Thresholds for breather solutions of the Discrete Nonlinear Schrödinger Equation with saturable and power nonlinearity*

J. Cuevas,
Departamento de Fisica Aplicada I, Escuela Universitaria Politénica,
C/ Virgen de Africa, 7, University of Sevilla,
41011 Sevilla, Spain

J. C. Eilbeck,
Maxwell Institute and Department of Mathematics,
Heriot-Watt University,
Edinburgh EH14 4AS, Scotland

N. I. Karachalios,
Department of Mathematics, University of the Aegean,
Karlovassi, 83200 Samos, Greece

Abstract

We consider the question of existence of periodic solutions (called breather solutions or discrete solitons) for the Discrete Nonlinear Schrödinger Equation with saturable and power nonlinearity. Theoretical and numerical results are proved concerning the existence and nonexistence of periodic solutions by a variational approach and a fixed point argument. In the variational approach we are restricted to DNLS lattices with Dirichlet boundary conditions. It is proved that there exists parameters (frequency or nonlinearity parameters) for which the corresponding minimizers satisfy explicit upper and lower bounds on the power. The numerical studies performed indicate that these bounds behave as thresholds for the existence of periodic solutions. The fixed point method considers the case of infinite lattices. Through this method, the existence of a threshold is proved in the case of saturable nonlinearity and an explicit theoretical estimate which is independent on the dimension is given. The numerical studies, testing the efficiency of the bounds derived by both methods, demonstrate that these thresholds are quite sharp estimates of a threshold value on the power needed for the the existence of a breather solution. This it justified by the consideration of limiting cases with respect to the size of the nonlinearity parameters and nonlinearity exponents.

1 Introduction

This work concerns the Discrete Nonlinear Schrödinger Equation (DNLS)

$$\mathrm{i}\dot{\psi}_n + \epsilon(\Delta_d\psi)_n - \beta F(|\psi_n|^2)\psi_n = 0, \quad \beta \in \mathbb{R}, \tag{1.1}$$

on a finite lattice supplemented with Dirichlet boundary conditions, and on infinite lattices ($n = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N$). We concentrate on two examples of nonlinearities,

$$F(|z|^2) = \frac{1}{1+|z|^2} \quad \text{and} \quad F(|z|^2) = |z|^{2\sigma}, \tag{1.2}$$

the *saturable* and *power nonlinearity* respectively.

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Note that we use the word *power* in two different senses in this paper, in *power nonlinearity* as above, and for a conserved quantity of the system (1.1), defined as

$$\mathcal{P}[\phi] = \sum_n |\phi_n|^2. \tag{1.3}$$

We present some theoretical and numerical results related to the existence of time periodic solutions, having the form

$$\psi_n(t) = e^{-\mathrm{i}\Omega t}\phi_n, \quad \Omega \in \mathbb{R}. \tag{1.4}$$

Substitution of the expression (1.4) into (1.1) with the nonlinearities (1.2), shows that $\phi = \{\phi_n\}_{n\in\mathbb{Z}^N}$, satisfies the system of algebraic equations

$$\Omega\phi_n = -\epsilon(\Delta_d\phi)_n + \beta F(|\phi_n|^2)\phi_n, \quad \beta \in \mathbb{R}. \tag{1.5}$$

Solutions given by the expression (1.4), are called stationary wave solutions. Localized solutions fulfilling $|\phi_n| \to 0$ as $|n| \to \infty$, are known as discrete solitons or breathers[1]. The problem of existence and properties of nonlinear localized modes in DNLS lattices, has attracted considerable research interest [5, 11]. For recent studies on the saturable DNLS or its cubic-quintic approximation, we refer to [3, 8, 19, 14, 1, 15, 20]. In these references, as well as in [4, 16] for the continuum models, remarkable properties and differences between models with power nonlinearities are reported. Although the case of the fundamental localized solutions assumes that ϕ_n is real [20], the results that we present here consider the existence and nonexistence of nontrivial breather solutions where ϕ_n is in general complex. The existence of nontrivial breather solutions for DNLS (1.1) will be established by variational methods. More precisely, we apply direct variational methods [2] to appropriate constrained minimization problems. This approach has been used to the focusing N-dimensional DNLS equation with a power nonlinearity

$$\mathrm{i}\dot{\psi}_n + \epsilon(\Delta_d\psi)_n + \beta|\psi_n|^{2\sigma}\psi_n = 0, \quad \beta > 0, \tag{1.6}$$

in infinite lattices [21]. The results of [21] not only establish the existence of nontrivial breather solutions, but in addition the existence of a global minimum – an excitation threshold – in one of the fundamental conserved energy quantities, the power (or norm). This minimum requires the nonlinearity exponent to be greater than or equal to a certain critical value, depending on the lattice dimension. More precisely, it is proved in [21, Theorem 3.1, pg. 678], that if $0 < \sigma < \frac{2}{N}$, spatially localized solutions (1.4) with $\Omega < 0$ of arbitrary small power exist, while if $\sigma \geq \frac{2}{N}$, there exists a ground state excitation threshold $\mathcal{P}_{\mathrm{thresh}}$. The result of [21], resolved the conjecture for ground state breathers of [6]. The second conserved energy quantity associated with (1.6), is the Hamiltonian

$$\mathcal{H}_\sigma[\phi] = \epsilon(-\Delta_d\phi, \phi)_2 - \frac{\beta}{\sigma+1}\sum_{n\in\mathbb{Z}^N}|\phi_n|^{2\sigma+2}, \quad \beta > 0.$$

A *ground state* is a minimizer of the variational problem

$$\inf\left\{\mathcal{H}_\sigma[\phi] \quad : \mathcal{P}[\phi] = R^2\right\},$$

where $(\cdot,\cdot)_2$ stands for the ℓ^2-scalar product. The existence of the excitation threshold was proved with the help of a delicate discrete interpolation inequality similar to the Gagliardo-Nirenberg inequality of the continuous case. It is proved in [21] that the inequality

$$\sum_{n\in\mathbb{Z}^N}|\phi_n|^{2\sigma+2} \leq C\left(\sum_{n\in\mathbb{Z}^N}|\phi_n|^2\right)^\sigma(-\Delta_d\phi, \phi)_2, \tag{1.7}$$

holds for $\sigma \geq \frac{2}{N}$. The excitation threshold is related to the best constant of (1.7). The ground state solution has frequency ω^* and power $\mathcal{P}_{\mathrm{thresh}}$.

After the preliminary results of Section 2, in Section 3 we consider the DNLS equation with *saturable nonlinearity.* Following the results mentioned above, for the DNLS equation with power nonlinearity (1.6), by

[1]We wish to reserve the term "breathers" throughout the text, for localized solutions of the form (1.4), although this term is not strictly valid in the case of a finite lattice: this is due to the fact that the lower bounds for the power of solutions (1.4) derived in this work in the case of finite lattice, are also valid for breather solutions.

the application of the variational approach to the saturable DNLS, we derive both the existence of nontrivial breather solutions, as well some bounds on the power of the minimizers. The variational study considers the saturable DNLS, *supplemented with Dirichlet boundary conditions*. Although this is a simpler case in comparison with the infinite lattice (where one has to deal with the lack of compactness [21]), this case is of importance especially for numerical simulations. Since the infinite lattice cannot be modelled numerically, numerical investigations normally consider finite lattices with Dirichlet or periodic boundary conditions. We note that the choice of boundary conditions only matters, if a localized pulse is moving and collides with a boundary.

In our study of the DNLS equation with saturable nonlinearity, we distinguish between defocusing ($\beta < 0$) and focusing ($\beta > 0$) nonlinearity. For the defocusing case (Section 3.1), we consider two variants of minimization problems: seeking for nontrivial breather solutions $\phi_n(t) = e^{\mathrm{i}\omega t}\phi_n$, of prescribed frequency $\omega > 0$, in the first variant we consider a minimization problem for the energy functional

$$\mathcal{E}_\omega[\phi] := \epsilon(-\Delta_d\phi, \phi)_2 + \omega \sum_{|n|\leq K} |\phi_n|^2,$$

subject to a constraint on the logarithmic part of the saturable Hamiltonian of the DNLS (1.1), defined by

$$\mathcal{H}[\phi] = \epsilon(-\Delta_d\phi, \phi)_2 + \beta \sum_{|n|\leq K} \log(1 + |\phi_n|^2).$$

This proves the existence of a nontrivial minimizer $\hat{\phi}$ of $\mathcal{E}_\omega$ (linear energy) and the existence of $\beta < 0$ as a Lagrange multiplier, satisfying $-\beta > \omega > 0$ such that $\psi_n(t) = e^{\mathrm{i}\omega t}\hat{\phi}_n$ is a solution of the saturable DNLS (1.1) with this β as a nonlinearity parameter. We note that, *in contrast* to the DNLS with power nonlinearity (1.6), the frequency of the breather is limited by the condition $\Lambda := -\beta > \omega$, due to the resonance with linear modes. Also in contrast with the power nonlinearity case, the parameter β cannot be scaled out. Due to this fact, the result is of interest, justifying the minimization of the linear energy, and the existence of a parameter β for which this minimum is attained.

The second variant for the defocusing case considers, for given $\beta = -\Lambda < 0$, the constrained minimization problem for the Hamiltonian

$$\inf\left\{\mathcal{H}[\phi] \quad : \mathcal{P}[\phi] = R^2\right\},$$

that is, we study the existence of the nontrivial breather solution as a ground state. This approach proves the existence of a nontrivial minimizer ϕ^*, at least in the parameter regime $\Lambda > 2\epsilon N$, and the existence of a frequency $\omega > 0$ (as a Lagrange multiplier), satisfying $\Lambda > \omega$, such that $\psi_n(t) = e^{\mathrm{i}\omega t}\phi_n^*$ is a stationary wave solution. Moreover, it is proved in this parameter regime that there exist frequencies, such that the corresponding nontrivial breather solution satisfy an upper bound for the total power, depending on the parameters Λ, ϵ, N. The first numerical study performed on this parameter regime, for the behaviour of breather power [2], justifies the existence of a range of frequencies, for which the upper bound of the power of the corresponding breather solution is satisfied.

Section 3.2, is devoted to the focusing saturable nonlinearity $\beta > 0$. For this case, the existence of a nontrivial breather solution $\psi_n(t) = e^{-i\Omega t}\tilde{\phi}_n$, is proved similarly to Section 3.1, by considering the constrained minimization problem for the Hamiltonian (a ground state). Through the application of the variational method, a simple relation involving the frequency Ω, β and the power $\mathcal{P}[\tilde{\phi}] = R^2$ is derived, which in terms of $\mathcal{P}$, provides a local lower bound on the power of the minimizer. The result actually states that there exists $\Omega > 0$ satisfying this lower bound. We would like at this point to distinguish between the term excitation threshold in the sense of [21] and lower bounds, used throughout the text: The variational methods used here, establish the existence of minimizers of the energy functionals and the existence of parameters (β or Ω) as Lagrange multipliers which are associated with these minimizers. Furthermore they provide theoretically some *local* (in the sense that they depend on Ω and β) lower bounds on the power of the minimizers, with respect to the variation of these parameters and not the existence of excitation thresholds on the power in the sense of [21]. To this end, the numerical studies performed, investigate the efficiency of the lower bounds, as well their possible behaviour as the parameters Ω and β vary. As a result, the second numerical study performed for the lower bound on the power of the focusing saturable case, verifies that this bound is actually the *smallest value of the power* below which we should not expect the existence of a nontrivial breather solution for arbitrary given ω and β and dimension N. We call such smallest values *thresholds* on the existence of periodic solutions.

[2] We consider, in all the numerical studies throughout the paper, the power of single-site breathers (also known as Sievers–Takeno modes[18]), which are the breathers with the smallest power

In the case of the focusing saturable nonlinearity $\beta > 0$, the numerical studies verify that for some parameter values, the bound predicts the trend of the behaviour of the numerically computed power. Moreover the numerical study in 2D-lattices, shows that the breather solution of the focusing saturable DNLS demonstrates a similar behaviour to that of the focusing DNLS with power nonlinearity, with respect to the existence of excitation thresholds: power decreases as the frequency increases until it reaches a minimum value at a certain frequency, *an excitation threshold in the sense of [21].* This behaviour should occur in higher dimensional lattices, and is observed in [20]. It is worth taking into account that the saturable nonlinearity can be approximated by a power one with $\sigma = 1$ for small values of $|\phi_n|$. This relation holds when Ω is close to β in the saturable case, and can justify the similarities between the saturable and power nonlinearities related to thresholds.

In Section 4, we apply an alternative method to derive a threshold on the power of the breather solution of the saturable DNLS in the focusing case $\beta > 0$. We use a fixed point argument which was also used in [10]. This approach is for the saturable DNLS, considered in *infinite lattices* (although similar estimates can be obtained in the case of Dirichlet boundary conditions). Replacing the saturable nonlinearity in the equation by its exact Taylor polynomial of order m, it is possible to derive the threshold for the power for both the DNLS with the saturable nonlinearity and the cubic-quintic approximation. The threshold appears to be the positive root of a polynomial equation. In the case of the 1D-lattice, the numerical study verifies first that the power decreases as frequency increases, as it was predicted by the lower bound derived by the variational method of Section 3. Also, the numerical power approaches the predicted threshold derived by the fixed point argument as the frequency increases up to the limit β. In comparison with the bound derived by the variational method, the latter also reaches the threshold derived by the fixed point argument, as the frequency increases. Especially for large values of the parameter β, the theoretical estimates are proved to be quite sharp for large values of frequencies. For 2D-lattices we also observe numerically the appearance of the excitation threshold in the sense of [21]. In the focusing case, it follows from the numerical study that an increase of the dimension as well as of the nonlinearity parameter is required for this excitation threshold to appear.

Section 5 is devoted to some theoretical and numerical results, related to the DNLS equation with *power nonlinearity.* We consider first the case of the defocusing ($\beta > 0$) DNLS, seeking for breather solutions $\psi_n(t) = e^{-\mathrm{i}\Omega t}$, $\Omega > 0$. By applying the same variational approach as for the saturable DNLS, we derive lower bounds on the power of the minimizers, depending on the dimension of the lattice. The numerical study demonstrates that these lower bounds can serve actually as thresholds on the existence of breather solutions. The approach of minimizing the linear energy functional

$$\mathcal{E}_\Omega[\phi] := \epsilon(-\Delta_d\phi, \phi)_2 - \Omega \sum_{|n|\leq K} |\phi_n|^2, \ \ \Omega > 0,$$

appears to be useful also in the power case as the numerical study indicates, since the derived lower bound on the power gives a result slightly closer to the real power.

