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ATTRACTORS FOR THE STOCHASTIC 3D NAVIER–STOKES EQUATIONS

PEDRO MARÍN-RUBIO

Dpto. Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla, Apdo. de Correos 1160,
41080 Sevilla, Spain
pedro@numer.us.es

JAMES C. ROBINSON

Mathematics Institute, University of Warwick,
Coventry, CV4 7AL, UK
jcr@maths.warwick.ac.uk

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In a 1997 paper, Ball defined a generalised semiflow as a means to consider the solutions of equations without (or not known to possess) the property of uniqueness. In particular he used this to show that the 3D Navier–Stokes equations have a global attractor provided that all weak solutions are continuous from $(0, \infty)$ into L^2. In this paper we adapt his framework to treat stochastic equations: we introduce a notion of a stochastic generalised semiflow, and then show a similar result to Ball's concerning the attractor of the stochastic 3D Navier–Stokes equations with additive white noise.

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1. Introduction

For certain deterministic differential equations, most notably the three-dimensional Navier–Stokes equations, we can prove the existence of solutions but are unable to prove their uniqueness. Nevertheless, if we still wish to consider such equations within the framework of a dynamical system there are various approaches available.

In particular there have been various attempts to apply the theory of global attractors to the 3D Navier–Stokes equations, despite the unresolved problem of uniqueness. There are two results which require no additional hypotheses: Foias and Temam [18] constructed a set, consisting of strong solutions, that attracts all weak solutions in the weak topology of the natural phase space; and Sell [27] analysed the induced single-valued flow on the phase space consisting of all solutions of the equation (an element of this space is a complete trajectory in the original phase space), showing that this has a global attractor.

However, the two results that are nearest to the standard theory require the assumption of (unproved) hypotheses: Foias and Temam [17] showed that the existence of globally defined strong solutions (which are then unique) automatically implies the existence of a global attractor, while Ball [5] deduced the same result assuming that the weak solutions trace out continuous trajectories in the phase space.

Under this condition on the solutions, Ball recast the 3D Navier–Stokes equations as a generalised (multivalued) semiflow, and showed that this generalised semiflow has a global attractor. Another general setting for such multivalued flows has recently been developed by Melnik and Valero [24], and in the past by workers in control theory (see Kloeden [21] for a summary).

In this paper we consider a stochastic version of the 3D Navier–Stokes equations: as for the deterministic equations, existence is known, but the issue of uniqueness is again unresolved (see Bensoussan and Temam [6], for example). Two of the above treatments of the deterministic 3D equations have been extended to treat the stochastic case: that of Foias and Temam [17] by Crauel and Flandoli [15] (regularity implies the existence of an attractor) and that of Sell by Flandoli and Schmalfuß [16] (an attractor in the "path space"). By working in a phase space formed of *solutions* of the stochastic 3D Navier–Stokes equations ("the path space"), Flandoli and Schmalfuß are able, despite the lack of uniqueness, to work with single-valued cocycles of the standard sort (as discussed in Sec. 2). Following the treatment of Ball [5] and insisting that we remain in the original phase space, we are required in what follows to introduce a multivalued framework. It is this that makes the problems of measurability and invariance of the attractor signficantly more involved. (For a more general framework for stochastic equations without uniqueness see Caraballo *et al.* [7]).

We begin the paper with a brief summary in Sec. 2 of the standard random dynamical systems framework in which the theory of random attractors can be developed. In Sec. 3 we recall the definition of weak solutions of the stochastic 3D Navier–Stokes equations due to Flandoli and Schmalfuß [16]. We then introduce in Sec. 4 the notion of a generalised stochastic semiflow, and prove that the weak solutions of the 3D NSE form a stochastic semiflow if and only if they are each continuous from $(0, \infty)$ into the natural phase space H. Section 5 develops the general theory of attractors for stochastic generalised semiflows, and this is then applied to the 3D NSE (under the continuity assumption on the solutions) in Sec. 6.

While the applications to the stochastic 3D Navier–Stokes equations were the motivating factor for our work, we believe that the main contribution of this paper is to offer a workable definition of a stochastic generalised semiflow, and a corresponding theory of global attractors.

2. Single-Valued Random Dynamical Systems and Their Attractors

We now recall the definition of a random dynamical system and a random attractor (for more background on random dynamical systems see Arnold [3]). We cover only the case of continuous time here.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\theta_t : \Omega \to \Omega,\ t \in \mathbb{R}\}$ a family of measure-preserving transformations such that $(t, \omega) \mapsto \theta_t \omega$ is measurable, $\theta_0 = \mathrm{id}$, and $\theta_{t+s} = \theta_t \theta_s$ for all $s, t \in \mathbb{R}$. The flow θ_t together with the corresponding probability space,

$$(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$$

is called a (*measurable*) *dynamical system.*

Let (X, d) be a Polish space (a complete separable metric space) equipped with its Borel σ-algebra $\mathcal{B}$. A continuous *random dynamical system* (RDS) on X is a measurable map

$$\begin{aligned} \varphi \colon \mathbb{R}^+ \times \Omega \times X &\to X \\ (t, \omega, x) &\mapsto \varphi(t, \omega)x \end{aligned}$$

such that $\mathbb{P}$-a.s.

(i) $\varphi(0, \omega) = \mathrm{id}$ on X
(ii) $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$ for all $t, s \in \mathbb{R}^+$ (cocycle property)
(iii) $\varphi(t, \omega) \colon X \to X$ is continuous.

The concept of a random attractor for such systems was first introduced by Crauel and Flandoli [14] and Schmalfuß [28], with notable developments in Crauel *et al.* [13]. This attractor is a random compact set that is invariant and attracting "in the pullback sense". In order to define these concepts more precisely we denote by $\mathrm{dist}(\cdot, \cdot)$ the Hausdorff semidistance in X,

$$\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)\,.$$

A random compact set $\{K(\omega)\}_{\omega \in \Omega}$ is a family of compact sets indexed by ω such that for every $x \in X$ the map $\omega \mapsto \mathrm{dist}(x, K(\omega))$ is measurable with respect to $\mathcal{F}$; and we say that a random set $\mathcal{A}(\omega)$ is attracting if for all deterministic bounded sets $B \subset X$ we have

$$\lim_{t \to \infty} \mathrm{dist}(\varphi(t, \theta_{-t}\omega)B, \mathcal{A}(\omega)) = 0 \quad \mathbb{P}\text{-almost surely}\,.$$

Since $\varphi(t, \theta_{-t}\omega)u_0$ can be interpreted as the position at time zero of the trajectory which was at u_0 at time $-t$, this *pullback attraction* is essentially attraction "from $t = -\infty$".

A random compact set $\mathcal{A}(\omega)$ is said to be a *random attractor* for the RDS φ if it is both attracting (as above) and invariant, that is

$$\varphi(t,\omega)\mathcal{A}(\omega)=\mathcal{A}(\theta_t\omega) \quad \text{for all } t\geq 0 \quad \mathbb{P}\text{-a.s.}$$

The standard result that provides the existence of random attractors is similar to that from the deterministic theory (e.g. Babin and Vishik [4], Hale [19], Ladyzhenskaya [22], Robinson [26], Temam [30]): the following elegant formulation is due to Crauel [12].

