

# The persistence of synchronization under environmental noise

BY TOMÁS CARABALLO<sup>a</sup> AND PETER E. KLOEDEN<sup>b</sup>

<sup>a</sup> *Departamento de Ecuaciones Diferenciales y Análisis Numérico,  
Universidad de Sevilla, Apdo. de Correos 1160,  
41080-Sevilla, Spain  
E-mail: caraball@us.es*

<sup>b</sup> *Fachbereich Mathematik, Johann Wolfgang Goethe Universität  
D-60054 Frankfurt am Main, Germany  
E-mail: kloeden@math.uni-frankfurt.de*

It is shown that the synchronization of dissipative systems persists when they are disturbed by additive noise no matter how large the intensity of the noise provided asymptotically stable stationary stochastic solutions are used instead of asymptotically stable equilibria.

**Keywords:** Synchronization, additive noise, random attractor, stationary stochastic process, one-sided Lipschitz dissipative condition

## 1. Introduction

Synchronization of coupled systems is a very well known phenomenon in biology and physics, and also in the social sciences. A readable descriptive account of its diversity of occurrence can be found in the recent book of Strogatz (2003), which contains an extensive list of references. The synchronization of coupled dissipative systems has been investigated mathematically in the case of autonomous systems by Afraimovich and Rodrigues (1998), Carvalho *et al.* (1998) and Rodrigues (1996), both for asymptotically stable equilibria and general attractors, such as chaotic attractors. Analogous results also hold for nonautonomous systems (Kloeden (2003)), but require a new concept of a nonautonomous attractor.

In this note we investigate the effect of additive noise on the synchronization of coupled dissipative systems with asymptotically stable equilibria, which results in a coupled system of Ito stochastic differential equations. Such noise is often considered as modelling background environmental effects. We show that synchronization persists independently of noise intensity in terms of asymptotically stable stationary stochastic solutions rather than equilibria. Such asymptotically stable stationary stochastic solutions are a special case of random attractors (see Crauel & Flandoli (1994), Crauel *et al.* (1997), Schmalfuss (1992)) which are a probabilistic counterpart of the deterministic nonautonomous attractor used in Kloeden (2003).

We present some required background material in Section 2 and then formulate the problem of synchronization of dissipative systems for both the deterministic and stochastic cases in Section 3. In Section 4 we show that the uncoupled systems with additive noise have random attractors consisting of single stationary stochastic

processes and then establish the corresponding result for coupled systems in Section 5. Finally, the asymptotic behaviour of the coupled systems in the limit as the coupling parameter becomes arbitrarily large are presented in Section 6.

## 2. Background material

We need some background notation and results. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space.

Following Arnold (1998) a random dynamical system (RDS)  $(\theta, \phi)$  on  $\Omega \times \mathbb{R}^d$  consists of a metric dynamical system  $\theta$  on  $\Omega$  and a cocycle mapping  $\phi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ . Essentially (and sufficient for our purposes here),  $\theta$  represents the driving noise process and  $\phi$  the state space evolution of the system. For example, for a stochastic differential equation on  $\mathbb{R}^d$  with a two-sided scalar Wiener process  $W_t$ , i.e. defined for all  $t \in \mathbb{R}$ ,  $\theta$  is defined by  $\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(\cdot)$  on a canonical sample space  $\Omega = C_0(\mathbb{R}, \mathbb{R})$  where, by definition,  $W_t(\omega) := \omega(t), t \in \mathbb{R}$ , and  $\phi$  is defined by  $\phi(t, \omega, x_0) = X_t^{x_0}(\omega)$ , the solution of the SDE starting at  $X_0^{x_0}(\omega) = x_0$ . See the expositions in Arnold (1998) and Kloeden *et al.* (1999) for more details.

A family  $\hat{A} = \{A(\omega), \omega \in \Omega\}$  of nonempty measurable compact subsets  $A(\omega)$  of  $\mathbb{R}^d$  is called  $\phi$ -invariant if  $\phi(t, \omega, A(\omega)) = A(\theta_t \omega)$  for all  $t \geq 0$  and is called a *random attractor* if in addition it is pathwise pullback attracting in the sense that

$$H_d^*(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)) \rightarrow 0 \quad \text{as } t \rightarrow -\infty$$

for all suitable (i.e. in a given attracting universe as, for instance, in Kloeden *et al.* (1999)) families of  $\hat{D} = \{D(\omega), \omega \in \Omega\}$  of nonempty measurable bounded subsets  $D(\omega)$  of  $\mathbb{R}^d$ . Here  $H_d^*$  is the Hausdorff semi-distance on  $\mathbb{R}^d$ . The following result (cf. Arnold (1998), Kloeden *et al.* (1999), Schmalfuss (1992)) ensures the existence of a random attractor.