We would like to summarize by pointing out the main differences with the results of [21]. The results of [21] prove the existence of an excitation threshold on the power of periodic solutions of the focusing DNLS with power nonlinearity which appears for the case $\sigma > 2/N$, as well the existence of a frequency $\omega^* > 0$ on which this threshold value on the power is achieved. The corresponding solution $\psi_n(t) = e^{i\omega^* t}\phi_n$ is a ground state having power $\mathcal{P}_{\mathrm{thresh}}$ – the excitation threshold value.

The thresholds proved by the fixed-point argument are *explicitly given threshold values which are independent of the dimension.* There are explicit estimates satisfied for *any* periodic solution with frequency $\beta > \Omega > 0$ and for any $N \geq 1$ in the focusing saturable nonlinearity, and for any $\omega > 0$ in the case of the focusing power nonlinearity and for any $\sigma > 0, N \geq 1$. They are thresholds in the sense that *no periodic localized solution can have power less than the prescribed estimates.* A characteristic example for the justification of this claim as well as its usefulness, is provided by the numerical study regarding the focusing power nonlinearity. *Even in the case $\sigma < 2/N$, where the excitation threshold [21] does not exist*, a periodic solution cannot have power less than the derived estimate. This "global character" of the estimates is revealed when one considers "limiting" cases of small (large) values of $\sigma < 2/N$ (large values of $\sigma \geq 2/N$ – the case of excitation threshold). The numerical studies verify that for small (large) values of frequencies the threshold is not only satisfied but is also a quite sharp estimate of the real power of the corresponding periodic solutions. Especially in the case $\sigma > 2/N$, the numerical studies demonstrate the fact that the excitation threshold $\mathcal{P}_{\mathrm{thresh}}$, which is not known explicitly, satisfies the derived lower bound. Thus this bound should not be viewed as a prediction of the excitation threshold in the case of $\sigma \geq 2/N$, nor as a theoretical prediction of the numerical power of periodic solutions, but as prediction of the smallest power a periodic solution can have, for any ω, σ and $N \geq 1$. The same global property is shared by the estimate for the saturable nonlinearity, as the numerical study in the case of large values of β in the defocusing case shows. We conclude with the remark that the numerical studies for the focusing saturable nonlinearity, as well as for the defocusing power nonlinearity, show that the local estimates derived by the variational methods also predict the smallest value of the power for arbitrary given values of parameters β, σ, ω and N.

2 Preliminaries

For the convenience of the reader, we recall some basic results on sequence spaces and their finite dimensional subspaces as well on discrete operators, that will be used in the sequel (see also [10, 21]).

For some positive integer N, we consider the complex sequence spaces

$$\ell^p = \left\{ \begin{array}{c} \phi = \{\phi_n\}_{n\in\mathbb{Z}^N},\ n = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N,\ \ \phi_n \in \mathbb{C}, \\ ||\phi||_p = \left(\sum_{n\in\mathbb{Z}^N} |\phi_n|^p\right)^{\frac{1}{p}} < \infty \end{array} \right\}. \tag{2.1}$$

(The following elementary embedding relation [17] holds between ℓ^p spaces

$$\ell^q \subset \ell^p, \quad ||\phi||_p \leq ||\phi||_q \ \ 1 \leq q \leq p \leq \infty, \tag{2.2}$$

in contrast with the $L^p(\Omega)$-spaces, if $\Omega \subset \mathbb{R}^N$ has finite measure. For $p = 2$, we get the usual Hilbert space of square-summable sequences, which becomes a real Hilbert space if endowed with the real scalar product

$$(\phi, \psi)_2 = \mathrm{Re} \sum_{n\in\mathbb{Z}^N} \phi_n \overline{\psi_n}, \ \ \phi,\ \psi \in \ell^2. \tag{2.3}$$

Note that any $\phi_n \in \ell^p$, $1 \leq p < \infty$ satisfies $\lim_{|n|\to\infty} \phi_n = 0$, as assumed for spatially localized solutions (i.e. discrete solitons or breathers). The discrete Laplacian is defined for $\phi_n = \phi_{(n_1,n_2,\ldots,n_N)}$ as

$$\begin{aligned} (\Delta_d\psi)_{n\in\mathbb{Z}^N} \ &= \ \psi_{(n_1-1,n_2,\ldots,n_N)} + \psi_{(n_1,n_2-1,\ldots,n_N)} + \cdots + \psi_{(n_1,n_2,\cdots,n_N-1)} \\ &\quad -2N\psi_{(n_1,n_2,\ldots,n_N)} + \psi_{(n_1+1,n_2,\ldots,n_N)} \\ &\quad +\psi_{(n_1,n_2+1,\ldots,n_N)} + \cdots + \psi_{(n_1,n_2,\cdots,n_N+1)}, \end{aligned} \tag{2.4}$$

Now we consider the discrete operator $\nabla^+ : \ell^2 \to \ell^2$ defined by

$$\begin{aligned} (\nabla^+\psi)_{n\in\mathbb{Z}^N} \ &= \ \left\{\psi_{(n_1+1,n_2,\ldots,n_N)} - \psi_{(n_1,n_2,\ldots,n_N)}\right\} \\ &\quad + \ \left\{\psi_{(n_1,n_2+1,\ldots,n_N)} - \psi_{(n_1,n_2,\ldots,n_N)}\right\} \\ &\quad \vdots \\ &\quad + \ \left\{\psi_{(n_1,n_2,\ldots,n_N+1)} - \psi_{(n_1,n_2,\ldots,n_N)}\right\}, \end{aligned} \tag{2.5}$$

and $\nabla^- : \ell^2 \to \ell^2$ defined by

$$\begin{aligned} (\nabla^-\psi)_{n\in\mathbb{Z}^N} \ &= \ \left\{\psi_{(n_1-1,n_2,\ldots,n_N)} - \psi_{(n_1,n_2,\ldots,n_N)}\right\} \\ &\quad + \ \left\{\psi_{(n_1,n_2-1,\ldots,n_N)} - \psi_{(n_1,n_2,\ldots,n_N)}\right\} \\ &\quad \vdots \\ &\quad + \ \left\{\psi_{(n_1,n_2,\ldots,n_N-1)} - \psi_{(n_1,n_2,\ldots,n_N)}\right\}. \end{aligned} \tag{2.6}$$

Setting

$$(\nabla^+_\nu\psi)_{n\in\mathbb{Z}^N} = \psi_{(n_1,n_2,\ldots,n_{\nu-1},n_\nu+1,n_{\nu+1},\ldots,n_N)} - \psi_{(n_1,n_2,\ldots,n_N)}, \tag{2.7}$$

$$(\nabla^-_\nu\psi)_{n\in\mathbb{Z}^N} = \psi_{(n_1,n_2,\ldots,n_{\nu-1},n_\nu-1,n_{\nu+1},\ldots,n_N)} - \psi_{(n_1,n_2,\ldots,n_N)}, \tag{2.8}$$

we observe that the operator $-\Delta_d$ satisfies the relations

$$(-\Delta_d\psi_1, \psi_2)_2 \ = \ \sum_{\nu=1}^{N} (\nabla^+_\nu\psi_1, \nabla^+_\nu\psi_2)_2, \ \ \text{for all} \ \ \psi_1, \psi_2 \in \ell^2, \tag{2.9}$$

$$(\nabla^+_\nu\psi_1, \psi_2)_2 \ = \ (\psi_1, \nabla^-_\nu\psi_2)_2, \ \ \text{for all} \ \ \psi_1, \psi_2 \in \ell^2. \tag{2.10}$$

From (2.9), it is clear that $-\Delta_d : \ell^2 \to \ell^2$ defines a self adjoint operator on ℓ^2, and $-\Delta_d \geq 0$.

To formulate the DNLS equation, subject to Dirichlet boundary conditions, we consider the finite dimensional subspaces of ℓ^p for a positive integer K, defined by

$$\ell^p(\mathbb{Z}_K^N) = \{\phi \in \ell^p \ : \ \phi_n = 0 \ \text{ for } \ |n| > K\}. \tag{2.11}$$

We have $\ell^p(\mathbb{Z}_K^N) \equiv \mathbb{C}^{(2K+1)^N}$ endowed with the norms (2.2)-finite sums. In this case, for any $1 \leq p \leq q \leq \infty$, there exist constants C_1, C_2 depending on K, such that

$$C_1||\psi||_p \leq ||\psi||_q \leq C_2||\psi||_p. \tag{2.12}$$

In the finite dimensional setting, the operator $-\Delta_d$ satisfies relations (2.9)-(2.10), and its principal eigenvalue $\lambda_1 > 0$ can be characterized as

$$\lambda_1 = \inf_{\substack{\phi \in \ell^2(\mathbb{Z}_K^N) \\ \phi \neq 0}} \frac{(-\Delta_d \phi, \phi)_2}{(\phi, \phi)_2} = \inf_{\substack{\phi \in \ell^2(\mathbb{Z}_K^N) \\ \phi \neq 0}} \frac{\sum_{\nu=1}^N ||\nabla_\nu^+ \phi||_2^2}{\sum_{|n| \leq K} |\phi_n|^2} \tag{2.13}$$

Hence (2.13) and (2.5)-(2.6) imply the inequality

$$\epsilon\lambda_1 \sum_{|n|\leq K} |\phi_n|^2 \leq \epsilon \sum_{\nu=1}^N ||\nabla_\nu^+ \phi||_2^2 \leq 4\epsilon N \sum_{|n|\leq K} |\phi_n|^2. \tag{2.14}$$

Then it follows from (2.14) that

$$\lambda_1 \leq 4N. \tag{2.15}$$

For example, in the case of an $1D$ lattice $n = 1, \ldots, K$, the eigenvalues of the discrete Dirichlet problem $-\Delta_d \phi = \lambda\phi$ with ϕ real, are given by

$$\lambda_n = 4 \sin^2 \left(\frac{n\pi}{4(K+1)} \right), \ \ n = 1, \ldots, K.$$

For a N-dimensional problem, the eigenvalues are:

$$\begin{aligned} \lambda_{(n_1,n_2,\ldots,n_N)} &= 4 \left[\sin^2 \left(\frac{n_1\pi}{4(K+1)} \right) + \sin^2 \left(\frac{n_2\pi}{4(K+1)} \right) + \ldots + \sin^2 \left(\frac{n_N\pi}{4(K+1)} \right) \right], \\ n_j &= 1, \ldots, K; \ \ j = 1, \ldots, N. \end{aligned}$$

In consequence, the principal eigenvalue of the discrete Dirichlet problem $-\Delta_d \phi = \lambda\phi$ with ϕ real, is given by

$$\lambda_1 \equiv \lambda_{(1,1,\ldots,1)} = 4N \sin^2 \left(\frac{\pi}{4(K+1)} \right).$$

3 The saturable nonlinearity: Constrained minimization problems-Dirichlet boundary conditions

3.1 A. Defocusing case $\beta < 0$: Periodic solutions $\psi_n(t) = e^{i\omega t}\phi_n, \omega > 0$

In this section we consider the existence of breather solutions of the saturable DNLS equation, for the case $\beta < 0$. For convenience we set

$$\beta = -\Lambda, \Lambda > 0.$$

Thus, we seek breather solutions for the DNLS equation

$$\mathrm{i}\dot{\psi}_n + \epsilon(\Delta_d \psi)_n + \Lambda \frac{\psi_n}{1 + |\psi_n|^2} = 0, \ \ \Lambda > 0, \tag{3.1}$$

of the form

$$\psi_n(t) = e^{i\omega t}\phi_n, \omega > 0. \tag{3.2}$$

In this case, the system (1.5) is rewritten as

$$-\epsilon(\Delta_d\phi)_n + \omega\phi_n = \Lambda\frac{\phi_n}{1+|\phi_n|^2}, \quad \Lambda > 0. \tag{3.3}$$

We note that in the case of the anticontinuum limit $\epsilon = 0$, it follows that the frequency of a non-trivial breather solution should satisfy

$$\Lambda > \omega. \tag{3.4}$$

The Hamiltonian $\mathcal{H}$ and the power $\mathcal{P}$, given by

$$\mathcal{H}[\phi] = \epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\phi||_2^2 - \Lambda\sum_{|n|\le K}\log(1+|\phi_n|^2), \tag{3.5}$$

$$\mathcal{P}[\phi] = \sum_{|n|\le K}|\phi_n|^2, \tag{3.6}$$

are quantities which are independent of time. We shall prove the existence of nontrivial breather solutions (3.2), by considering a constrained minimization problem. The system (3.3) will be considered as the Euler-Lagrange equation of the functional

$$\mathcal{H}_\omega[\phi] = \epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\phi||_2^2 + \omega\sum_{|n|\le K}|\phi_n|^2 - \Lambda\sum_{|n|\le K}\log(1+|\phi_n|^2), \tag{3.7}$$

involving $\mathcal{H}$ and $\mathcal{P}$. To produce the Euler-Lagrange equation (3.3) from the functionals $\mathcal{H}$ and $\mathcal{P}$, we shall use the following

Lemma 3.1 *Let $\phi \in \ell^2(\mathbb{Z}_K^N)$. Then the functional*

$$\mathcal{V}(\phi) = \sum_{|n|\le K}\log(1+|\phi_n|^2),$$

is a $\mathrm{C}^1(\ell^2(\mathbb{Z}_K^N),\mathbb{R})$ functional and

$$\langle\mathcal{V}'(\phi),\psi\rangle = 2\mathrm{Re}\sum_{|n|\le K}\frac{\phi_n}{1+|\phi_n|^2}\overline{\psi_n}, \quad \psi\in\ell^2(\mathbb{Z}_K^N). \tag{3.8}$$

Proof: We assume that $\phi,\, \psi \in \ell^2(\mathbb{Z}_K^N)$. Then for any $0 < s < 1$, we get

$$\begin{aligned}\frac{\mathcal{V}(\phi+s\psi)-\mathcal{V}(\phi)}{s} &= \frac{1}{s}\mathrm{Re}\sum_{|n|\le K}\int_0^1\frac{d}{d\theta}\log(1+|\phi_n+\theta s\psi_n|^2)d\theta \\ &= 2\mathrm{Re}\sum_{|n|\le K}\int_0^1\frac{\phi_n+s\theta\psi_n}{1+|\phi_n+s\theta\psi_n|^2}\overline{\psi_n}d\theta.\end{aligned} \tag{3.9}$$

From the inequality

$$\sum_{|n|\le K}\frac{|\phi_n+\theta s\psi_n|}{1+|\phi_n+\theta s\psi_n|^2}|\psi_n| \le \sum_{|n|\le K}(|\phi_n|+|\psi_n|)\,|\psi_n| \le (||\phi||_{\ell^2}+||\psi||_2)||\psi||_2, \tag{3.10}$$

we may let $s\to 0$, to get the existence of the Gateaux derivative (3.8) (discrete dominated convergence).