Theorem 1. *There exists a random attractor $\mathcal{A}(\omega)$ iff there exists a compact attracting set $K(\omega)$.*

3. Weak Solutions of the Stochastic 3D Navier–Stokes Equations

We will consider the Navier–Stokes equations with additive noise on a smooth bounded open domain $D\subset\mathbb{R}^3$:

$$\begin{aligned} \frac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p-f=\dot{W}_t &\quad \text{in } [0,\infty)\times D\,,\\ \text{div } u=0 &\quad \text{in } [0,\infty)\times D\,,\\ u=0 &\quad \text{in } [0,\infty)\times \partial D\,,\\ u(0,x)=u_0(x) &\quad x\in D\,; \end{aligned} \tag{3.1}$$

here ν is the kinematic viscosity, p the pressure, u the velocity field, f a time-independent body forcing field, and W_t a two-sided Wiener process which we will specify in more detail below. [Of course, the notation $\dot{W}_t$ is for convenience only, the main PDE of (3.1) needing to be interpreted in an integral sense, see below.] The presence of the stochastic term requires a modification of the deterministic definition of a weak solution. In what follows we adopt the approach of Flandoli and Schmalfuß [16], tailoring their presentation slightly to suit our purposes.

First we need to introduce some standard notation in order to treat the equation in the usual way (see Constantin and Foias [9], for example): let $\mathcal{V}$ denote the space of infinitely differentiable divergence-free three-dimensional vector fields on D with compact support contained in D. By H we denote the closure of $\mathcal{V}$ in $[L^2(D)]^3$ and by V its closure in $[H_0^1(D)]^3$; for the corresponding norms we will use $|\cdot|$ and $\|\cdot\|$ respectively. We can define a bilinear operator $B(u,v)\colon V\times V\to V'$, where V' is the dual of V, via

$$\langle B(u,v),z\rangle=\sum_{i,j=1}^{3}\int_D z_i(x)u_j(x)\frac{\partial v_i}{\partial x_j}(x)dx\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between V' and V.

If P is the orthogonal projection from $[L^2(D)]^3$ onto H (the "Leray projector"), then the Stokes operator A is defined by $Au=-P\Delta u$. We write $D(A)$ for the domain of A; $D(A)=[H^2(D)]^3\cap V$, and denote by λ_1 the first eigenvalue of A.

We take the white noise process W_t to be a two-sided Wiener process with values in V that can be expressed as

$$W_t = \sum_{j=1}^{\infty} c_j B_t^{(j)} w_j \,,$$

where $\{w_j\}$ is a complete set of eigenfunctions of A that are orthonormal in V, $B_t^{(j)}$ are mutually independent one-dimensional Brownian motions, and $\sum_{j=1}^{\infty} |c_j|^2 < \infty$. Thus, in the notation of Sec. 2, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the space $(C_0(\mathbb{R}; V), \mathcal{B}_{C_0(\mathbb{R};V)}, \mathbb{P})$ (with $\mathbb{P}$ the appropriate product of one-dimensional Wiener measures). The measure-preserving transformation θ_t can be represented by its action on realisations of the Wiener process as the shift operator with an appropriate adjustment to retain the condition that $W_{\theta_t \omega}(0) = 0$:

$$W_\cdot(\theta_t \omega) = W_{t+\cdot}(\omega) - W_t(\omega) \,.$$

In order to deal with the problems due to the white noise term, we will essentially make a substitution that allows us to consider the equation realisation-by-realisation: to do this we will use the unique stationary solution of the auxiliary Stokes equation

$$dz + [(A + \alpha)z]dt = dW_t \tag{3.2}$$

that is defined for all $t \in \mathbb{R}$, namely the Ornstein–Uhlenbeck process

$$z_\alpha(t) = \int_{-\infty}^{t} e^{(t-s)(-A-\alpha)} dW_s \,;$$

this is given more rigorously, $\mathbb{P}$-a.s., by

$$z_\alpha(t, \omega) = W_t(\omega) - \int_{-\infty}^{t} (A + \alpha) e^{(t-s)(-A-\alpha)} W_s(\omega) ds \tag{3.3}$$

(the inclusion of the parameter α will prove useful for our discussion of attractors, although it is not necessary in the proof of the existence of solutions). Various properties of z_α are recalled as and when we need them below.

Definition 1. Given $f \in [H^{-1}(D)]^3$ and a realisation $W_t(\omega)$ of the Wiener process that is continuous from $[0, \infty)$ into V, we say that $u \in L^2_{\mathrm{loc}}[0, \infty; H)$ is a weak solution of the Navier–Stokes equation (3.1) with noise ω if

- $u \in L^\infty_{\mathrm{loc}}[0, \infty; H) \cap L^2_{\mathrm{loc}}[0, \infty; V)$,
- $\frac{\partial}{\partial t}(u - W_t) \in L^{4/3}_{\mathrm{loc}}[0, \infty; V')$,
- for a.e. t and t_0 with $t \geq t_0 > 0$ and for $t_0 = 0$, we have

$$V_1(u, \omega)(t) \leq V_1(u, \omega)(t_0) \,, \tag{3.4}$$

where

$$V_1(u,\omega)(s) := e^{-\int_0^s(-\lambda_1+2C^*|z_\alpha(r;\omega)|_{L^4}^8)dr}|u(s)-z_\alpha(s;\omega)|^2$$
$$-\int_0^s e^{-\int_0^\sigma(-\lambda_1+2C^*|z_\alpha(r;\omega)|_{L^4}^8)dr}$$
$$\times 4\left[C_B^2|z_\alpha(\sigma;\omega)|_{L^4}^4+\frac{\alpha^2}{\lambda_1}|z_\alpha(\sigma;\omega)|^2+\|f\|_{V'}^2\right]d\sigma$$

and

$$V_2(u;\omega)(t) \le V_2(u;\omega)(t_0)\,, \tag{3.5}$$

where

$$V_2(u;\omega)(s) := |u(s)-z_\alpha(s;\omega)|^2+\int_0^s\|u(r)-z_\alpha(r;\omega)\|^2dr$$
$$-\int_0^s[2C^*|u(r)-z_\alpha(r;\omega)|^2|z_\alpha(r;\omega)|_{L^4}^8$$
$$+4C_b^2|z_\alpha(r;\omega)|_{L^4}^4+4\alpha^2|z_\alpha(r;\omega)|^2/\lambda_1+4\|f\|_{V'}^2]dr$$

- and for all $t \ge t_0 \ge 0$ and all $\phi \in V$

$$\langle u(t)-u(t_0),\phi\rangle+\int_{t_0}^t\langle A^{1/2}u(s),A^{1/2}\phi\rangle ds+\int_{t_0}^t\langle B(u(s),u(s)),\phi\rangle ds$$
$$=\langle W_t(\omega)-W_{t_0}(\omega),\phi\rangle+\int_{t_0}^t\langle f,\phi\rangle ds\,. \tag{3.6}$$

The constants C^* and C_B are defined in Flandoli and Schmalfuß [16], and are related to the constants occurring in two calculus inequalities.