**Theorem 2.1.** *Let  $(\theta, \phi)$  be an RDS on  $\Omega \times \mathbb{R}^d$ . If there exists a family  $\hat{B} = \{B(\omega), \omega \in \Omega\}$  of nonempty measurable compact subsets  $B(\omega)$  of  $\mathbb{R}^d$  and a  $T_{\hat{D}, \omega} \geq 0$  such that*

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \quad \forall t \geq T_{\hat{D}, \omega}$$

*for all families  $\hat{D} = \{D(\omega), \omega \in \Omega\}$  in the given attracting universe, then the RDS  $(\theta, \phi)$  has a random attractor  $\hat{A} = \{A(\omega), \omega \in \Omega\}$  with the component subsets defined for each  $\omega \in \Omega$  by*

$$A(\omega) = \bigcap_{s>0} \overline{\bigcup_{t \geq s} \phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}.$$

Note that if the random attractor consists of singleton sets, i.e  $A(\omega) = \{X^*(\omega)\}$  for some random variable  $X^*$ , then  $X_t^*(\omega) := X^*(\theta_t \omega)$  is a stationary stochastic process.

We also need the following lemmata.

**Lemma 2.2.** *Let  $\{x_n\}$  be a sequence in a complete metric space  $(X, d)$  such that every subsequence  $\{x_{n_i}\}$  has a subsequence  $\{x_{n_{i_j}}\}$  converging to a common limit  $x^*$ . Then the sequence  $\{x_n\}$  converges to  $x^*$ .*

*Proof.* If not, then there would exist an  $\eta > 0$  and a subsequence  $\{x_{n_i}\}$  such that  $d(x_{n_i}, x^*) \geq \eta$  for all  $i$ , which is not possible because  $\{x_{n_i}\}$  has a subsequence  $\{x_{n_{i_j}}\}$  converging to the  $x^*$ .  $\square$

**Lemma 2.3.** *Let  $W_t$  be a two-sided Wiener process, i.e. defined for all  $t \in \mathbb{R}$ . Then the integrals  $\nu \int_{-\infty}^t e^{-\nu(t-s)} dW_t(\omega)$  are pathwise uniformly bounded in  $\nu > 0$  on finite time intervals  $[T_1, T_2]$  of  $\mathbb{R}$  and the integrals*

$$\int_{T_1}^t e^{-\nu(t-s)} dW_t(\omega) \rightarrow 0 \quad \text{as } \nu \rightarrow \infty$$

*pathwise on finite time intervals  $[T_1, T_2]$  of  $\mathbb{R}$ .*

*Proof.* Integrating by parts (see Mao (1997)), we have

$$\int_{-\infty}^t e^{-\nu(t-s)} dW_s(\omega) = W_t(\omega) - \nu \int_{-\infty}^t e^{-\nu(t-s)} W_s(\omega) ds.$$

Then we use the sub-exponential growth of the paths of Wiener processes. For the second part we first notice that

$$\nu \int_{T_1}^t e^{-\nu(t-s)} W_t(\omega) ds = W_t(\omega) - e^{-\nu(t-T_1)} W_{T_1}(\omega)$$

for any  $\omega \in \Omega$ . Integrating again by parts it follows

$$\begin{aligned} \int_{T_1}^t e^{-\nu(t-s)} dW_s(\omega) &= W_t(\omega) - e^{-\nu(t-T_1)} W_{T_1}(\omega) - \nu \int_{T_1}^t e^{-\nu(t-s)} W_s(\omega) ds \\ &= e^{-\nu(t-T_1)} (W_t(\omega) - W_{T_1}(\omega)) \\ &\quad + \nu \int_{T_1}^t e^{-\nu(t-s)} (W_t(\omega) - W_s(\omega)) ds \end{aligned}$$

from which the result follows.  $\square$

### 3. Formulation of the problem

Suppose we have two autonomous ordinary differential equations in  $\mathbb{R}^d$ ,

$$\frac{dx}{dt} = f(x), \quad \frac{dy}{dt} = g(y), \quad (3.1)$$

which are sufficiently regular to ensure the forwards existence and uniqueness of solutions and satisfy one-sided dissipative Lipschitz conditions