To check that the functional $\mathcal{V}' : \ell^2 \to \ell^2$ is continuous, we consider a sequence $\phi_m \in \ell^2(\mathbb{Z}_K^N)$ such that $\phi_m \to \phi$ in $\ell^2(\mathbb{Z}_K^N)$. Then, by using an inequality very similar to (4.19) (see Section 4), we may show that $|\langle\mathcal{V}'(\phi_m)-\mathcal{V}'(\phi),\,\psi\rangle| \to 0$, as $m\to\infty$. The result is also valid in the case of an infinite lattice ($n\in\mathbb{Z}^N$). $\diamond$

The constrained minimization problem A.I The first variational problem we shall discuss is a constrained minimization problem for the energy quantity

$$\mathcal{E}_\omega[\phi] := \epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\phi||_2^2 + \omega\sum_{|n|\le K}|\phi_n|^2,\ \ \phi\in\ell^2(\mathbb{Z}_K^N),\ \ \omega>0, \tag{3.11}$$

for given $\omega>0$. We have the following

Theorem 3.2 *Let $\omega>0$ be given. Consider the variational problem on $\ell^2(\mathbb{Z}_K^N)$*

$$\inf\left\{\mathcal{E}_\omega[\phi]\ :\ \sum_{|n|\le K}\log(1+|\phi_n|^2)=R>0\right\}. \tag{3.12}$$

There exists a minimizer $\hat\phi\in\ell^2(\mathbb{Z}_K^N)$ for the variational problem (3.12) and a $\Lambda(R)>0$ such that $\Lambda(R)>\omega$, both satisfying the Euler-Lagrange equation

$$\begin{aligned} -\epsilon(\Delta_d\hat\phi)_n+\omega\hat\phi_n &= \Lambda\frac{\hat\phi_n}{1+|\hat\phi_n|^2},\ \ |n|\le K,\\ \hat\phi_n &= 0,\ \ |n|>K. \end{aligned}$$

Moreover, it holds that $\sum_{|n|\le K}\log(1+|\phi_n|^2)=R$.

Proof: Clearly $\mathcal{E}_\omega$ defines a $C^1(\ell^2(\mathbb{Z}_K^N),\mathbb{R})$ functional. The minimization problem we intend to solve is for the functional $\mathcal{E}_\omega$, restricted to the set

$$B_1=\left\{\phi\in\ell^2(\mathbb{Z}_K^N)\ :\ \sum_{|n|\le K}\log(1+|\phi_n|^2)=R>0\right\}, \tag{3.13}$$

It is not hard to check that the sequence $\{\phi_m\}_{m\in\mathbb{N}}\in B_1$ is bounded. Next, we consider a sequence $\{\phi_m\}_{m\in\mathbb{N}}\in B_1$, such that $\phi_m\to\phi$ as $m\to\infty$. We denote the nth coordinate of this sequence by $(\phi_m)_n$. Using (2.12), we observe that

$$\left|\sum_{|n|\le K}\log(1+|(\phi_m)_n|^2)-\sum_{|n|\le K}\log(1+|\phi_n|^2)\right|\le C||\phi_m-\phi||_{\ell^1(\mathbb{Z}_K^N)}\to 0\ \text{ as }\ m\to\infty. \tag{3.14}$$

Moreover, since $\phi_m\in B_1$, we find from (3.14) that

$$\sum_{|n|\le K}\log(1+|\phi_n|^2)=\lim_{m\to\infty}\sum_{|n|\le K}\log(1+|(\phi_m)_n|^2)=R.$$

Hence $\phi\in B_1$, which implies that B_1 is closed. The functional $\mathcal{E}_\omega$ is bounded from below on $\mathcal{B}_1$, since

$$\begin{aligned} \mathcal{E}_\omega[\phi] &= \epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\phi||_2^2+\omega\sum_{|n|\le K}|\phi_n|^2\\ &\ge\omega\sum_{|n|\le K}|\phi_n|^2\\ &\ge\omega\sum_{|n|\le K}\log(1+|\phi_n|^2)\ge\omega R. \end{aligned} \tag{3.15}$$

As we are restricted to the finite dimensional space $\ell^2(\mathbb{Z}_K^N)$, it follows that any minimizing sequence associated with the variational problem (3.12) is precompact: any minimizing sequence has a subsequence, converging to a minimizer. Thus $\mathcal{E}_\omega$ attains its infimum at a point $\hat\phi$ in B_1. We proceed in order to derive the variational equation for $\mathcal{E}_\omega$. Note that

$$\langle\mathcal{E}_\omega'[\phi],\psi\rangle=2\epsilon\sum_{\nu=1}^{N}(\nabla_\nu^+\phi,\nabla_\nu^+\psi)_2+2\omega\mathrm{Re}\sum_{|n|\le K}\phi_n\overline{\psi}_n. \tag{3.16}$$

By considering the $C^1(\ell^2(\mathbb{Z}_K^N),\mathbb{R})$ (see Lemma 3.8)

$$\mathcal{L}_R[\phi] = \sum_{|n|\leq K} \log(1+|\phi_n|^2) - R, \tag{3.17}$$

we observe that for any $\phi \in B_1$

$$\langle \mathcal{L}_R'[\phi], \phi\rangle = 2\sum_{|n|\leq K} \frac{|\phi_n|^2}{1+|\phi_n|^2} > 0. \tag{3.18}$$

The Regular Value Theorem ([2, Section 2.9], [9, Appendix A,pg. 556]) implies that the set $B_1 = \mathcal{L}_R^{-1}(0)$ is a C^1-submanifold of $\ell^2(\mathbb{Z}_K^N)$. By applying the Lagrange multiplier rule, we find the existence of a parameter $\Lambda = \Lambda(R) \in \mathbb{R}$, such that

$$\begin{aligned}\left\langle \mathcal{E}_\omega'[\hat{\phi}] - \Lambda\mathcal{L}_R'[\hat{\phi}], \psi \right\rangle &= 2\epsilon\sum_{\nu=1}^{N} (\nabla_\nu^+\hat{\phi}, \nabla_\nu^+\psi)_2 + 2\omega \mathrm{Re}\sum_{|n|\leq K}\hat{\phi}_n\overline{\psi}_n \\ &- 2\Lambda\mathrm{Re}\sum_{|n|\leq K}\frac{\hat{\phi}_n}{1+|\hat{\phi}_n|^2}\overline{\psi}_n = 0, \ \text{ for all } \ \psi\in\ell^2(\mathbb{Z}_K^N).\end{aligned} \tag{3.19}$$

Setting $\psi = \hat{\phi}$ in (3.19), we find that

$$2\mathcal{E}_\omega[\hat{\phi}] = 2\epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\hat{\phi}||_2^2 + 2\omega\sum_{|n|\leq K}|\hat{\phi}_n|^2 = 2\Lambda\sum_{|n|\leq K}\frac{|\hat{\phi}_n|^2}{1+|\hat{\phi}_n|^2}. \tag{3.20}$$

Since $\hat{\phi}\in B_1$ cannot be identically zero and $\mathcal{E}_\omega[\hat{\phi}] > 0$, it follows from (3.20) that $\Lambda > 0$.

Rewriting (3.20) as

$$\epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\hat{\phi}||_2^2 = \Lambda\sum_{|n|\leq K}\frac{|\hat{\phi}_n|^2}{1+|\hat{\phi}_n|^2} - \omega\sum_{|n|\leq K}|\hat{\phi}_n|^2, \tag{3.21}$$

we find that

$$\epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\hat{\phi}||_2^2 \leq (\Lambda-\omega)\sum_{|n|\leq K}|\hat{\phi}_n|^2. \tag{3.22}$$

The lhs of (3.22) is positive, and we find that

$$\Lambda(R) > \omega. \tag{3.23}$$

Therefore condition (3.4) is justified. From (3.19), there exists $\Lambda > 0$, such that the minimizer $\hat{\phi}\in B_1$ solves the equation

$$(-\epsilon\Delta_{\mathrm{d}}\phi, \psi)_2 + \omega(\phi,\psi)_2 = \Lambda\left(\frac{\phi}{1+|\phi|^2}, \psi\right)_2, \ \text{ for all } \ \psi\in\ell^2(\mathbb{Z}_K^N). \tag{3.24}$$

The above formula, is clearly equivalent to the Euler-Lagrange equation (3.3), that is, any solution of (3.3) is a solution of (3.24), and vice versa. $\diamond$

The constrained minimization problem A.II In the first minimization problem, we derived, under sufficient conditions for given $\omega > 0$, both the existence of $\beta = -\Lambda < 0$ as a Lagrange multiplier, and the existence of a nontrivial $\hat{\phi}\in\ell^2(\mathbb{Z}_K^N)$, such that (3.2) is a breather solution of (3.1). Here, by studying a different minimization problem, we shall derive some sufficient conditions depending on the given $\beta = -\Lambda < 0$, the lattice spacing ϵ and the dimension of the lattice N. These provide both the existence of a parameter $\omega > 0$ and a nontrivial ϕ^*, such that (3.41) is a solution of (1.1) involving this ω as the frequency of the breather solution. This alternative variational approach for the existence of breather solutions (3.2) for the DNLS equation (3.1), is to minimize the Hamiltonian $\mathcal{H}$, constrained to the set

$$B = \left\{\phi\in\ell^2(\mathbb{Z}_K^N)\ :\ \sum_{|n|\leq K}|\phi_n|^2 = R^2 > 0\right\}. \tag{3.25}$$

Theorem 3.3 *Let $\Lambda, \epsilon, N > 0$ be chosen such that*

$$\Lambda > 4\epsilon N. \tag{3.26}$$

Consider the variational problem on $\ell^2(\mathbb{Z}_K^N)$

$$\inf\left\{\mathcal{H}[\phi]\ : \mathcal{P}[\phi] = R^2 > 0\right\}. \tag{3.27}$$

Assuming that

$$R^2 < \frac{\Lambda - 4\epsilon N}{4\epsilon N}, \tag{3.28}$$

there exists a minimizer $\phi^* \in \ell^2(\mathbb{Z}_K^N)$ for the variational problem (3.27) and $\omega = \omega(R) > 0$ such that $\Lambda > \omega(R)$, both satisfying the Euler-Lagrange equation

$$\begin{aligned} -\epsilon(\Delta_d \phi^*)_n + \omega\phi_n^* &= \Lambda\frac{\phi_n^*}{1+|\phi_n^*|^2}, \quad |n| \leq K, \\ \phi_n^* &= 0, \quad |n| > K. \end{aligned}$$

Moreover, it holds that $\mathcal{P}[\phi^*] = R^2$.

Proof: We note first that $\mathcal{H} : \ell^2(\mathbb{Z}_K^N) \to \mathbb{R}$, is bounded from below, since

$$\mathcal{H}[\psi] \geq -\Lambda \sum_{|n|\leq K} \log(1+|\phi_n|^2) \geq -\Lambda \sum_{|n|\leq K} |\phi_n|^2 = -\Lambda R^2. \tag{3.29}$$

Again, the finite dimensionality of the problem implies that any minimizing sequence associated with the variational problem (3.25) is precompact, and that any minimizing sequence has a subsequence, which converges to a minimizer. Therefore, we conclude that $\mathcal{H} : B \to \mathbb{R}$ attains its infimum at a point $\phi^* \in B$. The next step is to derive the variational equation (1.5). To this end, by using Lemma 3.1, we observe that

$$\langle \mathcal{H}'[\phi], \psi\rangle = 2\epsilon \sum_{\nu=1}^{N} (\nabla_\nu^+\phi, \nabla_\nu^+\psi)_2 - 2\Lambda \mathrm{Re} \sum_{|n|\leq K} \frac{\phi_n}{1+|\phi_n|^2}\overline{\psi}_n. \tag{3.30}$$

We consider next the functional $\mathcal{N}_R : \ell^2(\mathbb{Z}_K^N) \to \mathbb{R}$

$$\mathcal{N}_R[\phi] := \sum_{|n|\leq K} |\phi_n|^2 - R^2. \tag{3.31}$$

Clearly, $\mathcal{N}_R$ is $C^1(\ell^2(\mathbb{Z}_K^N), \mathbb{R})$ and

$$\langle \mathcal{N}_R'[\phi], \psi\rangle = 2\mathrm{Re} \sum_{|n|\leq K} \phi_n \overline{\psi}_n. \tag{3.32}$$

Moreover, for any $\phi \in B$, we have

$$\langle \mathcal{N}_R'[\phi], \phi\rangle = 2 \sum_{|n|\leq K} |\phi_n|^2 = 2R^2 \neq 0. \tag{3.33}$$

Therefore, we may apply the Regular Value Theorem, to show that the set $B = \mathcal{N}_R^{-1}(0)$ is a C^1-submanifold of $\ell^2(\mathbb{Z}_K^N)$. It follows from (3.30), (3.32) and the rule of Lagrange multipliers, that there exists a Lagrange multiplier $\Omega = \Omega(R) \in \mathbb{R}$, such that

$$\begin{aligned} \langle \mathcal{H}'[\phi^*] - \Omega\mathcal{N}_R'[\phi^*], \psi\rangle &= 2\epsilon \sum_{\nu=1}^{N} (\nabla_\nu^+\phi^*, \nabla_\nu^+\psi)_2 - 2\Lambda \mathrm{Re} \sum_{|n|\leq K} \frac{\phi_n^*}{1+|\phi_n^*|^2}\overline{\psi}_n \\ &\quad -2\Omega \mathrm{Re} \sum_{|n|\leq K} \phi_n^* \overline{\psi_n} = 0, \quad \text{for all } \psi \in \ell^2(\mathbb{Z}_K^N). \end{aligned} \tag{3.34}$$

Thus, for $\psi = \phi^*$, we find that

$$2\mathcal{U}(\phi^*) = 2\Omega \sum_{|n|\leq K} |\phi^*|^2 = 2\Omega R^2, \tag{3.35}$$

where the functional $\mathcal{U} : \ell^2(\mathbb{Z}_K^N) \to \mathbb{R}$ is defined by

$$\mathcal{U}(\phi) := \epsilon \sum_{\nu=1}^{N} ||\nabla_\nu^+ \phi||_2^2 - \Lambda \sum_{|n|\leq K} \frac{|\phi_n|^2}{1+|\phi_n|^2}.$$

