# Random attractors for stochastic $2D$-Navier-Stokes equations in some unbounded domains

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**Abstract**

We show that the stochastic flow generated by the 2-dimensional Stochastic Navier-Stokes equations with rough noise on a Poincaré-like domain has a unique random attractor. One of the technical problems associated with the rough noise is overcomed by the use of the corresponding Cameron-Martin (or reproducing kernel Hilbert) space. Our results complement the result by Brzeźniak and Li [10] who showed that the corresponding flow is asymptotically compact and also generalize Caraballo et al. [12] who proved existence of a unique attractor for the time-dependent deterministic Navier-Stokes equations.

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## 1. Introduction

The analysis of infinite dimensional Random Dynamical Systems (RDS) is now an important branch in the study of qualitative properties of stochastic PDEs. From the first papers of Brzeźniak et al. [7], and Crauel and Flandoli [17], the use of the notions of random and  attractors have been used in many papers to give crucial information on the asymptotic behaviour of random (Brzeźniak et al. [7]), stochastic (Arnold [2], Crauel and Flandoli [17], Crauel [19]) and non-autonomous PDEs (Schmalfuss [35], Kloeden and Schmalfuss [27], Caraballo et al. [12]). Given a probability space, a random attractor is a compact random set, invariant for the associated RDS and attracting every bounded random set in its basis of attraction (see Definition 2.6).

The main general result on random attractors relies heavily on the existence of a random compact attracting set, see Crauel et al. [20]. But this condition was only shown to be true when the embedding $V \hookrightarrow H$ is compact, i.e. when our stochastic PDE is set in a *bounded* domain. In the deterministic case, this difficulty was solved by different methods, see Abergel [1], Ghidaglia [24] or Rosa [34] for the autonomous case and Łukaszewicz and Sadowski [33] or Caraballo et al. [12] for the non-autonomous one. Recently, these methods have been also generalized to a stochastic framework, see Brzeźniak and Li [8], [9],[10], Bates et al. [3],[4], Wang [38]. In particular, in Brzeźniak and Li [10], a deep work is provided for the existence of a stochastic flow and its asymptotic behaviour related to a 2D stochastic Navier-Stokes equations in an unbounded domain with a very general irregular additive white noise. Moreover, in [10] sufficient conditions for the existence of a unique random global attractor are proposed. This is the main subject that we will develop in this work.

Indeed, we study the asymptotic behaviour of solutions to the following problem. Let $\mathcal{O} \subset \mathbb{R}^2$ be an open set, not necessarily bounded, with sufficiently regular boundary $\partial\mathcal{O}$, and suppose that $\mathcal{O}$ satisfies the Poincaré inequality, i.e., there exists a constant $C > 0$ such that

$$C \int_{\mathcal{O}} \varphi^2 \, d\xi \leq \int_{\mathcal{O}} |\nabla \varphi|^2 \, d\xi \quad \text{for all } \varphi \in H_0^1(\mathcal{O}),$$

and consider the Navier-Stokes equations (NSE) in $\mathcal{O}$ with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f + \dfrac{dW(t)}{dt} \quad \text{in} \quad (0, +\infty) \times \mathcal{O}, \\ \operatorname{div} u \ = 0 \quad \text{in} \quad (0, +\infty) \times \mathcal{O}, \\ u = 0 \ \text{ on } (0, +\infty) \times \partial\mathcal{O}, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, $u$ is the velocity field of the fluid, $p$ the pressure, $u_0$ the initial velocity field, and $f$ a given external force field. Here

$W(t), t \in \mathbb{R}$, is a two-sided cylindrical Wiener process in H with its Reproducing Kernel Hilbert Space (RKHS) K satisfying Assumption A.1 below, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that following [10] we allow our driving noise to be much rougher than in previous works in the literature, see for instance Crauel and Flandoli [17] or Kloeden and Langa [28], for which it is possible to prove that there exists a random dynamical system associated to our model. Indeed, the rougher the noise the closer the model is to reality. Landau and Lifshitz in their fundamental 1959 work [30, Chapter 17] proposed to study NSEs under additional stochastic small fluctuations. Consequently the authors consider the classical balance laws for mass, energy and momentum forced by a random noise, to describe the fluctuations, in particular local stresses and temperature, which are not related to the gradient of the corresponding quantities. In [31, Chapter 12] the same authors then derive correlations for the random forcing by following the general theory of fluctuations. One of the requirements on the noise they impose is that the noise is either spatially uncorrelated or correlated as little as possible. It is known that spatially uncorrelated noise corresponds to the Wiener process with RKHS $L^2$ and if the RKHS of the Wiener process is the Sobolev space $H^{s,2}$, then the smaller the $s$ the less correlated noise is. In other words, the less regular spatially is the less correlated it is. Note that our Wiener process includes a finite dimensional Brownian Motion as a special case.

