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THE EXPONENTIAL STABILITY OF NEUTRAL STOCHASTIC DELAY PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we analyse the almost sure exponential stability and ultimate boundedness of the solutions to a class of neutral stochastic semilinear partial delay differential equations. This kind of equations arises in problems related to coupled oscillators in a noisy environment, or in viscoeslastic materials under random or stochastic influences.

1. **Introduction.**

In this paper we are mainly interested in the analysis of the exponential stability of some types of stochastic neutral partial functional differential equations. To motivate and justify our work, let us first describe the general framework in which our study will be carried out.

Let Z be a Banach space, $C_r = C([-r,0],Z)$ the Banach space of all continuous mappings ϕ from $[-r,0]$ into Z where $r>0$ is a real number. Let $A : Z \to Z$ be a closed operator and let $A_0 : Z \to Z$ be a bounded linear operator. Define the functional difference operator

$$D(\cdot) : [0,\infty) \times C_r \to Z$$

by

$$D(t)\varphi = \varphi(0) - g(t,\varphi),\ t \in [0,\infty), \varphi \in C_r,$$

where $g : [0,\infty) \times C_r \to Z$ is a continuous function. Let $G : [0,\infty) \times Z \to Z$ be a continuous function and let $X_t \in C_r$ be defined by $X_t(\theta) = X(t+\theta)$ for $\theta \in [-r,0]$. If $g \neq 0$, then the following stochastic problem

$$\begin{cases} d\,[D(t)X_t] = [AX(t) + A_0 X_t]dt + G(t,X_t)dW(t), t \geq 0, \\ X(t) = \phi(t), t \in [-r,0], \phi \in C_r, \end{cases} \tag{1}$$

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where, for simplicity, $W(t)$ is a standard Brownian motion, is termed a stochastic neutral functional differential equation.

When $Z = \mathbb{R}^n$ and $G \equiv 0$, the deterministic neutral equation has been extensively studied in the literature (see, e.g. [10], [11], [13], [12] and the references therein). As for the stochastic version (i.e., for $G \neq 0$), one can find some results in [14] and [18] amongst others.

However, when Z is an infinite-dimensional Hilbert space, i.e. when operators A, A_0, and G involve partial derivatives, only a few results have been obtained in this field despite the importance and interest of the model (1). In this respect, it is worth mentioning that this kind of neutral equations arises from problems related to coupled oscillators in a noisy environment, or in problems of viscoeslastic materials under random or stochastic influences (see [24] for a description of these problems in the deterministic case). To the best of our knowledge, there exist only a few papers already published in this field. To be more precise, in [16] (see also [17]) it is considered a linear version of (1) in the particular case in which the delays are constant, and some stability properties of the mild solutions are analysed in a similar way as Dakto proved in [8] in the deterministic case, while in [9] it is studied the existence and uniqueness of mild solutions to a semilinear model, as well as some results on the stability of the null solution. Here we are concerned with a non-autonomous semilinear model containing different types of finite delays (constant, variable, distributed, etc), treated within a variational formulation (as in [19], [1], [2]), and we analyse the ultimate boundedness and asymptotic behaviour of solutions even when zero is not a solution to our model. In addition, the partial differential operators appearing in the right-hand side of our equation are more general than the ones in [16] and [9].

Our main objective is to provide sufficient conditions ensuring the exponential pathwise stability of solutions to (1). Nevertheless, we will also prove a result on the existence and uniqueness of solution to our model by using the method of steps. In particular, our stability results can also be applied to stochastic delay partial differential equations, extending in some sense, some of the results previously proved in that field (see, e.g. [3] and [1]).

The contents of this paper are as follows. In Section 2 we recall some preliminaries on the Hilbert-valued stochastic integral. In Section 3 we discuss the existence and uniqueness of solution to our neutral semilinear stochastic partial differential equation. We present some auxiliary lemmata in Section 4. The moment exponential stability and ultimate exponential boundedness of the solutions, as well as the almost sure exponential stability of solutions are analysed in Section 5. Finally, in Section 6 we present an application example which illustrates the theory previously developed in this paper.

2. **Preliminaries.**

In this section we first introduce the variational framework in which our analysis will be developed. Let V and H be a reflexive Banach space and a separable Hilbert space with their respective norms $\|\cdot\|$ and $|\cdot|$, and such that

$$V \subset H \cong H^* \subset V^*,$$

where V is a dense subspace of H and the injections are continuous. We denote by $\langle \cdot, \cdot \rangle$ the duality between V and V^*, and by λ_1 a constant satisfying

$$\lambda_1 |u|^2 \leq ||u||^2, \quad \text{for all} \quad u \in V.$$

Assume that $\{\Omega, \mathcal{F}, P\}$ is a complete probability space, and let $\{\mathcal{F}_t\}_{t\geq 0}$ be an increasing and right continuous family of sub σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. We will denote $\mathcal{F}_t = \mathcal{F}_0$ for $t < 0$. Let $\{\beta^j(t),\ t \geq 0,\ j = 1, 2, ...\}$ be a given sequence of mutually independent standard real $\mathcal{F}_t$-Wiener processes defined on this space, and suppose given K, another separable Hilbert space, and $\{e_j;\ j = 1, 2, ...\}$, an orthonormal basis of K. We denote by $\{W(t);\ t \geq 0\}$, the cylindrical Wiener process with values in K defined formally as $W(t) = \sum_{j=1}^{\infty} \beta^j(t)e_j$.

It is well known that this series does not converge in K, but rather in any Hilbert space $\widetilde{K}$ such that $K \subset \widetilde{K}$, being the injection of K into $\widetilde{K}$ Hilbert-Schmidt (see e.g. Da Prato & Zabczyk [7] for more details).

Given real numbers $a < b$, and a separable Hilbert space $\mathcal{H}$ we will denote by $M^2_{\mathcal{F}_t}(a, b; \mathcal{H})$ the space of all processes $X \in L^2(\Omega \times (a, b), \mathcal{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; \mathcal{H})$ (where $\mathcal{B}((a, b))$ denotes the Borel $\sigma-$algebra on (a, b)) such that $X(t)$ is $\mathcal{F}_t-$measurable a.e. $t \in (a, b)$. The space $M^2_{\mathcal{F}_t}(a, b; \mathcal{H})$ is a closed subspace of $L^2(\Omega \times (a, b), \mathcal{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; \mathcal{H})$.

We will write $L^2_{\mathcal{F}_t}(\Omega; C([a, b]; \mathcal{H}))$ to denote the space of all continuous and $\mathcal{F}_t$-progressively measurable $\mathcal{H}$-valued processes $\{\varphi_t;\ a \leq t \leq b\}$ satisfying $E\left(\sup_{a\leq t\leq b} \|\varphi_t\|^2_{\mathcal{H}}\right) < \infty$.

For our separable Hilbert space H, with scalar product $(\cdot, \cdot)$, let us denote by $\mathcal{L}^2_0(K; H)$ the separable Hilbert space of Hilbert-Schmidt operators from K into H, and by $((\cdot, \cdot))_{\mathcal{L}^2_0}$ and $\|\cdot\|_{\mathcal{L}^2_0}$ the scalar product and its associated norm in $\mathcal{L}^2_0(K; H)$, where for all R and S in $\mathcal{L}^2_0(K; H)$,

$$((R, S))_{\mathcal{L}^2_0} = \sum_{j=1}^{\infty} (Re_j, Se_j).$$

For any process $\Psi \in M^2_{\mathcal{F}_t}(0, T; \mathcal{L}^2_0(K; H))$, we define the stochastic integral of Ψ with respect to the cylindrical Wiener process $W(t)$, denoted by $\int_0^t \Psi(s)\, dW(s)$, $0 \leq t \leq T$, as the unique continuous H-valued $\mathcal{F}_t$-martingale such that for all $h \in H$,

$$(\int_0^t \Psi(s)\, dW(s), h) = \sum_{j=1}^{\infty} \int_0^t (\Psi(s)e_j, h)\, d\beta^j(s), \quad 0 \leq t \leq T,$$

where the integral with respect to $\beta^j(s)$ is understood in the sense of Itô, and the series converges in $L^2(\Omega; C([0, T]))$. See e.g. Da Prato & Zabczyk [7] for the properties of the stochastic integral so defined. In particular, we note that if $\Psi \in M^2_{\mathcal{F}_t}(0, T; \mathcal{L}^2_0(K; H))$ and $\phi \in L^2(\Omega; L^\infty(0, T; H))$ is $\mathcal{F}_t$-progressively measurable, then the series $\sum_{j=1}^{\infty} \int_0^t (\Psi(s)e_j, \phi(s))\, d\beta^j(s)$, $0 \leq t \leq T$, converges in $L^1(\Omega; C([0, T]))$, and defines a real valued continuous $\mathcal{F}_t$-martingale. We will use the notation

$$\int_0^t (\Psi(s), \phi(s)dW(s)) = \sum_{j=1}^{\infty} \int_0^t (\Psi(s)e_j, \phi(s))\, d\beta^j(s), \quad 0 \leq t \leq T.$$

