_1(\phi)\nabla\theta] \quad \text{in} \quad Q, \\ c_t = K_2(\Delta c + M\nabla \cdot [c(1-c)\nabla\phi]) \quad \text{in} \quad Q, \\ 0 \le c \le 1 \quad \text{in} \quad Q, \end{cases} \tag{1}$$

where $Q = \Omega \times (0,\infty)$, being Ω an open connected bounded subset of $\mathbb{R}^N$ with $N = 2$ or 3, and with smooth boundary $\partial\Omega$. The order parameter (phase-field) ϕ is the state variable characterizing the different phases; the function θ represents the temperature; the concentration $c \in [0,1]$ denotes the fraction of one of the two materials in the mixture. The parameter $\alpha > 0$ is the relaxation scaling; the parameter $\beta \ge 0$ is given by $\beta = \varepsilon[s]/(3\sigma)$, where $\varepsilon > 0$ is a measure of the interface width, σ the surface tension and $[s]$ the entropy density difference between phases; $C_V > 0$ is the specific heat; the constant $l > 0$ is the latent heat; θ_A and θ_B are the respective melting temperatures of each of the two materials in the alloy; $K_2 > 0$ is the solute diffusivity; M is a constant related to the slopes of solidus and liquidus lines; and K_1 denotes the thermal conductivity. This physical parameter is assumed, as in [18, 5, 20], to be a function depending on the order parameter ϕ.

In the previous paper [5] global existence of weak solutions for such model was proved under Neumann boundary conditions.

Several results are available on the asymptotic behaviour in time of phase-field models for pure materials; concerning with Neumann boundary conditions, we can cite [6, 10, 17, 18, 25] among others. In particular, Laurençot [18] was able to describe the omega-limit sets of singletons in a precise way. More exactly, they also are singletons, namely, stationary points of the problem. However, there exists obvious differences between his model and that in [5], that do not allow the same treatment; in particular, an energy decay result was known explicitly in [18], but not for the alloy model in [5].

The asymptotic behaviour of the dynamical systems related to problem (1) fulfilled with Neumann boundary conditions was investigated in [20] in the framework of the theory of attractors. The Neumann boundary conditions require Poincaré-Wirtinger inequality in estimates and lead to a result of existence of attractor in a level-set formulation, similarly to the results in [6].

Besides the above comments concerning to phase-field models with Neumann boundary conditions, we must cite that there exist many references in the literature involving phase-field models with Dirichlet boundary conditions, see for instance [4, 1, 22, 14].

However, as commented before, in the literature there exist more involved studies on the structure of omega-limit sets, and even for single trajectories, for this kind of problems. Most of the results on stability and asymptotic convergence towards stationary solutions uses in an essential way the fact that there exists a natural energy functional that plays the role of a Lyapunov functional for the solutions. Nevertheless, it is not clear wether system (1) possesses a global Lyapunov functional (as in [18]) or, more specifically, if it satisfies a Łojasiewicz-Simon property (as in [1]).

Our goal in this paper is two-fold: on the one hand, we aim to study existence of solutions to problem (1) fulfilled with Dirichlet boundary conditions. On the other hand, we analyze deeply the long-time behaviour of the problem. The choice of Dirichlet boundary conditions will show to provide a more dissipative structure, in contrast with the results in [20]. Firstly, we prove the existence of global attractors for the two natural associated dynamical systems (depending on the regularity of the initial data) and the relationship among them. As far as uniqueness of solution for the considered system is unknown, we must use multi-valued dynamical systems for our approach. Secondly, and following some ideas from [18], we describe more precisely the elements belonging to the omega-limit sets. Due to the troublesome nonlinearities on the system, we are able to perform the analysis of the omega-limit sets under suitable assumptions.

It is convenient to rewrite the system by performing a change of variables in (1). Namely, we introduce the enthalpy variable $u = C_V(\theta - \theta_B) + \frac{l}{2}\phi$. We fulfill the system with homogeneous Dirichlet boundary conditions in the three unknowns ϕ, u, and c, and complete it with initial conditions, leading to the following problem

$$\begin{cases} \alpha\varepsilon^2\phi_t - \varepsilon^2\Delta\phi = \dfrac{1}{2}(\phi-\phi^3) + \dfrac{\beta}{C_V}(u - \dfrac{l}{2}\phi) + \beta(\theta_B - \theta_A)c \quad \text{in} \quad Q, \\ C_V u_t = \nabla\cdot[K_1(\phi)\nabla u] - \dfrac{l}{2}\nabla\cdot[K_1(\phi)\nabla\phi] \quad \text{in} \quad Q, \\ c_t = K_2(\Delta c + M\nabla\cdot[c(1-c)\nabla\phi]) \quad \text{in} \quad Q, \\ 0 \le c \le 1 \quad \text{in} \quad Q, \\ \phi = 0, \quad u = 0, \quad c = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ \phi(0) = \phi_0, \quad u(0) = u_0, \quad c(0) = c_0. \end{cases} \tag{2}$$

Observe that in this way we are assuming that the original variable θ, which is the controlled temperature on $\partial\Omega$, has the constant value θ_B.

We assume that K_1 is a (globally) Lipschitz continuous function and there exist positive constants $\underline{k}_1$, $\overline{k}_1$ such that

$$0 < \underline{k}_1 \le K_1(r) \le \overline{k}_1 \quad \forall\, r \in \mathbb{R}. \tag{3}$$

This condition will be assumed all through this paper.

The paper is organized as follows. In Section 2 we establish existence and regularity properties of weak solutions to problem (2). In Section 3 we recall briefly some abstract results concerning multi-valued dynamical systems and the study of their long-time behaviour, in particular, on existence of global attractors. These results will be used later, in Section 4, for the above problem. Namely, absorbing, compact, and continuity properties of the associated semiflows are established. In particular, we deduce that the global attractors in the two natural phase-spaces of the problem, which are $(L^2)^3$ and $H_0^1 \times (L^2)^2$, coincide thanks to the regularizing property of the model. Some other results related to omega-limits are also explored in this paragraph. Section 5 is devoted to a more involved study of the structure of the omega-limits. Actually, we improve some estimates that lead to identify the limiting problem (which is the natural stationary system associated to the model) satisfied by any element in an omega-limit.

2. Existence of solutions

Let us firstly introduce some notation which will be used hereafter all through the paper.

We will denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and its associated norm in $L^2(\Omega)$ or in $(L^2(\Omega))^N$. Otherwise, the norm in other spaces will be fully specified. The duality product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$.

The first eigenvalue of $-\Delta$ with Dirichlet boundary conditions will be denoted by λ_1, so that we have the Poincaré inequality $|f|^2 \le \lambda_1^{-1}|\nabla f|^2$ for any $f \in H_0^1(\Omega)$.

We define the concept of weak solution to system (2).

Definition 1. *We say that the triplet* (ϕ, u, c) *is a weak solution to system* (2) *in* $(0,T)$ *for* $T > 0$, *if*

(i) $\phi \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)) \cap L^4(0,T;L^4(\Omega))$, $\phi_t \in L^2(0,T;H^{-1}(\Omega)) + L^{4/3}(0,T;L^{4/3}(\Omega))$, $\phi(0) = \phi_0$,

(ii) $u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $u_t \in L^2(0,T;H^{-1}(\Omega))$, $u(0) = u_0$,

(iii) $c \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $c_t \in L^2(0,T;H^{-1}(\Omega))$, $c(0) = c_0$, $0 \le c \le 1$ *a.e. in* $\Omega \times (0,T)$,

and satisfies the equations

$$\begin{aligned} & \alpha\varepsilon^2 \int_0^T \langle \phi_t(t), \eta(t)\rangle dt + \varepsilon^2 \int_0^T (\nabla\phi(t), \nabla\eta(t))dt \\ = \; & \frac{1}{2}\int_0^T (\phi(t) - \phi^3(t), \eta(t))dt + \frac{\beta}{C_V}\int_0^T (u(t) - \frac{l}{2}\phi(t), \eta(t))dt \\ & + \beta(\theta_B - \theta_A)\int_0^T (c(t), \eta(t))dt, \end{aligned}$$

for any $\eta \in L^2(0,T;H_0^1(\Omega)) \cap L^4(0,T;L^4(\Omega))$,

$$\begin{aligned} & C_V \int_0^T \langle u_t(t), \eta(t)\rangle dt + \int_0^T (K_1(\phi(t))\nabla u(t), \nabla\eta(t))dt \\ = \; & \frac{l}{2}\int_0^T (K_1(\phi(t))\nabla\phi(t), \nabla\eta(t))dt, \end{aligned}$$

for any $\eta \in L^2(0,T;H_0^1(\Omega))$, *and*

$$\int_0^T \langle c_t(t), \eta(t)\rangle dt + K_2 \int_0^T (\nabla c(t), \nabla \eta(t)) dt \\ + K_2 M \int_0^T (c(t)(1-c(t))\nabla\phi(t), \nabla\eta(t))dt = 0,$$

for any $\eta \in L^2(0,T;H_0^1(\Omega))$.

Additionally, we will say that $(\phi, u, c) : Q \to \mathbb{R}^3$ *is a global solution to system* (2) *if its restriction to* $(0,T)$ *is a weak solution for any* $T > 0$.

We state now a result on existence of weak solutions to problem (2).

Theorem 2. *Let be given* $(\phi_0, u_0) \in (L^2(\Omega))^2$, *and* $c_0 \in L^2(\Omega;[0,1])$, *i.e.,* $c_0 \in L^2(\Omega)$ *such that* $0 \le c_0(x) \le 1$ *a.e. in* Ω. *Then there exists at least one weak solution* (ϕ, u, c) *to system* (2) *in* $(0,T)$ *for any* $T > 0$.