On the other hand, Caraballo et al. [12] introduced a concept of a asymptotically compact cocycle, which was successfully used to prove the existence of attractors for a 2D non-autonomous Navier-Stokes equations, and later has been also used to prove existence of random attractors for stochastic lattice dynamical systems in Bates et al. [3], stochastic reaction-diffusion equations in Bates et al. [4] and a stochastic Benjamin-Bona-Mahony equation in Wang [38], all of them related to unbounded domains. In this paper, we will use the same concept, which generalizes the one in [10], to prove the existence and the uniqueness of global random attractors for our stochastic 2D Navier-Stokes equations with irregular noise in Poincaré unbounded domains. In this sense, our main result implies that the stochastic flow associated to our model is asymptotically compact (see Proposition 3.1). It is remarkable that we also prove the measurability of our random attractor, which is usually missed in the literature.

**Notation 1.1.** *By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^-$, $\mathbb{R}$ and $\mathbb{R}_+$ we will denote respectively the sets of natural numbers (which includes the zero), of integers, of non-positive integers, of real numbers and of all non-negative real numbers. By $\mathcal{B}(X)$, where $X$ is a topological space, we will denote the $\sigma$-field of all Borel subsets of $X$.*

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## 2. Stochastic 2$D$-Navier-Stokes equations with additive noise in unbounded domains

### 2.1. Statement of the problem

Let $\mathcal{O} \subset \mathbb{R}^2$ be an open set, not necessarily a bounded one. We denote by $\partial\mathcal{O}$ the boundary of $\mathcal{O}$. We will always assume that the closure $\bar{\mathcal{O}}$ of the set $\mathcal{O}$ is manifold with boundary of $C^\infty$ class, whose boundary is equal to $\partial\mathcal{O}$, i.e. we will assume that $\mathcal{O}$ satisfies the condition (7.10) from [32, chapter I]:

$$(7.10) \left\{ \begin{array}{l} \partial\mathcal{O} \text{ is a 1-dimensional infinitely differentiable manifold, } \mathcal{O} \text{ being} \\ \text{locally on one side of } \mathcal{O}. \end{array} \right.$$

We will also assume that $\mathcal{O}$ is a Poincaré domain, i.e. that there exists a constant $\lambda_1 > 0$ such that the following Poincaré inequality is satisfied

$$\lambda_1 \int_{\mathcal{O}} \varphi^2\, dx \le \int_{\mathcal{O}} |\nabla\varphi|^2\, dx \quad \text{for all } \varphi \in H_0^1(\mathcal{O}). \tag{2.1}$$

In order to formulate our problem in an abstract framework let us recall the definitions of the following usual functional spaces.

$$\begin{aligned} \mathbb{L}^2(\mathcal{O}) &= L^2(\mathcal{O}, \mathbb{R}^2), \\ \mathbb{H}^k(\mathcal{O}) &= H^{k,2}(\mathcal{O}, \mathbb{R}^2),\ k \in \mathbb{N}, \\ \mathcal{V} &= \left\{ u \in C_0^\infty(\mathcal{O}, \mathbb{R}^2);\ \operatorname{div} u\ = 0 \right\}, \\ H &= \text{the closure of } \mathcal{V} \text{ in } \mathbb{L}^2(\mathcal{O}), \\ \mathbb{H}_0^1(\mathcal{O}) &= \text{the closure of } C_0^\infty(\mathcal{O}, \mathbb{R}^2) \text{ in } \mathbb{H}^1(\mathcal{O}), \\ V &= \text{the closure of } \mathcal{V} \text{ in } \mathbb{H}^1(\mathcal{O}). \end{aligned}$$