$$\begin{aligned} \langle x_1 - x_2, f(x_1) - f(x_2) \rangle &\leq -L|x_1 - x_2|^2, \\ \langle y_1 - y_2, g(y_1) - g(y_2) \rangle &\leq -L|y_1 - y_2|^2, \end{aligned} \quad (3.2)$$

on  $\mathbb{R}^d$  for some  $L > 0$ , and thus have unique equilibria  $\bar{x}$  and  $\bar{y}$ , respectively, which are globally asymptotically stable (cf. Kloeden (2004)). Notice that the continuity of  $f$  and  $g$ , and the one-sided dissipative Lipschitz conditions (3.2) ensure the forwards existence and uniqueness of solutions to (3.1).

Consider now the dissipatively coupled system

$$\frac{dx}{dt} = f(x) + \nu(y - x), \quad \frac{dy}{dt} = g(y) + \nu(x - y) \quad (3.3)$$

with  $\nu > 0$ . It can be shown (see Afraimovich and Rodrigues (1998), and Carvalho *et al.* (1998)) that this also has a unique equilibrium  $(\bar{x}^\nu, \bar{y}^\nu)$ , which is globally asymptotically stable. Moreover,  $(\bar{x}^\nu, \bar{y}^\nu) \rightarrow (\bar{z}, \bar{z})$  as  $\nu \rightarrow \infty$ , where  $\bar{z}$  is the unique globally asymptotically stable equilibrium of the ‘‘averaged’’ system

$$\frac{dz}{dt} = \frac{1}{2} (f(z) + g(z)). \quad (3.4)$$

This phenomena is known as *synchronization*. Analogous results hold for more general autonomous attractors (cf. Afraimovich and Rodrigues (1998), Carvalho *et al.* (1998)) as well as for nonautonomous systems (Kloeden (2003)) with appropriately defined nonautonomous attractors.

The aim of this note is to show that this synchronization effect is preserved under additive noise provided equilibria are replaced by stationary random solutions. Specifically, we consider two Ito stochastic differential equations in  $\mathbb{R}^d$ ,

$$dX_t = f(X_t) dt + \alpha dW_t^1, \quad dY_t = g(Y_t) dt + \beta dW_t^2, \quad (3.5)$$

where  $\alpha, \beta \in \mathbb{R}_+^d$  are constant vectors with no components equal to zero,  $W_t^1, W_t^2$  are independent two-sided scalar Wiener processes<sup>†</sup>, and  $f, g$  are as above, in particular, satisfying the one-sided dissipative Lipschitz conditions (3.2). We will show in the next section that each of these stochastic systems has a pathwise asymptotically stable random attractor consisting of a stationary random variable. For example, for linear drift terms, i.e the SDEs

$$dX_t = -X_t dt + \alpha dW_t^1, \quad dY_t = -Y_t dt + \beta dW_t^2, \quad (3.6)$$

these random variables are given explicitly by

$$\bar{X}_t = \alpha e^{-t} \int_{-\infty}^t e^s dW_s^1, \quad \bar{Y}_t = \beta e^{-t} \int_{-\infty}^t e^s dW_s^2. \quad (3.7)$$

The synchronized system corresponding to SDEs (3.5) reads

$$\begin{aligned} dX_t &= (f(X_t) + \nu(Y_t - X_t)) dt + \alpha dW_t^1, \\ dY_t &= (g(Y_t) + \nu(X_t - Y_t)) dt + \beta dW_t^2. \end{aligned} \quad (3.8)$$

It will be shown that this system is dissipative and has a unique stationary solution  $(\bar{X}_t^\nu, \bar{Y}_t^\nu)$ , which is pathwise globally asymptotically stable with

$$(\bar{X}_t^\nu, \bar{Y}_t^\nu) \rightarrow (\bar{Z}_t^\infty, \bar{Z}_t^\infty) \quad \text{as } \nu \rightarrow \infty,$$

pathwise on finite time intervals  $[T_1, T_2]$  of  $\mathbb{R}$ , where  $\bar{Z}_t^\infty$  is the unique pathwise globally asymptotically stable stationary solution of the ‘‘averaged’’ SDE

$$dZ_t = \frac{1}{2} (f(Z_t) + g(Z_t)) dt + \frac{1}{2} \alpha dW_t^1 + \frac{1}{2} \beta dW_t^2. \quad (3.9)$$

<sup>†</sup> alternatively, one could take  $\alpha, \beta$  scalar valued and  $W_t^1, W_t^2$  vector valued

For example, for the linear SDEs (3.6) we have

$$dZ_t = -Z_t dt + \frac{1}{2}\alpha dW_t^1 + \frac{1}{2}\beta dW_t^2$$

with

$$\bar{Z}_t^\infty = \frac{1}{2}e^{-t} \left( \alpha \int_{-\infty}^t e^s dW_s^1 + \beta \int_{-\infty}^t e^s dW_s^2 \right) = \frac{1}{2} (\bar{X}_t + \bar{Y}_t),$$

this averaged expression being due to the special linear structure.