Observe that if we drop all the terms involving W_t and z_α then (3.4) and (3.5) are consequences of the standard deterministic Leray energy inequality for weak solution (see Constantin and Foias [9], for example). Flandoli and Schmalfuß show that the definition is independent of α.

Almost every realisation of the Wiener process has trajectories that satisfy $W_t \in C_0(\mathbb{R};V)$, so certainly $W_t \in C_0([0,\infty),V)$ as required by the definition of a weak solution above. Given this observation, Flandoli and Schmalfuß [16, Proposition 2.2, p. 368] proved the existence of such weak solutions for the 3D stochastic Navier–Stokes equations:

Proposition 1. *For almost every ω, given $u_0 \in H$ and $f \in [H^{-1}(D)]^3$ there exists a weak solution of the Navier–Stokes equation with noise ω such that $u(0) = u_0$.*

(We note here that we have taken $f \in [H^{-1}(D)]^3$, rather than the more usual assumption that $f \in V'$, in line with the results of Langa *et al.* [23]: although

Definition 1 makes sense for $f \in V'$, it is not possible to recover the pressure p in any meaningful way unless $f \in [H^{-1}(D)]^3$. See Simon [29] for similar results for the deterministic Navier–Stokes equations.)

4. Generalised Stochastic Semiflows and the 3D Stochastic NSE

Following Ball's approach for deterministic semiflows without uniqueness [5] we now give a definition of a generalised stochastic semiflow, and show that this is applicable to the 3D Navier–Stokes equations with an additive noise, provided that we assume that the solutions are continuous from $(0, \infty)$ into H.

4.1. *Generalised stochastic semiflows*

We recover Ball's definition of a generalised semiflow if we remove from the following all dependence on ω (and the corresponding references to Ω). In order for our definition to make sense we impose the additional assumption that Ω is a Polish space. Because the primary application of the theory of random dynamical systems is to stochastic ordinary and partial differential equations, we believe that this does not overly limit the applicability of the concept. Generally we would expect Ω to be identified with the canonical (two-sided) Wiener space: in our application of the definition to the stochastic 3D Navier–Stokes equations we identify Ω with the space of all continuous paths from $\mathbb{R}$ into V, and say that $\omega_n \to \omega$ if

$$W_t(\omega_n) \to W_t(\omega) \quad \text{in } V$$

uniformly on compact subintervals of $\mathbb{R}$.

Definition 2. A *stochastic generalised semiflow* (SGS) $\mathcal{G}$ *on* X *with noise* Ω is a family of pairs

$$\{(\varphi, \omega) | \varphi\colon [0, +\infty) \to X,\ \omega \in \Omega\}$$

(called solutions) satisfying the following assumptions:

(S1) Existence: $\mathbb{P}$-a.s. in ω, for each $z \in X$ there exists at least one $(\varphi, \omega) \in \mathcal{G}$ with $\varphi(0) = z$.

(S2) Translates of solutions are solutions: If $(\varphi, \omega) \in \mathcal{G}$ and $\tau \geq 0$, then $(\varphi^\tau, \theta_\tau \omega) \in \mathcal{G}$, where $\varphi^\tau(t) := \varphi(\tau + t)$.

(S3) Generalised cocycle property: If (φ, ω) and $(\psi, \theta_t \omega)$ belong to $\mathcal{G}$, with $\psi(0) = \varphi(t)$, then $(\phi, \omega) \in \mathcal{G}$, where

$$\phi(\tau) := \begin{cases} \varphi(\tau) & \text{for } 0 \leq \tau \leq t\,, \\ \psi(\tau - t) & \text{for } \tau > t\,. \end{cases}$$

(S4) Upper semicontinuity with respect to initial data: if $(\varphi_n, \omega_n) \in \mathcal{G}$ with $\varphi_n(0) \to z$ and $\omega_n \to \omega$, then there exists a subsequence $(\varphi_{n'}, \omega_{n'})$ and a pair $(\varphi, \omega) \in \mathcal{G}$ with $\varphi(0) = z$ such that $\varphi_{n'}(t) \to \varphi(t)$ for every $t \geq 0$.

Given a stochastic generalised semiflow it is possible to define a "generalised cocycle" using the set of all attainable states,

$$\Phi(t,\omega)E = \{\varphi(t)\colon (\varphi,\omega)\in\mathcal{G},\ \varphi(0)\in E\}\,. \tag{4.1}$$

We now show that this has similar properties to the standard kind of cocycle (cf. Sec. 2).

Proposition 2. *Let $\mathcal{G}$ be a SGS, then $\mathbb{P}$-a.s.*

$$\Phi(0,\omega)E = E \quad \textit{for all } E\subset X \tag{4.2}$$

and

$$\Phi(t+s,\omega)E = \Phi(t,\theta_s\omega)\Phi(s,\omega)E \quad \textit{for all } t,s\in\mathbb{R}_+,\ E\subset X\,. \tag{4.3}$$

Moreover, every $\Phi(t,\omega)$ has compact values and is a multivalued upper semi-continuous mapping, i.e. for every neighbourhood N of $\Phi(t,\omega)x$, there exists a neighbourhood M of x such that $\Phi(t,\omega)M\subset N$.

Proof. Equality (4.2) is obvious by definition. In order to prove (4.3) consider $x\in\Phi(t+s,\omega)E$; then there exists $(\varphi,\omega)\in\mathcal{G}$ such that $x=\varphi(t+s)$ with $\varphi(0)\in E$. Now, $(\varphi^s,\theta_s\omega)$ is an element of $\mathcal{G}$ by (S2): therefore $x=\varphi^s(t)$, i.e. $x\in\Phi(t,\theta_s\omega)\varphi^s(0)$ and $\varphi^s(0)=\varphi(s)\in\Phi(s,\omega)\varphi(0)$.

For the other inclusion, let $x\in\Phi(t,\theta_s\omega)\Phi(s,\omega)E$. Then there exists an element $z\in\Phi(s,\omega)E$ with $x\in\Phi(t,\theta_s\omega)z$, and so there exist pairs $(\varphi,\theta_s\omega)$ and (ψ,ω) in $\mathcal{G}$ such that $x=\varphi(t)$ with $\varphi(0)=\psi(s)$ and $\psi(0)\in E$. We now use property (S3), and consider

$$\phi(\tau) := \begin{cases} \psi(\tau) & \text{for } 0\le\tau\le s\,,\\ \varphi(\tau-s) & \text{for } s<\tau\,. \end{cases}$$

Then $(\phi,\omega)\in\mathcal{G}$ and so $x\in\Phi(t+s,\omega)\psi(0)\subset\Phi(t+s,\omega)E$.