We endow the set $H$ with the inner product $(\cdot, \cdot)$ and the norm $|\cdot|$ induced by $\mathbb{L}^2(\mathcal{O})$. Thus, we have

$$(u, v) = \sum_{j=1}^{2} \int_{\mathcal{O}} u_j(x) v_j(x)\, dx,$$

Since the set $\mathcal{O}$ is a Poincaré domain, the norms on $V$ induced by $\mathbb{H}^1(\mathcal{O})$ and $\mathbb{H}_0^1(\mathcal{O})$ are equivalent. The latter norm and the associated inner product will be denoted by $\|\cdot\|$ and $(\!(\cdot, \cdot)\!)$, respectively. They satisfy the following equality

$$(\!(u, v)\!) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx,\ u, v \in \mathbb{H}_0^1(\mathcal{O}).$$

Since the space V is densely and continuously embedded into H, by identifying H with its dual H′, we have the following embeddings

$$\mathrm{V} \subset \mathrm{H} \cong \mathrm{H}' \subset \mathrm{V}'. \tag{2.2}$$

Let us observe here that, in particular, the spaces V, H and V′ form a Gelfand triple.

We will denote by $|\cdot|_{V'}$ and $\langle\cdot,\cdot\rangle$ the norm in $V'$ and the duality pairing between $V$ and $V'$, respectively.

The presentation of the Stokes operator is standard and we follow here the one given in [10]. We begin with defining a bilinear form $a : \mathrm{V}\times\mathrm{V}\to\mathbb{R}$ by

$$a(u,v) := (\nabla u, \nabla v), \quad u, v \in \mathrm{V}. \tag{2.3}$$

Since obviously the form $a$ coincides with the $(\!(\cdot,\cdot)\!)$ scalar product in V, it is V-continuous, i.e. it satisfies $|a(u,u)| \le C\|u\|^2$ for some $C > 0$ and all $u \in \mathrm{V}$. Hence, by the Riesz Lemma, there exists a unique linear operator $\mathcal{A} : \mathrm{V} \to \mathrm{V}'$, such that $a(u,v) = \langle \mathcal{A}u, v\rangle$, for $u, v \in \mathrm{V}$. Moreover, since the $\mathcal{O}$ is a Poincaré domain, the form $a$ is V-coercive, i.e. it satisfies $a(u,u) \ge \alpha\|u\|^2$ for some $\alpha > 0$ and all $u \in \mathrm{V}$. Therefore, in view of the Lax-Milgram theorem, see for instance Temam [37, Theorem II.2.1], the operator $\mathcal{A} : \mathrm{V} \to \mathrm{V}'$ is an isomorphism.

Next we define an unbounded linear operator A in H as follows.

$$\begin{cases} D(\mathrm{A}) & := \{u \in \mathrm{V} : \mathcal{A}u \in \mathrm{H}\} \\ \mathrm{A}u & := \mathcal{A}u,\ u \in D(\mathrm{A}). \end{cases} \tag{2.4}$$

It is now well established that under some assumptions[5] related to the regularity of the domain $\mathcal{O}$, the space $D(\mathrm{A})$ can be characterized in terms of the Sobolev spaces. For example, see [25], where only the 2-dimensional case is studied but the result is also valid in the 3-dimensional case, if $\mathcal{O} \subset \mathbb{R}^2$ is a uniform $C^2$-class Poincaré domain, then with $\mathrm{P} : \mathbb{L}^2(\mathcal{O}) \to \mathrm{H}$ being the orthogonal projection, we have

$$\begin{cases} D(\mathrm{A}) & := \mathrm{V} \cap \mathbb{H}^2(\mathcal{O}), \\ \mathrm{A}u & := -\mathrm{P}\Delta u, \quad u \in D(\mathrm{A}). \end{cases} \tag{2.5}$$

It is also a classical result, see e.g. Cattabriga [15] or Temam [37], p. 56, that A is a non-negative self adjoint operator in H. Moreover, see p. 57 in [37], $\mathrm{V} = D(\mathrm{A}^{1/2})$. Let us recall a result of Fujiwara–Morimoto [23] that the projection P extends to a bounded linear projection in the space $\mathbb{L}^q(D)$, $1 < q < \infty$.