In addition to the one-sided Lipschitz dissipative condition on the functions  $f$  and  $g$  we assume the following integrability condition:

There exists  $m_0 > 0$  such that for any  $m \in (0, m_0]$ , and any continuous function  $u : \mathbb{R} \rightarrow \mathbb{R}^d$  with sub-exponential growth it follows

$$\int_{-\infty}^t e^{ms} |f(u(s))|^2 ds < +\infty, \quad \int_{-\infty}^t e^{ms} |g(u(s))|^2 ds < +\infty. \quad (3.10)$$

Without loss of generality, we can assume that  $L \leq m_0$ .

#### 4. The uncoupled systems with additive noise

We consider the first of the uncoupled equations in (3.5),

$$dX_t = f(X_t) dt + \alpha dW_t^1. \quad (4.1)$$

Its solution paths are generally not differentiable, so in order to use the one-sided dissipative Lipschitz condition (3.2) we consider the difference  $X_t - \bar{X}_t$  where  $\bar{X}_t$  is the Ornstein-Uhlenbeck stationary process (3.7) satisfying the first of the linear equations (3.6). This difference is pathwise differentiable since the paths  $X_t$  and  $\bar{X}_t$  are continuous and satisfy the integral equation

$$X_t - \bar{X}_t = X_0 - \bar{X}_0 + \int_0^t (f(X_s) - f(\bar{X}_s)) ds,$$

which, by the fundamental theorem of calculus, is thus equivalent to the differential expression

$$\frac{d}{dt} (X_t - \bar{X}_t) = f(X_t) - f(\bar{X}_t),$$

from which it follows that

$$\begin{aligned} \frac{d}{dt} |X_t - \bar{X}_t|^2 &= 2 \langle X_t - \bar{X}_t, f(X_t) - f(\bar{X}_t) \rangle + 2 \langle X_t - \bar{X}_t, f(\bar{X}_t) - f(\bar{X}_t) \rangle \\ &\leq -2L |X_t - \bar{X}_t|^2 + L |X_t - \bar{X}_t|^2 + \frac{1}{L} |f(\bar{X}_t) - f(\bar{X}_t)|^2 \end{aligned}$$

and thus

$$|X_t - \bar{X}_t|^2 \leq |X_{t_0} - \bar{X}_{t_0}|^2 e^{-L(t-t_0)} + \frac{e^{-Lt}}{L} \int_{t_0}^t e^{Ls} |f(\bar{X}_s) - f(\bar{X}_s)|^2 ds.$$

Pathwise pullback convergence (i.e. as  $t_0 \rightarrow -\infty$ ) gives pullback absorption (cf. Theorem 2.1)

$$|X_t - \bar{X}_t|^2 \leq R_X^2(\theta_t \omega) := 1 + \frac{e^{-Lt}}{L} \int_{-\infty}^t e^{Ls} |f(\bar{X}(\theta_s \omega)) + \bar{X}(\theta_s \omega)|^2 ds$$

for all  $t \geq T_{\widehat{D}(\omega)}$  for appropriate families  $\widehat{D}(\omega)$  of bounded sets  $\{D(\theta_t \omega), t \in \mathbb{R}\}$  of initial conditions.

The integrals here exist due to assumption (3.10) and the fact that Ornstein-Uhlenbeck processes inherit the sub-exponential growth of their Wiener processes. Thus

$$|X_t(\omega) - \bar{X}_t(\omega)| \leq R_X(\theta_t \omega), \quad \forall t \geq T_{\widehat{D}(\omega)},$$

and so

$$|X_t(\omega)| \leq |\bar{X}_t(\omega)| + R_X(\theta_t \omega), \quad \forall t \geq T_{\widehat{D}(\omega)},$$

which means that this system has a random attractor  $\widehat{A} = \{A(\omega), \omega \in \Omega\}$ . But the difference of any two solutions satisfies the differential inequality

$$\frac{d}{dt} |X_t^1 - X_t^2|^2 \leq -2L |X_t^1 - X_t^2|^2,$$

which means all solutions converge pathwise to each other and thus the random attractor sets are singleton sets  $A(\omega) = \{X^*(\omega)\}$ , i.e. the random attractor is formed by a stationary random process  $X_t^*(\omega) = X^*(\theta_t \omega)$  which pathwise attracts all other solutions.