Since (S4) holds, it is obvious that every $\Phi(t,\omega)$ has compact values. For the upper semicontinuity it is easy to see (arguing by contradiction) that $\Phi(t,\omega)$ is ε-u.s.c. [for each x, for every $\varepsilon>0$ there exists a $\delta>0$ such that the image under $\Phi(t,\omega)$ of the δ ball about x lies within an ε neighbourhood of $\Phi(t,\omega)(x)$]; since $\Phi(t,\omega)$ takes compact values this implies that it is upper semicontinuous (cf. Aubin and Cellina [1]). ☐

In order to prove the existence of a random attractor for such a generalised stochastic semiflow, we will require a strengthening of the property (S4) which also includes some compactifying properties of the generalised cocycle $\Phi(t,\omega)$. We say that $\mathcal{G}$ is a compactifying generalised stochastic semiflow (KSGS) if (S1–3) and (S4*) hold, where for (S4*) we require the same as (S4) but when the initial data converges only weakly:

(S4*) Upper semicontinuity with respect to weakly converging initial data: if $(\varphi_n, \omega_n) \in \mathcal{G}$ with $\varphi_n(0) \rightharpoonup z$ and $\omega_n \to \omega$, then there exists a subsequence $(\varphi_{n'}, \omega_{n'})$ and a pair $(\varphi, \omega) \in \mathcal{G}$ with $\varphi(0) = z$ such that $\varphi_{n'}(t) \to \varphi(t)$ for every $t > 0$.

4.2. *A stochastic generalised semiflow for the 3D stochastic NSE*

We denote by $\mathcal{G}_{\rm SNS}$ the set of all pairs (φ, ω), where φ is a weak solution of the stochastic 3D Navier–Stokes equations associated to a realisation ω of the noise. We have the following result, after Proposition 7.4 of Ball [5]. Note that the topology on Ω is that discussed at the beginning of Sec. 4.1.

Proposition 3. *The following are equivalent*:

(i) $\mathcal{G}_{\rm SNS}$ *is a compactifying stochastic generalised semiflow.*
(ii) $\mathbb{P}$-*a.s. in* ω, *each weak solution* u (*associated with* ω) *is continuous from* $(0, \infty)$ *to* H.
(iii) $\mathbb{P}$-*a.s. in* ω, *each weak solution* u (*associated with* ω) *is continuous from* $[0, \infty)$ *to* H.

We note here that in the particular case of the stochastic 3D NSE, (S4) and (S4*) are equivalent.

Proof. (i) implies (ii). We follow the proof of Theorem 2.1 in Ball's paper, which combines a version of Lusin's theorem with the properties of weak solutions to deduce (ii) by contradiction.

More precisely, suppose that $\mathcal{G}_{\rm SNS}$ is a generalised stochastic semiflow on H, and let ω lie within the set of full measures given by (S1) such that there exists at least one weak solution. Now choose φ to be any one of these weak solutions, and let us assume that it is not continuous from $(0, \infty) \to H$. It follows that for some finite time interval $I = (a, a + \delta)$, there exists a $t_0 \in J \equiv (a + \delta/3, a + 2\delta/3)$ and $h_j \to 0^+$ such that

$$\varphi(t_0 + h_j) \not\to \varphi(t_0)\,. \tag{4.4}$$

We will deduce a contradiction from the above claim. In order to do that we will show that all h_j-shifts of φ are continuous almost everywhere and then use the weak continuity of the solution.

Since $\varphi \in C([0, \infty); H_w)$ it is weakly measurable, and by Pettis' theorem (see p. 73 of Hille and Phillips [20]) φ is strongly measurable. Lusin's theorem (see Oxtoby [25] for example) can be applied to ensure the existence of a family of closed sets $F_j \subset J$ with $\text{meas}(J \backslash F_j) \leq 1/j^2$ such that the restriction of φ to F_j is continuous. We now define $E_j = J \cap F_j \cap (F_j - h_j)$, where h_j are the times in (4.4).

Clearly $\text{meas}(J\backslash E_j) \leq 2/j^2$, and so

$$\text{meas}\left(J\backslash \bigcup_{n\geq 1}\bigcap_{j\geq n} E_j\right) = \text{meas}\left(\bigcap_{n\geq 1}\bigcup_{j\geq n}(J\backslash E_j)\right)$$

$$\leq \text{meas}\left(\bigcup_{j\geq n_0}(J\backslash E_j)\right) \leq \sum_{j\geq n_0} 2/j^2 \to 0 \quad \text{as } n_0 \to \infty\,,$$

from which we deduce that almost every $t \in J$ belongs to $\bigcup_{n\geq 1}\bigcap_{j\geq n} E_j$, i.e. $\varphi(\cdot + h_j)$ is continuous and so $\varphi(t + h_j) \to \varphi(t)$ for almost every $t \in J$.

Now take t_1 and t_2 in J, with $t_1 < t_0 < t_2$ and $\varphi(t_i + h_j) \to \varphi(t_i)$ as $j \to \infty$.

Since $(\varphi(t_1 + \cdot), \theta_{t_1}\omega) \in \mathcal{G}_{\text{SNS}}$ and $(\varphi(t_1 + h_j + \cdot), \theta_{t_1+h_j}\omega) \in \mathcal{G}_{\text{SNS}}$ with

$$\varphi(t_1 + h_j + 0) \to \varphi(t_1) \quad \text{when } h_j \to 0\,,$$

using (S4) we can deduce the existence of a subsequence and a pair $(\psi, \theta_{t_1}\omega) \in \mathcal{G}_{\text{SNS}}$ with $\psi(0) = \varphi(t_1)$ and

$$\varphi(t_1 + h_\mu + t) \to \psi(t) \quad \text{for all } t \geq 0$$

when $\mu \to \infty$.

In particular, $\psi(t) = \varphi(t_1 + t)$ for almost every $t \in (0, a + 2\delta/3 - t_1)$. Using (S3) we can guarantee that $(\phi, \theta_{t_1}\omega) \in \mathcal{G}_{\text{SNS}}$, where

$$\phi(t) = \begin{cases} \varphi(t + t_1) & 0 \leq t \leq t_2 - t_1\,, \\ \psi(t) & t > t_2 - t_1\,. \end{cases}$$

Since the weak solutions of the Navier–Stokes equations are weakly continuous, the semi-flow has the property (in Ball's terminology) of unique representatives, namely that since $(\varphi(t_1 + \cdot), \theta_{t_1}\omega)$ and $(\phi(\cdot), \theta_{t_1}\omega)$ coincide for almost every $t \in (0, \infty)$ we must have $\phi(t) = \varphi(t_1 + t)$ for all $t \geq 0$. In particular, we deduce that $\varphi(t_0 + h_\mu)$ tends to $\phi(t_0 - t_1) = \varphi(t_0)$ as $\mu \to \infty$, a contradiction.