Consider the trilinear form $b$ on $V \times V \times V$ given by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad u, v, w \in V.$$

Indeed, $b$ is a continuous bilinear form and, see for instance [36], Lemma 1.3, p.163 and Temam [37],

$$\begin{aligned} & b(u,v,v) = 0, \quad \text{for } u \in \mathrm{V}, v \in \mathbb{H}_0^{1,2}(\mathcal{O}), \\ & b(u,v,w) = -b(u,w,v), \quad \text{for } u \in \mathrm{V}, v, w \in \mathbb{H}_0^{1,2}(\mathcal{O}), \end{aligned} \tag{2.6}$$

[5] These assumptions are satisfied in our case

$$|b(u,v,w)| \le C \begin{cases} |u|^{1/2}|\nabla u|^{1/2}|\nabla v|^{1/2}|\mathrm{A}v|^{1/2}|w|, & u \in \mathrm{V}, v \in D(\mathrm{A}), w \in \mathrm{H}, \\ |u|^{1/2}|\mathrm{A}u|^{1/2}|\nabla v||w|, & u \in D(\mathrm{A}), v \in \mathrm{V}, w \in \mathrm{H}, \\ |u||\nabla v||w|^{1/2}|\mathrm{A}w|^{1/2}, & u \in \mathrm{H}, v \in \mathrm{V}, w \in D(\mathrm{A}), \\ |u|^{1/2}|\nabla u|^{1/2}|\nabla v||w|^{1/2}|\nabla w|^{1/2}, & u,v,w \in \mathrm{V} \end{cases} \tag{2.7}$$

for some $C > 0$. Define next a bilinear map $B : V \times V \to V'$ by

$$\langle B(u,v), w\rangle = b(u,v,w), \quad u,v,w \in V,$$

and a homogenous polynomial of second degree $B : V \to V'$ by

$$B(u) = B(u,u),\ u \in V.$$

Let us also recall [10, Lemma 4.2].

**Lemma 2.1.** *The trilinear map $b : \mathrm{V} \times \mathrm{V} \times \mathrm{V} \to \mathbb{R}$ has a unique extension to a bounded trilinear map from $\mathbb{L}^4(\mathcal{O}) \times (\mathbb{L}^4(\mathcal{O}) \cap \mathrm{H}) \times \mathrm{V}$ and from $\mathbb{L}^4(\mathcal{O}) \times \mathrm{V} \times \mathbb{L}^4(\mathcal{O})$ to $\mathbb{R}$. Moreover, $B$ maps $\mathbb{L}^4(\mathcal{O}) \cap \mathrm{H}$ (and so $\mathrm{V}$) into $\mathrm{V}'$ and*

$$\|B(u)\|_{\mathrm{V}'} \le C_1|u|^2_{\mathbb{L}^4(\mathcal{O})} \le 2^{1/2}C_1|u||\nabla u| \le C_2|u|^2_V, \quad u \in \mathrm{V}. \tag{2.8}$$

*Proof.* It it enough to observe that from the Hölder inequality we have the following inequality

$$|b(u,v,w)| \le C|u|_{\mathbb{L}^4(\mathcal{O})}|\nabla v|_{\mathbb{L}^2(\mathcal{O})}|w|_{\mathbb{L}^4(\mathcal{O})}, \quad u,v,w \in \mathbb{H}^{1,2}_0(\mathcal{O}). \tag{2.9}$$

□

*2.2. Attractors for random dynamical systems*

**Definition 2.2.** *A triple $\mathfrak{T} = (\Omega, \mathcal{F}, \vartheta)$ is called a* ***measurable dynamical system*** *(DS) iff $(\Omega, \mathcal{F})$ is a measurable space and $\vartheta : \mathbb{R} \times \Omega \ni (t,\omega) \mapsto \vartheta_t\omega \in \Omega$ is a measurable map such that $\vartheta_0 =$identity, and for all $t, s \in \mathbb{R}$, $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$. A quadruple $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is called* ***metric DS*** *iff $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathfrak{T}' := (\Omega, \mathcal{F}, \vartheta)$ is a measurable DS such that for each $t \in \mathbb{R}$, the map $\vartheta_t : \Omega \to \Omega$ is $\mathbb{P}$-preserving.*

**Definition 2.3.** *Suppose that $\mathrm{X}$ is a Polish space, i.e. a metrizable complete separable topological space, $\mathcal{B}$ is its Borel $\sigma-$field and $\mathfrak{T}$ is a metric DS. A map $\varphi : \mathbb{R}_+ \times \Omega \times \mathrm{X} \ni (t,\omega,x) \mapsto \varphi(t,\omega,x) \in \mathrm{X}$ is called a* ***measurable random dynamical system*** *(RDS) (on $\mathrm{X}$ over $\mathfrak{T}$), iff*

*(i)* $\varphi$ *is* $(\mathcal{B}(\mathbb{R}_+) \bigotimes \mathcal{F} \bigotimes \mathcal{B}, \mathcal{B})$*-measurable;*
*(ii)* $\varphi$ *is a* $\vartheta$*-cocycle, i.e.*

$$\varphi(t+s,\omega,x) = \varphi\big(t, \vartheta_s\omega, \varphi(s,\omega,x)\big).$$