If in addition $\phi_0 \in H_0^1(\Omega)$, *then, for any solution* (ϕ,u,c) *in* $(0,T)$, *one has that*

$$\phi \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H_0^1(\Omega)), \quad \phi_t \in L^2(0,T;L^2(\Omega)),$$

and ϕ *satisfies the first equation in* (2) *a.e. in* $\Omega \times (0,T)$.

Proof. Let us fix $T > 0$. The existence of weak solution to (2) can be related to an auxiliary problem. To this end, let Π be the function

$$\Pi(r) = \begin{cases} 0 & \text{if } r < 0, \\ r & \text{if } 0 \le r \le 1, \\ 1 & \text{if } r > 1, \end{cases}$$

and consider the following problem

$$\begin{cases} \alpha\varepsilon^2\phi_t - \varepsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \frac{\beta}{C_V}(u - \frac{l}{2}\phi) + \beta(\theta_B - \theta_A)c \text{ in } \Omega\times(0,T), \\ C_V u_t = \nabla\cdot[K_1(\phi)\nabla u] - \frac{l}{2}\nabla\cdot[K_1(\phi)\nabla\phi] \quad \text{in } \Omega\times(0,T), \\ c_t = K_2(\Delta c + M\nabla\cdot[\Pi(c)(1-\Pi(c))\nabla\phi]) \quad \text{in } \Omega\times(0,T), \\ \phi(0) = \phi_0 \quad u(0) = u_0, \quad c(0) = c_0 \quad \text{in } \Omega\times(0,T), \\ \phi = 0, \quad u = 0, \quad c = 0 \quad \text{on } \partial\Omega\times(0,T). \end{cases} \tag{4}$$

A weak solution to this auxiliary problem is that given in Definition 1, but with the natural modification, according to the new third equation in (4). Namely, c must satisfy now

$$\int_0^T \langle c_t(t), \eta(t)\rangle dt + K_2 \int_0^T (\nabla c(t), \nabla \eta(t)) dt \\ + K_2 M \int_0^T (\Pi(c(t))(1-\Pi(c(t)))\nabla\phi(t), \nabla\eta(t))dt = 0,$$

for any $\eta \in L^2(0,T;H_0^1(\Omega))$.

Let us split the proof into three steps.

Step 1. Assume that (ϕ, u, c) is a weak solution of (4), and that $0 \leq c_0 \leq 1$ a.e. in Ω. Then, $0 \leq c \leq 1$, therefore $\Pi(c) = c$, and so this triplet is indeed a weak solution for (2).

First, we prove that if $c_0 \leq 1$ a.e. in Ω, then $c(t) \leq 1$ for all $t \in [0, T]$ and a.e. in Ω. Let us consider the positive part of $(c-1)$, namely $(c-1)^+ = \max(c-1, 0)$. According to [11], we have that

$$\nabla(c-1)^+ = \begin{cases} \nabla c & \text{if } c - 1 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Multiplying the third equation in (4) by $(c-1)^+(s)\chi_{(0,t)}(s)$ and integrating in $(0, t)$, we have

$$\begin{aligned} & \frac{1}{2}|(c-1)^+(t)|^2 + K_2 \int_0^t |\nabla(c-1)^+(s)|^2 ds \\ = \; & \frac{1}{2}|(c_0-1)^+|^2 - K_2 M \int_0^t \int_\Omega \Pi(c(s))(1 - \Pi(c(s)))\nabla\phi(s)\nabla(c-1)^+(s)dxds. \end{aligned}$$

Since $c_0 \leq 1$, one has that $|(c_0-1)^+| = 0$. Moreover, if $c < 1$, it follows that $\nabla(c-1)^+ = 0$ and, if $c \geq 1$, we have that $\Pi(c) = 1$ and thus, $1 - \Pi(c) = 0$. So the last integral vanishes and we can conclude that

$$|(c-1)^+(t)|^2 \leq 0 \quad \forall\, t \in [0, T].$$

Therefore, $(c-1)^+(t) = 0$ for all $0 \leq t \leq T$ and a.e. in Ω, which implies that $c(t) \leq 1$ for all $0 \leq t \leq T$ and a.e. in Ω.

Next, we prove that if $c_0 \geq 0$ a.e. in Ω, then $c(t) \geq 0$ for all $t \in [0, T]$ and a.e. in Ω. For this we consider the negative part of c, namely $c^- = \max(-c, 0)$. Observe that now we have that (see [11])

$$\nabla c^- = \begin{cases} 0 & \text{if } c \geq 0, \\ -\nabla c & \text{otherwise.} \end{cases}$$

Multiplying the third equation in (4) by $-c^-(s)\chi_{(0,t)}(s)$, after integration in $(0, t)$ we obtain

$$\begin{aligned} & \frac{1}{2}|c^-(t)|^2 + K_2 \int_0^t |\nabla c^-(s)|^2 ds \\ = \; & \frac{1}{2}|c_0^-|^2 + K_2 M \int_0^t \int_\Omega \Pi(c(s))(1 - \Pi(c(s)))\nabla\phi(s)\nabla c^-(s)dxds. \end{aligned}$$

Similarly as before, since $c_0 \geq 0$ we have that $|c_0^-| = 0$. Moreover, if $c \geq 0$ the last integral vanishes and, if $c < 0$, we have that $\Pi(c) = 0$, thus, the last integral also vanishes. Therefore, we deduce

$$|c^-(t)|^2 \leq 0 \quad \forall\, t \in [0, T].$$

Hence, $c^-(t) = 0$ for all $0 \leq t \leq T$ and a.e. in Ω, which implies that $c(t) \geq 0$ for all $0 \leq t \leq T$ and a.e. in Ω.

This proves the claim in Step 1. So, in order to obtain the existence of solutions, we only have to check the following claim.

Step 2. There exists at least one weak solution to (4).

We apply the classical Faedo-Galerkin method. Let us sketch the main ideas. First, we introduce the Galerkin approximations. Let $\{w_n\}_{n\in\mathbb{N}}$ be a Hilbert basis of $L^2(\Omega)$ composed by eigenfunctions of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions, $V^k = span\{w_1, .., w_k\}$, and P^k be the orthogonal projection from $L^2(\Omega)$ onto V^k.

For each $m \in \mathbb{N}$, we consider the approximate problem of finding

$$\phi^m(t) = \sum_{j=1}^{m} \phi_{mj}(t) w_j, \quad u^m(t) = \sum_{j=1}^{m} u_{mj}(t) w_j, \quad c^m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j$$

satisfying

$$\begin{cases} \alpha\varepsilon^2(\phi_t^m, w_j) + \varepsilon^2(\nabla\phi^m, \nabla w_j) \\ = \dfrac{1}{2}\big(\phi^m - (\phi^m)^3, w_j\big) + \dfrac{\beta}{C_V}\big(u^m - \dfrac{l}{2}\phi^m, w_j\big) + \beta(\theta_B - \theta_A)\big(c^m, w_j\big), \\ C_V(u_t^m, w_j) + \big(K_1(\phi^m)\nabla u^m, \nabla w_j\big) = \dfrac{l}{2}\big(K_1(\phi^m)\nabla\phi^m, \nabla w_j\big), \\ (c_t^m, w_j) + K_2(\nabla c^m, \nabla w_j) + K_2 M\big(\Pi(c^m)(1 - \Pi(c^m))\nabla\phi^m, \nabla w_j\big) = 0, \end{cases} \tag{5}$$

for $1 \le j \le m$, and

$$\phi^m(0) = P^m\phi_0, \quad u^m(0) = P^m u_0, \quad c^m(0) = P^m c_0. \tag{6}$$

Problem (5)-(6) is in fact an initial value problem for a system of $3m$ ordinary differential equations. Since the non-linear terms are locally Lipschitz continuous functions, the above problem has a unique solution (ϕ^m, u^m, c^m) defined on some maximal time interval $[0, T^m)$ with $0 < T^m \le T$. Next, we derive some estimates which show in particular that all solutions are well-defined in $[0, T]$.

Multiplying the first equation in (5) by $\phi_{mj}(t)$ and summing from $j = 1$ to m, we deduce that for all $t \in (0, T^m)$

$$\begin{aligned} &\alpha\varepsilon^2 \frac{d}{dt}|\phi^m|^2 + 2\varepsilon^2|\nabla\phi^m|^2 \\ = \quad & |\phi^m|^2 - \|\phi^m\|^4_{L^4(\Omega)} + \frac{2\beta}{C_V}(u^m, \phi^m) - \frac{\beta l}{C_V}|\phi^m|^2 + 2\beta(\theta_B - \theta_A)(c^m, \phi^m) \\ \le \quad & -\|\phi^m\|^4_{L^4(\Omega)} + C_1\big(|\phi^m|^2 + |u^m|^2 + |c^m|^2\big), \end{aligned} \tag{7}$$

where C_1 is a constant independent of m.

Multiplying the second equation in (5) by $u_{mj}(t)$ and summing from $j = 1$ to m, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u^m|^2 + \frac{1}{C_V}(K_1(\phi^m)\nabla u^m, \nabla u^m) &= \frac{l}{2C_V}(K_1(\phi^m)\nabla\phi^m, \nabla u^m) \\ &\le \frac{\underline{k}_1}{2C_V}|\nabla u^m|^2 + \frac{\overline{k}_1^2 l^2}{8C_V\underline{k}_1}|\nabla\phi^m|^2, \end{aligned}$$

and using again (3) we have that

$$\frac{1}{2}\frac{d}{dt}|u^m|^2 + \frac{\underline{k}_1}{2C_V}|\nabla u^m|^2 \le \frac{\overline{k}_1^2 l^2}{8C_V\underline{k}_1}|\nabla\phi^m|^2. \tag{8}$$

Multiplying the third equation in (5) by $c_{mj}(t)$ and summing from $j=1$ to m, gives

$$\frac{1}{2}\frac{d}{dt}|c^m|^2 + K_2|\nabla c^m|^2 = -K_2 M\big(\Pi(c^m)(1-\Pi(c^m))\nabla\phi^m, \nabla c^m\big)$$
$$\leq \frac{K_2}{2}|\nabla c^m|^2 + \frac{K_2 M^2}{2}|\nabla \phi^m|^2, \tag{9}$$

where we have used that $0 \leq \Pi(c^m)(1-\Pi(c^m)) \leq 1$.