An analogous situation holds for the second equation of the uncoupled equations in (3.5).

## 5. The synchronized system with additive noise

To show that the synchronized system (3.8) is strongly dissipative, we use the OU processes

$$X_t^{*,\nu} = \alpha e^{-\nu t} \int_{-\infty}^t e^{\nu s} dW_s^1, \quad Y_t^{*,\nu} = \beta e^{-\nu t} \int_{-\infty}^t e^{\nu s} dW_s^2, \quad (5.1)$$

which are the stationary solutions of the linear equations

$$dX_t = -\nu X_t dt + \alpha dW_t^1, \quad dY_t = -\nu Y_t dt + \beta dW_t^2. \quad (5.2)$$

The differences of the solutions of (3.8) and these stationary solutions are pathwise differentiable and satisfy the system of random differential equations

$$\begin{aligned} \frac{d}{dt} (X_t - X_t^{*,\nu}) &= f(X_t) + \nu(Y_t - X_t) + \nu X_t^{*,\nu} \\ \frac{d}{dt} (Y_t - Y_t^{*,\nu}) &= g(Y_t) + \nu(X_t - Y_t) + \nu Y_t^{*,\nu}, \end{aligned}$$

which, with  $U_t^\nu := X_t - X_t^{*,\nu}$  and  $V_t^\nu := Y_t - Y_t^{*,\nu}$ , is equivalent to

$$\frac{d}{dt}U_t^\nu = f(U_t^\nu + X_t^{*,\nu}) + \nu(V_t^\nu - U_t^\nu) + \nu Y_t^{*,\nu}, \quad \frac{d}{dt}V_t^\nu = g(Y_t) + \nu(U_t^\nu - V_t^\nu) + \nu X_t^{*,\nu}.$$

Thus

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|U_t^\nu|^2 + |V_t^\nu|^2) &= \langle U_t^\nu, f(U_t^\nu + X_t^{*,\nu}) - f(X_t^{*,\nu}) \rangle + \langle V_t^\nu, g(Y_t + Y_t^{*,\nu}) - g(Y_t^{*,\nu}) \rangle \\ &\quad + \langle U_t^\nu, f(X_t^{*,\nu}) + \nu Y_t^{*,\nu} \rangle + \langle V_t^\nu, g(Y_t^{*,\nu}) + \nu X_t^{*,\nu} \rangle \\ &\quad - \nu \langle U_t^\nu - V_t^\nu, U_t^\nu - V_t^\nu \rangle \\ &\leq -L (|U_t^\nu|^2 + |V_t^\nu|^2) + \frac{L}{2} (|U_t^\nu|^2 + |V_t^\nu|^2) \\ &\quad + \frac{2}{L} |f(X_t^{*,\nu}) + \nu Y_t^{*,\nu}|^2 + \frac{2}{L} |g(Y_t^{*,\nu}) + \nu X_t^{*,\nu}|^2, \end{aligned}$$

and hence

$$\frac{d}{dt} (|U_t^\nu|^2 + |V_t^\nu|^2) \leq -L (|U_t^\nu|^2 + |V_t^\nu|^2) + \frac{4}{L} |f(X_t^{*,\nu}) + \nu Y_t^{*,\nu}|^2 + \frac{4}{L} |g(Y_t^{*,\nu}) + \nu X_t^{*,\nu}|^2.$$

This means that  $|U_t^\nu(\omega)|^2 + |V_t^\nu(\omega)|^2$  is pathwise absorbed by the family  $\widehat{B}^\nu = \{B^\nu(\omega), \omega \in \Omega\}$  of closed balls in  $\mathbb{R}^{2d}$  centered on the origin and of radius  $R_\nu(\omega)$ , where  $R_\nu^2(\omega)$  is defined by

$$1 + \frac{4}{L} \int_{-\infty}^0 e^{Ls} \left( |f(X_s^{*,\nu}(\theta_s \omega) + \nu Y_s^{*,\nu}(\theta_s \omega))|^2 ds + |g(Y_s^{*,\nu}(\theta_s \omega) + \nu X_s^{*,\nu}(\theta_s \omega))|^2 \right) ds.$$