For an appropriate choice of R, ϵ, Λ, we can show that $\mathcal{U}[\phi^*] < 0$: From (2.14), we have

$$\epsilon \sum_{\nu=1}^{N} ||\nabla_\nu^+ \phi||_2^2 \leq 4\epsilon N ||\phi||_2^2. \tag{3.36}$$

Then, noting that $|\phi_n^*|^2 \leq \sum_{|n|\leq K} |\phi_n^*|^2$, and using (3.36), we observe that

$$\mathcal{U}(\phi^*) \leq 4\epsilon N \sum_{|n|\leq K} |\phi_n^*|^2 - \Lambda \sum_{|n|\leq K} \frac{|\phi_n^*|^2}{1+||\phi_n^*||_2^2} = \left(4\epsilon N R^2 - \Lambda \frac{R^2}{1+R^2}\right). \tag{3.37}$$

Therefore $\mathcal{U}[\phi^*] < 0$, if

$$4\epsilon N(1+R^2) < \Lambda. \tag{3.38}$$

which in terms of R, gives (3.28). We shall consider equation (3.35), for the choice of parameters (3.38): since $\phi^* \in B$ cannot be identically zero, and $\mathcal{U}[\phi^*] < 0$, it follows that $\Omega(R) < 0$. Thus, we may set

$$\Omega(R) = -\omega(R), \;\; \omega(R) > 0. \tag{3.39}$$

The inequality (3.22) is still applicable, to verify that $\omega(R)$ satisfies condition (3.4). We get from (3.34) and (3.39), that there exists $\omega(R) > 0$, such that the minimizer $\phi^* \in B$ solves the equation (3.24). $\diamond$

Numerical Study. The result of Theorem 3.2, establishes the existence of a nontrivial $\hat{\phi}$, for a given $\omega > 0$, and the existence of $\Lambda > 0$, satisfying $\Lambda > \omega$ such that $\psi_n(t) = e^{i\omega t}\hat{\phi}_n$, solves the (3.1) equation, with such a Λ as a nonlinearity parameter. In Theorem 3.2, there are no further relations assumed between ω, ϵ, N.

On the other hand, the result of Theorem 3.3 has the following implementation concerning the existence of breather solutions: for given Λ, ϵ, N satisfying condition (3.26),

$$\Lambda > 4\epsilon N,$$

there exists some $\omega > 0$ such that $\Lambda > \omega$ and $\phi^* \in \ell^2(\mathbb{Z}_K^N)$ not identically zero, solving the Euler-Lagrange equation (3.3). Hence, if (3.26) holds, there exists $\omega > 0$, such that $\psi_n(t) = e^{i\omega t}\phi_n^*$, is a solution for the DNLS equation (3.1), with power

$$\mathcal{P}[\psi] = R^2 < \frac{\Lambda - 4\epsilon N}{4\epsilon N}.$$

The statement of the Theorem 3.3 could be useful in the sense that the parameter regime (3.26) establishes the existence of a range for frequencies $\omega > 0$, such that the corresponding breather solutions of the DNLS equation (3.1) have power satisfying the upper bound (3.28).

A numerical study has been performed to study the behaviour of the power of the breather solution in the parameter regime (3.26), to test the result of Theorem 3.3 and the upper bound (3.28). There is an extra condition for the existence of breathers arising from the condition of non-resonance with linear modes, which is that

$$\Lambda > 4N\epsilon - \omega. \tag{3.40}$$

Clearly, (3.40) is satisfied when (3.26) is assumed. The numerical power verifies that there exists a range of frequencies such that the corresponding breather solutions have power satisfying the upper bound (3.28): first we have depicted the power versus the frequency for breathers with $\Lambda = 2$ and $\epsilon = 0.2$ in a 1-dimensional lattice

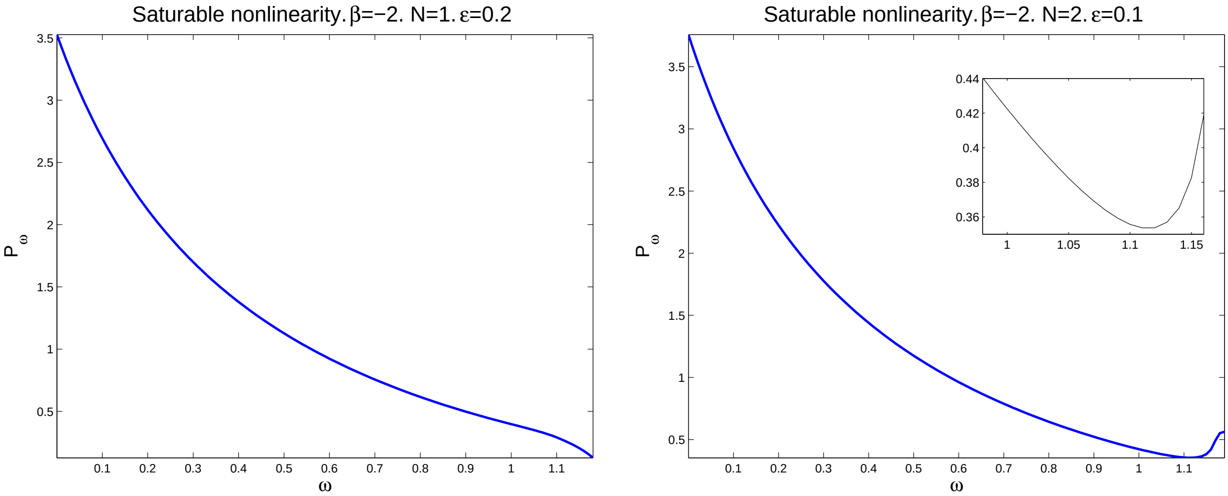

Figure 1: Power vs frequency of the defocusing DNLS with saturable nonlinearity for the cases (a) $\Lambda = 2, \epsilon = 0.2, N = 1$ (b) $\Lambda = 2, \epsilon = 0.1, N = 2$. The predicted value by the upper bound (3.28), in both cases is $\mathcal{P} = 1.5$ (see text). In case (b), we observe the existence of a minimum of the total power (excitation threshold)

.

(see Fig. 1). From the theoretical prediction, an $\omega > 0$ should exist satisfying $\Lambda > \omega$, with a power which should be always smaller than 1.5. From the figure, it can be deduced that the prediction is satisfied for all $\omega > 0.353$. We have also considered the case $\Lambda = 2$, $\epsilon = 0.1$, for $N = 2$. Similarly, for the 2D-lattice, the breathers solution of frequency $\omega > 0.830$, satisfy the theoretical upper bound $\mathcal{P} = 1.5$. The numerical study in the 2D-case reveals the existence of an excitation threshold for the defocusing saturable DNLS as in the case of the power nonlinearity [21]: power decreases as frequency increases, attaining a minimum value for a certain value of frequency, as shown in the inset of Figure 1 (b). As the frequency increases further, it seems that the power reaches a "threshold value". In the case of higher dimensional lattices, the upper bound (3.28), could be even more useful, as an estimate *from above* of the excitation threshold as well as of the "threshold" value of the increased power reached, as the frequency increases further up to the resonant limit.

3.2 B. Focusing case $\beta > 0$: Periodic solutions $\psi_n(t) = e^{-i\Omega t}\phi_n, \Omega > 0$

This section considers the existence of breather solutions of the saturable DNLS equation in the focusing case $\beta > 0$. We look for breather solutions of the form

$$\psi_n(t) = e^{-i\Omega t}\phi_n, \quad \Omega > 0 \tag{3.41}$$

By considering again the case of the anticontinuum limit $\epsilon = 0$, it follows that the frequency of a nontrivial breather solution (3.41) satisfies

$$\beta > \Omega. \tag{3.42}$$

We choose to consider the minimization problem for the Hamiltonian

$$\mathcal{H}[\phi] \quad = \quad \epsilon \sum_{\nu=1}^{N} ||\nabla_\nu^+ \phi||_2^2 + \beta \sum_{|n|\leq K} \log(1 + |\phi_n|^2), \ \ \beta > 0, \tag{3.43}$$

constrained on the set B given by (3.25), since the approach of the variational problem A.I does not seem to be applicable in this case.

Theorem 3.4 *Let $\beta > 0$ be given, and consider the following variational problem on $\ell^2(\mathbb{Z}_K^N)$*

$$\inf \left\{ \mathcal{H}[\phi] \ : \mathcal{P}[\phi] = R^2 > 0 \right\}. \tag{3.44}$$

There exists a minimizer $\tilde{\phi} \in \ell^2(\mathbb{Z}_K^N)$ for the variational problem (3.27) and $\Omega = \omega(R) > 0$ such that $\beta > \Omega(R)$, both satisfying the Euler-Lagrange equation

$$\begin{aligned} -\epsilon(\Delta_d \tilde{\phi})_n - \Omega\tilde{\phi}_n &= -\beta\frac{\tilde{\phi}_n}{1+|\tilde{\phi}_n|^2}, \quad |n| \le K, \\ \tilde{\phi}_n &= 0, \quad |n| > K. \end{aligned}$$

Moreover, it holds that $\mathcal{P}[\tilde{\phi}] = R^2$ and

$$\Omega > \frac{\beta}{1+R^2}. \tag{3.45}$$

Proof: To see that $\mathcal{H}$, is bounded from below, this time we use the inequality (2.13): we have

$$\begin{aligned} \mathcal{H}[\phi] &\ge \epsilon\sum_{\nu=1}^{N} ||\nabla_\nu^+ \phi||_2^2 \\ &\ge \epsilon\lambda_1 \sum_{|n|\le K} |\phi_n|^2 \ge \epsilon\lambda_1 R^2. \end{aligned} \tag{3.46}$$

The existence of the minimizer $\tilde{\phi} \in \ell^2(\mathbb{Z}_K^N)$, and of the Lagrange multiplier $\Omega \in \mathbb{R}$, can be derived by using the same variational arguments as in case A: since

$$\begin{aligned} \left\langle \mathcal{H}'[\tilde{\phi}] - \Omega\mathcal{N}_R'[\tilde{\phi}], \psi \right\rangle &= 2\epsilon\sum_{\nu=1}^{N}(\nabla_\nu^+\tilde{\phi}, \nabla_\nu^+\psi)_2 + 2\beta\mathrm{Re}\sum_{|n|\le K}\frac{\tilde{\phi}_n}{1+|\tilde{\phi}_n|^2}\overline{\psi}_n \\ &\quad -2\Omega\mathrm{Re}\sum_{|n|\le K}\tilde{\phi}_n\overline{\psi_n} = 0, \quad \text{for all } \psi \in \ell^2(\mathbb{Z}_K^N). \end{aligned} \tag{3.47}$$

Setting $\psi = \tilde{\phi}$, we get the equation

$$2\mathcal{U}[\tilde{\phi}] := 2\epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\tilde{\phi}||_2^2 + 2\beta\sum_{|n|\le K}\frac{|\tilde{\phi}_n|^2}{1+|\tilde{\phi}_n|^2} = 2\Omega\sum_{|n|\le K}|\tilde{\phi}_n|^2, \tag{3.48}$$

and since $\tilde{\phi} \ne 0$ and $\mathcal{U}[\tilde{\phi}] > 0$, we find that $\Omega(R) > 0$. To justify condition (3.42), we note by using (3.36) and (3.48), that for arbitrary $\epsilon > 0$

$$(4\epsilon N + \beta)\sum_{|n|\le K}|\tilde{\phi}_n|^2 > \Omega\sum_{|n|\le K}|\tilde{\phi}_n|^2,$$

implying that

$$\epsilon + \frac{\beta}{4N} > \frac{\Omega}{4N}, \quad \text{for any } \epsilon > 0,$$

thus

$$\beta \ge \Omega > 0. \tag{3.49}$$

Since $\mathcal{P}[\tilde{\phi}] = R^2$, we get from (3.48), that

$$\begin{aligned} 2\epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\tilde{\phi}||_2^2 &= 2\Omega\sum_{|n|\le K}|\tilde{\phi}_n|^2 - 2\beta\sum_{|n|\le K}\frac{|\tilde{\phi}_n|^2}{1+|\tilde{\phi}_n|^2} \\ &\le 2\Omega\sum_{|n|\le K}|\tilde{\phi}_n|^2 - \sum_{|n|\le K}\frac{|\tilde{\phi}_n|^2}{1+||\tilde{\phi}_n||_2^2} \\ &\le 2R^2\left(\Omega - \frac{\beta}{1+R^2}\right). \end{aligned} \tag{3.50}$$

Let us assume that $R > 0$ and

$$\Omega \leq \frac{\beta}{1 + R^2}. \tag{3.51}$$

Then, (3.50) and (2.14) imply that $R = 0$, and in this case (3.51) turns out that $\beta \leq \Omega$. Thus, we have a contradiction both with the assumption $R > 0$ as well as with (3.42). Therefore for $R > 0$, we should have $\beta > \Omega$ and (3.45). $\diamond$

On the other hand, the inequality (3.45), in terms of the power, it can be rewritten

$$R^2 > \frac{\beta}{\Omega} - 1. \tag{3.52}$$

Thus the result of Theorem 3.4 shows that, for given $\beta > 0$, there exists some $\Omega > 0$, satisfying $\beta > \Omega$, and a nontrivial $\tilde{\phi} \in \ell^2(\mathbb{Z}_K^N)$, such that $\psi_n(t) = e^{-i\Omega t}\tilde{\phi}_n$, is a breather solution of (1.1), with power satisfying the lower bound (3.52). Let us observe that the rhs of (3.52) predicts that the power should be a decreasing function of the frequency Ω, as the frequency increases to the resonant limit β. In the next section, we shall derive a threshold value for the power of breather solutions for the focusing case $\beta > 0$, by using a fixed point argument. The lower bound of this section as well as the fixed point threshold, will be tested numerically.