Now, adding (7), (8) multiplied by $\frac{4C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}$, and (9) multiplied by $\frac{\varepsilon^2}{K_2M^2}$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\Big(\alpha\varepsilon^2|\phi^m|^2 + \frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u^m|^2 + \frac{\varepsilon^2}{2K_2M^2}|c^m|^2\Big)\\
&+\|\phi^m\|^4_{L^4(\Omega)} + \varepsilon^2|\nabla\phi^m|^2 + \frac{2\varepsilon^2\underline{k}_1^2}{\overline{k}_1^2 l^2}|\nabla u^m|^2 + \frac{\varepsilon^2}{2M^2}|\nabla c^m|^2\\
\leq\ & C_1\big(|\phi^m|^2 + |u^m|^2 + |c^m|^2\big),
\end{aligned}$$

for any $m \geq 1$.

It follows from Gronwall Lemma that

$$|\phi^m(t)|^2 + |u^m(t)|^2 + |c^m(t)|^2 \leq C_2(T) \quad \forall t \in [0, T^m),\ \forall m \geq 1.$$

So we can take $T^m = T$ and we infer from the above estimates, after integration in time, that

$$\int_0^T \big(\|\phi^m(s)\|^4_{L^4(\Omega)} + |\nabla\phi^m(s)|^2 + |\nabla u^m(s)|^2 + |\nabla c^m(s)|^2\big)ds$$
$$\leq C_3\big(|\phi_0|^2 + |u_0|^2 + |c_0|^2\big) + C_2(T),$$

where $C_3 > 0$ is independent of m.

From the uniform estimates obtained above, we deduce that there exist functions $\xi \in L^{4/3}(0,T;L^{4/3}(\Omega))$ and $(\phi,u,c) \in L^2(0,T;(H^1_0(\Omega))^3) \cap L^\infty(0,T;(L^2(\Omega))^3)$ with $\phi \in L^4(0,T;L^4(\Omega))$, and that from the sequence $\{(\phi^m,u^m,c^m)\}_m$ we can extract a subsequence (cf. e.g., [19]), relabelled the same, such that

$$\begin{aligned}
(\phi^m,u^m,c^m) &\rightharpoonup (\phi,u,c) \text{ weakly in } L^2(0,T;(H^1_0(\Omega))^3),\\
(\phi^m,u^m,c^m) &\overset{*}{\rightharpoonup} (\phi,u,c) \text{ weakly-star in } L^\infty(0,T;(L^2(\Omega))^3),\\
\phi^m &\rightharpoonup \phi \text{ weakly in } L^4(0,T;L^4(\Omega)),\\
(\phi^m)^3 &\rightharpoonup \xi \text{ weakly in } L^{4/3}(0,T;L^{4/3}(\Omega)).
\end{aligned}$$

From the first equation in (5), $\{\phi^m_t\}_m$ is bounded in $L^2(0,T;H^{-1}(\Omega)) + L^{4/3}(0,T;L^{4/3}(\Omega))$; and from the second and third equations in (5) it follows that the sequences $\{u^m_t\}_m$ and $\{c^m_t\}_m$ are bounded in $L^2(0,T;H^{-1}(\Omega))$.

Then, by taking into account that $H^1_0(\Omega)$ is compactly embedded into $L^2(\Omega)$, we conclude (up to a subsequence) that

$$(\phi^m,u^m,c^m) \to (\phi,u,c) \text{ strongly in } L^2(0,T;(L^2(\Omega))^3) \text{ and a.e. in } \Omega\times(0,T),$$

and, by using [19, Lem.1.3, p.12], one obtains that $\xi = \phi^3$.

Hence we can pass to limit in (5)-(6), noting that K_1 and Π are bounded Lipschitz continuous functions (cf. [20] for similar arguments) and find out that there exists at least one weak solution for (4) in $(0,T)$, which was the claim of Step 2.

Step 3. Regularity of the solution.

Finally, if $\phi_0 \in H_0^1(\Omega)$ we observe that the sequence $\{\phi_0^m = P^m\phi_0\}_m$ converges to ϕ_0 in $H_0^1(\Omega)$. This allows us to obtain another estimate for $\{\phi^m\}_m$. Multiplying the first equation in (5) by $\lambda_j\phi_{mj}$, where λ_j is the eigenvalue associated to w_j, and summing from $j=1$ to m, we deduce that

$$\begin{aligned}
&\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\nabla\phi^m|^2 + \varepsilon^2|\Delta\phi^m|^2 + \frac{3}{2}\int_\Omega(\phi^m)^2|\nabla\phi^m|^2dx + \frac{\beta l}{2C_V}|\nabla\phi^m|^2\\
=\;& -\frac{1}{2}(\phi^m,\Delta\phi^m) - \frac{\beta}{C_V}(u^m,\Delta\phi^m) - \beta(\theta_B-\theta_A)(c^m,\Delta\phi^m)\\
\le\;& \frac{1}{4\varepsilon^2}|\phi^m|^2 + \frac{\beta^2}{C_V^2\varepsilon^2}|u^m|^2 + \frac{\beta^2(\theta_B-\theta_A)^2}{\varepsilon^2}|c^m|^2 + \frac{3\varepsilon^2}{4}|\Delta\phi^m|^2 \quad \text{in } (0,T).
\end{aligned}$$

From (2) and the above inequality, we deduce that the sequence $\{\phi^m\}_m$ is bounded in $L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))$. In the limit we obtain the desired regularity for ϕ. The proof is then complete. □

Remark 3. *From the above result, it is clear that by a recursive procedure of concatenation on intervals, for instance, of the form $(T,2T)$, $(2T,3T)$, etc, there exist global solutions to* (2) *for any initial datum $(\phi_0,u_0,c_0)\in(L^2(\Omega))^2\times L^2(\Omega;[0,1])$.*

As a consequence of the previous result we have a regularizing effect in the problem.

Proposition 4. *Any global solution (ϕ,u,c) of* (2) *with initial datum $(\phi_0,u_0,c_0)\in (L^2(\Omega))^2\times L^2(\Omega;[0,1])$ satisfies*

$$\phi\in C((0,\infty);H_0^1(\Omega))\cap L^2(\epsilon,T;H^2(\Omega)) \quad \forall\,\epsilon,T \text{ such that } 0<\epsilon<T,$$

and moreover, (ϕ,u,c) satisfies the first equation in (2) *a.e. in Q.*

Proof. Since the first component, ϕ, of any solution (ϕ,u,c) to (2), satisfies $\phi\in L^2(0,T;H_0^1(\Omega))$, we may consider a positive time $\tau<T$ (almost sure) such that $\phi(\tau)\in H_0^1(\Omega)$, whence the solution in $[\tau,T]$ becomes more regular.

The uniqueness of solution for the first equation, with u and c fixed, concludes the proof. □

3. Abstract results for multi-valued dynamical systems

In this section we summarize some basic results on the existence of global attractors for multi-valued dynamical systems. In order to do that, firstly we recall some basic definitions on multi-valued dynamical systems and the study of their long-time behaviour (we refer to [21] for a more detailed exposition on this topic).

For a given metric space $(\mathcal{X},d)$, we will denote by $P(\mathcal{X})$, $B(\mathcal{X})$, $C(\mathcal{X})$, and $K(\mathcal{X})$ the classes of nonempty subsets of $\mathcal{X}$, nonempty and bounded subsets of $\mathcal{X}$, nonempty and closed subsets of $\mathcal{X}$, and nonempty and compact subsets of $\mathcal{X}$ respectively. We will denote by $\mathrm{dist}_\mathcal{X}(\cdot,\cdot)$ the Hausdorff semidistance between elements of $P(\mathcal{X})$, given by $\mathrm{dist}_\mathcal{X}(A,B)=\sup_{a\in A}\inf_{b\in B}d(a,b)$.

Definition 5. *A multi-valued map $\mathcal{G} : \mathbb{R}_+ \times \mathcal{X} \to P(\mathcal{X})$ is called a multi-valued semiflow on $(\mathcal{X}, d)$, and will be denoted by $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$, if*

(a) $\mathcal{G}(0, \cdot) = \mathrm{Id}$ *(identity map),*

(b) *for any pair t_1, $t_2 \geq 0$ and for all $x \in \mathcal{X}$,*

$$\mathcal{G}(t_1 + t_2, x) \subset \mathcal{G}(t_1, \mathcal{G}(t_2, x)), \text{ where } \mathcal{G}(t, A) = \bigcup_{a\in A} \mathcal{G}(t, a).$$

When the above inclusion is an equality, it is said that the multi-valued semiflow is strict.

We say that a multi-valued map $F : \mathcal{X} \to P(\mathcal{X})$ is upper semicontinuous if for every $x \in \mathcal{X}$ and every neighborhood M of $F(x)$, there exists a neighborhood N of x such that $F(y) \subset M$ for any $y \in N$.

We recall several important concepts in the study of a (multi-valued) dynamical system.