Hence, by Theorem 2.1, the synchronized system has a random attractor  $\widehat{A}^\nu = \{A^\nu(\omega), \omega \in \Omega\}$  with  $A^\nu(\omega) \subset B^\nu(\omega)$ . But the difference  $(\Delta X_t, \Delta Y_t) = (X_t^1 - X_t^2, Y_t^1 - Y_t^2)$  of any pair of solutions satisfies the system of random differential equations

$$\frac{d}{dt} \Delta X_t = f(X_t^1) - f(X_t^2) + \nu(\Delta Y_t - \Delta X_t), \quad \frac{d}{dt} \Delta Y_t = g(Y_t^1) - g(Y_t^2) + \nu(\Delta X_t - \Delta Y_t),$$

so

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|\Delta X_t|^2 + |\Delta Y_t|^2) &= \langle \Delta X_t, f(X_t^1) - f(X_t^2) \rangle + \langle \Delta Y_t, g(Y_t^1) - g(Y_t^2) \rangle \\ &\quad - \nu \langle \Delta X_t - \Delta Y_t, \Delta X_t - \Delta Y_t \rangle \\ &\leq -L (|\Delta X_t|^2 + |\Delta Y_t|^2), \end{aligned}$$

from which we obtain

$$|\Delta X_t(\omega)|^2 + |\Delta Y_t(\omega)|^2 \leq (|\Delta X_0(\omega)|^2 + |\Delta Y_0(\omega)|^2) e^{-2Lt},$$

which means all solutions converge pathwise to each other as  $t \rightarrow \infty$ . Thus the random attractor consists of singleton sets formed by an ordered pair of stationary processes  $(\bar{X}_t^\nu(\omega), \bar{Y}_t^\nu(\omega))$ .

## 6. The synchronized stationary solutions as $\nu \rightarrow \infty$

**Lemma 6.1.**  $\bar{X}_t^\nu(\omega) - \bar{Y}_t^\nu(\omega) \rightarrow 0$  as  $\nu \rightarrow \infty$  pathwise on any bounded time interval  $[T_1, T_2]$  of  $\mathbb{R}$ .

*Proof.* Subtracting the second from the first equation of (3.8) gives

$$\begin{aligned} d(\bar{X}_t^\nu - \bar{Y}_t^\nu) &= (-2\nu(\bar{X}_t^\nu - \bar{Y}_t^\nu) + f(\bar{X}_t^\nu) - g(\bar{Y}_t^\nu)) dt \\ &\quad + \alpha dW_t^1 - \beta dW_t^2, \end{aligned}$$

or, with  $D_t^\nu = \bar{X}_t^\nu - \bar{Y}_t^\nu$ ,

$$d(D_t^\nu e^{2\nu t}) = e^{2\nu t} (f(\bar{X}_t^\nu) - g(\bar{Y}_t^\nu)) dt + \alpha e^{2\nu t} dW_t^1 - \beta e^{2\nu t} dW_t^2,$$

so pathwise

$$\begin{aligned} |D_t^\nu| &\leq e^{-2\nu T_1} |D_{T_1}^\nu| + \int_{T_1}^t e^{-2\nu(t-s)} (|f(\bar{X}_s^\nu)| + |g(\bar{Y}_s^\nu)|) ds \\ &\quad + |\alpha| \left| \int_{T_1}^t e^{-2\nu(t-s)} dW_s^1 \right| + |\beta| \left| \int_{T_1}^t e^{-2\nu(t-s)} dW_s^2 \right|. \end{aligned} \quad (6.1)$$

By Lemma 2.3 we see that the radius  $R_\nu(\theta_t \omega)$  is pathwise uniformly bounded on each bounded time interval  $[T_1, T_2]$ , so  $|X_t^\nu(\omega)|$ ,  $|Y_t^\nu(\omega)|$  and  $|D_0^\nu(\omega)|$  are pathwise uniformly bounded on each bounded time interval  $[T_1, T_2]$ . Then by Lemma 2.3 and condition (3.10) we see that all of the integrals in (6.1) converge to zero as  $\nu \rightarrow \infty$  pathwise on the bounded time interval  $[T_1, T_2]$ .  $\square$

**Theorem 6.2.**  $(\bar{X}_t^{\nu_n}, \bar{Y}_t^{\nu_n}) \rightarrow (Z_t^\infty, Z_t^\infty)$  pathwise uniformly on bounded time intervals  $[T_1, T_2]$  for any sequence  $\nu_n \rightarrow \infty$ , where  $Z_t^\infty$  is the stationary stochastic solution of the averaged SDE

$$dZ_t = \frac{1}{2} (f(Z_t) + g(Z_t)) dt + \frac{1}{2} \alpha dW_t^1 + \frac{1}{2} \beta dW_t^2. \quad (6.2)$$