4 Thresholds for periodic solutions of the saturable DNLS by a fixed point argument-Infinite lattices: focusing case $\beta > 0$

We repeat here the fixed point argument of [10] to derive a threshold on the power, for the non-existence of non-trivial breather solutions for (1.1). The approach covers the case of an infinite lattice ($n \in \mathbb{Z}^N$). We consider the case where the parameters $\beta > \Omega > 0$ are given, and we investigate conditions on the non-existence of non-trivial solutions of the form

$$\psi_n(t) = e^{-\mathrm{i}\Omega t}\phi_n, \quad \beta > \Omega > 0. \tag{4.1}$$

Note that ϕ satisfies (1.5), rewritten as

$$-\epsilon(\Delta_d\phi)_n - \Omega\phi_n = -\beta\frac{\phi_n}{1 + |\phi_n|^2}, \quad \beta > \Omega > 0. \tag{4.2}$$

For the convenience of the reader we state [22, Theorem 18.E, pg. 68] (Theorem of Lax and Milgram), which as for the case of the 2σ-power nonlinearity [10], will be used to establish existence of solutions for an auxiliary linear system of algebraic equations related to (4.2).

Theorem 4.1 *Let X be a Hilbert space and $\mathbf{A} : X \to X$ be a linear continuous operator. Suppose that there exists $c^* > 0$ such that*

$$\mathrm{Re}(\mathbf{A}u, u)_X \geq c^*||u||_X^2, \quad \textit{for all } u \in X. \tag{4.3}$$

Then for given $f \in X$, the operator equation $\mathbf{A}u = f$, $u \in X$, has a unique solution

Recall that for $f(x) = \frac{1}{1+x^2}, x \in \mathbb{R}$, the following identity holds

$$f(x) = 1 - x^2 + x^4 + \cdots + (-1)^n x^{2n} + \frac{(-1)^{m+1}x^{2(m+1)}}{1 + x^2}, \quad \text{for all } x \in \mathbb{R}, \tag{4.4}$$

which coincides with the Taylor polynomial of order m. Applying (4.4) for $x = |\zeta|, \zeta \in \mathbb{C}$ we rewrite the saturable nonlinearity as

$$F(\zeta) = \frac{\zeta}{1 + |\zeta|^2} = \left(1 - |\zeta|^2 + |\zeta|^4 + \cdots + (-1)^m|\zeta|^{2m} + \frac{(-1)^{m+1}|\zeta|^{2(m+1)}}{1 + |\zeta|^2}\right)\zeta \text{ for all } \zeta \in \mathbb{C}. \tag{4.5}$$

Using (4.5), equation (4.2) can be rewritten as

$$-\epsilon(\Delta_d\phi)_n + (\beta - \Omega)\phi_n = F_*(\phi_n) + T_*(\phi_n), \tag{4.6}$$

where

$$F_*(\phi_n) := \beta\left(|\phi_n|^2\phi_n - |\phi_n|^4\phi_n + \cdots + (-1)^{m+1}|\phi_n|^{2m}\phi_n\right), \tag{4.7}$$

$$T_*(\phi_n) := \beta\frac{(-1)^{m+2}|\phi_n|^{2(m+1)}\phi_n}{1+|\phi_n|^2}. \tag{4.8}$$

Setting

$$\delta := \beta - \Omega > 0, \tag{4.9}$$

we observe that the (linear and continuous) operator $\mathcal{T}_\delta : \ell^2 \to \ell^2$, defined as

$$(\mathcal{T}_\delta\phi)_{n\in\mathbb{Z}^N} = -\epsilon(\Delta_d\phi)_{n\in\mathbb{Z}^N} + \delta\phi_n, \tag{4.10}$$

satisfies condition (4.3) if (4.9) holds, since

$$(\mathcal{T}_\delta\phi, \phi)_2 = \epsilon\sum_{\nu=1}^{N}||\nabla_\nu^+\phi||_2^2 + \delta||\phi||_2^2 \geq \delta||\phi||_2^2 \text{ for all } \phi \in \ell^2. \tag{4.11}$$

Next, setting $F_{2m}(\zeta) = |\zeta|^{2m}\zeta$, we may define from F_{2m}, a map $\mathbf{F}_{2m} : \ell^2 \to \ell^2$. We have that

$$||\mathbf{F}_{2m}(z)||_{\ell^2}^2 \leq \sum_{n\in\mathbb{Z}^N}|z_n|^{4m+2} \leq ||z||_2^{4m+2}. \tag{4.12}$$

Then, writing $F_* = \beta\sum_{j=1}^m F_{2j}$, we may also define from F_* a nonlinear map $\mathbf{F}_* : \ell^2 \to \ell^2$, since

$$||\mathbf{F}_*(z)||_{\ell^2} \leq \beta\sum_{j=1}^{m}||\mathbf{F}_{2j}(z)||_{\ell^2} \leq \beta\sum_{j=1}^{m}||z||_2^{2j+1}. \tag{4.13}$$

Similarly, from the remainder term T_*, we may define a map $\mathbf{T}_* : \ell^2 \to \ell^2$: we have

$$||\mathbf{T}_*(\phi)||_{\ell^2}^2 \leq \beta^2\sum_{n\in\mathbb{Z}^N}|\phi_n|^{4m+4} \leq \beta^2||\phi||_{\ell^2}^{4m+4}. \tag{4.14}$$

Hence the assumptions of Theorem 4.1, are satisfied and the auxiliary linear problem

$$(\mathcal{T}_\delta\phi)_{n\in\mathbb{Z}^N} = \mathcal{K}(z_n), \ \ (\mathcal{K}(z))_{n\in\mathbb{Z}^N} := (\mathbf{F}_*(z)))_{n\in\mathbb{Z}^N} + (\mathbf{T}_*(z))_{n\in\mathbb{Z}^N}, \tag{4.15}$$

has a unique solution.

We proceed with the definition of the map $\mathcal{L}$ on which the fixed point argument will be applied. For any given $z \in \ell^2$, we define the map $\mathcal{L} : \ell^2 \to \ell^2$, by $\mathcal{L}(z) := \phi$, where ϕ is the unique solution of the operator equation (4.15). Thus, $\mathcal{L} : \ell^2 \to \ell^2$ is well defined. We will verify next that $\mathcal{L} : \mathcal{B}_R \to \mathcal{B}_R$ and is a contraction, satisfying the hypotheses of the Banach fixed point theorem.

Let $\zeta,\ \xi \in \mathcal{B}_R$ such that $\phi = \mathcal{L}(\zeta)$, $\psi = \mathcal{L}(\xi)$. The difference $\chi := \phi - \psi$ satisfies the equation

$$(\mathcal{T}_\delta\chi)_{n\in\mathbb{Z}^N} = (\mathcal{K}(\zeta))_{n\in\mathbb{Z}^N} - (\mathcal{K}(\xi))_{n\in\mathbb{Z}^N}, \tag{4.16}$$

where

$$(\mathcal{K}(\zeta))_{n\in\mathbb{Z}^N} - (\mathcal{K}(\xi))_{n\in\mathbb{Z}^N} = \{(\mathbf{F}_*(\zeta))_{n\in\mathbb{Z}^N} - (\mathbf{F}_*(\xi))_{n\in\mathbb{Z}^N}\} + \{(\mathbf{T}_*(\zeta))_{n\in\mathbb{Z}^N} - (\mathbf{T}_*(\xi))_{n\in\mathbb{Z}^N}\}.$$

The map $\mathbf{F}_* : \ell^2 \to \ell^2$ is locally Lipschitz: We recall that for any $F \in \mathrm{C}(\mathbb{C},\mathbb{C})$ which takes the form $F(z) = g(|\zeta|^2)\zeta$, with g real and sufficiently smooth, the following relation holds

$$F(\zeta) - F(\xi) = \int_0^1 \left\{(\zeta-\xi)(g(r) + rg'(r)) + (\overline{\zeta}-\overline{\xi})\Phi^2 g'(r)\right\} d\theta, \tag{4.17}$$

for any $\zeta,\ \xi \in \mathbb{C}$,where $\Phi = \theta\zeta + (1-\theta)\xi$, $\theta \in (0,1)$ and $r = |\Phi|^2$ (see [7, pg. 202]). Applying (4.17) for the case of $F_{2m}(\zeta) = |\zeta|^{2m}\zeta$, one finds that

$$F_{2m}(\zeta) - F_{2m}(\xi) = \int_0^1[(m+1)(\zeta-\xi)|\Phi|^{2m} + m(\overline{\zeta}-\overline{\xi})\Phi^2|\Phi|^{2m-2}]d\theta. \tag{4.18}$$

Assuming that $\zeta,\,\xi\in\mathcal{B}_R$, and noting that $||\Phi||_2\leq R$, we get from (4.18) the inequality

$$\begin{aligned}\sum_{n\in\mathbb{Z}^N}|\mathbf{F}_{2m}(\zeta_n)-\mathbf{F}_{2m}(\xi_n)|^2 &\leq \beta^2(2m+1)^2\sum_{n\in\mathbb{Z}^N}\left\{\int_0^1|\Phi_n|^{2m}|\zeta_n-\xi_n|d\theta\right\}^2\\ &\leq \beta^2(2m+1)^2\sum_{n\in\mathbb{Z}^N}\left\{\int_0^1||\Phi||_2^{2m}|\zeta_n-\xi_n|d\theta\right\}^2\\ &\leq \beta^2(2m+1)^2\sum_{n\in\mathbb{Z}^N}\left\{\int_0^1R^{2m}|\zeta_n-\xi_n|d\theta\right\}^2\\ &=\beta^2(2m+1)^2R^{4m}\sum_{n\in\mathbb{Z}^N}|\zeta_n-\xi_n|^2. \end{aligned}\tag{4.19}$$

Application of (4.17) to the remainder term, where

$$g(r)=\frac{(-1)^{m+2}r^{m+1}}{1+r},$$

implies that

$$\begin{aligned}T_*(\zeta)-T_*(\xi) &= \int_0^1(\zeta-\xi)\left[\frac{(-1)^{m+2}(m+2)|\Phi|^{2m+2}}{1+|\Phi|^2}-\frac{(-1)^{m+2}|\Phi|^{2m+4}}{(1+|\Phi|^2)^2}\right]d\theta\\ &+\int_0^1(\overline{\zeta}-\overline{\xi})\Phi^2\left[\frac{(-1)^{m+2}(m+1)|\Phi|^{2m}}{1+|\Phi|^2}-\frac{(-1)^{m+2}|\Phi|^{2m+2}}{(1+|\Phi|^2)^2}\right]d\theta.\end{aligned}\tag{4.20}$$

Then working similarly as for the derivation of (4.19), we get that

$$\begin{aligned}\sum_{n\in\mathbb{Z}^N}|\mathbf{T}_*(\zeta_n)-\mathbf{T}_*(\xi_n)|^2 &\leq \beta^2\sum_{n\in\mathbb{Z}^N}\left\{\int_0^1\left[(2m+3)|\Phi_n|^{2m+2}+2|\Phi_n|^{2m+4}\right]|\zeta_n-\xi_n|d\theta\right\}^2\\ &\leq \beta^2\sum_{n\in\mathbb{Z}^N}\left\{\int_0^1\left[(2m+3)||\Phi||_{\ell^2}^{2m+2}+2||\Phi||_{\ell^2}^{2m+4}\right]|\zeta_n-\xi_n|d\theta\right\}^2\\ &\leq \beta^2\left[(2m+3)R^{2m+2}+2R^{2m+4}\right]^2\sum_{n\in\mathbb{Z}^N}|\zeta_n-\xi_n|^2.\end{aligned}\tag{4.21}$$

From inequalities (4.19) and (4.21), we set

$$\begin{aligned}L_1(R) &= \sum_{j=1}^m(2j+1)R^{2j},\\ L_2(R) &= (2m+3)R^{2(m+1)}+2R^{2(m+2)},\\ L(R) &= L_1(R)+L_2(R).\end{aligned}\tag{4.22}$$

Then combining (4.19) and (4.21), we observe that the map $\mathcal{K}:\ell^2\to\ell^2$ is locally Lipschitz, satisfying

$$||\mathcal{K}(\zeta)-\mathcal{K}(\xi)||_{\ell^2}\leq\beta L(R)||\zeta-\xi||_{\ell^2}.\tag{4.23}$$

Taking now the scalar product of (4.16) with χ in ℓ^2 and using (4.23), we have that

$$\begin{aligned}\epsilon\sum_{\nu=1}^N||\nabla_\nu^+\chi||_{\ell^2}^2+\delta||\chi||_2^2 &\leq ||\mathcal{K}(\zeta)-\mathcal{K}(\xi)||_2||\chi||_2\\ &\leq \beta L(R)||\zeta-\xi||_2||\chi||_2\\ &\leq \frac{\delta}{2}||\chi||_2^2+\frac{\beta^2}{2\delta}L^2(R)||\zeta-\xi||_2^2.\end{aligned}\tag{4.24}$$

From (4.24), we obtain the inequality

$$||\chi||_2^2=||\mathcal{L}(z)-\mathcal{L}(\xi)||_2^2\leq\frac{\beta^2}{\delta^2}L^2(R)||\zeta-\xi||_2^2.\tag{4.25}$$

Since $\mathcal{L}(0) = 0$, we observe that the map $\mathcal{L} : \mathcal{B}_R \to \mathcal{B}_R$ and is a contraction, having a unique fixed point – the trivial one – if

$$L(R) < \frac{\delta}{\beta}. \tag{4.26}$$

We consider the polynomial function

$$\Pi(R) := L(R) - \frac{\delta}{\beta}. \tag{4.27}$$

A threshold value for the existence of nontrivial breather solutions can be derived from condition (4.26): the polynomial equation $\Pi(R) = 0$, has exactly two real roots $R^* < 0 < R_*$, such that $R_* = -R^*$. Thus

$$\Pi(R) < 0 \ \text{ for every } \ R \in (0, R_*),$$

that is, condition (4.26) is satisfied if $R \in (0, R_*)$. We summarize our results in the following

Theorem 4.2 *We assume that the parameters $\epsilon > 0$ and $\beta, \Omega > 0$ are given such that*

$$\beta > \Omega.$$

Let $R_* > 0$ denote the unique positive root of the polynomial equation $\Pi(R) = 0$, where $\Pi(R)$ and $L(R)$ are given by (4.27) and (4.22) respectively. Then a breather solution (4.1) of (1.1), must have power bigger than

$$\mathcal{P}_{\beta,\Omega} := R_*^2(\beta, \Omega) \tag{4.28}$$

Cubic-Quintic approximation The saturable nonlinearity can be approximated by a cubic-quintic approximation ($m = 2$ in Taylor's formula (4.5)).