Definition 6. *Let be given a multi-valued semiflow $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$.*

(a) *A bounded set $B_0 \subset \mathcal{X}$ is said to be absorbing for the semiflow if for any $B \in B(\mathcal{X})$, there exists a time $T(B) \geq 0$ such that $\mathcal{G}(t, B) \subset B_0$ for all $t \geq T(B)$.*

(b) *The semiflow is said asymptotically compact if for any $B \in B(\mathcal{X})$ and any sequence $\{t_n\}_n$ with $t_n \to \infty$, any sequence $\{x_n\}_n$, with $x_n \in \mathcal{G}(t_n, B)$, possesses a converging subsequence in $\mathcal{X}$.*

(c) *The omega-limit of a set $B \subset \mathcal{X}$, denoted by $\omega(B)$, can be defined as follows:*

$$\omega(B) = \big\{x \in \mathcal{X} : \exists \{t_n\}_n \subset \mathbb{R}_+, \{b_n\}_n \subset B, \{x_n\}_n \subset \mathcal{X}, \\ t_n \to \infty, x_n \in \mathcal{G}(t_n, b_n), x = \lim_{n\to\infty} x_n \text{ in } \mathcal{X}\big\}.$$

(d) *A global attractor $\mathcal{A} \subset \mathcal{X}$ is a compact set, which is invariant, i.e., $\mathcal{G}(t, \mathcal{A}) = \mathcal{A}$ for all $t \geq 0$, and that attracts all bounded sets, that is,*

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{X}}(\mathcal{G}(t, B), \mathcal{A}) = 0 \qquad \forall B \in B(\mathcal{X}).$$

Remark 7. *Let be given a multi-valued semiflow $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$.*

(a) *For a subset $B \in \mathcal{P}(\mathcal{X})$, the set $\omega(B)$ can be described as (cf.* [21, Lem.1]*):*

$$\omega(B) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \mathcal{G}(s, B)}^{\mathcal{X}}.$$

(b) *In case that a global attractor exists, we have the following consequences:*

(i) *It is unique. Moreover, it is minimal among all closed sets attracting each bounded set.*

(ii) *For any $\varepsilon > 0$, the set $B_{0,\varepsilon} = \{x \in \mathcal{X} : d(x, \mathcal{A}) \leq \varepsilon\}$ is absorbing.*

(iii) *For any $B \in B(\mathcal{X})$, $\omega(B)$ is a nonempty compact subset of $\mathcal{X}$ and it is the minimal closed set that attracts B. Indeed, the global attractor can be characterized as $\mathcal{A} = \overline{\bigcup_{B\in B(\mathcal{X})} \omega(B)}^{\mathcal{X}}$.*

(iv) *If $B_0 \in B(\mathcal{X})$ is absorbing for the semiflow, then $\mathcal{A} = \omega(B_0)$.*

For our purpose, it will be enough to consider the next result (cf. [21, Th.3, Rmk.8]).

Theorem 8. *Let $(\mathcal{X}, d)$ be a metric space, and $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$ be an asymptotically compact strict multi-valued semiflow with a bounded absorbing set. Suppose also that $\mathcal{G}(t,\cdot) : \mathcal{X} \to C(\mathcal{X})$ is upper semicontinuous for any $t \geq 0$. Then $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$ possesses the global attractor $\mathcal{A}$.*

Remark 9. *Observe that when a multi-valued semiflow $(\mathcal{X}, \{\mathcal{G}(t)\}_{t\geq 0})$ possesses a bounded absorbing set and there exists a time $T > 0$ such that for any $B \in B(\mathcal{X})$, the set $\mathcal{G}(T, B)$ is relatively compact in $\mathcal{X}$, then, the semiflow is asymptotically compact.*

4. Global attractors for a phase-field model

In this section we study the existence of multi-valued semiflows for problem (2), and associated attractors. To this end, we check that sufficient conditions of the preceding paragraph are fulfilled. The proofs are similar to the case of Neumann boundary conditions (for details see [20]). However, for the sake of completeness, some of them are included (or sketched) here.

Let us introduce $D(\phi_0, u_0, c_0)$ the set of global solutions to (2) with initial conditions $(\phi_0, u_0, c_0) \in (L^2(\Omega))^2 \times L^2(\Omega; [0, 1])$. This set is well-defined thanks to Theorem 2 and Remark 3.

We define

$$G(t, \phi_0, u_0, c_0) = \{(\phi(t), u(t), c(t)) \ : \ (\phi, u, c) \in D(\phi_0, u_0, c_0)\},$$

which is well defined by the continuity in time of solutions. Indeed, Theorem 2 combined with Proposition 4 allows to construct two multi-valued semiflows, with the same map, but from different suitable metric spaces into themselves.

Proposition 10. *The following pairs define two strict multi-valued semiflows:*

$$\left((L^2(\Omega))^2 \times L^2(\Omega; [0, 1]), \{G(t)\}_{t\geq 0}\right),$$

and

$$\left(H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega; [0, 1]), \{G(t)\}_{t\geq 0}\right).$$

Next result provides estimates for the solutions with non-regular and regular data. These estimates will allow to obtain absorbing properties for the respective multi-valued semiflows.

Let us also observe that any global solution satisfies that $c \in [0, 1]$ a.e. in Q, so $c \in L^\infty(Q)$, and therefore estimates for this third variable are not necessary.

Proposition 11. *Consider a triplet (ϕ_0, u_0, c_0) in any of the phase-spaces of Proposition 10. Then, the following estimates hold for any global solution (ϕ, u, c) to problem (2).*

(a) *There exist positive constants C_4 and C_5 such that if $\phi_0 \in L^2(\Omega)$, then*

$$\begin{aligned} &\frac{\alpha\varepsilon^2}{2}|\phi(t)|^2 + \frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u(t)|^2 \\ \leq \ &\left(\frac{\alpha\varepsilon^2}{2}|\phi_0|^2 + \frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u_0|^2\right)e^{-C_5 t} + \frac{C_4}{C_5} \quad \forall t \geq 0. \end{aligned}$$

(b) *There also exist positive constants* C_6, C_7, *and* C_8, *such that if* $\phi_0 \in H_0^1(\Omega)$, *then*

$$\frac{\alpha\varepsilon^2}{2}|\phi(t)|^2 + \frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u(t)|^2 + \frac{\alpha C_6\varepsilon^2}{2}|\nabla\phi(t)|^2$$
$$\leq \left(\frac{\alpha\varepsilon^2}{2}|\phi_0|^2 + \frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u_0|^2 + \frac{\alpha C_6\varepsilon^2}{2}|\nabla\phi_0|^2\right)e^{-C_7 t} + \frac{C_8}{C_7} \quad \forall t \geq 0.$$

Proof. We split the proof in two parts, concerning the two claims in the statement.

Step 1: Claim (a). Multiplying the first equation in (2) by ϕ and applying twice the Young inequality with δ an arbitrary positive constant to be fixed later, we obtain

$$\begin{aligned} &\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\phi|^2 + \varepsilon^2|\nabla\phi|^2 + \frac{1}{2}\int_\Omega(\phi^4 - \phi^2)dx + \frac{\beta l}{2C_V}|\phi|^2 \\ = &\int_\Omega\left(\frac{\beta}{C_V}u\phi + \beta(\theta_B - \theta_A)c\phi\right)dx \\ \leq &\ \delta|u|^2 + \frac{1}{4}\|\phi\|^4_{L^4(\Omega)} + \frac{\beta^4|\Omega|}{16C_V^4\delta^2} + \frac{1}{4}|\phi|^2 + |\beta(\theta_B - \theta_A)c|^2 \\ \leq &\ \delta|u|^2 + \frac{1}{4}\|\phi\|^4_{L^4(\Omega)} + \frac{1}{4}|\phi|^2 + C_3 \quad a.e.\ t > 0, \end{aligned}$$

where

$$C_3 = \frac{\beta^4|\Omega|}{16C_V^4\delta^2} + 2\beta^2(\theta_B^2 + \theta_A^2)|\Omega|.$$

Arranging terms, we arrive at

$$\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\phi|^2 + \varepsilon^2|\nabla\phi|^2 + \frac{1}{4}\int_\Omega(\phi^4 - 3\phi^2)dx + \frac{\beta l}{2C_V}|\phi|^2 \leq \delta|u|^2 + C_3 \quad a.e.\ t > 0. \tag{10}$$

Now, multiplying the second equation in (2) by u and using assumption (3) of boundedness for K_1, we obtain

$$\frac{C_V}{2}\frac{d}{dt}|u|^2 + \underline{k}_1|\nabla u|^2 \leq \frac{\overline{k}_1^2 l^2}{8\underline{k}_1}|\nabla\phi|^2 + \frac{\underline{k}_1}{2}|\nabla u|^2 \quad a.e.\ t > 0.$$

So, arranging terms and multiplying by $\frac{2\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}$ we conclude

$$\frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}\frac{d}{dt}|u|^2 + \frac{2\varepsilon^2\underline{k}_1^2}{\overline{k}_1^2 l^2}|\nabla u|^2 \leq \frac{\varepsilon^2}{2}|\nabla\phi|^2 \quad a.e.\ t > 0, \tag{11}$$

which added to (10) gives

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2 + \frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u|^2\right) + \frac{\varepsilon^2}{2}|\nabla\phi|^2 \\ &+\frac{1}{4}\int_\Omega(\phi^4 - 3\phi^2)dx + \frac{\beta l}{2C_V}|\phi|^2 + \frac{2\varepsilon^2\underline{k}_1^2}{\overline{k}_1^2 l^2}|\nabla u|^2 \leq \delta|u|^2 + C_3 \quad a.e.\ t > 0. \end{aligned}$$

By the Poincaré inequality, taking $\delta = \frac{\underline{k}_1^2\varepsilon^2\lambda_1}{\overline{k}_1^2 l^2}$ above, and combined with the inequality $x^4 - 3x^2 \geq x^2 - 4$ for all $x \in \mathbb{R}$, we can deduce

$$\frac{d}{dt}\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2 + \frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u|^2\right) + \frac{\varepsilon^2}{2}|\nabla\phi|^2 + \left(\frac{1}{4} + \frac{\beta l}{2C_V}\right)|\phi|^2 + \frac{\varepsilon^2\underline{k}_1^2\lambda_1}{\overline{k}_1^2 l^2}|u|^2 \leq |\Omega| + C_3 =: C_4 \quad a.e.\ t > 0. \tag{12}$$

Now the statement in Claim (a) can be obtained after an estimate from below and by choosing

$$0 < C_5 = \min\left\{\frac{1}{\alpha\varepsilon^2}\left(\frac{1}{2} + \frac{\beta l}{C_V}\right), \frac{\underline{k}_1\lambda_1}{2C_V}\right\}.$$

Step 2: H^1-estimate for ϕ. By using the extra regularity obtained in second part of Theorem 2, we will obtain another estimate for ϕ that complements the obtained in Claim (a) to conclude Claim (b).