*The map $\varphi$ is said to be* ***continuous*** *iff, for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$, $\varphi(t,\omega,\cdot) : \mathrm{X} \to \mathrm{X}$ is continuous. Similarly, $\varphi$ is said to be time continuous iff, for all $\omega \in \Omega$ and $x \in \mathrm{X}$, the map $\varphi(\cdot,\omega,x) : \mathbb{R}_+ \to \mathrm{X}$ is continuous.*

The notion of a random set is presented following [7], see also Crauel [19] and Definition 2.3 in [10]. For two non-empty sets $A, B \subset X$, where $(\mathrm{X}, d)$ is a Polish space, we put

$$d(A,B) = \sup_{x \in A} d(x,B) \quad \text{and} \quad \rho(A,B) = \max\{d(A,B), d(B,A)\}.$$

The latter is called the Hausdorff distance. It is known that $\rho$ restricted to the family $\mathfrak{CB}(X)$ (the family of all non-empty closed and bounded subsets of X) is a distance, see Castaing and Valadier [14]. From now on, let $\mathfrak{X}$ be the $\sigma-$field on $\mathfrak{CB}$ generated by open sets with respect to the Hausdorff metric $\rho$, e.g. [7], [14] or Crauel [19].

**Definition 2.4.** *Let us assume that $(\Omega, \mathcal{F})$ is a measurable space and $(\mathrm{X}, d)$ is a Polish space. A set valued map $C : \Omega \to \mathfrak{CB}(X)$ is said to be measurable iff $C$ is $(\mathcal{F}, \mathcal{X})$-measurable. Such a map $C$ will often be called a* ***closed and bounded random set*** *on $X$. A closed and bounded random set $C$ on $X$ will be called a* ***compact random set*** *on $X$ iff for each $\omega \in \Omega$, $C(\omega)$ is a compact subset of $X$.*

**Remark 2.5.** Let $f : X \mapsto \mathbb{R}_+$ be a continuous function on the Polish space $X$, and $R : \Omega \mapsto \mathbb{R}_+$ an $\mathcal{F}$-measurable random variable. If the set $C_{f,R}(\omega) := \{x : f(x) \leq R(\omega)\}$ is non-empty for each $\omega \in \Omega$, then $C_{f,R}$ is a closed and bounded random set (see [16] Proposition 1.3.6).

We denote by $\mathcal{F}^u$ the $\sigma$-algebra of universally measurable sets associated to the measurable space $(\Omega, \mathcal{F})$, see Crauel's monograph [19] for the definition and basic properties.

To our best knowledge, the following definition appeared for the first time in the fundamental work by Fladoli and Schmalfuss [22], see Definition 3.4.

**Definition 2.6.** *A random set $A : \Omega \to \mathfrak{CB}(X)$ is a* ***random $\mathfrak{D}$-attractor*** *iff*

*(i) $A$ is a compact random set,*

*(ii) $A$ is $\varphi$-invariant, i.e., $P$-a.s.*

$$\varphi(t, \omega)A(\omega) = A(\vartheta_t \omega)$$

*(iii) $A$ is $\mathfrak{D}$-attracting, in the sense that, for all $D \in \mathfrak{D}$ it holds*

$$\lim_{t \to \infty} d(\varphi(t, \vartheta_{-t}\omega)D(\vartheta_{-t}\omega), A(\omega)) = 0.$$

**Definition 2.7.** *We say that a RDS $\vartheta$-cocycle $\varphi$ on $X$ is $\mathfrak{D}$-**asymptotically compact** iff for each $D \in \mathfrak{D}$, for every $\omega \in \Omega$, for any positive sequence $(t_n)$ such that $t_n \to \infty$ and for any sequence $\{x_n\}$ such that*

$$x_n \in D(\vartheta_{-t_n}\omega), \quad \textit{for all } n \in \mathbb{N},$$

*the following set is pre-compact in* X*:*

$$\{\varphi(t_n, \vartheta_{-t_n}\omega, x_n) : n \in \mathbb{N}\}.$$

We now write the result on the existence of a random $\mathfrak{D}$-attractor, see [3, 4, 12, 38].