Let $r > 0$ denote a positive real number and let $C_r = C([-r, 0], H)$ be the Banach space of all continuous mappings from $[-r, 0]$ into H equipped with the sup-norm. Given $u \in C([-r, +\infty), H)$, for any $t \geq 0$, we denote by u_t the element in C_r defined by

$$u_t(\theta) = u(t + \theta), \quad \theta \in [-r, 0].$$

In this paper we consider the exponential stability and exponential ultimate boundedness of solutions to the following neutral stochastic delay partial differential equation:

$$\begin{cases} d\left[X(t) - k(t, X(t-r))\right] = \left[A(t)X(t) + f(t, X_t)\right] dt + g(t, X_t) dW(t) \\ X(t) = \varphi(s),\ -r \le t \le 0, \end{cases} \tag{2}$$

where $\varphi(s, \cdot)$ is $\mathcal{F}_0-$measurable for all $s \in [-r, 0]$, $A(t) : V \longrightarrow V^*$ is a linear operator for almost all $t \in [0, \infty)$, $k : [-r, \infty) \times V \longrightarrow V$ is a globally Lipschitz function, and $f : (0, \infty) \times C_r \to V^*$ and $g : (0, \infty) \times C_r \longrightarrow \mathcal{L}_0^2(K, H)$ are continuous and globally Lipschitz mappings.

3. **Existence and uniqueness of solutions.** Although the existence and uniqueness of solutions of a problem more general than (2) (for instance, in which the term k takes the form $k(t, X_t)$ and $A(t, \cdot)$ is a family of nonlinear operators) will be the aim of our forthcoming paper [5], we will content ourselves now with this case since, as far as we know, there are only a few works dealing with stochatic neutral partial differential equations (e.g. Liu [16] and [9] for the existence of mild solutions to semilinear neutral equations), but none of them analyse the variational framework in the present paper.

On the other hand, it is remarkable that, allowing the initial datum to be defined in the interval $[-2r, 0]$ instead of $[-r, 0]$, a suitable method of steps will give us the existence and uniqueness of solutions in a straighforward way. Bearing in mind that our main objective in this paper is to study the asymptotic behaviour of (2), we will consider this situation.

Let $A(t) : V \to V^*$ be a family of linear operators defined a.e. $t \in (0, +\infty)$ such that $A(\cdot) \in L^\infty(0, T; \mathcal{L}(V, V^*))$ for all $T > 0$, and satisfying the following coercivity assumption:

(C) there exist $\alpha > 0$ and $\lambda \in \mathbb{R}$ such that

$$-2\langle A(t)u, u\rangle + \lambda |u|^2 \ge \alpha \|u\|^2, \quad \forall u \in V,\ \text{a.e.}\, t \in (0, +\infty).$$

Let $f : \Omega \times (0, +\infty) \times C_r \to V^*$ and $g : \Omega \times (0, +\infty) \times C_r \to \mathcal{L}_0^2(K, H)$ be two families of nonlinear operators defined a.e. $t \in (0, +\infty)$ such that:

$(f.1)$ $\forall \phi \in C_r$, the stochastic process $f(\cdot, \phi)$ is $\mathcal{F}_t-$progressively measurable,
$(f.2)$ $f(\cdot, 0) \in M^2_{\mathcal{F}_t}(0, T; V^*)$, for all $T > 0$,
$(f.3)$ there exists $C_f > 0$ such that

$$\|f(t, \phi_1) - f(t, \phi_2)\|_*^2 \le C_f\, |\phi_1 - \phi_2|_{C_r}^2, \quad \forall \phi_1, \phi_2 \in C_r,\ \text{a.e.}\, t \in (0, +\infty),$$

$(g.1)$ $\forall \phi \in C_r$, the stochastic process $g(\cdot, \phi)$ is $\mathcal{F}_t-$progressively measurable,
$(g.2)$ $g(t, 0) \in M^2_{\mathcal{F}_t}(0, T; \mathcal{L}_0^2(K; H))$ for all $T > 0$,
$(g.3)$ there exists $C_g > 0$ such that

$$\|g(t, \phi_1) - g(t, \phi_2)\|_{\mathcal{L}_0^2}^2 \le C_g\, |\phi_1 - \phi_2|_{C_r}^2, \quad \forall \phi_1, \phi_2 \in C_r,\ \text{a.e.}\, t \in (0, +\infty),$$

Let us now consider the problem for any $T > 0$,

$$\begin{cases} u \in M^2_{\mathcal{F}_t}(-r, T; V) \cap L^2(\Omega; C(-r, T; H)), \\ u(t) = \psi(0) + \int_0^t A(s)u(s)\, ds + \int_0^t (f(s, u_s) + \widetilde{f}(s))\, ds \\ \qquad\qquad + \int_0^t (g(s, u_s) + \widetilde{g}(s))\, dW(s),\ t \in [0, T], \\ u(t) = \psi(t),\ t \in [-r, 0]. \end{cases} \tag{3}$$

Caraballo et al. proved in [1] the following result*:

Theorem 1. *Assume that hypotheses* (C), $(f.1)$-$(f.3)$ *and* $(g.1)$-$(g.3)$ *hold. Then, for every* $\psi \in M^2_{\mathcal{F}_0}(-r,0;V)\cap L^2(\Omega;C_r)$, $\widetilde{f} \in M^2_{\mathcal{F}_t}(0,T;V^*)$ *and* $\widetilde{g} \in M^2_{\mathcal{F}_t}(0,T;\mathcal{L}^2_0(K;H))$, *there exists a unique solution* u *to the problem (3).*

Based on this result we can prove the existence and uniqueness of solutions to our problem (2) but for an initial value in $[-2r,0]$.

Theorem 2. *Assume that* A, f *and* g *satisfy assumptions in Theorem 1 for any* $T>0$. *Suppose that* $k:[-r,+\infty)\times H \to H$ *is a globally Lipschitz function (with respect to its second variable) such that* $k([-r,+\infty)\times V)\subset V$, *and is also globally Lipschitz in* V, *i.e., there exists* $L_k>0$ *such that*

$$|k(t,u)-k(t,v)| \le L_k|u-v|, \quad \text{for all} \;\; u,v\in H, t\ge -r,$$
$$||k(t,u)-k(t,v)|| \le L_k||u-v||, \quad \text{for all} \;\; u,v\in V, t\ge -r,$$

and it satisfies that, $k(\cdot,0)\in L^2(-r,T;V)$ *for all* $T>0$, *and for each* $v\in H$, *the mapping* $k(\cdot,v)\in C([-r,+\infty);H)$. *Then, for any initial datum* $\varphi \in M^2_{\mathcal{F}_0}(-2r,0;V)\cap L^2(\Omega;C_{2r})$ *there exists a unique solution to the problem*

$$\begin{cases} X\in M^2_{\mathcal{F}_t}(-2r,T;V)\cap L^2(\Omega;C([-2r,T];H) \;\; \text{for all} \;\; T>0, \\ d\left[X(t)-k(t,X(t-r))\right] = \left[A(t)X(t)+f(t,X_t)\right]dt + g(t,X_t)dW(t), \\ X(t)=\varphi(s), \; -2r\le t\le 0, \end{cases} \tag{4}$$

or, in its integral form,

$$\begin{cases} X(t)-k(t,X(t-r)) = \varphi(0)-k(0,\varphi(-r)) \\ \qquad\qquad + \int_0^t \left[A(s)X(s)+f(s,X_s)\right]ds + \int_0^t g(s,X_s)dW(s), \quad t\ge 0, \\ X(t)=\varphi(t), \quad -2r\le t\le 0. \end{cases} \tag{5}$$