(ii) implies (iii). Since each solution u is continuous from $(0, \infty) \to H$, we only have to prove the continuity at $t = 0$. But it is straightforward given (3.5), taking $t_0 = 0$, to check that given any sequence $t_j \to 0^+$:

$$|u(0) - z_\alpha(0, \omega)| \leq \liminf |u(t_j) - z_\alpha(t_j, \omega)|$$

$$\leq \limsup |u(t_j) - z_\alpha(t_j, \omega)| \leq |u(0) - z_\alpha(0, \omega)|\,.$$

From this, the continuity of z_α and the weak convergence of $u(t)$ to $u(0)$ (following from the weak continuity of solutions), we deduce (iii).

(iii) implies (i). Condition (S1) is ensured by Proposition 1 and (S3) is easy to obtain; without (ii) or (iii) the translation property (S2) needs not hold, due to violation of the energy inequality (cf. Ball [5]) although Eq. (3.6) is satisfied by translates (cf. Lemma 5.1 in Flandoli and Schmalfuß [16]). However, assuming (iii), each weak solution associated to (an appropriate) ω is continuous from $[0, \infty) \to H$;

hence $V_i(u,\omega)(t)$ $(i = 1, 2)$ are continuous for all $t \geq 0$ and therefore non-increasing, from which (S2) follows.

So, we concentrate on proving (S4*): take a sequence $(u_n, \omega_n) \in \mathcal{G}_{\mathrm{SNS}}$ such that $u_n(0) \to z$ in H and $W_t(\omega_n) \to W_t(\omega)$ uniformly on compact subintervals of $\mathbb{R}$. We have to prove that there exists a subsequence (u_μ, ω_μ) and $(u,\omega) \in \mathcal{G}_{\mathrm{SNS}}$ with $u(0) = z$ and $u_\mu(t) \to u(t)$ for all $t \geq 0$. (We will deal with a fixed time interval $[0, T]$ and the result for $[0, \infty)$ will follow by a diagonal argument.)

Since we are concerned with solutions of the equation corresponding to various different realisations of the noise (ω_n), in order to apply Definition 1 we have to use various different processes z_α. Since these are given by

$$z_\alpha(t,\omega) = W_t(\omega) - \int_{-\infty}^{t} (A+\alpha)e^{(t-s)(-A-\alpha)}W_s(\omega)ds$$

(this was (3.3)) it follows that $z_\alpha(t, \omega_n)$ converges uniformly on compact time intervals to $z_\alpha(t,\omega)$ as $n \to \infty$. Using this observation we can derive estimates for $u_n(t) - z_\alpha(t;\omega_n)$ that are uniform in n. By (3.4) we have

$$\begin{aligned}|u_n(t) - z_\alpha(t;\omega_n)|^2 \leq{}& e^{\int_0^t(-\lambda_1 + 2C^*|z_\alpha(s;\omega_n)|^8_{L^4})ds}|u_n(0) - z_\alpha(0;\omega_n)|^2 \\ &+ \int_0^t e^{\int_\sigma^t(-\lambda_1+2C^*|z_\alpha(s;\omega_n)|^8_{L^4})ds} \\ &\times 4\left[C_B^2|z_\alpha(\sigma;\omega_n)|^4_{L^4} + \frac{\alpha^2}{\lambda_1}|z_\alpha(\sigma;\omega_n)|^2 + \|f\|^2_{V'}\right]d\sigma\,.\end{aligned}$$

Thus for some constant $C_T(\alpha;\omega) > 0$

$$|u_n(\cdot) - z_\alpha(\cdot;\omega_n)|_{L^\infty(0,T;H)} \leq C_T(\alpha;\omega) \quad \text{for all } n\,. \tag{4.5}$$

On the other hand, from (3.5) we obtain

$$\begin{aligned}&|u_n(t) - z_\alpha(t;\omega_n)|^2 + \int_0^t \|u_n(s) - z_\alpha(s;\omega_n)\|^2 ds \\ &\leq |u_n(0) - z_\alpha(0;\omega_n)|^2 + \int_0^t \Bigg[2C^*|u_n(\sigma) - z_\alpha(\sigma;\omega_n)|^2|z_\alpha(\sigma;\omega_n)|^8_{L^4} \\ &\quad + 4C_B^2|z_\alpha(\sigma;\omega_n)|^4_{L^4} + \frac{4\alpha^2}{\lambda_1}|z_\alpha(\sigma;\omega_n)|^2 + 4\|f\|^2_{V'}\Bigg]d\sigma\,,\end{aligned}$$

from whence, using (4.5), we obtain (adjusting the definition of C_T if necessary)

$$\|u_n(\cdot) - z_\alpha(\cdot;\omega_n)\|_{L^2(0,T;V)} \leq C_T(\alpha;\omega) \quad \text{for all } n\,.$$

Now it is standard to deduce that

$$\left\|\frac{d}{dt}[u_n(\cdot) - z_\alpha(\cdot;\omega_n)]\right\|_{L^{4/3}(0,T;V')} \leq C_T(\alpha;\omega) \quad \text{for all } n$$

(once more changing C_T suitably).

Applying well-known compactness results (see Constantin and Foias [9], for example), we can ensure the existence of a subsequence (which we do not relabel) and a function $v \in C([0,T];H_w) \cap L^2(0,T;V)$ such that, setting $u = v + z_\alpha$,

$$\begin{aligned} u_n(t) - z_\alpha(t;\omega_n) &\rightharpoonup u(t) - z_\alpha(t;\omega) \quad \text{in } H\,, \quad \forall\, t \in [0,T]\,, \\ u_n(t) - z_\alpha(t;\omega_n) &\rightharpoonup u(t) - z_\alpha(t,\omega) \quad \text{in } L^2(0,T;V)\,, \\ \frac{d}{dt}[u_n(\cdot) - z_\alpha(\cdot;\omega_n)] &\rightharpoonup \frac{d}{dt}[u(\cdot) - z_\alpha(\cdot;\omega)] \quad \text{in } L^{4/3}(0,T;V')\,, \\ u_n &\to u \quad \text{in } L^2(0,T;H)\,, \\ u_n(t) &\to u(t) \quad \text{in } H \quad \text{a.e. } t \in (0,T)\,. \end{aligned} \tag{4.6}$$

Because of (3.2) and convergence in the above senses, it is easy to check that $u(t)$ satisfies (3.6) with initial data $z = \lim_{\text{weak}} u_n(0)$. Passing to the limit, inequality (3.4) is also straightforward. Following Ball, writing (3.5) for each term in the subsequence and then taking limits on the left using the weak lower semicontinuity and on the right using (4.6) we obtain (3.5) for $u(t) - z_\alpha(t;\omega)$. Thus u is a weak solution corresponding to noise ω, and so by assumption u is continuous into H; therefore $V_2(u_n,\omega_n)(\cdot)$ and $V_2(u,\omega)(\cdot)$ are also continuous and, since each function is decreasing and

$$V_2(u_n,\omega_n)(\cdot) \to V_2(u,\omega)(\cdot) \quad \text{a.e. } t > 0\,,$$

we may ensure that the above convergence occurs for every t. This implies that $|u_n(t)| \to |u(t)|$, which along with the weak convergence of $u_n(t)$ to $u(t)$ gives us the required strong convergence of u_n to u in H for every $t > 0$ and (S4) holds. □

5. Attractors for Generalised Stochastic Semiflows

Ball [5] proves the existence of a global attractor for a generalised semiflow if and only if the semiflow is pointwise dissipative [there is a bounded set B_0 such that for any $\varphi \in G$, $\varphi(t) \in B_0$ for all sufficiently large t] and asymptotically compact [for any sequence $\varphi_j \in G$ with $\varphi_j(0)$ bounded, and for any sequence $t_j \to \infty$, the sequence $\varphi_j(t_j)$ has a convergent subsequence]. We prove a similar result here for a compactifying SGS, but the details are different since we also have to take into account the random element.