Multiplying the first equation in (2) by $-\Delta\phi$ (this is formally, and must be done rigourously by using the Galerkin approximations as in the proof of Theorem 2), we obtain

$$\begin{aligned}&\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\nabla\phi|^2 + \varepsilon^2|\Delta\phi|^2 + \frac{3}{2}\int_\Omega \phi^2|\nabla\phi|^2 dx + \frac{1}{2}\int_\Omega \phi\Delta\phi dx + \frac{\beta l}{2C_V}|\nabla\phi|^2\\ = \ & -\frac{\beta}{C_V}\int_\Omega u\Delta\phi dx - \beta(\theta_B - \theta_A)\int_\Omega c\Delta\phi dx \quad a.e.\ t > 0.\end{aligned}$$

By using again the Young inequality and the fact that $c \in [0, 1]$, we deduce

$$\begin{aligned}&\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\nabla\phi|^2 + \frac{\varepsilon^2}{2}|\Delta\phi|^2 + \frac{\beta l}{2C_V}|\nabla\phi|^2\\ \leq\ & \frac{\beta^2}{C_V^2\varepsilon^2}|u|^2 + \frac{\beta^2(\theta_B - \theta_A)^2|\Omega|}{\varepsilon^2} - \frac{1}{2}\int_\Omega \phi\Delta\phi dx\\ \leq\ & \frac{\beta^2}{C_V^2\varepsilon^2}|u|^2 + \frac{\beta^2(\theta_B - \theta_A)^2|\Omega|}{\varepsilon^2} + \frac{\varepsilon^2}{4}|\Delta\phi|^2 + \frac{1}{4\varepsilon^2}|\phi|^2 \quad a.e.\ t > 0.\end{aligned}$$

In particular, neglecting one term in the left hand side, multiplying the result by a suitable constant C_6 to be fixed later on, and adding to (12), it yields

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\alpha\varepsilon^2}{2}|\phi|^2 + \frac{2C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u|^2 + \frac{\alpha C_6\varepsilon^2}{2}|\nabla\phi|^2\right)\\ &+\left(\frac{1}{4} + \frac{\beta l}{2C_V} - \frac{C_6}{4\varepsilon^2}\right)|\phi|^2 + \left(\frac{\varepsilon^2\underline{k}_1^2\lambda_1}{\overline{k}_1^2 l^2} - \frac{\beta^2 C_6}{C_V^2\varepsilon^2}\right)|u|^2 + \left(\frac{\varepsilon^2}{2} + \frac{\beta C_6 l}{2C_V}\right)|\nabla\phi|^2\\ \leq\ & C_4 + \frac{\beta^2}{\varepsilon^2}C_6(\theta_B - \theta_A)^2|\Omega| =: C_8 \quad a.e.\ t > 0.\end{aligned}$$

Now, it is easy to conclude the statement in Claim (b) by choosing

$$0 < C_6 < \min\left\{\varepsilon^2\left(1 + \frac{2\beta l}{C_V}\right), \frac{C_V^2\varepsilon^4\underline{k}_1^2\lambda_1}{\beta^2\overline{k}_1^2 l^2}\right\},$$

and then

$$C_7 = \min\left\{ \frac{1}{\alpha\varepsilon^2}\left(\frac{1}{2}+\frac{\beta l}{C_V}-\frac{C_6}{2\varepsilon^2}\right), \frac{\overline{k}_1^2 l^2}{2C_V\varepsilon^2\underline{k}_1}\left(\frac{\varepsilon^2\underline{k}_1^2\lambda_1}{\overline{k}_1^2 l^2}-\frac{\beta^2 C_6}{C_V^2\varepsilon^2}\right), \frac{\beta l}{\alpha C_V\varepsilon^2}+\frac{1}{\alpha C_6}\right\}.$$

□

As an immediate consequence of Proposition 11 and from the fact that c takes values in $[0,1]$, we have the following result.

Corollary 12. *The multi-valued semiflows*

$$\left((L^2(\Omega))^2 \times L^2(\Omega;[0,1]), \{G(t)\}_{t\geq 0}\right),$$

and

$$\left(H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1]), \{G(t)\}_{t\geq 0}\right).$$

have bounded absorbing sets in their respective phase-spaces.

We give another result that will be useful for the analysis of the compact properties of the semiflows, and also for the study of their long time behaviour.

Proposition 13. *Let be given $T > 0$ and a bounded set B from $(L^2(\Omega))^2 \times L^2(\Omega;[0,1])$. Then, $G(T,B)$ is bounded in $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1])$.*

Proof. It is a consequence of Step 2 in Proposition 11, and analogous arguments to those in [20, Prop.16]. □

Remark 14. *From the above result, observe that any absorbing set for the semiflow $(H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1]), \{G(t)\}_{t\geq 0})$ is also absorbing for the semiflow $((L^2(\Omega))^2 \times L^2(\Omega;[0,1]), \{G(t)\}_{t\geq 0})$.*

Now we establish a compactness property for the semiflows.

Lemma 15. *Consider any sequence $\{(\phi^n, u^n, c^n)\}_n$ of global solutions of (2) with respective initial data $\{(\phi_0^n, u_0^n, c_0^n)\}_n \subset (L^2(\Omega))^2 \times L^2(\Omega;[0,1])$, and satisfying that $(\phi_0^n, u_0^n, c_0^n) \rightharpoonup (\phi_0, u_0, c_0)$ weakly in $(L^2(\Omega))^3$. Let us also fix a value $t^* > 0$. Then, $c_0 \in L^2(\Omega;[0,1])$ and there exist a subsequence $\{(\phi^\mu, u^\mu, c^\mu)\}_\mu$ and a global solution of (2) (ϕ, u, c) with initial datum (ϕ_0, u_0, c_0), such that*

(a) *the following convergences hold for all $T > 0$:*

$$\begin{aligned}
(\phi^\mu, u^\mu, c^\mu) &\rightharpoonup (\phi, u, c) \quad \text{weakly in } L^2(0,T;(H_0^1(\Omega))^3),\\
(\phi_t^\mu, u_t^\mu, c_t^\mu) &\rightharpoonup (\phi_t, u_t, c_t) \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega))+L^{4/3}(0,T;L^{4/3}(\Omega))\\
&\qquad\qquad \times(L^2(0,T;H^{-1}(\Omega)))^2,
\end{aligned}$$

(b) $(\phi^\mu(t^*), u^\mu(t^*), c^\mu(t^*)) \to (\phi(t^*), u(t^*), c(t^*))$ *strongly in* $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1])$.

(c) *If moreover $\phi_0^n \rightharpoonup \phi_0$ weakly in $H_0^1(\Omega)$, then one also has for all $T > 0$ that*

$$\begin{aligned}
(\phi^\mu, u^\mu, c^\mu) &\rightharpoonup (\phi, u, c) \quad \text{weakly in } L^2(0,T;H^2(\Omega)\times(H_0^1(\Omega))^2),\\
(\phi_t^\mu, u_t^\mu, c_t^\mu) &\rightharpoonup (\phi_t, u_t, c_t) \quad \text{weakly in } L^2(0,T;L^2(\Omega)\times(H^{-1}(\Omega))^2).
\end{aligned}$$

Proof. Observe that proceeding analogously as for the estimates of the Galerkin approximations in Theorem 2, we may obtain uniform estimates for the sequence $\{(\phi^n, u^n, c^n)\}_n$. These estimates allow to conclude claims (a) and (c).

Claim (b) can be proved by using an energy method making the most of the continuity of the solutions, and some inequalities obtained in the proof of Proposition 11. For more details in a close result we refer to [20, Lem.17]. □

A direct consequence from the previous results is the following

Corollary 16. *The semiflows associated to problem* (2) *given in Proposition 10, are asymptotically compact in their respective metrics.*

Using the above result, combined with Lemma 15, we deduce some properties for the multi-valued map G.

Corollary 17. *The multi-valued semiflow* $(X, \{G(t)\}_{t\geq 0})$, *where* X *can be* $H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1])$, *or* $(L^2(\Omega))^2\times L^2(\Omega;[0,1])$, *possesses the following properties:*

(a) *It has compact values, i.e.,* $G:\mathbb{R}_+\times X\to K(X)$.
(b) *For each fixed* $t\geq 0$, $G(t,\cdot):X\to K(X)$ *is upper semicontinuous.*

Proof. Claim (a) follows from Lemma 15 applied to a singleton. Claim (b) is not difficult to prove by contradiction, using again Lemma 15. □

As a consequence of the above results, we are able to establish the existence of global attractors for the considered semiflows, analogously as in [20, Th.22].