Proof. Let us first consider the problem for t in the interval $[0,r]$. Let us denote by $h(t)=k(t,\varphi(t-r))$ for $t\in[-r,r]$. Then, problem (5) can be rewritten as

$$\begin{cases} X(t)=X(0)+h(t)-h(0)+\int_0^t\left[A(s)X(s)+f(s,X_s)\right]ds \\ \qquad\qquad + \int_0^t g(s,X_s)dW(s), \quad t\in[0,r], \\ X(t)=\varphi(t), \quad -2r\le t\le 0, \end{cases} \tag{6}$$

and making the change of variables $Y(t)=X(t)-h(t)$, we obtain

$$\begin{cases} Y(t)=Y(0)+\int_0^t\left[A(s)Y(s)+A(s)h(s)+f(s,Y_s+h_s)\right]ds \\ \qquad\qquad + \int_0^t g(s,Y_s+h_s)dW(s), \quad t\in[0,r], \\ Y(t)=\varphi(t)-k(t,\varphi(t-r)), \quad -r\le t\le 0. \end{cases} \tag{7}$$

It is straightforward to check that the operators and the initial value in this problem satisfy the assumptions in Theorem 1, so that we can ensure the existence of a unique solution $Y(\cdot)$ of (7) in $M^2_{\mathcal{F}_t}(-r,r;V)\cap L^2(\Omega;C(-r,r;H))$, and thus a unique solution $X(\cdot)=Y(\cdot)+h(\cdot)$ of (5) in $[-r,r]$.

It is clear that we can iterate again in the interval $[r,2r]$ and, recursively, in any $[nr,(n+1)r]$ for $n\in\mathbb{N}$. Thus there exists a unique solution in $[-r,+\infty)$. □

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*In [1] the result was proved for a standard Wiener process but it can be easily extended to the actual situation.

In what follows we assume that our problem (2) possesses a unique solution defined globally in time (what may happen under different assumptions than those in Theorem 2).

4. **Some auxiliary lemmata.**

In this section we prove several lemmata which will be helpful for our later stability analysis. We start with a result which is based on an inequality for real numbers.

Lemma 1. *Let $0 < c < 1$. Then, for any $u, v \in H$, it follows that*

$$|u|^2 \leq \frac{1}{1-c}|u-v|^2 + \frac{1}{c}|v|^2. \tag{8}$$

Proof. Observe that, for any $u, v \in H$, if follows from the Young inequality that

$$\begin{aligned}(u,u) &= (u-v, u-v) + 2\,(u,v) - (v,v)\\ &\leq |u-v|^2 - |v|^2 + c|u|^2 + \frac{1}{c}|v|^2,\end{aligned}$$

whence (8) holds. □

Given a $H-$valued stochastic process $X(t),\ t \geq -r$ with $X(s) = \varphi(s), -r \leq s \leq 0$, let us first assume that there exists a positive real number $B_0 > 0$ such that

$$\sup_{-r \leq t \leq 0} E|\varphi(t)|^2 \leq B_0. \tag{9}$$

Then, we have the following result.

Lemma 2. *Let (9) hold and let M_δ and M_γ be nonnegative constants. Assume that the continuous function $k : [-r, \infty) \times H \to H$ satisfies the following inequality*

$$|k(t,x)|^2 \leq c^2|x|^2 + \delta(t),\ 0 < c < 1, t \geq -r, \tag{10}$$

where $\delta : [-r, +\infty) \to [0, +\infty)$ is a continuous function with $0 \leq \delta(t) \leq M_\delta$.
If a stochastic process $X(t),\ t \geq -r$ with $X(s) = \varphi(s), -r \leq s \leq 0$, satisfies the following inequality:

$$E|X(t) - k(t, X(t-r))|^2 \leq M_0 e^{-\theta t} + \gamma(t),\ M_0 > 0, \theta > 0,\ t \geq 0, \tag{11}$$

where $\gamma : [-r, +\infty) \to [0, +\infty)$ is another continuous function with $0 \leq \gamma(t) \leq M_\gamma$, then there exist positive real numbers $\rho > 0$ and $M_1 > 0$ such that

$$E|X(t)|^2 \leq M_1 e^{-\rho t} + c_0,\ t \geq 0,$$

where $c_0 = \frac{M_\delta}{c(1-c)} + \frac{M_\gamma}{(1-c)^2}$.

Proof. By Lemma 1 we have that

$$E|X(t)|^2 \leq \frac{1}{1-c}E|X(t) - k(t, X(t-r))|^2 + \frac{1}{c}E|k(t, X(t-r))|^2,$$

and, by (10)-(11),

$$E|X(t)|^2 \leq \frac{M_0}{1-c}e^{-\theta t} + cE|X(t-r)|^2 + \frac{M_\delta}{c} + \frac{M_\gamma}{1-c}.$$

Since $0 < c < 1$, we can take a $\beta \in (c, 1)$ such that $0 < \frac{1}{r} \log \frac{\beta}{c} < \theta$. Let $\alpha = \frac{1}{r} \log \frac{\beta}{c}$. We note that $c < ce^{\alpha r} = \beta < 1$ and $c^j e^{-\alpha(t-jr)} = \beta^j e^{-\alpha t}$. For any fixed $t > 0$, there exists an integer $n \geq 0$ such that $-r \leq t - nr < 0$. Thus, since $0 < \alpha < \theta$,

$$\begin{aligned}
E|X(t)|^2 &\leq \frac{M_0}{1-c} e^{-\alpha t} + cE|X(t-r)|^2 + \frac{M_\delta}{c} + \frac{M_\gamma}{1-c} \\
&\leq \frac{M_0}{1-c} \left(\sum_{j=0}^{n-1} c^j e^{-\alpha(t-jr)} \right) + c^n E|\varphi(t-nr)|^2 + \left(\sum_{j=0}^{n-1} c^j \right) \left(\frac{M_\delta}{c} + \frac{M_\gamma}{1-c} \right) \\
&\leq \frac{M_0}{1-c} \left(\sum_{j=0}^{n-1} \beta^j \right) e^{-\alpha t} + c^n B_0 + \left(\sum_{j=0}^{n-1} c^j \right) \left(\frac{M_\delta}{c} + \frac{M_\gamma}{1-c} \right).
\end{aligned}$$

On the other hand, since $c = \beta e^{-\alpha r}$ and $-r \leq t - nr < 0$, we have that

$$c^n = \beta^n e^{-n\alpha r} < \beta^n e^{-\alpha t} < e^{-\alpha t}.$$

Hence we obtain that

$$E|X(t)|^2 \leq \left(\frac{M_0 \beta_0}{1-c} + B_0 \right) e^{-\alpha t} + c_0,$$

where $\beta_0 = \frac{1}{1-\beta}$. This completes the proof of the lemma. $\square$

Next we assume that

$$E\left(\sup_{-r \leq t \leq 0} |\varphi(t)|^2 \right) \leq B_1. \tag{12}$$

Then we can establish the following lemma.