Following the result for single-valued RDS of Crauel [12] (see Theorem 1, above) we will give a necessary and sufficient condition for the existence of an attractor for our generalised stochastic semiflow [the precise definition of an "attractor" in this case is given just before Theorem 2, below], namely the existence of a compact attracting set, i.e. a set $K(\omega)$ such that if D is a deterministic bounded set,

$$\text{dist}(\Phi(t,\theta_{-t}\omega)D, K(\omega)) \to 0 \quad \text{as } t \to \infty\,,$$

where Φ is the generalised cocycle defined in (4.1).

The existence of such a compact attracting set (note that we do not require to be measurable) implies (and motivates) the following concept: a generalised semiflow is said to be *asymptotically compact* if $\mathbb{P}$-a.s., for every bounded set D and sequences $t_n \to \infty$ and $x_n \in \Phi(t_n, \theta_{-t_n}\omega)D$, there exists a subsequence in $\{x_n\}$ which converges.

As a first step define the Ω-limit set of a fixed deterministic set D by

$$\Omega_D(\omega) = \bigcap_{T>0} \overline{\bigcup_{t>T} \Phi(t, \theta_{-t}\omega)D}\,.$$

We now prove some basic properties of these sets.

Lemma 1. *Let $\mathcal{G}$ be an asymptotically compact SGS. For any non-empty closed bounded deterministic set D, $\mathbb{P}$-a.s. $\Omega_D(\omega)$ is non-void, compact and the minimal closed set that attracts D*:

$$\lim_{t\to\infty} \operatorname{dist}(\Phi(t, \theta_{-t}\omega)D, \Omega_D(\omega)) = 0\,. \tag{5.1}$$

Moreover, it is negatively invariant, that is

$$\Omega_D(\theta_t\omega) \subseteq \Phi(t, \omega)\Omega_D(\omega) \quad \textit{for all } t \geq 0\,.$$

If D is also compact, then $\Omega_D(\omega)$ is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{F}$. The same result holds for weakly compact sets D provided that $\mathcal{G}$ is also a compactifying SGS.

Proof. Let us first check that $\Omega_D(\omega)$ is non-void. Consider any element $d \in D$ and a sequence $\{t_n\}$ with $t_n \to \infty$. By (S1) there exist solutions $(\varphi_n, \theta_{-t_n}\omega) \in \mathcal{G}$ such that $\varphi_n(0) = d$; the asymptotic compactness implies that there is a subsequence $\{\varphi_\mu(t_\mu)\}$ that converges to an element $z \in \Omega_D(\omega)$, and hence $\Omega_D(\omega)$ is non-void. The set $\Omega_D(\omega)$ is obviously closed; we will see, by a Cantor diagonalization argument, that it is also compact: given a sequence $\{y_n\} \subset \Omega_D(\omega)$, there exist sequences $t_n \to \infty$ and pairs $(\varphi_n, \theta_{-t_n}\omega) \subset \mathcal{G}$ with $\{\varphi_n(0)\} \subset D$ and $d(\varphi_n(t_n), y_n) < 1/n$. Using the asymptotic compactness again there exists a subsequence $\varphi_\mu(t_\mu) \to z$ and so $y_\mu \to z \in \Omega_D(\omega)$.

We omit the proof of the attraction property of $\Omega_D(\omega)$ (by contradiction) and of its minimality, since these follow closely the equivalent arguments from the single-valued case.

First we prove the negative invariance of Ω_D. To this end, consider $y \in \Omega_D(\theta_t\omega)$. We have to check that $y \in \Phi(t,\omega)\Omega_D(\omega)$. Since $y \in \Omega_D(\theta_t\omega)$ there exist a sequence $t_n \to \infty$ and a sequence of pairs $\{(\varphi_n, \theta_{-t_n+t}\omega)\} \subset \mathcal{G}$ with $\varphi_n(0) \in D$ such that $y_n = \varphi_n(t_n)$ converges to y. Now, take $n \geq n_0$ such that $t_n \geq t$ for all $n \geq n_0$. Observe that $x_n = \varphi_n(t_n - t) \in \Phi(t_n - t, \theta_{-(t_n-t)}\omega)D$ and, using the cocycle property from Proposition 2, that $y_n \in \Phi(t,\omega)x_n$. Using the asymptotic compactness property there is a subsequence (which we do not relabel) $x_n \to x \in \Omega_D(\omega)$. On the other hand, $y_n = \varphi_n^{t_n-t}(t)$ with $\varphi_n^{t_n-t}(0) = x_n$. By (S4) there exist another subsequence (which we do not relabel) and a pair $(\varphi, \omega) \in \mathcal{G}$

with $\varphi(0) = x$ and $\varphi_n^{t_n - t}(s) \to \varphi(s)$ for all $s \geq 0$. In particular, with $s = t$ we see that $\varphi_n^{t_n-t}(t) = y_n \to \varphi(t)$, and so $y = \varphi(t) \in \Phi(t,\omega)x \subset \Phi(t,\omega)\Omega_D(\omega)$.

We now prove that $\Omega_D(\omega)$ is measurable if D is compact. As a first step, observe that for every such set D the map $\Phi(\cdot,\cdot)D : (0,\infty) \times \Omega \mapsto \mathcal{K}(H)$ (the compact subsets of H) is measurable w.r.t. the completion of $\mathcal{B}(0,+\infty)\otimes\mathcal{F}$ using the product measure of the measure on Borel subsets of $(0,\infty)$ and $\mathbb{P}$.

Indeed, by Theorem A.2 in Bensoussan and Temam [6] (see also Theorem III.30 in Castaing and Valadier [8], Theorem 8.1.4 in Aubin and Frankowska [2], or Proposition 2.4 in Crauel [10]) it is enough to check that $\Phi(\cdot,\cdot)D$ has closed graph, i.e. given $(\varphi_n, \omega_n) \in \mathcal{G}$ such that $\varphi_n(0) \in D$, $\omega_n \to \omega$, $t_n \to t > 0$ and $\varphi_n(t_n) \to y$, we must have $y \in \Phi(t,\omega)D$.