We consider first the approximation without taking into account the remainder term (the term $L_2(R)$ does not appear in the polynomial equation). Stationary wave solutions (4.1) satisfy the infinite system of algebraic equations

$$-\epsilon(\Delta_d \phi)_n - \Omega \phi_n = -\beta(1 - |\phi_n|^2 + |\phi_n|^4)\phi_n, \ \ n \in \mathbb{Z}^N, \ \ \beta > \Omega > 0. \tag{4.29}$$

In this case, the threshold value is $\mathcal{P}_{\beta,\Omega} = R_*^2$, where R_* is the root of the quadratic equation $\Pi(R) = 3R^2 + 5R^4 - \frac{\delta}{\beta} = 0$, $\delta = \beta - \Omega$,

$$\mathcal{P}_{\beta,\Omega} = {R_*}^2(\beta, \Omega) = -\frac{3}{10} + \frac{(29\beta - 20\Omega)^{1/2}}{10\beta^{1/2}}. \tag{4.30}$$

Setting, for example $\beta = 2$, $\Omega = 0.5$, we obtain the threshold value

$$\mathcal{P}_{2,0.5} \approx 0.189898.$$

Exact saturable nonlinearity For the exact saturable nonlinearity we should take into account the remainder term: we look for the root R^*, of the equation $\Pi(R) = 3R^2 + 5R^4 + 7R^6 + 2R^8 - \frac{\delta}{\beta} = 0$. For $\beta = 1$, $\Omega = 0.5$, we obtain

$$\mathcal{P}_{2,0.5} \approx 0.180917.$$

Note that the threshold value appears to be the same for parameters $\beta > \Omega > 0$, giving the same ratio $\frac{\beta - \Omega}{\beta}$.

Finite dimensional lattice We may also derive a threshold value, taking into account the finite dimensionality of the lattice, when the problem is supplemented with Dirichlet boundary conditions. We may replace the constant $\delta = \beta - \Omega > 0$ by the constant

$$\delta_1 = \epsilon\lambda_1 + \beta - \Omega,$$

where $\beta > \Omega > 0$. Therefore, in this case one has to work with the polynomial equation

$$\Pi(R) := L(R) - \frac{\delta_1}{\beta}. \tag{4.31}$$

For the case of the cubic quintic approximation, the threshold value (4.30), (without considering the remainder term), becomes

$$\mathcal{P}_{\beta,\Omega,D} = -\frac{3}{10} + \frac{[29\beta + 20(\epsilon\lambda_1 - \Omega)]^{1/2}}{10\beta^{1/2}}, \tag{4.32}$$

involving the principal eigenvalue $\lambda_1 > 0$ of the operator $-\Delta_d$ and lattice spacing ϵ.

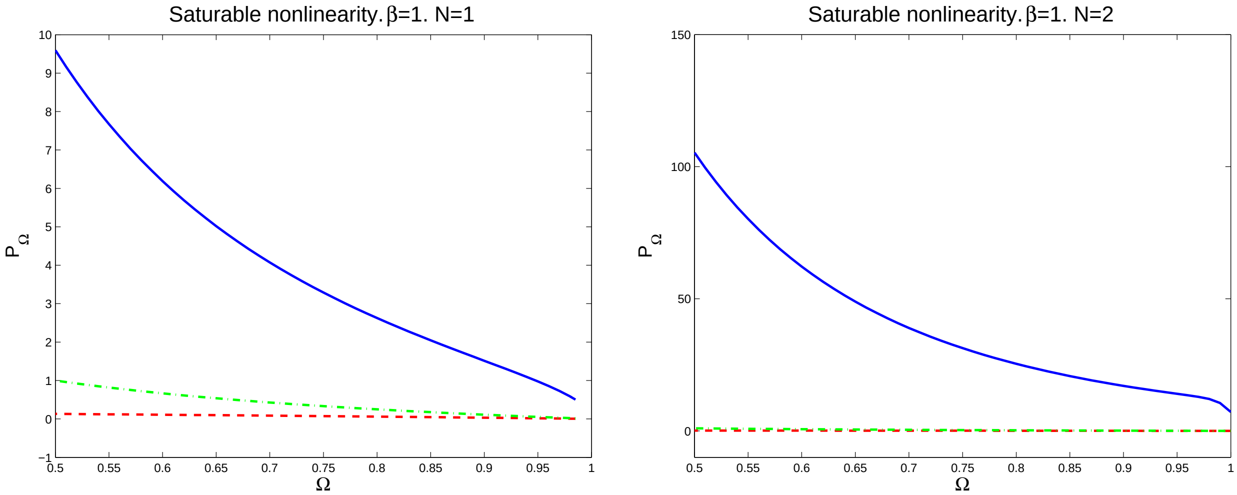

Figure 2: Dependence of the power with respect to the frequency for $1D$ and $2D$ lattices of the focusing DNLS with saturable nonlinearity for $\beta = 1$ and $\epsilon = 1$. Power decreases approaching the theoretical lower bounds when Ω tends to β.

Numerical study. We performed numerical studies to test the lower bound (3.52) derived in Theorem 3.3, and the threshold (4.28) derived in Theorem 4.2. In figures 2 and 3, we show the dependence of the power $\mathcal{P}_\Omega = \sum |\psi_n|^2$ with respect to the frequency Ω. In the figures, the solid line represents the numerically calculated power, while the dashed line corresponds to the analytical threshold defined as the root of the equation (4.31) for the finite with Dirichlet b.c. lattice. The dot-dashed line corresponds to the lower bound defined in (3.52). In most of the cases considered in our study, the difference between infinite and finite lattices is difficult to see, so the former has not been represented in the figures. As we cannot model an infinite lattice numerically, the numerical results for the power correspond to finite lattices with Dirichlet b.c. Note that, due to the resonance with linear modes, the frequencies of the breathers are limited by the condition $\Omega < \beta$. Besides, the continuation of the solutions is quite difficult close to this limit.

Figure 2 refers to the parameters $\beta = 1$ and $\epsilon = 1$. The figure verifies that we should not expect existence of breather solutions below the threshold value (4.28). As Ω increases to the limit β, power decreases, approaching both theoretical estimates. We also note that the lower bound (3.52) predicts the decrease of the power as frequency increases, approaching the threshold (4.28).

Figure 3 considers the case $\beta = 10$, $\epsilon = 1$ in 1D and 2D-lattices. We observe the increased accuracy of the qualitative and quantitative predictions of the variational lower bound (3.52), in the 1D-case. In the 2D-case an excitation threshold appears, i.e. there exist a minimum of the power (excitation threshold), and as frequency increases, the power increases, reaching a local maximum. This behaviour is in accordance with that described in the recent work [20]. In the focusing saturable DNLS, this behaviour seems to appear in the 2D-lattice for larger values of the parameter β, while in the defocusing case this behaviour seems to appear when only the dimension of the lattice is increased.

The "limiting case" with respect to the size of β, $\beta = 10$, clearly demonstrates that (4.28) is a quite sharp estimate of the threshold value on the power, for the existence of a breather in the focusing saturable nonlinearity. The corresponding breathers with large frequencies are real examples demonstrating that this estimate is the smallest value on the power of an existing breather. The same closeness appears for the variational lower bound (3.52).

5 DNLS equation with power nonlinearity

5.1 Thresholds for the defocusing DNLS $\beta > 0$: solutions $\psi_n(t) = e^{-\mathrm{i}\Omega t}\phi_n, \Omega > 0$.

In this subsection we derive a threshold on the power for the existence of non-trivial breather solutions for the DNLS equation with power nonlinearity (1.1), in the *defocusing case* $\beta > 0$. We seek breather solutions

$$\psi_n(t) = e^{-\mathrm{i}\Omega t}\phi_n, \quad \Omega > 0, \tag{5.1}$$

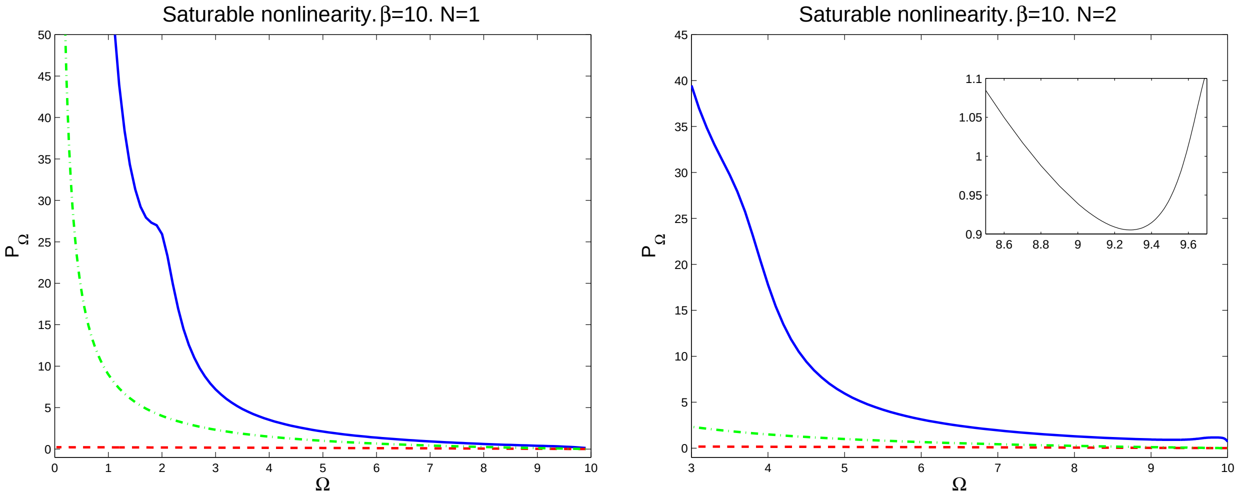

Figure 3: Dependence of the power with respect to the frequency for $1D$ and $2D$ lattices of the focusing DNLS with saturable nonlinearity for $\beta = 10$ and $\epsilon = 1$. Power decreases approaching the lower bounds when Ω tends to β. Increased accuracy of the variational lower bound (3.52), for $N = 1$. An excitation threshold appears in the 2D-lattice.

for the DNLS equation

$$\mathrm{i}\dot{\psi}_n + \epsilon(\Delta_d \psi)_n - \beta|\psi_n|^{2\sigma}\psi_n = 0, \quad \beta > 0. \tag{5.2}$$

Solutions (5.1) of DNLS equation (5.2), satisfy the equation

$$-\epsilon(\Delta_d \phi)_n - \Omega\phi_n = -\beta|\phi_n|^{2\sigma}\phi_n, \quad \Omega > 0, \ \beta > 0. \tag{5.3}$$

We consider the Hamiltonian for the defocusing DNLS (5.2)

$$\mathcal{H}_\sigma[\phi] = \epsilon\sum_{\nu=1}^{N} ||\nabla_\nu^+ \phi||_2^2 + \frac{\beta}{\sigma+1}\sum_{n\in\mathbb{Z}^N} |\phi_n|^{2\sigma+2}, \quad \beta > 0, \tag{5.4}$$

and the energy functional

$$\mathcal{E}_\Omega[\phi] = \epsilon\sum_{\nu=1}^{N} ||\nabla_\nu^+ \phi||_2^2 - \Omega\sum_{n\in\mathbb{Z}^N} |\phi_n|^2, \quad \Omega > 0. \tag{5.5}$$

We also recall that the derivative of the functional

$$\mathcal{L}_R[\phi] = \sum_{|n|\leq K} |\phi_n|^{2\sigma+2},$$

is given by (see also [10, Lemma 2.3])

$$\langle \mathcal{L}_R'[\phi], \psi\rangle = (2\sigma+2)\mathrm{Re}\sum_{|n|\leq K} |\phi_n|^{2\sigma}\phi_n\overline{\psi}, \ \text{for all} \ \ \psi \in \ell^2(\mathbb{Z}_K^N). \tag{5.6}$$

Existence of a nontrivial breather solution (5.1), will be derived by considering constrained minimization problems similar to those in Section 3 (see also [21]). We have the following

Theorem 5.1 A. *Consider the variational problem on* $\ell^2(\mathbb{Z}_K^N)$

$$\inf\left\{\mathcal{H}_\sigma[\phi] \ : \mathcal{P}[\phi] = R^2 > 0\right\}, \tag{5.7}$$

for some $\beta > 0$. There exists a minimizer $\phi^* \in \ell^2(\mathbb{Z}^N_K)$ for the variational problem (5.7) and $\omega(R) > 0$, such that $\Omega(R) > \epsilon\lambda_1$, with both satisfying the Euler-Lagrange equation

$$\begin{aligned} -\epsilon(\Delta_d\phi)_n &+ \beta|\phi_n|^{2\sigma}\phi_n = \Omega\phi_n, \ \ |n| \leq K, \\ \phi^*_n &= 0, \ \ |n| > K, \end{aligned}$$

with $\mathcal{P}[\phi^*] = R^2$.
B. *Consider the variational problem on $\ell^2(\mathbb{Z}^N_K)$*

$$\inf\left\{\mathcal{E}_\Omega[\phi] \ : \sum_{|n|\leq K}|\phi_n|^{2\sigma+2} = M > 0\right\}, \tag{5.8}$$

for some $\Omega > 0$. Assume that

$$\Omega > 4\epsilon N. \tag{5.9}$$

There exists a minimizer $\hat{\phi} \in \ell^2(\mathbb{Z}^N_K)$ for the variational problem (5.8) and $\beta(M) > 0$, both satisfying the Euler-Lagrange equation (5.8), and $\sum_{|n|\leq K}|\hat{\phi}_n|^{2\sigma+2} = M$. Assume that the power of the minimizer is $\mathcal{P}[\hat{\phi}] = R^2$. Then the power satisfies the lower bound

$$\left[\frac{\Omega - 4N\epsilon}{\beta(\sigma+1)}\right]^{\frac{1}{\sigma}} \leq R^2 := \mathcal{P}_1[\hat{\phi}]. \tag{5.10}$$

Proof: A. Relation (5.6) implies that the functional $\mathcal{H}_\sigma : \ell^2(\mathbb{Z}^N_K) \to \mathbb{R}$ is a C^1-functional. It is bounded from below, since from (2.13)

$$\mathcal{H}_\sigma[\phi] \geq \epsilon\sum_{\nu=1}^N ||\nabla^+_\nu\phi||^2_2 \geq \epsilon\lambda_1 R^2.$$

The same variational arguments of Section 3 imply the existence of a minimizer $\phi^* \in \ell^2(\mathbb{Z}^N_K)$ of $\mathcal{H}_\sigma$, and the Lagrange multiplier $\Omega(R) > 0$, such that

$$\begin{aligned} \langle\mathcal{H}'_\sigma[\phi^*] - \Omega\mathcal{N}'_R[\phi], \psi\rangle &= 2\epsilon\sum_{\nu=1}^N(\nabla^+_\nu\phi^*, \nabla^+_\nu\psi)_2 + 2\beta\mathrm{Re}\sum_{|n|\leq K}|\phi^*_n|^{2\sigma}\phi^*_n\overline{\psi}_n \\ &\quad -2\Omega\sum_{|n|\leq K}\phi^*_n\overline{\psi_n} = 0, \ \text{for all} \ \psi \in \ell^2(\mathbb{Z}^N_K). \end{aligned} \tag{5.11}$$

Then setting $\psi = \phi^*$ in (5.11), and by using (2.14), we obtain that

$$2\epsilon\lambda_1\sum_{|n|\leq K}|\phi^*_n|^2 \leq 2\epsilon\sum_{\nu=1}^N||\nabla^+_\nu\phi^*||^2_2 + 2\beta\sum_{|n|\leq K}|\phi^*_n|^{2\sigma+2} = 2\Omega\sum_{|n|\leq K}|\phi^*_n|^2, \tag{5.12}$$

which shows that $\Omega(R) > \epsilon\lambda_1 > 0$.