Lemma 3. *Let (12) hold and let N be a natural number. Assume that all the conditions of Lemma 2 are fulfilled except condition (11) which is replaced by*

$$E\left(\sup_{Nr \leq t \leq (N+1)r} |X(t) - k(t, X(t-r))|^2 \right) \leq M_0 e^{-\theta N} + \gamma(t), t \geq 0. \tag{13}$$

Then, there exist positive real numbers $\rho > 0$ and $M_2 > 0$ such that

$$E\left(\sup_{Nr \leq t \leq (N+1)r} |X(t)|^2 \right) \leq M_2 e^{-\rho N} + c_0, \ t \geq 0.$$

Proof. By Lemma 1 it follows that

$$\begin{aligned}
&E\left(\sup_{N \leq t \leq N+1} |X(t)|^2 \right) \\
&\leq \frac{1}{1-c} E\left(\sup_{Nr \leq t \leq (N+1)r} |X(t) - k(t, X(t-r))|^2 \right) \\
&+ \frac{1}{c} E\left(\sup_{Nr \leq t \leq (N+1)r} |k(t, X(t-r))|^2 \right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}&E\left(\sup_{Nr\le t\le (N+1)r}|X(t)|^2\right)\\&\le \frac{M_0}{1-c}e^{-\theta Nr}+cE\left(\sup_{Nr\le t\le (N+1)r}|X(t-r)|^2\right)+\frac{M_\delta}{c}+\frac{M_\gamma}{1-c}.\end{aligned}$$

Since the real number c satisfies $0<c<1$, we can choose $\mu>0$ such that

$$0<\mu<\theta \text{ and } 0<\mu<\frac{1}{r}\log\frac{1}{c}.$$

Then we obtain that

$$\begin{aligned}E\left(\sup_{Nr\le t\le (N+1)r}|X(t)|^2\right)&\le \frac{M_0}{1-c}\left(\sum_{j=0}^{N-1}(ce^{\mu r})^j\right)e^{-\mu Nr}\\&\quad+c^N E\left(\sup_{Nr\le t\le (N+1)r}|X(t-Nr)|^2\right)+c_0\left(\sum_{j=0}^{N-1}c^j\right).\end{aligned}$$

Thus, since $c^N<e^{-\mu Nr}$, we obtain that

$$\begin{aligned}E\left(\sup_{Nr\le t\le (N+1)r}|X(t)|^2\right)&\le \frac{M_0e^{-\mu Nr}}{(1-c)(1-ce^{\theta r})}+c^NB_1+\frac{c_0}{1-c}\\&\le\left(\frac{M_0}{(1-c)(1-ce^{\mu r})}+B_1\right)e^{-\mu Nr}+\frac{c_0}{1-c},\end{aligned}$$

which completes the proof of the lemma. □

Now, thanks to these previous lemmata, we can discuss the ultimate exponential boundedness and the exponential stability of the solutions to the neutral stochastic delay differential equation (2).

5. **Exponential stability under a coercivity condition.** It is known that some kind of coercivity conditions may play the rôle of stability criteria (see, e.g., [3] or [2]) in dealing with the analysis of stability of solutions to partial differential equations (either deterministic or stochastic). In this section we will show that the same is true in our neutral case. However, we will formulate this condition under an integral form which is, in fact, weaker. To this end, we assume that the operators in the neutral stochastic delay partial differential equation (2) satisfy the following condition:

There exists $m_0>0$ such that for each $m\in[0,m_0]$ there exist constants $\alpha_1(m)>0$, $\alpha_2(m)\ge 0$ and a function $\xi(\cdot,m):\mathbb{R}_+\to\mathbb{R}_+$ such that for all $T>0$ and all $u\in L^2(-r,T;V)\cap C(-r,T;H)$ it holds

$$\begin{aligned}&\int_{t_0}^{t}e^{ms}\left[2\langle u(s)-k(s,u(s-r)),A(s)u(s)+f(s,u_s)\rangle+||g(s,u_s)||^2_{\mathcal{L}_0^2}\right]\,ds\\&\le-\alpha_1(m)\int_{t_0}^{t}e^{ms}||u(s)||^2\,ds+\alpha_2(m)\int_{t_0-r}^{t_0}e^{ms}||u(s)||^2\,ds+\xi(t,m),\qquad(14)\end{aligned}$$

for a.e. $t \in (t_0, T)$, and all $t_0 \in [0, T]$. We assume in this section that $k(t, \cdot) : H \to H$ is continuous, maps V into itself, satisfies (10) and

$$||k(t,u)||^2 \leq c^2 ||u||^2 + \delta(t),\ 0 < c < 1, \quad \text{for } u \in V, \tag{15}$$

for the same constant c and function δ as in (10).

Remark 1. Observe that a stronger condition implying (14) is the following: There exist a constant $\alpha_1 > 0$ and a nonnegative function $\xi_1(\cdot)$ such that, for any $\phi \in C_r$ it follows

$$2\langle \phi(0) - k(t, \phi(-r)), A(t)\phi(0) + f(t, \phi)\rangle + ||g(t, \phi)||^2_{\mathcal{L}^2_0} \leq -\alpha_1 ||\phi(0)||^2 + \xi_1(t). \tag{16}$$

Notice that (16) seems to be a strong condition since it indicates that the final value $\phi(0)$ plays a dominant rôle in the analysis, what in addition may restrict the possibilities for the operators f and g. Nevertheless, some types of integral inequalities imposed on f and g allow that more general cases fall within this framework. Indeed, assume that f and g satisfy the following assumptions:
There exists $m_0 > 0$ such that for all $m \in [0, m_0]$, $T > 0$, and any $u, v \in L^2(-r, T; V) \cap C(-r, T; H)$

$$\int_{t_0}^t e^{ms} ||f(s, u_s) - f(s, v_s)||_*^2 ds \leq L_f^2 \int_{t_0 - r}^t e^{ms} |u(s) - v(s)|^2 \, ds, \tag{17}$$

$$\int_{t_0}^t e^{ms} ||g(s, u_s) - g(s, v_s)||_{\mathcal{L}^2_0}^2 ds \leq L_g^2 \int_{t_0 - r}^t e^{ms} |u(s) - v(s)|^2 \, ds. \tag{18}$$

We remark that these conditions are fulfilled for functions f, g which contain constant, variable or distributed bounded delays (see [4] for some illustrative examples). In addition, let us suppose that k satisfies (10), (15) and that A satisfies (C) with $\lambda = 0$ (or with $\alpha - \lambda\lambda_1^{-1} > 0$). Then

$$\begin{aligned}
I &= \int_{t_0}^t e^{ms} \left[2\langle u(s) - k(s, u(s-r)), A(s)u(s) + f(s, u_s)\rangle + ||g(s, u_s)||^2_{\mathcal{L}^2_0} \right] \, ds \\
&\leq \int_{t_0}^t 2e^{ms} \langle u(s), A(s)u(s)\rangle \, ds + \int_0^t 2e^{ms} \langle u(s), f(s, u_s)\rangle \, ds \\
&\quad - \int_{t_0}^t 2e^{ms} \langle k(s, u(s-r)), A(s)u(s)\rangle \, ds \\
&\quad - \int_{t_0}^t 2e^{ms} \langle k(s, u(s-r)), f(s, u_s)\rangle \, ds + \int_{t_0}^t e^{ms} ||g(s, u_s)||^2_{\mathcal{L}^2_0} \, ds \\
&\leq -\alpha \int_{t_0}^t e^{ms} ||u(s)||^2 \, ds + \underbrace{\int_{t_0}^t 2e^{ms} ||u(s)|| ||f(s, u_s)||_* \, ds}_{=I_1} \\
&\quad + \underbrace{\int_{t_0}^t 2e^{ms} ||k(s, u(s-r))|| ||A(s)u(s)||_* \, ds}_{=I_2} \\
&\quad + \underbrace{\int_{t_0}^t 2e^{ms} ||k(s, u(s-r))|| ||f(s, u_s)||_* \, ds}_{=I_3} + \underbrace{\int_{t_0}^t e^{ms} ||g(s, u_s)||^2_{\mathcal{L}^2_0} \, ds}_{=I_4}.
\end{aligned}$$