Theorem 18. *The multi-valued semiflows* $((L^2(\Omega))^2\times L^2(\Omega;[0,1]),\{G(t)\}_{t\geq 0})$ *and* $(H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1]),\{G(t)\}_{t\geq 0})$ *possess global attractors* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *respectively. Moreover, they coincide:* $\mathcal{A}_1=\mathcal{A}_2$.

Proof. The existence of attractors is a consequence of Corollaries 12, 16, and 17, and by applying Theorem 8.

The equality of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a consequence of the following facts. Both attractors are compact (and therefore bounded) in their respective topologies, and invariant for their respective semiflows.

Since the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is continuous, $\mathcal{A}_2$ is bounded in $(L^2(\Omega))^2\times L^2(\Omega;[0,1])$, and then it holds

$$\lim_{t\to\infty}\operatorname{dist}_{(L^2)^3}(G(t,\mathcal{A}_2),\mathcal{A}_1)=0,$$

but $G(t,\mathcal{A}_2)=\mathcal{A}_2$, and so this means that

$$\operatorname{dist}_{(L^2)^3}(\mathcal{A}_2,\mathcal{A}_1)=0.$$

So, $\mathcal{A}_2\subset\mathcal{A}_1$.

For the other inclusion, observe that for any positive time $T>0$, by Proposition 13, $G(T,\mathcal{A}_1)$ is bounded in $H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1])$. Then it holds

$$\lim_{t\to\infty}\operatorname{dist}_{H_0^1\times(L^2)^2}(G(t,G(T,\mathcal{A}_1)),\mathcal{A}_2)=0,$$

but $G(t,G(T,\mathcal{A}_1))=\mathcal{A}_1$, and so this means that

$$\operatorname{dist}_{H_0^1\times(L^2)^2}(\mathcal{A}_1,\mathcal{A}_2)=0.$$

Thus, $\mathcal{A}_1\subset\mathcal{A}_2$ and the proof is completed. □

Actually, we may complete the above result with a more detailed description of the relation of omega-limit sets in both semiflows. For brevity, let us denote by $\omega_1(\cdot)$ and $\omega_2(\cdot)$ the omega-limit sets in the spaces $(L^2(\Omega))^2\times L^2(\Omega;[0,1])$ and $H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega;[0,1])$ respectively.

Corollary 19. *For any bounded sets* $B_1 \in B((L^2(\Omega))^2 \times L^2(\Omega;[0,1]))$ *and* $B_2 \in B(H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1]))$, *the following equalities hold*

$$\omega_1(B_1) = \omega_2(G(t,B_1)) \quad \forall\, t > 0,$$
$$\omega_1(B_2) = \omega_2(B_2).$$

Proof. All the omega-limit sets in the statement are well-defined by Theorem 18 and Remark 7 (b)(iii).

Observe that by Proposition 13, and since $\omega_1(B_1) = \omega_1(G(t,B_1))$ for any $t > 0$, then the first equality is a consequence of the second. So, we only have to prove that $\omega_1(B_2) = \omega_2(B_2)$.

In order to do that, firstly observe that the inclusion to the right is trivial by the continuous injection of $H_0^1(\Omega)$ into $L^2(\Omega)$.

For the opposite, consider sequences $\{b_n\}_n \subset B_2$, $\{t_n\}_n \subset (0,\infty)$ with $t_n \uparrow \infty$, and $\{(\phi(t_n), u(t_n), c(t_n))\}_n$, with $(\phi(t_n), u(t_n), c(t_n)) \in G(t_n, b_n)$ such that $(\phi(t_n), u(t_n), c(t_n)) \to (\phi_\infty, u_\infty, c_\infty)$ in $(L^2(\Omega))^3$.

Secondly, observe that $G(t_n, B_2) = G(t_n - t_1, G(t_1, B_2))$ for all $n \geq 2$, $G(t_1, B_2)$ is bounded in $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1])$ by Proposition 13, and so, by Corollary 16, there exists a converging subsequence $\{(\phi(t_{n'}), u(t_{n'}), c(t_{n'}))\}$ to some element $(\tilde{\phi}_\infty, \tilde{u}_\infty, \tilde{c}_\infty) \in \omega_2(B_2)$. By uniqueness of the limit, one deduces that $(\phi_\infty, u_\infty, c_\infty) = (\tilde{\phi}_\infty, \tilde{u}_\infty, \tilde{c}_\infty)$ and so the inclusion $\omega_1(B_2) \subset \omega_2(B_2)$ holds. □

Remark 20. *An alternative proof of the equality of the attractors in Theorem 18 follows from Remark 14, Remark 7 (iv), and the above result.*

5. Structure of the omega-limit sets

As mentioned in the Introduction, because of the strong coupling of the model, the convergence of solutions towards the equilibria is difficult to determine using standard methods. However, we show that under suitable assumptions any weak solution converges to an equilibrium.

More precisely, in this section we obtain a characterization of the elements belonging to the omega-limit sets of both semiflows given in Proposition 10. Actually, by the regularizing effect of the problem, we have seen (cf. Corollary 19 above) that they are related.

In order to achieve a more complete description of the omega-limit sets we will use one technique employed in [16] (see also [18]). To this end, we improve some estimates obtained previously for the solutions.

Proposition 21. *Assume that the following relation among the coefficients in the model holds,*

$$\varepsilon^2 \lambda_1 > \left(\frac{\beta^2 \overline{k}_1^2 l^2}{2C_V^2 \varepsilon^2 \underline{k}_1^2 \lambda_1} + \frac{2\beta^2 M^2 (\theta_B - \theta_A)^2}{\varepsilon^2 \lambda_1} + 1 - \frac{\beta l}{C_V} \right)^+, \tag{13}$$

where $(\cdot)^+$ *denotes the positive part, i.e.,* $(\cdot)^+ = \max\{\cdot, 0\}$.

Then, there exists a constant C *such that the following inequality holds for any global solution* (ϕ, u, c) *of* (2) *with initial datum* $(\phi_0, u_0, c_0) \in H_0^1(\Omega) \times L^2(\Omega) \times$

$L^2(\Omega;[0,1])$:

$$|\phi(t)|^2+|u(t)|^2+|c(t)|^2+|\nabla\phi(t)|^2 + \int_0^t (|\nabla\phi(s)|^2+|\phi_t(s)|^2+|\nabla u(s)|^2+|\nabla c(s)|^2)ds \le C(\phi_0,u_0,c_0) \quad \forall t\ge 0.$$

The constant C only depends on the norms of (ϕ_0,u_0,c_0), but is independent of t.

Proof. We will obtain now several inequalities using different test functions in the system (2).

We begin by taking ϕ as test function in the first equation in (2) and applying the Young inequality with arbitrary positive constants c_1 and c_2 to be fixed later. So we obtain

$$\begin{aligned} &\frac{\alpha\varepsilon^2}{2}\frac{d}{dt}|\phi|^2+\varepsilon^2|\nabla\phi|^2+\frac{1}{2}\int_\Omega(\phi^4-\phi^2)dx+\frac{\beta l}{2C_V}|\phi|^2 \\ \le\ &\frac{\beta^2}{2c_1C_V^2}|u|^2+\frac{c_1}{2}|\phi|^2+\frac{\beta^2(\theta_B-\theta_A)^2}{2c_2}|c|^2+\frac{c_2}{2}|\phi|^2 \quad a.e.\ t>0. \end{aligned} \tag{14}$$

By multiplying the first equation in (2) by ϕ_t and applying again the Young inequality, we also have

$$\begin{aligned} &\alpha\varepsilon^2|\phi_t|^2+\frac{\varepsilon^2}{2}\frac{d}{dt}|\nabla\phi|^2+\frac{1}{2}\frac{d}{dt}\left(\int_\Omega\left(\frac{\phi^4}{4}-\frac{\phi^2}{2}\right)dx\right)+\frac{\beta l}{4C_V}\frac{d}{dt}|\phi|^2 \\ \le\ &\frac{\beta^2}{\alpha C_V^2\varepsilon^2}|u|^2+\frac{\beta^2(\theta_B-\theta_A)^2}{\alpha\varepsilon^2}|c|^2+\frac{\alpha\varepsilon^2}{2}|\phi_t|^2 \quad a.e.\ t>0. \end{aligned} \tag{15}$$

By multiplying the third equation in (2) by c, and since $c\in[0,1]$, one deduces similarly as above that

$$\frac{1}{2}\frac{d}{dt}|c|^2+K_2|\nabla c|^2 \le \frac{K_2M^2}{2}|\nabla\phi|^2+\frac{K_2}{2}|\nabla c|^2 \quad a.e.\ t>0.$$

So, arranging terms and multiplying by $\frac{\varepsilon^2}{2K_2M^2}$, we arrive at

$$\frac{\varepsilon^2}{4K_2M^2}\frac{d}{dt}|c|^2+\frac{\varepsilon^2}{4M^2}|\nabla c|^2 \le \frac{\varepsilon^2}{4}|\nabla\phi|^2 \quad a.e.\ t>0. \tag{16}$$

Recall that taking u as test function in the second equation in (2), using the boundedness of K_1, arranging terms and multiplying by $\frac{2\underline{k}_1\varepsilon^2}{\overline{k}_1^2 l^2}$ we have previously obtained inequality (11).