B. The functional $\mathcal{E}_\Omega$ is bounded from below: the equivalence of norms (2.12), implies the existence of a N-dependent constant C_2, such that

$$||\phi||^2_2 \leq C_2^2||\phi||^2_{2\sigma+2}, \ \text{for all} \ \phi \in \ell^2(\mathbb{Z}^N_K). \tag{5.13}$$

Then using (5.13), we find that

$$\mathcal{E}_\Omega[\phi] \geq -\Omega\sum_{|n|\leq K}|\phi_n|^2 \tag{5.14}$$

$$\geq -\Omega C_2^2\left(\sum_{|n|\leq K}|\phi_n|^{2\sigma+2}\right)^{\frac{1}{\sigma+1}} \tag{5.15}$$

$$\geq -\Omega C_2^2 M^{\frac{1}{\sigma+1}}. \tag{5.16}$$

Again the existence of the minimizer $\hat{\phi}$ and of the Lagrange multiplier $\lambda(M) \in \mathbb{R}$ can be obtained by the same arguments as in Section 3. Moreover by using (5.6), we have that

$$\begin{aligned}\left\langle \mathcal{E}'_\Omega[\hat{\phi}] - \lambda \mathcal{L}'_R[\hat{\phi}], \psi \right\rangle &= 2\epsilon \sum_{\nu=1}^{N} (\nabla_\nu^+ \hat{\phi}, \nabla_\nu^+ \psi)_2 - 2\Omega \mathrm{Re} \sum_{|n|\leq K} \hat{\phi}_n \overline{\psi}_n \\ &\quad -2(\sigma+1)\lambda \sum_{|n|\leq K} |\hat{\phi}_n|^{2\sigma} \hat{\phi}_n \overline{\psi_n} = 0, \quad \text{for all } \psi \in \ell^2(\mathbb{Z}_K^N). \end{aligned} \tag{5.17}$$

Setting $\psi = \hat{\phi}$ in (5.17), we obtain

$$2\mathcal{E}_\Omega[\hat{\phi}] = 2\epsilon \sum_{\nu=1}^{N} ||\nabla_\nu^+ \hat{\phi}||_2^2 - 2\Omega \sum_{|n|\leq K} |\hat{\phi}_n|^2 = 2(\sigma+1)\lambda \sum_{|n|\leq K} |\hat{\phi}|^{2\sigma+2}. \tag{5.18}$$

We observe that

$$\mathcal{E}_\Omega[\hat{\phi}] \leq 4\epsilon N \sum_{|n|\leq K} |\hat{\phi}_n|^2 - \Omega \sum_{|n|\leq K} |\hat{\phi}_n|^2. \tag{5.19}$$

Thus $\mathcal{E}_\Omega[\hat{\phi}] < 0$ if (5.9) is satisfied. Note that due to the estimate (2.15), the condition (5.9) implies that

$$\Omega > \epsilon \lambda_1. \tag{5.20}$$

Then assuming (5.9), we find that $\lambda(M) < 0$. We set $\lambda = -\beta$, $\beta > 0$. Finally, we assume that the power of the nontrivial minimizer $\hat{\phi}$ is $\sum_{|n|\leq K} |\hat{\phi}_n|^2 = R^2$. Then, returning to (5.18), and by using (2.2) which holds also in the finite dimensional lattice, we get

$$\begin{aligned} 2\Omega R^2 &= 2\epsilon \sum_{\nu=1}^{N} ||\nabla_\nu^+ \hat{\phi}||_2^2 + 2(\sigma+1)\beta \sum_{|n|\leq K} |\hat{\phi}_n|^{2\sigma+2} \\ &\leq 8\epsilon N \sum_{|n|\leq K} |\hat{\phi}_n|^2 + 2(\sigma+1)\beta \sum_{|n|\leq K} |\hat{\phi}_n|^{2\sigma+2} \\ &\leq 8\epsilon N \sum_{|n|\leq K} |\hat{\phi}_n|^2 + 2(\sigma+1)\beta \left(\sum_{|n|\leq K} |\hat{\phi}_n|^2 \right)^{\sigma+1} \\ &\leq 8\epsilon N R^2 + 2(\sigma+1)\beta R^{2\sigma+2}. \end{aligned} \tag{5.21}$$

Under condition (5.9), inequality (5.21) implies the lower bound (5.10). $\diamond$

A threshold for the power could be also derived from the case A. of Theorem 5.1: working exactly as for the derivation of that in (5.21), we find from (5.12), that

$$\Omega R^2 \leq 4\epsilon N R^2 + \beta R^{2\sigma+2}. \tag{5.22}$$

Thus, in the case of Theorem 5.1 A, and under the hypothesis that the Lagrange multiplier $\Omega(R)$ of the case A is taking values $\Omega > 4\epsilon N$, we find from (5.22) that

$$\left[\frac{\Omega - 4N\epsilon}{\beta} \right]^{\frac{1}{\sigma}} \leq R^2 := \mathcal{P}_2[\phi^*]. \tag{5.23}$$

Comparing with (5.10), it readily follows that

$$\mathcal{P}_1[\hat{\phi}] < \mathcal{P}_2[\phi^*]. \tag{5.24}$$

5.2 Numerical study for the defocusing DNLS with power nonlinearity.

Similarly to the results of Section 3, for the saturable DNLS, it seems interesting to test the behaviour of the lower bounds (5.10) and (5.23), as thresholds for the existence of breather solutions for the defocusing DNLS

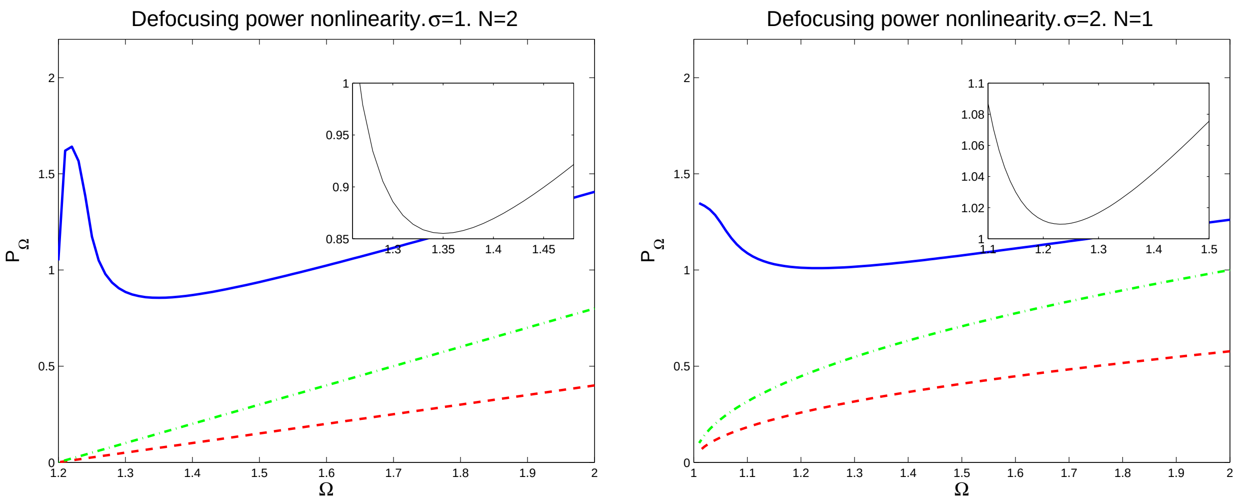

Figure 4: Numerical power for the defocusing DNLS with power nonlinearity and (a) $\sigma = 1$, $N = 2$, $\epsilon = 0.15$ (b) $\sigma = 2$, $N = 1$, $\epsilon = 0.25$. The lower bounds (5.10) and (5.23), dot-dashed and dashed lines respectively, serve as thresholds for the existence of nontrivial breather solutions.

(5.2). Theorem 5.1 A implies that, for a given $\beta > 0$, the existence of a frequency $\Omega > \epsilon\lambda_1$ (as a Lagrange multiplier), and of a nontrivial minimizer ϕ^* of the Hamiltonian (5.4), such that the corresponding breather solution $\psi_n(t) = e^{-\mathrm{i}\Omega t}\phi_n^*$ has a power satisfying the lower bound (5.23), in the case where Ω is assumed to be such that $\Omega > 4\epsilon N$. On the other hand, Theorem 5.1 B implies, for a given $\Omega > 4\epsilon N$, the existence of $\beta > 0$ and of a nontrivial minimizer $\hat{\phi}$ of the energy functional (5.5), such that the corresponding breather solution $\psi_n(t) = e^{-\mathrm{i}\Omega t}\hat{\phi}_n$, has a power satisfying the lower bound (5.10).

The numerical study is for 1D and 2D-lattices. For the case $N = 1$ we consider $\epsilon = 0.25$, and for the case $N = 2$ we consider $\epsilon = 0.15$. We study first values of σ satisfying $\sigma \geq \frac{2}{N}$. In Figure 4, the study refers to the cases $\sigma = 1$, $N = 2$, and $\sigma = 2$, $N = 1$ respectively. We observe first that the numerical power of the solutions fulfils both lower bounds (5.23), (5.10). Moreover we observe that they can be considered also as *thresholds* on the power of periodic solutions with frequencies $\Omega > 4\epsilon N$. This fact is revealed by the case $\sigma = 10$, $N = 1$ for which the lower bound (5.10) is proved to be a quite sharp estimate of the power for large frequencies. The same satisfactory accuracy of the theoretical estimates (5.10) and (5.23) is observed also in case of $\sigma = 10$, $N = 1$ considered in Figure 5. We note that the phonon band of the defocusing DNLS equation extends to the interval $[0, 4\epsilon N]$. Then breathers frequencies must lie in the intervals $\Omega > 4\epsilon N$, or $\Omega < 0$. It is the former case which we consider in this paragraph. The numerical studies for the case $\sigma < 2/N$ where the excitation threshold [21] do no exist, seem to fully justify the argument that the lower bounds (5.23), (5.10) can be used as thresholds on the power for the existence of breather solutions. The results for $\sigma < 2/N$ are demonstrated in Figures 6 and 7. Figure 6 considers the case $\sigma = 0.1$ for $N = 1$ and $N = 2$ and (5.23), (5.10) are clearly sharp estimates of the smallest value of the power a breather solution can have. This is justified by the numerical power of breather solutions with small frequencies. The numerical power for the case $\sigma = 1, N = 1$ also fulfils the theoretical estimates.

We remark that the approach of minimizing the linear energy appeared to be useful also in the defocusing DNLS with power nonlinearity since the lower bound (5.10) provides in general better quantitative predictions of the numerical power if compared with (5.23).

5.3 The focusing DNLS with power nonlinearity ($\beta < 0$): solutions $\phi_n(t) = e^{-i\Omega t}\phi_n$, $\Omega < 0$.

We conclude with a numerical study of the focusing DNLS with power nonlinearity. Thus we shall consider solutions with frequencies $\Omega < 0$ (the latter case of the phonon band condition). As in Section 3, we use for convenience, the notation $\beta = -\Lambda$, $\Lambda > 0$, and $\Omega = -\omega$, $\omega > 0$.

Instead of the threshold estimate [10, (2.38), pg. 126] derived by the fixed point argument, for breather

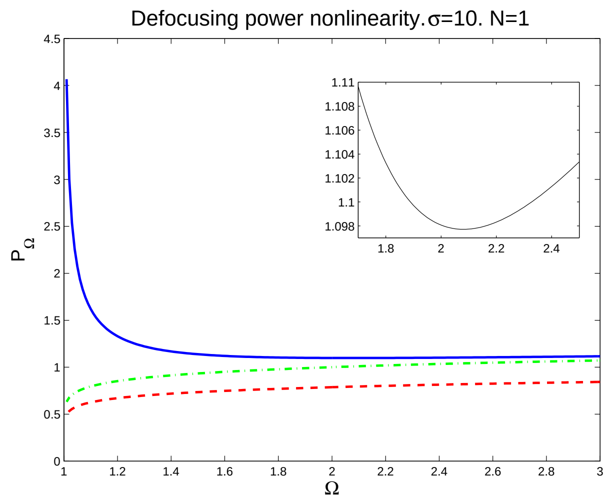

Figure 5: Numerical power for the defocusing DNLS with power nonlinearity and $\sigma = 10$, $N = 1$, $\epsilon = 0.25$. Both lower bounds (5.10) and (5.23), dot-dashed and dashed lines respectively, are fulfilled as threshold values.