Let us now estimate the integrals $I_i, \quad i = 1, 2, 3, 4$, by using the assumptions on A, f, g, k, and Hölder's and Young's inequalities. To start, we assume that $A(\cdot) \in L^\infty\left((0,+\infty), \mathcal{L}(V,V^*)\right)$ and denote $a = ||A||_{L^\infty((0,+\infty),\mathcal{L}(V,V^*))}$.
First, for positive constants $l_1 > 0$, $l_2 > 0$, to be chosen later, we have

$$
\begin{aligned}
I_1 &\leq \int_{t_0}^{t} e^{ms}\left(l_1||u(s)||^2 + \frac{1}{l_1}||f(s,u_s)||_*^2\right)\, ds \\
&\leq l_1\int_{t_0}^{t} e^{ms}||u(s)||^2\, ds + \frac{2}{l_1}\int_{t_0}^{t} e^{ms}\left(||f(s,u_s)-f(s,0)||_*^2 + ||f(s,0)||_*^2\right)\, ds \\
&\leq l_1\int_{t_0}^{t} e^{ms}||u(s)||^2\, ds + \frac{2\lambda_1^{-1}L_f^2}{l_1}\int_{t_0-r}^{t} e^{ms}||u(s)||^2\, ds + \frac{2}{l_1}\int_{t_0}^{t} e^{ms}||f(s,0)||_*^2\, ds \\
&\leq \left(l_1 + \frac{2\lambda_1^{-1}L_f^2}{l_1}\right)\int_{t_0}^{t} e^{ms}||u(s)||^2\, ds \\
&\qquad + \underbrace{\frac{2\lambda_1^{-1}L_f^2}{l_1}}_{=\alpha_{21}(m)}\int_{t_0-r}^{t_0} e^{ms}||u(s)||^2\, ds + \underbrace{\frac{2}{l_1}\int_{t_0}^{t} e^{ms}||f(s,0)||_*^2\, ds}_{=\xi_1(t,m)},
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\leq \int_{t_0}^{t} e^{ms}\left(l_2||k(s,u(s-r))||^2 + \frac{1}{l_2}||A(s)u(s)||_*^2\right)\, ds \\
&\leq l_2\int_{t_0}^{t} e^{ms}||k(s,u(s-r))||^2\, ds + \frac{a^2}{l_2}\int_{t_0}^{t} e^{ms}||u(s)||^2\, ds \\
&\leq \frac{a^2}{l_2}\int_{t_0}^{t} e^{ms}||u(s)||^2\, ds + l_2\int_{t_0}^{t} e^{ms}\left(c^2||u(s-r)||^2 + \delta(s)\right)\, ds \\
&\leq \left(\frac{a^2}{l_2} + c^2 l_2 e^{mr}\right)\int_{t_0}^{t} e^{ms}||u(s)||^2\, ds + \underbrace{l_2\int_{t_0}^{t} e^{ms}\delta(s)\, ds}_{=\xi_2(t,m)} \\
&\qquad + \underbrace{l_2 c^2 e^{mr}}_{=\alpha_{22}(m)}\int_{t_0-r}^{t_0} e^{ms}||u(s)||^2\, ds.
\end{aligned}
$$

As for I_3, we choose another positive constant l_3 and observe that

$$
\begin{aligned}
I_3 &\leq \int_{t_0}^{t} e^{ms}\left(l_3||k(s,u(s-r))||^2 + \frac{1}{l_3}||f(s,u_s)||_*^2\right)\, ds \\
&\leq l_3\int_{t_0}^{t} e^{ms}\left(c^2||u(s-r)||^2 + \delta(s)\right)\, ds + \frac{1}{l_3}\int_{t_0}^{t} e^{ms}||f(s,u_s)||_*^2\, ds \\
&\leq c^2 e^{mr} l_3\int_{t_0}^{t} e^{ms}||u(s)||^2\, ds + \frac{1}{l_3}\int_{t_0}^{t} e^{ms}||f(s,u_s)||_*^2\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+ l_3 \int_{t_0}^{t} e^{ms}\delta(s)\ ds + l_3 c^2 e^{mr}\int_{t_0-r}^{t_0} e^{ms}||u(s)||^2\ ds\\
&\leq c^2 e^{mr} l_3 \int_{t_0}^{t} e^{ms}||u(s)||^2\ ds + \frac{2\lambda_1^{-1}L_f^2}{l_3}\int_{t_0-r}^{t} e^{ms}||u(s)||^2\ ds\\
&+ \frac{2}{l_3}\int_{t_0}^{t} e^{ms}||f(s,0)||_*^2\ ds\\
&+ l_3 \int_{t_0}^{t} e^{ms}\delta(s)\ ds + l_3 c^2 e^{mr}\int_{t_0-r}^{t_0} e^{ms}||u(s)||^2\ ds\\
&\leq \left(c^2 e^{mr} l_3 + \frac{2\lambda_1^{-1}L_f^2}{l_3}\right)\int_{t_0}^{t} e^{ms}||u(s)||^2\ ds\\
&+ \underbrace{\left(\frac{2\lambda_1^{-1}L_f^2}{l_3} + l_3 c^2 e^{mr}\right)}_{=\alpha_{23}(m)}\int_{t_0-r}^{t_0} e^{ms}||u(s)||^2\ ds\\
&+ \underbrace{\frac{2}{l_3}\int_{t_0}^{t} e^{ms}||f(s,0)||_*^2 ds + l_3\int_{t_0}^{t} e^{ms}\delta(s)\ ds}_{=\xi_3(t,m)}
\end{aligned}
$$

and, finally, we have for I_4 that

$$
\begin{aligned}
I_4 &= \int_{t_0}^{t} e^{ms}||g(s,u_s)||_{\mathcal{L}_0^2}^2\ ds\\
&\leq 2\int_{t_0}^{t} e^{ms}\left(||g(s,u_s)-g(s,0)||_{\mathcal{L}_0^2}^2 + ||g(s,0)||_{\mathcal{L}_0^2}^2\right)\ ds\\
&\leq 2L_g^2\lambda_1^{-1}\int_{t_0-r}^{t} e^{ms}||u(s)||^2\ ds + 2\int_{t_0}^{t} e^{ms}||g(s,0)||_{\mathcal{L}_0^2}^2\ ds\\
&\leq 2L_g^2\lambda_1^{-1}\int_{t_0}^{t} e^{ms}||u(s)||^2\ ds\\
&\quad + \underbrace{2L_g^2\lambda_1^{-1}}_{=\alpha_{24}(m)}\int_{t_0-r}^{t_0} e^{ms}||u(s)||^2\ ds + \underbrace{2\int_{t_0}^{t} e^{ms}||g(s,0)||_{\mathcal{L}_0^2}^2\ ds.}_{=\xi_4(t,m)}
\end{aligned}
$$

Consequently, for any choice of positive constants l_1, l_2 and l_3 we have that

$$
\begin{aligned}
I &\leq \int_{t_0}^{t} e^{ms}\left(\left[-\alpha + C(l_1,l_2,l_3,m,L_f,L_g,\lambda_1,r)\right]||u(s)||^2\right)\ ds\\
&+ \alpha_2(m)\int_{t_0-r}^{t_0} e^{ms}||u(s)||^2\ ds + \xi(t,m),
\end{aligned}
$$

where

$$
C(l_1,l_2,l_3,m,L_f,L_g,\lambda_1,r) = l_1 + 2\lambda_1^{-1}L_f^2\left(\frac{1}{l_1}+\frac{1}{l_3}\right) + \frac{a^2}{l_2} + (l_2+l_3)\,c^2 e^{mr} + 2L_g^2\lambda_1^{-1},
$$

$$
\alpha_2(m) = \sum_{i=1}^{4}\alpha_{2i}(m) \quad \text{and} \quad \xi(t,m) = \sum_{i=1}^{4}\xi_i(t,m).
$$

So, for any suitable choice of the constants l_i we can have different assumptions on the coefficients of our problem so that $-\alpha_1(m) = -\alpha + C(l_1, l_2, l_3, m, L_f, L_g, \lambda_1, r) < 0$, and thus assumption (14) is satisfied. Observe that the best chioce for these constants (i.e. the ones which impose less restrictions on the smallness of operators f, g and k is achieved when

$$l_1 = \sqrt{2}\lambda_1^{-1/2} L_f, \quad l_2 = ac^{-1}e^{-mr/2}, \quad l_3 = \sqrt{2}\lambda_1^{-1/2} L_f c^{-1} e^{-mr/2}$$

what yields that

$$-\alpha_1(m) = -\alpha + 2\sqrt{2}\lambda_1^{-1/2} L_f \left(1 + ce^{mr/2}\right) + 2ace^{mr/2} + 2L_g^2\lambda_1^{-1}. \qquad (19)$$