Using the compactness of D there is a subsequence (which we relabel) with $\varphi_n(0) \to z \in D$. It follows using (S4) that for a further sequence (which again we do not relabel) there exists a pair $(\varphi,\omega) \in \mathcal{G}$ such that $\varphi_n(t) \to \varphi(t)$ for all $t > 0$. Theorem 2.2 in Ball [5] now shows that this convergence is in fact uniform on compact subintervals of $(0,\infty)$, from whence $\varphi_n(t_n) \to \varphi(t) \in \Phi(t,\omega)D$, proving the required measurability of Φ.

From this observation the measurability of $\Omega_D(\omega)$ for compact D is a consequence of the Projection theorem (cf. Castaing and Valadier [8], Theorem III.23; see also [7, 14]).

The same argument applies unchanged if D is weakly compact and $\mathcal{G}$ is a KSGS, since then the weak convergence of $\varphi_n(0) \rightharpoonup z \in D$ is sufficient, using (S4*), to obtain a pair $(\varphi,\omega) \in \mathcal{G}$ such that $\varphi_n(t) \to \varphi(t)$ for all $t > 0$. □

By taking the union of all possible Ω-limit sets,

$$A(\omega) = \overline{\bigcup_{D \text{ bounded}} \Omega_D(\omega)}\,, \tag{5.2}$$

we obtain, as proved below, the minimal invariant compact random set that attracts all bounded sets (in the sense of (5.1)). We term this set the *global attractor.*

Theorem 2. *A compactifying generalised stochastic semiflow has a global attractor $A(\omega)$ if and only if it has a compact attracting set $K(\omega)$. In this case $A(\omega)$ is given by* (5.2).

Proof. The condition is clearly necessary. We now prove that the condition is sufficient, noting that the existence of a compact attracting set implies that the SGS is asymptotically compact. Since $A(\omega)$ is closed, compactness follows since it must be a subset of the compact attracting set $K(\omega)$.

The negative invariance of $A(\omega)$ is similarly straightforward, given the results of Lemma 1:

$$\begin{aligned} A(\theta_t\omega) = \overline{\bigcup_D \Omega_D(\theta_t\omega)} &\subset \overline{\Phi(t,\omega)\bigcup_D \Omega_D(\omega)} \\ &\subset \overline{\Phi(t,\omega)\overline{\bigcup_D \Omega_D(\omega)}} = \Phi(t,\omega)A(\omega)\,; \end{aligned}$$

the first inclusion follows from the negative invariance of $\Omega_D(\omega)$, while the final equality is valid since $\Phi(t,\omega)$ is upper semicontinuous (Proposition 2) and the upper semicontinuous image of a compact set is once again compact, and therefore closed.

Taking the countable sequence of weakly compact sets $B_n = \overline{B_H(0,n)}$, we can write

$$A(\omega) = \overline{\bigcup_n \Omega_{B_n}(\omega)}\,,$$

and so A is clearly measurable since each of the $\Omega_{B_n}(\omega)$ is measurable because $\mathcal{G}$ is a KSGS.

A straightforward proof of the positive invariance of $A(\omega)$ is not possible without the strong assumption of lower semicontinuity[a] of the solutions with respect to their initial data. Instead we borrow the following nice argument due to Crauel [10, 11] and extended by Caraballo *et al.* [7]. This proves the positive invariance using the negative invariance, maximality, and measurability of A.

Consider a fixed $t \in \mathbb{R}_+$. We will prove that the set $A'(\omega) = \Phi(t,\theta_{-t}\omega)A(\theta_{-t}\omega)$ is a negatively invariant random compact set, and consequently, $A'(\omega) \subset A(\omega)$, from which the result follows.

To prove that $A'(\theta_s\omega) \subset \Phi(s,\omega)A'(\omega)$ for $s \in \mathbb{R}_+$, by the definition of A', it suffices to show that

$$\Phi(t,\theta_{s-t}\omega)A(\theta_{s-t}\omega) \subset \Phi(s,\omega)\Phi(t,\theta_{-t}\omega)A(\theta_{-t}\omega)\,. \tag{5.3}$$

Using the negative invariance of $A(\omega)$, we have $A(\theta_{s-t}\omega) \subset \Phi(s,\theta_{-t}\omega)A(\theta_{-t}\omega)$. Applying $\Phi(t,\theta_{s-t}\omega)$ to this last inclusion, and using the cocycle property (4.3) of Φ (Proposition 2), inclusion (5.3) follows.

Since $\mathbb{P}$-a.s., $A(\theta_{-t}\omega)$ is a compact set, and $\Phi(t,\theta_{-t}\omega)$ is u.s.c. and has compact values, $A'(\omega)$ is also compact.

For the measurability of $A'(\omega)$ (again with respect to $\mathcal{F}^0$, the $\mathbb{P}$-completion of $\mathcal{F}$) we suppose that $t > 0$ (if $t = 0$, it is trivial since $A'(\omega) = A(\omega)$ is measurable). By Theorem 8.1.4 (Characterisation theorem) in Aubin and Frankowska [2], we have to check that for every closed set $C \subset H$, the set $M = \{\omega | A'(\omega) \cap C \neq \emptyset\}$ belongs to $\mathcal{F}^0$. It is easy to see that $M = \theta_{-t}\{\omega | (\omega, A(\omega)) \cap M_1 \neq \emptyset\}$, where $M_1 = \{(\omega,x) | \Phi(t,\omega)x \cap C \neq \emptyset\}$.

Since $\Phi(t,\cdot)(\cdot)$ is u.s.c. [this follows from (S4)], it is immediate that M_1 is closed in $\Omega \times H$ [here make the most of the topological structure we have imposed upon Ω]. On the other hand, the map $\omega \mapsto (\omega, A(\omega))$ is measurable, because it is the composition of the Carathéodory identity map on $\Omega \times H$, and the closed measurable map $\omega \mapsto A(\omega)$. Thus $M \in \mathcal{F}^0$ as desired. □

We note here that, although for technical reasons (in order to avoid having to deal with the universal sigma algebra), we have used the $\mathbb{P}$-completion of $\mathcal{F}$, it is

[a] A multivalued map F is said to be lower semi-continuous in x if for every $y \in F(x)$ and every sequence $x_n \to x$, there exist values $y_n \in F(x_n)$ with $y_n \to y$.

possible (cf. Lemma 2.7 in Crauel [10]) to obtain a random compact set measurable w.r.t. $\mathcal{F}$ which is equal to $A(\omega)$ $\mathbb{P}$-a.s.

It is interesting that the following result (cf. Crauel [11]) also holds in this multivalued case.