**Theorem 2.8.** *Assume that $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a metric DS, $\mathrm{X}$ is a Polish space, $\mathfrak{D}$ is a nonempty class of closed and bounded random sets on $X$ and $\varphi$ is a continuous, $\mathfrak{D}$-asymptotically compact RDS on $\mathrm{X}$ (over $\mathfrak{T}$). Assume that there exists a $\mathfrak{D}$-absorbing closed and bounded random set $B$ on $X$, i.e., given $D \in \mathfrak{D}$ there exists $t(D))$ such that $\varphi(t, \vartheta_t\omega)D(\vartheta_{-t}\omega) \subset B(\omega)$ for all $t \geq t(D)$. Then, there exits an $\mathcal{F}^u$-measurable $\mathfrak{D}$-attractor $A$ given by*

$$A(\omega) = \Omega_B(\omega), \quad \omega \in \Omega, \tag{2.10}$$

*with*

$$\Omega_B(\omega) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}, \quad \omega \in \Omega.$$

**Remark 2.9.** One should mention here that a related paper [28] is about 2D Navier-Stokes in **bounded** domains and and with much more regular noise and its results do not imply those from the current paper. On the other hand Theorem 4.6 from that paper is applicable to Stochastic NSEs in 2-D unbounded domains, instead of Theorem 2.8 provided one can prove that the corresponding system is asymptotically compact and has random bounded absorbing set. That's is what we do in our paper, with a small but important difference that our class of families of random sets with respect to which AC and absorption hold are different. Our Theorem 2.8 on the existence of attractor is generalization (or modification, if one prefers) of the above Theorem 4.6 to the case considered in the present paper.

*Proof.* The existence of $A(\omega)$ follows from Theorem 7 in [12]. We only need to prove the measurability claim. For this we will follow a slight modification of the proof of Proposition 1.6.2 in [16], see also the proof of Lemma 2.3 in [7]. Observe that evidently, for every $\omega \in \Omega$,

$$\Omega_B(\omega) = \bigcap_{n \in \mathbb{Z}_+} \overline{\gamma_B^n(\omega)},$$

where, by definition,

$$\gamma_B^n(\omega) = \bigcup_{t \geq n} \varphi\big(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)\big).$$

Let us fix $x \in X$. Since $\gamma_B^n(\omega) \subset \gamma_B^{n+1}(\omega) \subset \overline{\gamma_B^{n+1}(\omega)} \subset \overline{\gamma_B^n(\omega)}$, we have that

$$d\big(x, \gamma_B^n(\omega)\big) \leq d\big(x, \gamma_B^{n+1}(\omega)\big) \leq d\big(x, \Omega_B(\omega)\big),$$

and therefore there exists the $\lim\limits_{n \to \infty} d\big(x, \gamma_B^n(\omega)\big)$, and

$$\lim_{n \to \infty} d\big(x, \gamma_B^n(\omega)\big) \leq d\big(x, \Omega_B(\omega)\big), \quad \forall\, \omega \in \Omega.$$

Let us fix $\omega \in \Omega$, and take $x_n \in \gamma_B^n(\omega)$ such that

$$d(x, x_n) \leq d\big(x, \gamma_B^n(\omega)\big) + \frac{1}{n}, \quad n \in \mathbb{N}.$$

Since $\varphi$ is a $\mathfrak{D}$-asymptotically compact RDS on X, there exists a subsequence $n_k = n_k(\omega)$ and an element $y = y(\omega)$ such that $x_{n_k} \to y$. Evidently, $y \in \Omega_B(\omega)$. Therefore,

$$d\big(x, \Omega_B(\omega)\big) \leq d(x, y) = \lim_{k\to\infty} d(x, x_{n_k}) \leq \lim_{n\to\infty} d\big(x, \gamma_B^n(\omega)\big).$$

Thus, we get

$$d\big(x, \Omega_B(\omega)\big) = \lim_{n\to\infty} d\big(x, \gamma_B^n(\omega)\big) \quad \forall\, \omega \in \Omega,$$

and consequently, observing that by Proposition 1.5.1 in [16] the map $\omega \mapsto d\big(x, \gamma_B^n(\omega)\big)$ is $\mathcal{F}^u$-measurable, we obtain that the map $\omega \mapsto d\big(x, \Omega_B(\omega)\big)$ is also $\mathcal{F}^u$-measurable. $\square$

**Remark 2.10.** If $\mathfrak{D}$ contains every bounded and closed nonempty deterministic subsets of $X$, then as a consequence of this theorem, of Theorem 2.1 in [20], and of Corollary 5.8 in [18], we obtain that the random attractor $A$ is given by

$$A(\omega) = \overline{\bigcup_{C\subset X} \Omega_C(\omega)} \quad \mathbb{P} - a.s., \tag{2.11}$$

where the union in (2.11) is made for all bounded and closed nonempty deterministic subsets $C$ of $X$.