By multiplying (15) by a constant $c_3 > 0$ to be fixed later on, and adding to (14), (11), and (16), we conclude

$$\begin{aligned}
&\frac{d}{dt}\left(\Big(\frac{\alpha\varepsilon^2}{2}+\frac{lc_3\beta}{4C_V}\Big)|\phi|^2+\frac{C_V\varepsilon^2\underline{k}_1}{\overline{k}_1^2 l^2}|u|^2+\frac{\varepsilon^2}{4K_2M^2}|c|^2+\frac{c_3\varepsilon^2}{2}|\nabla\phi|^2\right)\\
&+\frac{c_3}{2}\frac{d}{dt}\left(\int_\Omega\left(\frac{\phi^4}{4}-\frac{\phi^2}{2}\right)dx\right)+\frac{\varepsilon^2}{2}|\nabla\phi|^2+\frac{\beta l}{2C_V}|\phi|^2+\frac{1}{2}\int_\Omega\phi^4dx\\
&+\frac{\alpha c_3\varepsilon^2}{2}|\phi_t|^2+\frac{\varepsilon^2\underline{k}_1^2}{\overline{k}_1^2 l^2}|\nabla u|^2+\frac{\varepsilon^2}{4M^2}|\nabla c|^2\\
\le\ &\Big(\frac{\beta^2}{2c_1C_V^2}+\frac{\beta^2c_3}{\alpha C_V^2\varepsilon^2}\Big)|u|^2+\Big(\frac{\beta^2(\theta_B-\theta_A)^2}{2c_2}+\frac{\beta^2c_3(\theta_B-\theta_A)^2}{\alpha\varepsilon^2}\Big)|c|^2\\
&+\frac{(c_1+c_2+1)}{2}|\phi|^2\quad a.e.\ t>0.
\end{aligned}\tag{17}$$

Now, we rearrange coefficients using the Poincaré inequality suitably, in such a manner that positive terms and positive coefficients remain in the left hand side and the right hand side be zero.

But for that, before to use the Poincaré inequality, we must care about two possible cases for the coefficients of $|\phi|^2$ and, if necessary, of $|\nabla\phi|^2$. These two possibilities can be abbreviated into one expression using the positive part function $(\cdot)^+$, which appears below.

Thus, denoting for short by $d_i > 0$, $i = 1, \dots, 4$, to the coefficients in the first four derivatives in (17), depending on c_3, and by the above arguments, we arrive at

$$\begin{aligned}
&\frac{d}{dt}(d_1|\phi|^2+d_2|u|^2+d_3|c|^2+d_4|\nabla\phi|^2)+\frac{c_3}{2}\frac{d}{dt}\left(\int_\Omega\left(\frac{\phi^4}{4}-\frac{\phi^2}{2}\right)dx\right)\\
&\qquad+\frac{\alpha c_3\varepsilon^2}{2}|\phi_t|^2+D_1|\nabla u|^2+D_2|\nabla c|^2+D_3|\nabla\phi|^2\le 0\quad a.e.\ t>0,
\end{aligned}\tag{18}$$

where

$$\begin{aligned}
D_1&=\frac{\varepsilon^2\underline{k}_1^2}{\overline{k}_1^2l^2}-\left(\frac{\beta^2}{2c_1C_V^2}+\frac{\beta^2c_3}{\alpha C_V^2\varepsilon^2}\right)\lambda_1^{-1},\\
D_2&=\frac{\varepsilon^2}{4M^2}-\left(\frac{\beta^2(\theta_B-\theta_A)^2}{2c_2}+\frac{\beta^2c_3(\theta_B-\theta_A)^2}{\alpha\varepsilon^2}\right)\lambda_1^{-1},\\
D_3&=\frac{\varepsilon^2}{2}-\left(\frac{c_1+c_2+1}{2}-\frac{\beta l}{2C_V}\right)^+\lambda_1^{-1}.
\end{aligned}\tag{19}$$

Now, from (13) it is possible to choose $c_i > 0$, $i = 1, \dots, 3$, such that $D_j > 0$ for $j = 1, \dots, 3$. For clarity, the proof of this fact is postponed to an auxiliary result (cf. Lemma 22 below). So, once we have fixed such $c_i > 0$, and denoting

$$d_5=\min\{\alpha c_3\varepsilon^2/2, D_1, D_2, D_3\}>0,$$

from (18), integrating in time and using the inequality $x^4 - 2x^2 \geq x^2 - 9/4$ for all $x \in \mathbb{R}$, we conclude

$$\begin{aligned}
&\Big(d_1 + \frac{c_3}{8}\Big)|\phi|^2 + d_2|u|^2 + d_3|c|^2 + d_4|\nabla\phi|^2 \\
&+d_5\int_0^t \Big\{|\nabla\phi|^2 + |\phi_t|^2 + |\nabla u|^2 + |\nabla c|^2\Big\}ds \\
\leq\ & d_1|\phi_0|^2 + \frac{c_3}{8}\|\phi_0\|^4_{L^4(\Omega)} + d_2|u_0|^2 + d_3|c_0|^2 + d_4|\nabla\phi_0|^2 + \frac{9}{32}c_3|\Omega| \quad \forall t \geq 0,
\end{aligned}$$

whence the claim follows. □

Lemma 22. *Condition* (13) *is equivalent to the possibility of choosing positive values* $\{c_i\}_{i=1}^3$, *such that* $\{D_i\}_{i=1}^3$, *given in* (19), *are positive.*

Proof. First at all, observe that c_3 appears in D_1 and D_2 in a proportional way, which means that we can take $c_3 > 0$ as small as desired, and we have the result provided that c_1 and c_2 are chosen such that

$$\begin{aligned}
R_1 &= \frac{\varepsilon^2\underline{k}_1^2}{\overline{k}_1^2 l^2} - \frac{\beta^2}{2c_1C_V^2\lambda_1}, \\
R_2 &= \frac{\varepsilon^2}{2M^2} - \frac{\beta^2(\theta_B-\theta_A)^2}{c_2\lambda_1}, \\
R_3 &= \varepsilon^2\lambda_1 - \Big(c_1 + c_2 + 1 - \frac{\beta l}{C_V}\Big)^+
\end{aligned}$$

are positive.

Assume that (13) holds. We must prove that it is possible to choose $c_1, c_2 > 0$ such that $R_i > 0$ for $i = 1, \ldots, 3$.

Observe that R_1 and R_2 are strictly increasing functions in relation to c_1 and c_2 respectively. Moreover, there exist positive values c_1^* and c_2^* such that $R_i = 0$, and therefore $R_i > 0$ if and only if $c_i > c_i^*$, for $i = 1, 2$.

Then, for any choice $c_i > c_i^*$, since the positive part function is non-decreasing, we have that

$$\Big(c_1^* + c_2^* + 1 - \frac{\beta l}{C_V}\Big)^+ \leq \Big(c_1 + c_2 + 1 - \frac{\beta l}{C_V}\Big)^+.$$

From this inequality and condition (13), it holds that for sufficiently close values $c_i > c_i^*$, $i = 1, 2$, then R_3 is positive, as we desired to prove.

Let us prove the opposite implication. Assume that there exist values c_1, $c_2 > 0$ such that $R_i > 0$ for $i = 1, \ldots, 3$. We must prove that (13) holds.

Using again the monotonicity of the positive part function $(\cdot)^+$, the proof is analogous to the above, but going back into the steps. □

A direct consequence of the Proposition 21 is the following.

Corollary 23. *Under the assumptions of Proposition 21, there exists a constant* $C > 0$, *only depending on the norms of* ϕ_0, u_0, *and* c_0, *such that any global solution*

(ϕ, u, c) *with such initial datum satisfies*

$$\begin{aligned}\|\phi_t\|_{L^2(0,\infty;L^2(\Omega))} + \|\phi\|_{L^\infty(0,\infty;H_0^1(\Omega))} + \|u\|_{L^2(0,\infty;H_0^1(\Omega))} \\ +\|u\|_{L^\infty(0,\infty;L^2(\Omega))} + \|c\|_{L^2(0,\infty;H_0^1(\Omega))} &\leq C, \\ \|u_t\|_{L^2(0,\infty;H^{-1}(\Omega))} + \|c_t\|_{L^2(0,\infty;H^{-1}(\Omega))} &\leq C.\end{aligned}$$

Proof. First estimate is an immediate consequence of Proposition 21. Second estimate follows from the above and second and third equations in (2). □

We have the following description of the omega-limit set of trajectories which contains the solutions of the steady state system naturally associated to the evolution problem.

Theorem 24. *Assume that* (13) *is satisfied. Then, for any* $B \in B((L^2(\Omega))^2 \times L^2(\Omega;[0,1]))$ *and any* $(\phi_\infty, u_\infty, c_\infty) \in \omega_1(B)$, *it holds that*

$$\phi_\infty \in H^4(\Omega) \cap H_0^1(\Omega), \quad u_\infty \in H^2(\Omega) \cap H_0^1(\Omega), \quad c_\infty \in H^3(\Omega) \cap H_0^1(\Omega),$$

and they satisfy

$$\begin{cases} -\varepsilon^2 \Delta \phi_\infty = \dfrac{1}{2}(\phi_\infty - \phi_\infty^3) + \dfrac{\beta}{C_V}(u_\infty - \dfrac{l}{2}\phi_\infty) + \beta(\theta_B - \theta_A)c_\infty \quad in \quad \Omega, \\ \nabla \cdot [K_1(\phi_\infty)\nabla u_\infty] = \dfrac{l}{2}\nabla \cdot [K_1(\phi_\infty)\nabla \phi_\infty] \quad in \quad \Omega, \\ -K_2(\Delta c_\infty + M\nabla \cdot [c_\infty(1-c_\infty)\nabla\phi_\infty]) = 0 \quad in \quad \Omega, \\ 0 \leq c_\infty \leq 1 \quad in \quad \Omega. \end{cases} \tag{20}$$

Proof. By Corollary 19, it is clear that we may assume that $B \in B(H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1]))$. Let $(\phi_\infty, u_\infty, c_\infty) \in \omega_2(B)$ and $t_n \to \infty$ such that

$$(\phi(t_n), u(t_n), c(t_n)) \to (\phi_\infty, u_\infty, c_\infty) \text{ in } H_0^1(\Omega) \times (L^2(\Omega))^2. \tag{21}$$