*Proof.* Define

$$Z_t^\nu := \frac{1}{2} (\bar{X}_t^\nu + \bar{Y}_t^\nu) \quad \forall t \in \mathbb{R}$$

and observe that  $Z_t^\nu$  satisfies the equation

$$dZ_t^\nu = \frac{1}{2} (f(\bar{X}_t^\nu) + g(\bar{Y}_t^\nu)) dt + \frac{1}{2} \alpha dW_t^1 + \frac{1}{2} \beta dW_t^2.$$

Also define

$$\bar{Z}_t := \frac{1}{2} (\bar{X}_t + \bar{Y}_t) \quad \forall t \in \mathbb{R}$$

where  $\bar{X}_t$  and  $\bar{Y}_t$  are the OU processes satisfying the linear SDEs (3.6).

The difference  $Z_t^\nu - \bar{Z}_t$  is pathwise continuous and thus by Lemma 2.3 and the definitions of the respective absorbing sets, equi-bounded w.r.t.  $\nu > 0$  on the bounded time interval  $[T_1, T_2]$ . Moreover  $Z_t^\nu - \bar{Z}_t$  is pathwise differentiable and satisfies the random differential expression

$$\frac{d}{dt} (Z_t^\nu - \bar{Z}_t) = \frac{1}{2} f(\bar{X}_t^\nu) + \frac{1}{2} g(\bar{Y}_t^\nu) + \frac{1}{2} \bar{X}_t + \frac{1}{2} \bar{Y}_t,$$



Since

$$\begin{aligned} \left| \frac{d}{dt} (Z_t^\nu(\omega) - \bar{Z}_t(\omega)) \right| &\leq \frac{1}{2} |f(\bar{X}_t^\nu(\omega))| + \frac{1}{2} |g(\bar{Y}_t^\nu(\omega))| + \frac{1}{2} |\bar{X}_t(\omega)| + \frac{1}{2} |\bar{Y}_t(\omega)| \\ &\leq M_{T_1, T_2}(\omega) < \infty \end{aligned}$$

by Lemma 2.3, we can use the Ascoli Theorem to conclude that for any sequence  $\nu_n \rightarrow \infty$ , there is a (possibly) random subsequence  $\nu_{n_j}(\omega) \rightarrow \infty$  such that  $Z_t^{\nu_{n_j}}(\omega) - \bar{Z}_t(\omega) \rightarrow Z_t^\infty(\omega) - \bar{Z}_t(\omega)$  as  $j \rightarrow \infty$ , and thus that  $Z_t^{\nu_{n_j}}(\omega) \rightarrow Z_t^\infty(\omega)$  as  $j \rightarrow \infty$ .

Now

$$\begin{aligned} Z_t^{\nu_{n_j}}(\omega) - \bar{Y}_t^{\nu_{n_j}}(\omega) &= \frac{1}{2} \left( \bar{X}_t^{\nu_{n_j}}(\omega) - \bar{Y}_t^{\nu_{n_j}}(\omega) \right) \rightarrow 0, \\ Z_t^{\nu_{n_j}}(\omega) - \bar{X}_t^{\nu_{n_j}}(\omega) &= \frac{1}{2} \left( \bar{Y}_t^{\nu_{n_j}}(\omega) - \bar{X}_t^{\nu_{n_j}}(\omega) \right) \rightarrow 0 \end{aligned}$$

as  $\nu_{n_j} \rightarrow \infty$ , so

$$\begin{aligned} \bar{X}_t^{\nu_{n_j}}(\omega) &= 2Z_t^{\nu_{n_j}}(\omega) - \bar{Y}_t^{\nu_{n_j}}(\omega) \rightarrow Z_t^\infty(\omega), \\ \bar{Y}_t^{\nu_{n_j}}(\omega) &= 2Z_t^{\nu_{n_j}}(\omega) - \bar{X}_t^{\nu_{n_j}}(\omega) \rightarrow Z_t^\infty(\omega) \end{aligned}$$