Proposition 4. *For a KSGS with a global attractor, the maximal invariant set that attracts all compact sets is also well defined, and is equal to $A(\omega)$ $\mathbb{P}$-a.s.*

Proof. Since $A(\omega)$ is a random compact set, it is possible to choose a non-random compact set which contains $A(\omega)$ with a probability as high as wished (cf. Proposition 2.15 in Crauel [10]). Denote by K_n a deterministic compact set such that $\mathbb{P}(\{\omega \in \Omega | A(\omega) \subset K_n\}) > 1 - 1/n$. By Theorem 2 and an adaptation of the Recurrence theorem of Poincaré for negatively invariant compact random sets in the multivalued case (cf. Caraballo *et al.* [7], Theorem 3), we deduce that $\mathbb{P}(\{\omega \in \Omega | A(\omega) \subset \Omega_{K_n}(\omega)\}) > 1 - 1/n$, and therefore $A(\omega) = \bigcup_n \Omega_{K_n}(\omega)$, whence follows the maximality of $A(\omega)$ among every negatively invariant random compact set, and we are done. □

6. Dissipativity and a Global Attractor for the 3D Stochastic NSE

For our particular example we can prove a stronger compactness condition than the existence of a compact attracting set; instead we prove the existence of a compact absorbing set, i.e. a compact set $K(\omega)$ such that $\mathbb{P}$-a.s. for every bounded deterministic set D there is a time $T_D(\omega)$ such that

$$\Phi(t, \theta_{-t}\omega)D \subset K(\omega)\,, \quad t \geq T_D(\omega)\,.$$

Firstly, following Crauel and Flandoli [15], we prove the same result but with $K(\omega)$ replaced by a bounded set $B(\omega)$. Taking $u(0) \in D$, and setting $t_0 = 0$ in (3.4) we obtain

$$\begin{aligned} |u(t) - z_\alpha(t;\theta_{-t}\omega)|^2 &\leq e^{\int_0^t (-\lambda_1 + 2C^*|z_\alpha(s;\theta_{-t}\omega)|_{L^4}^8)ds}|u(0) - z_\alpha(0;\theta_{-t}\omega)|^2 \\ &\quad + 4\int_0^t e^{\int_\sigma^t (-\lambda_1 + 2C^*|z_\alpha(s;\theta_{-t}\omega)|_{L^4}^8)ds}\bigg[C_B^2|z_\alpha(\sigma;\theta_{-t}\omega)|_{L^4}^4 \\ &\quad + \frac{\alpha^2}{\lambda_1}|z_\alpha(\sigma;\theta_{-t}\omega)|^2 + \|f\|_{V'}^2\bigg]d\sigma\,. \end{aligned} \tag{6.1}$$

We follow the idea given in step 5 in Crauel and Flandoli [15] among others. A simple ergodic argument (see Crauel and Flandoli [15] or Lemma 7.2 in Flandoli and Schmalfuß [16]) guarantees that

$$\liminf_{\alpha\to\infty} \limsup_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau |z_\alpha(s;\theta_{-\tau}\omega)|_{L^4}^8 ds = 0\,, \tag{6.2}$$

and hence for α sufficiently large there exists a $t_0(\omega)$ such that for all $t \geq t_0$

$$2C^* \frac{1}{t} \int_0^t |z_\alpha(s;\theta_{-t}\omega)|_{L^4}^8 ds < \frac{\lambda_1}{2}\,.$$

In particular, for almost every ω we have

$$\int_0^t (-\lambda_1 + 2C^*|z_\alpha(s;\theta_{-t}\omega)|_{L^4}^8)ds \leq -\frac{\lambda_1 t}{2} \quad \text{for } t \geq t_0(\omega)\,. \tag{6.3}$$

A bound for the first term on the right-hand side of (6.1) holds since $|u(0)|$ is bounded by assumption, and

$$z_\alpha(0,\theta_{-t}\omega) = z_\alpha(-t,\omega)$$

has polynomial growth (by Lemma 3.6(ii), p. 377 in Flandoli and Schmalfuß [16] or (10) in Crauel and Flandoli [15]), i.e.

$$\lim_{t\to-\infty} \frac{|z_\alpha(t,\omega)|_{L^4}^j}{|t|^j} = 0 \quad \text{for } j \in \mathbb{N}\,.$$

The second term on the right-hand side of (6.1) does not depend on D, and we need an estimate that is valid for all $t \geq 0$. In order to do that, we transform it by two changes of variables, $\sigma - t = s$ and $r - t = \rho$, as follows:

$$\int_0^t e^{\int_\sigma^t (-\lambda_1 + 2C^*|z_\alpha(s;\theta_{-t}\omega)|_{L^4}^8)ds} \left[C_B^2|z_\alpha(\sigma;\theta_{-t}\omega)|_{L^4}^4 + \frac{\alpha^2}{\lambda_1}|z_\alpha(\sigma;\theta_{-t}\omega)|^2 + \|f\|_{V'}^2\right]d\sigma$$

$$= \int_{-t}^0 e^{\int_s^0 (-\lambda_1 + 2C^*|z_\alpha(\rho;\omega)|_{L^4}^8)d\rho} \left[C_B^2|z_\alpha(s;\omega)|_{L^4}^4 + \frac{\alpha^2}{\lambda_1}|z_\alpha(s;\omega)|^2 + \|f\|_{V'}^2\right]ds\,.$$

By continuity, polynomial growth and ergodic argument (6.2), (6.3) (with another change of variables) for the process z_α, we can proceed and finish the proof exactly as in Crauel and Flandoli [15].

Thus, it follows that given $u(0)$ in a bounded set, there exists a random variable $r(\omega)$ and a time $t_1(\omega, |u(0)|)$ such that

$$|u(t)| \leq r(\omega) \quad \text{for all } t \geq t_1(\omega, |u(0)|)\,.$$

Write $B(\omega)$ for the ball of radius $r(\omega)$ in H.

Now we recall (cf. Ball [5]), as observed at the end of Lemma 1, that the proof of Proposition 3 also shows that $\mathcal{G}_{\text{SNS}}$ is compact, i.e. given ω, a sequence of solutions $(\varphi_n, \omega) \in \mathcal{G}_{\text{SNS}}$ with $\{\varphi_n(0)\}$ bounded has a subsequence that converges for all $t > 0$. In particular this means that $\Phi(1,\omega)$ is compact, $\mathbb{P}$-a.s., and so

$$K(\omega) = \Phi(1,\theta_{-1}\omega)B(\theta_{-1}\omega)$$

is a compact set. Due to its definition, $K(\omega)$ is absorbing, since

$$\Phi(t+1,\theta_{-1-t}\omega)D = \Phi(1,\theta_{-1}\omega)\Phi(t,\theta_{-1-t}\omega)D$$

and so for all $t \geq t_1(\theta_{-1}\omega, D)$

$$\Phi(t+1, \theta_{-1-t}\omega)D \subseteq \Phi(1, \theta_{-1}\omega)B(\theta_{-1}\omega) = K(\omega)\,.$$

Application of Theorem 2 now yields a global attractor for the 3D stochastic NSE:

Theorem 3. *If weak solutions of the stochastic 3D Navier–Stokes equations are continuous from* $(0, \infty)$ *into* H, *then the equations define a compactifying generalised stochastic semiflow which has a global attractor.*

7. Conclusion

We have extended the idea of a generalised semiflow due to Ball [5] to treat stochastic systems, and shown that such a generalised semiflow has a global attractor if and only if it has a compact attracting set.

These abstract ideas have been applied to the stochastic 3D Navier–Stokes equations, for which we have shown that, as in the deterministic case, continuity of solutions into the natural phase space H implies the existence of a global attractor.

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