For $n \geq 1$ and $t \in (0,1)$ we define

$$\phi_n(t) = \phi(t_n + t), \quad u_n(t) = u(t_n + t), \quad c_n(t) = c(t_n + t).$$

From Corollary 23 we infer some uniform estimates for the sequences $\{\phi_n\}_n$, $\{u_n\}_n$, and $\{c_n\}_n$. More precisely, there exists a constant $C > 0$ such that for all $n \geq 1$:

$$\begin{aligned}&\|\phi_{nt}\|_{L^2(0,1;L^2(\Omega))} + \|\phi_n\|_{L^\infty(0,1;H_0^1(\Omega))} + \|u_n\|_{L^2(0,1;H_0^1(\Omega))} + \|u_n\|_{L^\infty(0,1;L^2(\Omega))} \\ &+ \|c_n\|_{L^2(0,1;H_0^1(\Omega))} + \|u_{nt}\|_{L^2(0,1;H^{-1}(\Omega))} + \|c_{nt}\|_{L^2(0,1;H^{-1}(\Omega))} \leq C.\end{aligned} \tag{22}$$

Next, we identify the limit of ϕ_n, u_n, and c_n as n goes to ∞. We observe that for each $t \in [0,1]$, by Hölder inequality,

$$\begin{aligned}|\phi_n(t) - \phi(t_n)| &= \Big| \int_{t_n}^{t_n+t} \phi_t(s)ds \Big| \\ &\leq \Big(\int_{t_n}^{\infty} |\phi_t(s)|^2 ds \Big)^{1/2},\end{aligned}$$

and the right hand side converges to zero as $n \to \infty$ by Corollary 23. Similarly, we have that $u_n(t)$ and $c_n(t)$ converge to $u(t_n)$ and $c(t_n)$, respectively, in $H^{-1}(\Omega)$ uniformly in $t \in [0,1]$.

Now, we write

$$|\phi_n(t) - \phi_\infty| \leq |\phi_n(t) - \phi(t_n)| + |\phi(t_n) - \phi_\infty|,$$

and using (21) we conclude that

$$\phi_n \to \phi_\infty \quad \text{in } C([0,1];L^2(\Omega)).$$

In a similar way, we prove that

$$(u_n, c_n) \to (u_\infty, c_\infty) \quad \text{in } C([0,1];(H^{-1}(\Omega))^2).$$

We also have the following bounds

$$\begin{aligned}\int_0^1 |\phi_{nt}(s)|^2 ds &= \int_{t_n}^{t_n+1} |\phi_t(s)|^2 ds \\ &\leq \int_{t_n}^{\infty} |\phi_t(s)|^2 ds,\end{aligned}$$

$$\int_0^1 \|u_{nt}(s)\|^2_{H^{-1}(\Omega)} ds \leq \int_{t_n}^{\infty} \|u_t(s)\|^2_{H^{-1}(\Omega)} ds,$$

and

$$\int_0^1 \|c_{nt}(s)\|^2_{H^{-1}(\Omega)} ds \leq \int_{t_n}^{\infty} \|c_t(s)\|^2_{H^{-1}(\Omega)} ds,$$

so that

$$\begin{aligned}\phi_{nt} &\to 0 \ \text{ in } L^2(0,1;L^2(\Omega)),\\ (u_{nt}, c_{nt}) &\to 0 \ \text{ in } L^2(0,1;(H^{-1}(\Omega))^2).\end{aligned}$$

Moreover, from (22) and the Aubin-Lions lemma (e.g., cf. [19]), there exist subsequences of $\{\phi_n\}_n$, $\{u_n\}_n$, and $\{c_n\}_n$ (which we still denote the same) satisfying

$$\begin{aligned}\phi_n &\overset{*}{\rightharpoonup} \phi_\infty \quad \text{weakly-star in } \ L^\infty(0,1;H^1_0(\Omega)),\\ \phi_n &\to \phi_\infty \quad \text{a.e. } (0,1)\times\Omega,\\ (u_n, c_n) &\rightharpoonup (u_\infty, c_\infty) \quad \text{weakly in } \ L^2(0,1;(H^1_0(\Omega))^2),\\ (u_n, c_n) &\overset{*}{\rightharpoonup} (u_\infty, c_\infty) \quad \text{weakly-star in } \ L^\infty(0,1;(L^2(\Omega))^2),\\ (u_n, c_n) &\to (u_\infty, c_\infty) \quad \text{strongly in } \ L^2(0,1;(L^2(\Omega))^2).\end{aligned}$$

Finally, we prove that ϕ_∞, u_∞ and c_∞ satisfy (20). To this end, consider

$$\varrho \in \mathcal{D}(0,1) \text{ with } \int_0^1 \varrho(s) ds = 1 \ \text{ and } \ \zeta \in \mathcal{D}(\Omega).$$

Multiplying the first equation in (2) by $\varrho(t-t_n)\zeta(x)$, and integrating in $(0, t_n+1)$,

$$\begin{aligned}&\alpha\varepsilon^2 \int_0^{t_n+1} \langle \phi_t(t), \zeta\rangle \varrho(t-t_n) dt + \varepsilon^2 \int_0^{t_n+1} (\nabla\phi(t), \nabla\zeta)\varrho(t-t_n) dt \\ = \ &\frac{1}{2}\int_0^{t_n+1} (\phi(t) - \phi^3(t), \zeta)\varrho(t-t_n) dt + \frac{\beta}{C_V}\int_0^{t_n+1} (u(t) - \frac{l}{2}\phi(t), \zeta)\varrho(t-t_n) dt \\ &+\beta(\theta_B - \theta_A)\int_0^{t_n+1} (c(t), \zeta)\varrho(t-t_n) dt.\end{aligned}$$

By using the change of variable $s = t - t_n$,

$$\begin{aligned} & \alpha\varepsilon^2 \int_0^1 \langle \phi_{nt}(s), \zeta\rangle \varrho(s) ds + \varepsilon^2 \int_0^1 (\nabla\phi_n(s), \nabla\zeta)\varrho(s) ds \\ = \; & \frac{1}{2}\int_0^1 (\phi_n(s) - \phi_n^3(s), \zeta)\varrho(s) ds \\ & + \frac{\beta}{C_V}\int_0^1 (u_n(s) - \frac{l}{2}\phi_n(s), \zeta)\varrho(s) ds + \beta(\theta_B - \theta_A)\int_0^1 (c_n(s), \zeta)\varrho(s) ds. \end{aligned}$$

From the above convergences, and by [19, Lem.1.3, p.12], passing to the limit as $n \to \infty$ we obtain

$$\begin{aligned} & \varepsilon^2 \int_0^1 \varrho(s) ds (\nabla\phi_\infty, \nabla\zeta) \\ = \; & \frac{1}{2}\int_0^1 \varrho(s) ds (\phi_\infty - \phi_\infty^3, \zeta) + \frac{\beta}{C_V}\int_0^1 \varrho(s) ds (u_\infty - \frac{l}{2}\phi_\infty, \zeta) \\ & + \beta(\theta_B - \theta_A)\int_0^1 \varrho(s) ds (c_\infty, \zeta). \end{aligned}$$

Thus, first equation in (20) is satisfied.

Similarly, multiplying by the same function in the second equation in (2) and integrating, it yields

$$\begin{aligned} & C_V \int_0^1 \langle u_{nt}(s), \zeta\rangle \varrho(s) ds + \int_0^1 (K_1(\phi_n(s))\nabla u_n(s), \nabla\zeta)\varrho(s) ds \\ = \; & \frac{l}{2}\int_0^1 (K_1(\phi_n(s))\nabla\phi_n(s), \nabla\zeta)\varrho(s) ds. \end{aligned}$$

Since K_1 is Lipschitz continuous, we have that

$$K_1(\phi_n) \to K_1(\phi_\infty) \quad \text{strongly in} \quad L^2(0,1;L^2(\Omega)),$$

and passing to limit we obtain that second equation in (20) holds.

Next, since $0 \le c_n \le 1$ and $c_n \to c_\infty$ in $L^2(0,1;L^2(\Omega))$, by the Dominated Convergence Theorem, it follows that $c_n(1-c_n)$ converges to $c_\infty(1-c_\infty)$ in $L^p(0,1;L^p(\Omega))$ for any $p \in [1,\infty)$, and also $0 \le c_\infty \le 1$. Proceeding analogously as before in the concentration equation in (2), we can pass to the limit in

$$\begin{aligned} & \int_0^1 \langle c_{nt}(s), \zeta\rangle \varrho(s) ds + K_2 \int_0^1 (\nabla c_n(s), \nabla\zeta)\varrho(s) ds \\ & + K_2 M \int_0^1 (c_n(s)(1 - c_n(s))\nabla\phi_n(s), \nabla\zeta)\varrho(s) ds \quad = \quad 0, \end{aligned}$$

and deduce that third equation in (20) is satisfied.

Finally, concerning the regularity of the triplet $(\phi_\infty, u_\infty, c_\infty)$, let us observe that they belong not only to $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega;[0,1])$. By elliptic regularity and bootstrapping in (20), it is not difficult to deduce that ϕ_∞, and later u_∞ and c_∞, gain the regularity claimed in the statement. The proof is then complete. □

Remark 25. *If the function K_1 is more regular, the regularity of the elements in the omega-limit can be improved, again by a bootstrapping argument as above.*

Remark 26. *The above result identifies the possible cluster points of any single solution, but it does not indicate whether this limit set is properly a singleton. This last question may be answered by the study of the set of stationary states. Indeed, the answer is positive if, for instance, one can show that the number of solutions to* (20) *is discrete, or the system satisfies some kind of Łojasiewicz-Simon property. None of these properties seem to be easy to obtain here. In fact, they do not hold in several examples (cf.* [3, 12]*).*

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