Now, it is clear that there exists $m_0 > 0$ such that (19) holds for all $m \in [0, m_0]$ provided that

$$-\alpha + 2\sqrt{2}\lambda_1^{-1/2} L_f (1 + c) + 2ac + 2L_g^2\lambda_1^{-1} < 0.$$

Then we have

Lemma 4. *Assume (14), (15) and (10) with $\alpha_1(m) \geq \widetilde{\alpha}_1 > 0$ for all $m \in [0, m_0]$. Assume that the function δ in (10) and $\xi(\cdot, m)$ in (14) are bounded and that, for any initial value $\varphi \in M^2_{\mathcal{F}_0}(-r, 0; V) \cap L^2(\Omega; C_r)$ the corresponding solution $X(\cdot)$ to (2) is globally defined in the future. Then, there exists $\theta \in (0, \rho)$, and two nonnegative constants $M_2, M_3 > 0$ (which may depend on the initial datum φ) such that*

$$E|X(t) - k(t, X(t-r))|^2 \leq M_2 e^{-\theta t} + M_3, \quad \textit{for all } t \geq 0.$$

If, in addition, the function $\delta(\cdot)$ has subexponential decay to zero as t goes to ∞, i.e. if there exist constants $\widetilde{\delta} > 0$ and $\rho > 0$ such that

$$\delta(t) \leq \widetilde{\delta} e^{-\rho t}, \ t \geq 0, \qquad (20)$$

then, without loss of generality we assume $\theta < \rho$ and, there exists a constant $M_4 > 0$ such that

$$E|X(t) - k(t, X(t-r))|^2 \leq M_4 e^{-\theta t} \quad \textit{for all } t \geq 0.$$

Proof. Let $X(t)$ be the solution to (2) corresponding to the initial value φ. By the condition of the theorem we can choose $\theta \in (0, \rho)$ such that

$$-\widetilde{\alpha}_1\lambda_1 + 2\theta e^{r\theta}(1 + c^2) < 0.$$

Then, applying Itô's formula to the process $e^{\theta t}|X(t) - k(t, X(t-r))|^2$ we obtain

$$\begin{aligned} d[e^{\theta t}&|X(t) - k(t, X(t-r))|^2] \\ &= \theta e^{\theta t}|X(t) - k(t, X(t-r))|^2 \, dt \\ &\quad + e^{\theta t} d[|X(t) - k(t, X(t-r))|^2] \\ &= e^{\theta t}[\theta|X(t) - k(t, X(t-r))|^2 + ||g(t, X_t)||^2_{\mathcal{L}_0^2}] \, dt \\ &\quad + 2e^{\theta t} \langle X(t) - k(t, X(t-r)), A(t)X(t) + f(t, X_t) \rangle \, dt \\ &\quad + 2e^{\theta t} \left(X(t) - k(t, X(t-r)), g(t, X_t) dW(t) \right). \end{aligned}$$

Integrating, taking expectation and taking into account condition (14) we have

$$
\begin{aligned}
&e^{\theta t}E|X(t)-k(t,X(t-r))|^2\\
&=E|\varphi(0)-k(0,\varphi(-r))|^2+\int_0^t e^{\theta s}[\theta E|X(s)-k(s,X(s-r))|^2+E||g(s,X_s)||^2_{\mathcal{L}_0^2}]\ ds\\
&+\int_0^t 2e^{\theta s}E\left\langle X(s)-k(s,X(s-r)),A(s)X(s)+f(s,X_s)\right\rangle\ ds\\
&\le E|\varphi(0)-k(0,\varphi(-r))|^2+\int_0^t e^{\theta s}\theta E|X(s)-k(s,X(s-r))|^2\ ds\\
&-\alpha_1(\theta)\int_0^t e^{\theta s}E||X(s)||^2\ ds+\alpha_2(\theta)\int_{-r}^0 e^{\theta s}E||\varphi(s)||^2\ ds+\xi(t,\theta)\\
&\le E|\varphi(0)-k(0,\varphi(-r))|^2+2e^{r\theta}c^2\theta\int_{-r}^0 e^{\theta s}E|\varphi(s)|^2\ ds+\alpha_2(\theta)\int_{-r}^0 e^{\theta s}E||\varphi(s)||^2\ ds\\
&+\xi(t,\theta)+\int_0^t e^{\theta s}[-\alpha_1(\theta)E||X(s)||^2+2\theta e^{r\theta}(1+c^2)E|X(s)|^2+2\theta e^{r\theta}\delta(s)]\ ds\\
&\le E|\varphi(0)-k(0,\varphi(-r))|^2+2e^{r\theta}c^2\theta\int_{-r}^0 e^{\theta s}E|\varphi(s)|^2\ ds+\alpha_2(\theta)\int_{-r}^0 e^{\theta s}E||\varphi(s)||^2\ ds\\
&+\xi(t,\theta)+\int_0^t e^{\theta s}\left[[-\alpha_1(\theta)\lambda_1+2\theta e^{r\theta}(1+c^2)]E|X(s)|^2+2\theta e^{r\theta}\delta(s)]\right]\ ds\\
&\le E|\varphi(0)-k(0,\varphi(-r))|^2+2e^{r\theta}c^2\theta\int_{-r}^0 e^{\theta s}E|\varphi(s)|^2\ ds\\
&+\alpha_2(\theta)\int_{-r}^0 e^{\theta s}E||\varphi(s)||^2\ ds+\xi(t,\theta)+2\theta e^{r\theta}\int_0^t e^{\theta s}\delta(s)\ ds.
\end{aligned}
$$

If we now assume that $\delta(\cdot)$ and $\xi(\cdot,\theta)$ are bounded functions, then it follows that the existence of constants M_2 and M_3 such that

$$E|X(t)-k(t,X(t-r))|^2\le M_2e^{-\theta t}+M_3,\quad \text{for all } t\ge 0.$$

If, in addition, we assume that (20) holds, we then obtain that there exists a constant $M_4>0$ such that

$$E|X(t)-k(t,X(t-r))|^2\le M_4e^{-\theta t},\quad \text{for all } t\ge 0.$$

Hence, the proof of the lemma is complete. □

On account of Lemma 2 and Lemma 4 we have the following moment exponential stability theorem.

Theorem 3. *Assume that conditions (14) and (10) are satisfied with* $\alpha_1(m)\ge\widetilde{\alpha}_1>0$ *for all* $m\in[0,m_0]$. *If the initial datum* φ *satisfies* $E\int_{-r}^0||\varphi(s)||^2ds<\infty$, *and* $\xi(\cdot,m)$, $\delta(\cdot)$ *are bounded functions (as in Lemma 4), then its associate solution* $X(\cdot)$ *to (2) is exponentially ultimately bounded in mean square. If furthermore, condition (20) is satisfied, then any solution* $X(t)$ *converges to zero exponentially in mean square.*

Now, before establishing a sufficient condition for the exponential stability with probability one of the solutions to (2), we need an auxiliary technical lemma.

Lemma 5. *Assume that conditions (14), (15), (10) and (20) are satisfied. Then, there exists a constant $\theta > 0$ such that any solution $X(\cdot)$ to problem (2) satisfies*

$$\int_\tau^t E||X(s)||^2 \, ds \leq Ce^{-\theta\tau}, \quad \textit{for all } 0 \leq \tau \leq t < +\infty,$$

where the constant $C \geq 0$ may depend on the initial datum associated to the solution $X(\cdot)$.