*2.3. Stochastic Navier-Stokes equations with an additive noise*

The model we consider in this subsection is the same as the one studied in [10]. It was shown therein that the RDS generated by the stochastic NSEs below is asymptotically compact. We will strengthen that result by showing that

(i) it is $\mathfrak{D}$-asymptotically compact and

(ii) there exists a family $B \in \mathfrak{D}$ which is $\mathfrak{D}$-absorbing,

for a family $\mathfrak{D}$ of random closed and bounded sets to be defined below. Thus we will conclude that Theorem 2.8 is applicable.

Our aim in this subsection is to study the following stochastic Navier-Stokes equations in $\mathcal{O}$

$$\begin{cases} du + \{\nu \mathrm{A}u + B(u)\}\, dt = f\, dt + dW(t), \quad t \geq 0 \\ u(0) = x, \end{cases} \tag{2.12}$$

where we assume that $x \in \mathrm{H}$, $f \in \mathrm{V}'$ and $W(t), t \in \mathbb{R}$, is a two-sided cylindrical Wiener process in H with its Reproducing Kernel Hilbert Space (RKHS) K satisfying Assumption A.1 below (see Remark 6.1 in [10]) defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The following is our standing assumption.

**Assumption A.1.** $\mathrm{K} \subset \mathrm{H} \cap \mathbb{L}^4(\mathcal{O})$ *is a Hilbert space such that for some* $\delta \in (0, 1/2)$,

$$\mathrm{A}^{-\delta} : K \to \mathrm{H} \cap \mathbb{L}^4(\mathcal{O}) \text{ is } \gamma\text{-radonifying}. \tag{2.13}$$

Let us denote $\mathrm{X} = \mathrm{H} \cap \mathbb{L}^4(\mathcal{O})$ and let E be the completion of $X$ $A^{-\delta}(\mathrm{X})$ with respect to the image norm $|x|_{\mathrm{E}} = |A^{-\delta}x|_{\mathrm{X}}$, $x \in \mathrm{X}$. It is well known that E is a separable Banach space. For $\xi \in [0, 1/2)$ we set

$$\|\omega\|_{C^{\xi}_{1/2}(\mathbb{R})} = \sup_{t \neq s \in \mathbb{R}} \frac{|\omega(t) - \omega(s)|_{\mathrm{E}}}{|t-s|^{\xi}(1+|t|+|s|)^{1/2}}.$$

By $C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$ we will denote the set of all $\omega \in C(\mathbb{R}, \mathrm{E})$ such that $\omega(0) = 0$ and $\|\omega\|_{C^{\xi}_{1/2}(\mathbb{R})} < \infty$. It is easy to prove that the closure of $\{\omega \in C^{\infty}_0(\mathbb{R}, \mathrm{E}) : \omega(0) = 0\}$ in $C^{\xi}_{1/2}(\mathbb{R})$, denoted by $\Omega(\xi, \mathrm{E})$, is a separable Banach space[6].

Finally, we set

$$\|\omega\|_{C_{1/2}(\mathbb{R},\mathrm{E})} = \sup_{t \in \mathbb{R}} \frac{|\omega(t)|_{\mathrm{E}}}{1+|t|^{1/2}}$$

and denote by $C_{1/2}(\mathbb{R}, \mathrm{E})$ the space of all continuous functions $\omega : \mathbb{R} \to \mathrm{E}$ such that $\|\omega\|_{C_{1/2}(\mathbb{R},\mathrm{E})} < \infty$. Then the space $C_{1/2}(\mathbb{R}, \mathrm{E})$ endowed with the norm $\|\cdot\|_{C_{1/2}(\mathbb{R},\mathrm{E})}$ is a separable Banach space.