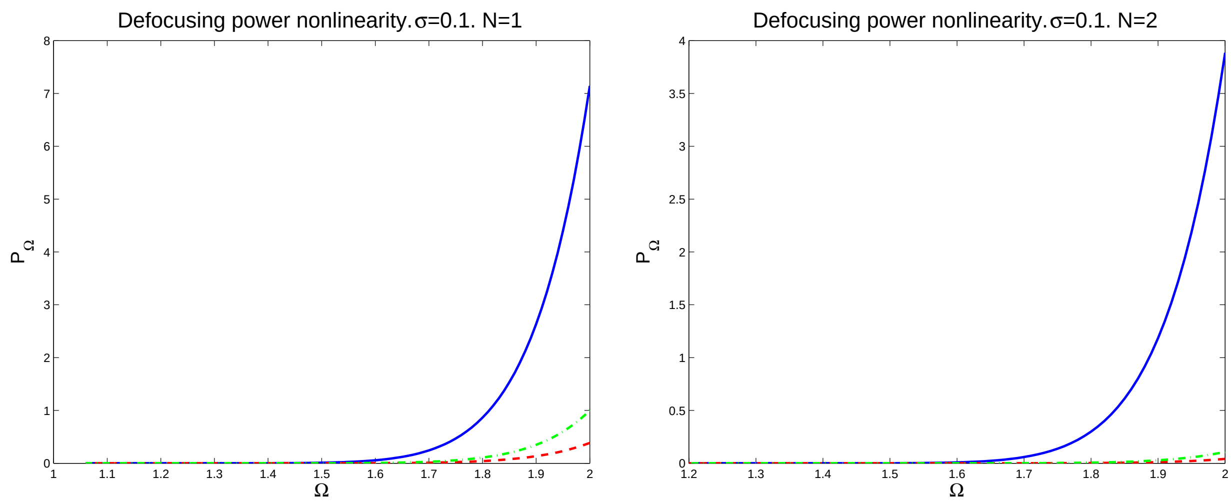

Figure 6: Numerical power for the defocusing DNLS with power nonlinearity case for (a) $\sigma = 0.1$, $N = 1$, $\epsilon = 0.25$ (b) $\sigma = 0.1$, $N = 2$, $\epsilon = 0.15$. The lower bounds (5.10) and (5.23), dot-dashed and dashed lines respectively, are fulfilled as threshold values.

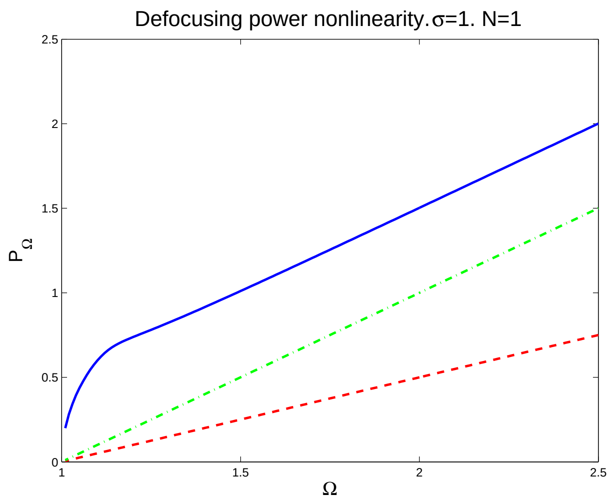

Figure 7: Numerical power for the defocusing DNLS with power nonlinearity case $\sigma = 1$, $N = 1$, $\epsilon = 0.25$. The lower bounds (5.10) and (5.23), dot-dashed and dashed lines respectively, are fulfilled as threshold values.

solutions

$$\psi_n(t) = e^{\mathrm{i}\omega t}\phi_n, \quad \omega > 0, \tag{5.25}$$

of the DNLS (1.6) in *infinite lattices*,

$$E^2_{\min} := \frac{1}{4}\left[\frac{\omega}{\Lambda(2\sigma+1)}\right]^{\frac{1}{\sigma}}, \tag{5.26}$$

we may derive a possibly improved one: by using (4.18) (which holds also for an infinite lattice) instead of [10, (2.44), pg. 127], we find that

$$\mathcal{P}_\omega := \left[\frac{\omega}{\Lambda(2\sigma+1)}\right]^{\frac{1}{\sigma}} = 4E^2_{\min}. \tag{5.27}$$

Although (5.27), holds also for the case of the Dirichlet boundary conditions, we may derive a threshold value taking into account the finite dimensionality of the problem: for solutions (5.25), we consider the operator $\mathcal{T}_\Omega : \ell^2(\mathbb{Z}^N_K) \to \ell^2(\mathbb{Z}^N_K)$, defined as

$$(\mathcal{T}_\omega\phi)_{|n|\leq K} \quad = \quad -\epsilon(\Delta_d\phi)_{|n|\leq K} + \omega\phi_n. \tag{5.28}$$

Clearly

$$\delta_1 := \epsilon\lambda_1 + \omega > 0, \tag{5.29}$$

and by using (2.13), we get that

$$(\mathcal{T}_\omega\phi, \phi)_2 = \epsilon\sum_{\nu=1}^{N}|\nabla^+_\nu\phi||^2_2 + \omega||\phi||^2_2 \geq \delta_1||\phi||^2_2 \;\text{ for all }\; \phi \in \ell^2(\mathbb{Z}^N_K). \tag{5.30}$$

Then, by using (4.19), we may derive the threshold value

$$\mathcal{P}_{\omega,D} := \left[\frac{\epsilon\lambda_1 + \omega}{\Lambda(2\sigma+1)}\right]^{\frac{1}{\sigma}}. \tag{5.31}$$

Throughout this study we have fixed $\epsilon = 1$. A first result is that the power of the solutions fulfils (5.27), that is, it is higher than

$$\mathcal{P}_\omega := \left[\frac{\omega}{\Lambda(2\sigma+1)}\right]^{\frac{1}{\sigma}}$$

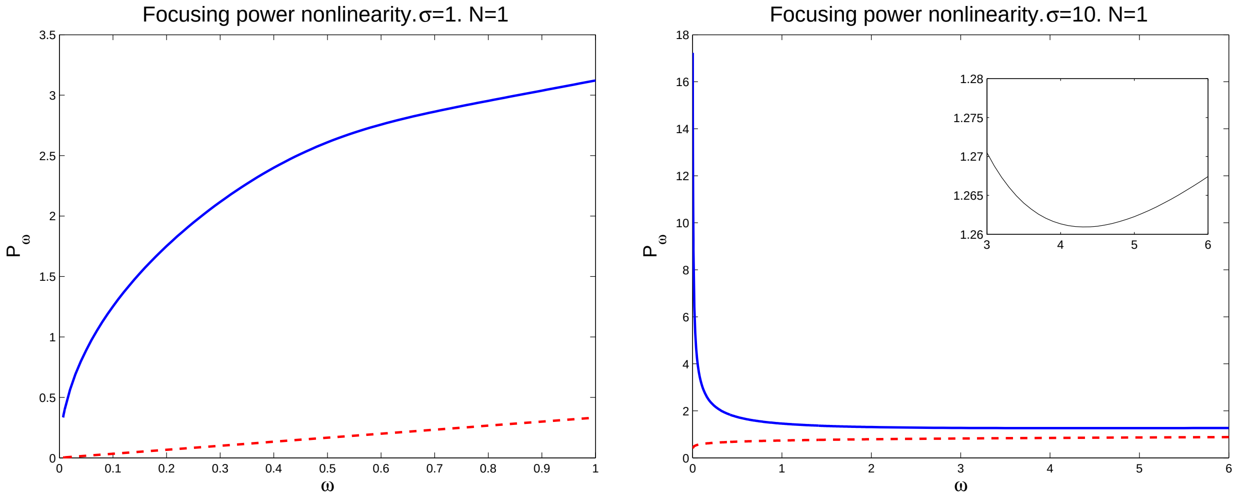

Figure 8: Numerical power for the focusing DNLS with power nonlinearity and (a) $\sigma = 1$, $N = 1$, $\epsilon = 1$ ($\sigma < \frac{2}{N}$-no excitation threshold), (b) $\sigma = 10$, $N = 1$, $\epsilon = 1$ ($\sigma > \frac{2}{N}$-excitation threshold). The inset in (b) shows a magnification of the region where the power reaches its minimum value. Dashed line corresponds to lower bound (5.31)

We mention first, that regarding the numerical study of the threshold (5.31), as in the case of the saturable nonlinearity, we have not observed remarkable improvement or differences in comparison with (5.27). Although the threshold (5.27) is independent of the dimension, it is interesting to compare this with the results of [21], related to the conditions on existence of excitation threshold which depends on the dimension and the nonlinearity exponent σ. In Figure 8, we present the numerical power (solid curve) for the cases $\sigma = 1$ and $\sigma = 10$, and $N = 1$. The values $\sigma = 1$ and $N = 1$ satisfy $\sigma < 2/N$, and the power fulfils the threshold (5.27) for existence. The case $\sigma = 10$ and $N = 1$ satisfies $\sigma > 2/N$, and as predicted in [21] and [6], the power approaches a minimum (the excitation threshold). The numerical power still fulfils the threshold (5.27). In the latter case, the corresponding breathers with large frequencies provide real examples demonstrating that (5.27) is a quite sharp estimate of the smallest value of the power a breather can have when $\sigma \geq 2/N$. Note that the dependence of this excitation threshold with respect to σ has been numerically calculated in [13].

In figure 9 we present the results for the cases $\sigma = 0.1, N = 1$ and $\sigma = 1, N = 2$. The first case satisfies $\sigma < 2/N$ and the second satisfies $\sigma \geq 2/N$, in its critical value $\sigma = \frac{2}{N}$. The threshold (5.27) is fulfilled in both cases. We observe that in the case $\sigma = 0.1, N = 1$, the threshold (5.27) is a sharp estimate of the power observed for small values of frequency. These breathers serve this time as real examples demonstrating that (5.27) predicts the smallest value of a breather also in the case $\sigma < 2/N$. In the second case we observe the appearance of the excitation threshold.

To show the global character of the estimate (5.27) as a threshold on the existence of breather solutions, we consider in Figure 10 the "limiting" cases with respect to the size of the nonlinearity exponent, $\sigma = 0.1$ and $\sigma = 10$, this time for $N = 2$. The first case is again an example for $\sigma < 2/N$ and the second for $\sigma > 2/N$. The threshold (5.27) is again fulfilled in both cases. We observe that in both cases the threshold (5.27) is still a quite sharp estimate of the smallest value of the power of an existing breather as the comparison with the power of breathers with small (large) values of frequency demonstrates.

It is worth noticing the existence of a maximum in the power for $\sigma = 1$, $N = 2$, as was predicted in [12].

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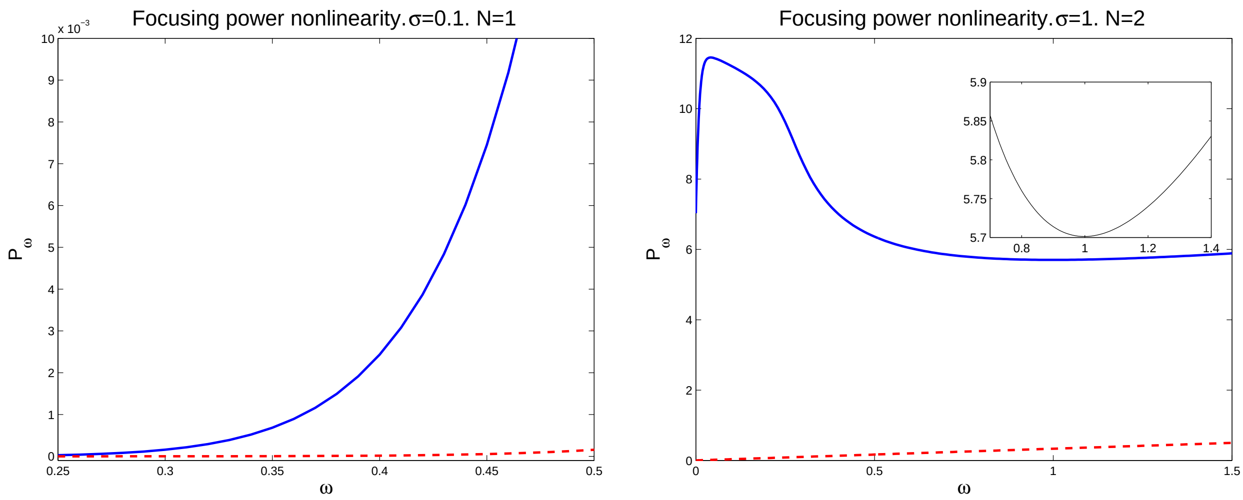

Figure 9: Numerical power for the focusing DNLS with power nonlinearity and (a) $\sigma = 0.1$, $N = 1$, $\epsilon = 1$ ($\sigma < \frac{2}{N}$-no excitation threshold), (b) $\sigma = 1$, $N = 2$, $\epsilon = 1$ ($\sigma = \frac{2}{N}$-excitation threshold). The inset in (b) shows a magnification of the region where the power reaches its minimum value. Dashed line corresponds to lower bound (5.31)

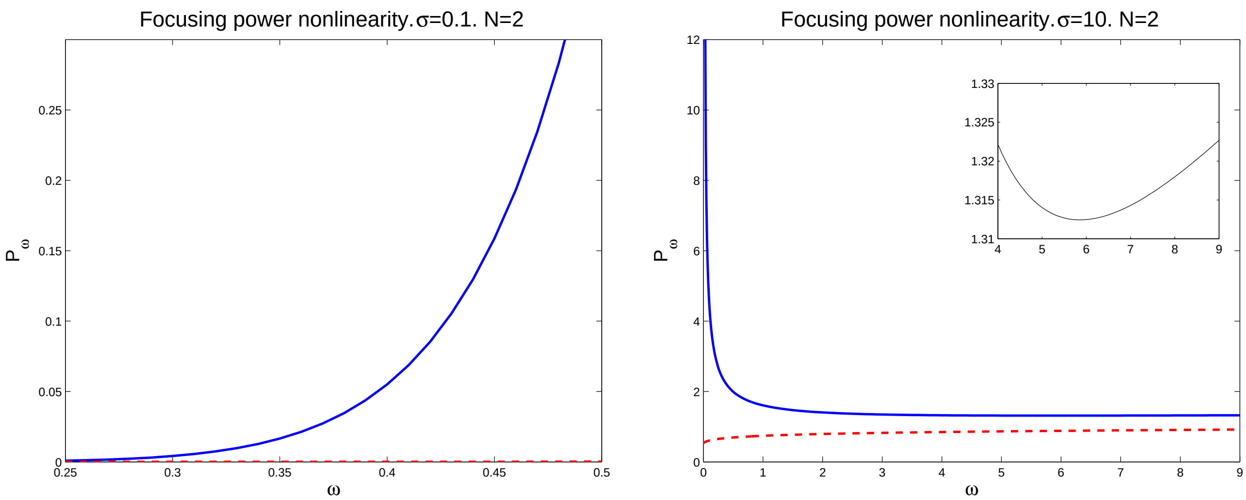

Figure 10: Numerical power for the focusing DNLS with power nonlinearity and (a) $\sigma = 0.1$, $N = 2$, $\epsilon = 1$ ($\sigma < \frac{2}{N}$-no excitation threshold), (b) $\sigma = 10$, $N = 2$, $\epsilon = 1$ ($\sigma > \frac{2}{N}$-excitation threshold). The inset in (b) shows a magnification of the region where the power reaches its minimum value. Dashed line corresponds to lower bound (5.31)

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