Proof. First, let us choose $\theta > 0$ as in the proof of Lemma 4. Then we can choose $\alpha_3(\theta) > 0$ and $\alpha_4(\theta) > 0$ such that $\alpha_1(\theta) = \alpha_3(\theta) + \alpha_4(\theta)$ and $-\alpha_3(\theta)\lambda_1 + 2\theta e^{r\theta}(1 + c^2) < 0$. Thus, it follows that

$$\begin{aligned} e^{\theta t} E|X(t) - k(t, X(t-r))|^2 &+ \alpha_4(\theta) \int_0^t e^{\theta s} E||X(s)||^2 \, ds \\ &\leq E|\varphi(0) - k(0, \varphi(-r))|^2 + 2e^{r\theta} c^2 \theta \int_{-r}^0 e^{\theta s} E|\varphi(s)|^2 \, ds \\ &+ \alpha_2(\theta) \int_{-r}^0 e^{\theta s} E||\varphi(s)||^2 \, ds + \xi(t,\theta) + 2\theta e^{r\theta} \int_0^t e^{\theta s} \delta(s) \, ds. \end{aligned}$$

So, for this positive θ we have that

$$\int_0^t e^{\theta s} E||X(s)||^2 \, ds \leq C, \quad \text{for all } \ t \geq 0.$$

Then, for $0 \leq \tau < t$, we easily conclude that

$$e^{\theta\tau} \int_\tau^t E||X(s)||^2 \, ds \leq \int_\tau^t e^{\theta s} E||X(s)||^2 \, ds \leq \int_0^t e^{\theta s} E||X(s)||^2 \, ds \leq C,$$

and the proof is now complete. □

Theorem 4. *Assume that conditions (14), (15), (10) and (20) are satisfied. In addition, assume that*

$$\int_{t_0}^t ||g(s, u_s)||^2_{\mathcal{L}_0^2} \, ds \leq g_0 \int_{t_0}^t |u(s)|^2 \, ds + g_1 \int_{t_0 - r}^{t_0} ||u(s)||^2 \, ds + g_2 e^{-\rho_3 t},$$

for some $\rho_3 > 0, g_0 > 0, g_1, g_2 \geq 0$, and for every $u \in C(-r, +\infty; H), t \geq 0$, and that there exist $\widetilde{\xi} \geq 0$ and $\rho > 0$ such that

$$\xi(t, 0) \leq \widetilde{\xi} e^{-\rho t}, \quad \textit{for all } \ t \geq 0. \tag{21}$$

Then, for any initial datum φ, the corresponding solution $X(\cdot)$ to (2) converges to zero exponentially almost surely. That is, there exist constants $\rho > 0$ (independent of φ), $M_1 > 0$, and a $T(\omega) > 0$ (which may depend both on φ) such that for a.e. $\omega \in \Omega$

$$|X(t)|^2 \leq M_1 e^{-\rho t}, \ \textit{for } t \geq T(\omega).$$

Proof. Let $X(t)$ be any fixed solution to (2). Let N be a natural number. By the Itô formula we have for any $t \geq Nr$

$$\begin{aligned}|X(t)-k(t,X(t-r))|^2 &= |X(Nr)-k(Nr,X(Nr-r))|^2\\ &\quad + 2\int_{Nr}^{t} \langle X(s)-k(s,X(s-r)), A(s)X(s)+f(s,X_s)\rangle \ ds\\ &\quad + \int_{Nr}^{t} ||g(s,X_s)||^2_{\mathcal{L}_0^2}\\ &\quad + 2\int_{Nr}^{t} (X(s)-k(s,X(s-r)), g(s,X_s)\ dW(s)).\end{aligned}$$

By (14) it follows

$$\begin{aligned}|X(t)-k(t,X(t-r))|^2 &\leq |X(Nr)-k(Nr,X(Nr-r))|^2\\ &\quad - \alpha_1(0)\int_{Nr}^{t} ||X(s)||^2\ ds + \alpha_2(0)\int_{Nr-r}^{Nr} ||X(s)||^2\ ds\\ &\quad + \xi(t,0) + 2\int_{Nr}^{t} (X(s)-k(s,X(s-r)), g(s,X_s)\ dW(s))\\ &\leq |X(Nr)-k(Nr,X(Nr-r))|^2 + \alpha_2(0)\int_{Nr-r}^{Nr} ||X(s)||^2\ ds\\ &\quad + \xi(t,0) + 2\int_{Nr}^{t} (X(s)-k(s,X(s-r)), g(s,X_s)\ dW(s))\end{aligned}$$

Next, by the Burkhölder-Davis-Gundy lemma

$$\begin{aligned}&2E\left(\sup_{Nr\leq t\leq (N+1)r}\left|\int_{Nr}^{t} (X(s)-k(s,X(s-r)), g(s,X_s)\ dW(s))\right|\right)\\ &\leq n_2\int_{Nr}^{(N+1)r} E||g(s,X_s)||^2_{\mathcal{L}_0^2} ds\\ &+ \frac{1}{2}E\left(\sup_{Nr\leq t\leq (N+1)r} |X(t)-k(t,X(t-r))|^2\right)\\ &\leq n_2\left[\int_{Nr}^{(N+1)r} g_0 E|X(s))|^2 ds + g_1\int_{Nr-r}^{Nr} E||X(s)||^2\ ds + g_2 e^{-\rho_3(N+1)r}\right]\\ &+ \frac{1}{2}E\left(\sup_{Nr\leq t\leq (N+1)r} |X(t)-k(t,X(t-r))|^2\right),\end{aligned}$$

where $n_2 > 0$ is a suitable constant. Then, we have

$$\begin{aligned}
&E\left(\sup_{Nr\le t\le (N+1)r} |X(t)-k(t,X(t-r))|^2\right)\\
&\le 2E \mid X(Nr)-k(Nr,X(Nr-r)) \mid^2\\
&\quad + 2n_2\left[\int_{Nr}^{(N+1)r} g_0 E|X(s))|^2 ds + g_1\int_{Nr-r}^{Nr} E||X(s)||^2\ ds + g_2 e^{-\rho_3(N+1)r}\right]\\
&\quad + \sup_{t\in[Nr,(N+1)r]} \xi(t,0)
\end{aligned}$$

Therefore by condition (21), Lemma 2, Lemma 4 and Lemma 5, there exist positive real numbers $M_5 > 0$, $\rho_1 > 0$ such that

$$E\left(\sup_{Nr\le t\le (N+1)r} |X(t)-k(t,X(t-r))|^2\right) \le M_5 e^{-\rho_1 N}$$

Thus by Lemma 3 we obtain that

$$E\left(\sup_{Nr\le t\le (N+1)r} |X(t)|^2\right) \le M_4 e^{-\rho_2 N},\ M_4>0,\ \rho_2>0.$$

Now, thanks to the Borel Cantelli lemma, it is a standard matter to prove the pathwise exponential stability. □

As a corollary of our previous results (i.e. by simply setting $k = 0$) we can establish a result for the exponential stability of solutions to the following stochastic delay partial differential equation (see [2] for similar results)

$$dX(t) = [A(t)X(t)) + f(t,X_t)]\,dt + g(t,X_t)dW(t) \tag{22}$$
$$X(t) = \varphi(s),\ -r\le t\le 0, \tag{23}$$

where $E\int_{-r}^0 ||\varphi(s)||^2 ds < \infty$ and the function g satisfies the same condition as in Theorem 4. Then we have

Corollary 1. *Let $\alpha_1, \eta > 0$ and $\xi_1(t) = \delta_0(t)e^{-\eta t}$ with $\delta_0(t)$ a bounded integrable function. Assume that there exists $m_0 > 0$ such that for all $m \in [0, m_0]$, $T > 0$, and any $u \in L^2(-r,T;V)\cap C(-r,T;H)$ the following inequality holds:*

$$\begin{aligned}
&\int_{t_0}^t e^{ms}\left[2\langle u(s), A(s)u(s)+f(s,u_s)\rangle + ||g(s,u_s)||^2_{\mathcal{L}_0^2}\right]\ ds\\
&\qquad \le -\int_{t_0}^t e^{ms}\alpha_1(m)||u(s)||^2\ ds + \int_{t_{0-r}}^{t_0} e^{ms}\alpha_2(m)||u(s)||^2\ ds + \xi_1(t),
\end{aligned}$$

for a.e. $t\in(t_0,T)$, and all $t_0\in[0,T)$. Then, any solution to the stochastic delay partial differential equation (22)-(23) converges to zero exponentially almost surely.

Remark 2. The reader is referred to the papers [2], [3] in order to compare Corollary 1 with the results in these papers.