as  $\nu_{n_j} \rightarrow \infty$ . Moreover,

$$\begin{aligned} Z_t^\nu - \bar{Z}_t &= Z_t^\nu - \bar{Z}_t + \frac{1}{2} \int_{T_1}^t f(\bar{X}_s^\nu) ds + \frac{1}{2} \int_{T_1}^t g(\bar{Y}_s^\nu) ds \\ &\quad + \frac{1}{2} \int_{T_1}^t \bar{X}_s ds + \frac{1}{2} \int_{T_1}^t \bar{Y}_s ds \end{aligned}$$

which converges pathwise to

$$\begin{aligned} Z_t^\infty &= Z_{T_1}^\infty + \frac{1}{2} \int_{T_1}^t f(Z_s^\infty) ds + \frac{1}{2} \int_{T_1}^t g(Z_s^\infty) ds \\ &\quad + \bar{Z}_t - \bar{Z}_{T_1} + \frac{1}{2} \int_{T_1}^t \bar{X}_s ds + \frac{1}{2} \int_{T_1}^t \bar{Y}_s ds \\ &= Z_{T_1}^\infty + \frac{1}{2} \int_{T_1}^t f(Z_s^\infty) ds + \frac{1}{2} \int_{T_1}^t g(Z_s^\infty) ds \\ &\quad + \frac{1}{2} \alpha \int_{T_1}^t dW_s^1 + \frac{1}{2} \beta \int_{T_1}^t dW_s^2, \end{aligned}$$

on the interval  $[T_1, T_2]$ , so  $Z_t^\infty$  is a solution of the SDE (6.2) for all  $t \in \mathbb{R}$ . The drift of this SDE satisfies the strongly dissipative one-sided Lipschitz condition (3.2), so it has a random attractor consisting of a singleton set formed by a stationary stochastic process which thus must be equal to  $Z_t^\infty$ .

Finally, we note that pathwise all possible subsequences here have the same limit, so by Lemma 2.2 every full sequence  $Z_t^{\nu_n}$  actually converges to  $Z_t^\infty$  as  $\nu_n \rightarrow \infty$ .  $\square$

As a straightforward consequence of the arguments in the previous proof we have

**Corollary 6.3.**  $(\bar{X}_t^\nu, \bar{Y}_t^\nu) \rightarrow (Z_t^\infty, Z_t^\infty)$  as  $\nu \rightarrow \infty$  pathwise on any bounded time interval  $[T_1, T_2]$  of  $\mathbb{R}$ .  $\square$

We would like to thank the anonymous referees for helpful comments and suggestions.

This work has been partially supported by the Ministerio de Ciencia y Tecnología (Spain) and FEDER (European Community) grant BFM2002-03068.

## References

- Arnold, L., 1998 *Random Dynamical Systems*, Springer-Verlag.
- Afraimovich, V.S., & Rodrigues, H.M., 1998 Uniform dissipativeness and synchronization of nonautonomous equations, In *International Conference on Differential Equations (Lisboa 1995)*, World Scientific Publishing, River Edge, N.J., pp. 3–17.
- Carvalho, A.N., Rodrigues, H.M. & Dlotko, T., 1998 Upper semicontinuity of attractors and synchronization, *J. Math. Anal. Applns.*, 220, 13–41.
- Crauel, H., Debussche, A., & Flandoli, F., 1997 Random attractors, *J. Dynam. Diff. Eq.*, 9, 307–341.
- Crauel, H., & Flandoli, F., 1994 Attractors for random dynamical systems, *Probab. Theory Related Fields*, 100, 365–393.
- Kloeden, P.E., 2003 Synchronization of nonautonomous dynamical systems. *Elect. J. Diff. Eqns.* 2003, 1–10.
- Kloeden, P.E., 2004 Nonautonomous attractors of switching systems, (preprint).
- Kloeden, P.E., Keller, H. & Schmalfuß, B., 1999 Towards a theory of random numerical dynamics. In *Stochastic Dynamics*. Editors: H. Crauel and V. M. Gundlach, Springer-Verlag, pp. 259–282.
- Mao, X., 1997 *Stochastic Differential Equations and their Applications*, Horwood Publishing, Chichester.
- Robinson, J.C., 2002 Stability of random attractors under perturbation and approximation, *J. Diff. Eqns.* 186, 652–669.
- Rodrigues, H.M., 1996 Abstract methods for synchronization and applications, *Applicable Anal.*, 62, 263–296.
- Schmalfuss, B., 1992 Backward cocycles and attractors of stochastic differential equations, In V. Reitmann, T. Riedrich and N Koks, editors, “International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behaviour,” 185–192.
- Strogatz, S., 2003 *Sync: The Emerging Science of Spontaneous Order*, Hyperion Press, New York.