6. **An example.** Although we could consider several applications to show how our theory can be applied, we have preferred to treat a simple situation in which a usual coercivity condition (which is easier to check than the integral condition (14)) ensures the asymptotic behaviour of the model. Indeed, let us consider the problem

$$\begin{aligned}&d\left[X(t)-k(t,X(t-r))\right]=\left[A(t)X(t)+f(t,X(t-r))\right]dt+g(t,X(t))dW(t)\\&X(t)=\varphi(s),\ -r\le t\le 0,\end{aligned}$$

where A and k satisfy the assumptions in the previous sections, $f:\mathbb{R}^+\times H\to V^*$ and $g:H\to\mathcal{L}_0^2(K,H)$ are Lipschitz continuous and it holds

$$\begin{aligned}&2\left\langle u-k(t,v),A(t)u+f(t,v)\right\rangle+||g(t,u)||^2_{\mathcal{L}_0^2}\\&\quad\le-\alpha_1||u||^2+\alpha_2|u|^2+\beta_1||v||^2+\beta_2|v|^2+\xi_1(t),\end{aligned}\tag{24}$$

for $u,v\in V$ where $\alpha_1>\beta_1\ge 0$ and $\alpha_2,\beta_2\ge 0$. If we assume that $(\alpha_1-\beta_1)\lambda_1>\alpha_2+\beta_2$, then (24) implies (14) since for any $u\in L^2(-r,T;V)\cap C(-r,T;H)$ it follows

$$\begin{aligned}&\int_{t_0}^t e^{ms}\left[2\left\langle u(s)-k(s,u(s-r)),A(s)u(s)+f(s,u(s-r))\right\rangle+||g(s,u(s))||^2_{\mathcal{L}_0^2}\right]\,ds\\
&\le\int_{t_0}^t e^{ms}\left(-\alpha_1||u(s)||^2+\alpha_2|u(s)|^2+\beta_1||u(s-r)||^2+\beta_2|u(s-r)|^2+\xi_1(s)\right)\,ds\\
&\le\int_{t_0}^t e^{ms}\left(-\alpha_1||u(s)||^2+\alpha_2|u(s)|^2+\xi_1(s)\right)\,ds\\
&+\int_{t_0}^t e^{ms}\left(\beta_1||u(s-r)||^2+\beta_2|u(s-r)|^2\right)\,ds\\
&\le\int_{t_0}^t e^{ms}\left(-\alpha_1||u(s)||^2+\alpha_2\lambda_1^{-1}||u(s)||^2+\xi_1(s)\right)\,ds\\
&+e^{mr}\int_{t_0-r}^{t-r} e^{ms}\left(\beta_1||u(s)||^2+\beta_2|u(s)|^2\right)\,ds\\
&\le\int_{t_0}^t e^{ms}\left(\left(-\alpha_1+\alpha_2\lambda_1^{-1}\right)||u(s)||^2+\xi_1(s)\right)\,ds\\
&+e^{mr}\int_{t_0-r}^{t-r} e^{ms}\left(\beta_1+\beta_2\lambda_1^{-1}\right)||u(s)||^2\,ds\\
&\le\int_{t_0}^t e^{ms}\left(\left(-\alpha_1+\alpha_2\lambda_1^{-1}+e^{mr}\left(\beta_1+\beta_2\lambda_1^{-1}\right)\right)||u(s)||^2+\xi_1(s)\right)\,ds\\
&+e^{mr}\int_{t_0-r}^{t_0} e^{ms}\left(\beta_1+\beta_2\lambda_1^{-1}\right)||u(s)||^2\,ds\\
&\le\int_{t_0}^t e^{ms}\left(\left(-\alpha_1+\alpha_2\lambda_1^{-1}+e^{mr}\left(\beta_1+\beta_2\lambda_1^{-1}\right)\right)||u(s)||^2\right)\,ds\\
&+\int_{t_0}^t e^{ms}\xi_1(s)\,ds+\left(\beta_1+\beta_2\lambda_1^{-1}\right)e^{mr}\int_{t_0-r}^{t_0} e^{ms}||u(s)||^2\,ds.\end{aligned}$$

As $(\alpha_1-\beta_1)\lambda_1>\alpha_2+\beta_2$, there exists $m_0>0$ such that for all $m\in[0,m_0)$ we have that

$$-\alpha_1+\alpha_2\lambda_1^{-1}+e^{mr}\left(\beta_1+\beta_2\lambda_1^{-1}\right)=-(\alpha_1-e^{mr}\beta_1-\lambda_1^{-1}\left(\alpha_2+e^{mr}\beta_2\right))=-\alpha_1(m)<0.$$

On the other hand, setting

$$\alpha_2(m) = \left(\beta_1 + \beta_2\lambda_1^{-1}\right) e^{mr} \quad \text{and} \quad \xi(t,m) = \int_0^t e^{ms}\xi_1(s)\, ds$$

we have the desired conclusion. Now, if

$$-\alpha_1 + \alpha_2\lambda_1^{-1} + \beta_1 + \beta_2\lambda_1^{-1} < 0, \tag{25}$$

it is obvious that there exists $m_0 > 0$ such that for all $m \in [0, m_0]$ it follows that $\alpha_1(m) > 0$ and our theory can be applied to ensure exponential stability of our problem.

Finally, we will especialise this situation to a more explicit one. Indeed, let $H = L^2(0,\pi)$, $V = H_0^1(0,\pi)$ and $V^* = H^{-1}(0,\pi)$, with the usual norms in the spaces H and V defined as $|\xi| = \left(\int_0^\pi \xi^2(x)dx\right)^{1/2}$ for $\xi \in H$, and $||u|| = \left(\int_0^\pi (\frac{\partial u}{\partial x})^2 dx\right)^{1/2}$ for $u \in V$, respectively.

Let us define

$$A(t) = \frac{\partial}{\partial x}\left(a(t,x)\frac{\partial}{\partial x}\right),$$

where $a(t,x)$ is measurable in $(0,+\infty)\times(0,\pi)$, and satisfies

$$0 < \nu \le a(t,x) \le \alpha \quad \text{in } (0,+\infty)\times(0,\pi).$$

The family of operators $A(t)$ satisfies $A(\cdot) \in L^\infty(0,T;\mathcal{L}(V,V^*))$ for all $T > 0$, and it is known that

$$\langle A(t)u, u\rangle \le -\nu\|u\|^2, \quad u \in V,$$

and $\lambda_1 = 1$.

Consider the following neutral stochastic delay differential equation.

$$\begin{cases} d(X(t,x) - cX(t-r,x)) \\ \qquad = \dfrac{\partial}{\partial x}\left(a(t,x)\dfrac{\partial X(t,x)}{\partial x}\right) + bX(t-r,x) + g(t,X(t,x))dW(t), \\ X(t,0) = X(t,\pi) = 0,\ t \ge 0, \\ X(t,x) = \varphi(t,x),\ \ t \in [-r,0], \end{cases} \tag{26}$$

where $W(t)$ is a cylindrical Wiener process with values in $L^2(0,\pi)$, $\varphi(t)$ is an adequate continuous square integrable process and $b, c \in \mathbb{R}$. We assume that

$$||g(t,u)||^2_{\mathcal{L}_0^2(L^2(0,\pi))} \le c_g|u|^2 + g_1(t),\ c_g > 0,\ t \ge -r,$$

$$|c| < 1 \quad \text{y} \quad 2\nu > |b|(2+|c|) + c_g + \alpha|c|, \tag{27}$$

with $0 \le g_1(t) \le g_0 e^{-\rho t}$, $g_0 > 0, \rho > 0$ for any $t \ge 0$. Let $k(t,v) = cv$ and $f(t,v) = bv$. Then we have

$$\begin{aligned} &2\langle u - k(t-r,v), A(t)u + f(t-r,v)\rangle + ||g(t,u)||^2_{L_2^0} \\ &= 2\langle u - cv, A(t)u + bv\rangle + ||g(t,u)||^2_{L_2^0} \\ &\le -2\nu||u||^2 + 2|b||u||v| + \alpha|c|||u||||v|| + |bc||v|^2 + c_g|u|^2 + g_1(t) \\ &\le -(2\nu - \frac{\alpha|c|}{2})||u||^2 + (|b| + c_g)|u|^2 + (|b| + |bc|)|v|^2 + \frac{\alpha|c|}{2}||v||^2 + g_1(t). \end{aligned}$$

Thanks to our condition (27), we can ensure that (25) holds and, therefore, any solution to (26) converges exponentially to zero almost surely.

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