By $\mathcal{F}$ we will denote the Borel $\sigma$-algebra on $\Omega(\xi)$. One can show by methods from [5], but see also [26] for a similar problem in the one dimensional case, that for $\xi \in (0, 1/2)$, there exists a Borel probability measure $\mathbb{P}$ on $\Omega(\xi)$ such that the canonical process $w = (w_t)_{t \in \mathbb{R}}$, defined by

$$w_t(\omega) := \omega(t), \quad \omega \in \Omega(\xi), \ t \in \mathbb{R}, \tag{2.14}$$

where $i_t : \Omega(\xi) \ni \gamma \mapsto \gamma(t) \in \mathrm{E}$, is the evaluation map at time $t$, is a two-sided Wiener process such that the Cameron-Martin, i.e. the Reproducing Kernel Hilbert, space of the Gaussian measure $\mathcal{L}(w_1)$ on E is equal to K.

For $t \in \mathbb{R}$, let $\mathcal{F}_t := \sigma\{w_s : s \leq t\}$. Since for each $t \in \mathbb{R}$ the map $z \circ i_t : \mathrm{E}^* \to L^2(\Omega(\xi), \mathcal{F}_t, \mathbb{P})$ satisfies $\mathbb{E}|z \circ i_t|^2 = t|z|^2_{\mathrm{K}}$, there exists a unique extension of $z \circ i_t$ to a bounded linear map $W_t : \mathrm{K} \to L^2(\Omega(\xi), \mathcal{F}_t, \mathbb{P})$. Moreover, the family $(W_t)_{t \in \mathbb{R}}$ is a H-cylindrical Wiener process on a filtered probability space $(\Omega(\xi), (\mathfrak{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$ in the sense of definition given in [11].

On the space $C_{1/2}(\mathbb{R}, \mathrm{X})$ we consider a flow $\vartheta = (\vartheta_t)_{t \in \mathbb{R}}$ defined by

$$\vartheta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}.$$

With respect to this flow the spaces $C^{\xi}_{1/2}(\mathbb{R})$ and $\Omega(\xi, \mathrm{E})$ are invariant and we will often denote by $\vartheta_t$ the restriction of $\vartheta_t$ to any one of these spaces.

It is obvious that for each $t \in \mathbb{R}$, $\vartheta_t$ preserves $\mathbb{P}$. In order to define an Ornstein-Uhlenbeck process we need to recall some analytic preliminaries from [10].

[6]But for $\xi \in (0, 1)$ $C^{\xi}_{1/2}(\mathbb{R})$ endowed with the norm $\|\cdot\|_{C^{\xi}_{1/2}(\mathbb{R})}$ is not separable.

**Proposition 2.11.** *Assume that $A$ is a generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ on a separable Banach space $\mathrm{X}$, such that for some $C>0$ and $\gamma>0$*

$$\|A^{1+\delta}e^{-tA}\|_{\mathcal{L}(\mathrm{X},\mathrm{X})} \leq Ct^{-1-\delta}e^{-\gamma t}, \quad t\geq 0. \tag{2.15}$$

*For $\xi\in(\delta,\frac{1}{2})$ there exists a unique linear and bounded map $\hat{z}: C^{\xi}_{1/2}(\mathbb{R},\mathrm{X})\to C_{1/2}(\mathbb{R},\mathrm{X})$ such that for any $\tilde{\omega}\in C^{\xi}_{1/2}(\mathbb{R},\mathrm{X})$*

$$\hat{z}(t)=\hat{z}(\tilde{\omega})(t)=\int_{-\infty}^{t} A^{1+\delta}e^{-(t-r)A}\big(\tilde{\omega}(t)-\tilde{\omega}(r)\big)\,dr, t\in\mathbb{R}. \tag{2.16}$$

*In particular, there exists a constant $C_2<\infty$ such that for any $\tilde{\omega}\in C^{\xi}_{1/2}(\mathbb{R},\mathrm{X})$*

$$|\hat{z}(\tilde{\omega})(t)|_{\mathrm{X}} \leq C_2(1+|t|^{1/2})\|\tilde{\omega}\|, \quad t\in\mathbb{R}. \tag{2.17}$$

*Moreover, the above results are valid with the space $C^{\xi}_{1/2}(\mathbb{R},\mathrm{X})$ replaced by $\Omega(\xi,\mathrm{X})$.*

*Proof.* See Proposition 6.2 in [10]. The last part follows as $\Omega(\xi,\mathrm{X})$ is a closed subset of $C^{\xi}_{1/2}(\mathbb{R},\mathrm{X})$. □

The following results are respectively Corollary 6.4, Theorem 6.6 and Corollary 6.8 from [10].

**Corollary 2.12.** *Under the assumptions of Proposition 2.11, for all $-\infty<a<b<\infty$ and $t\in\mathbb{R}$, the map*

$$C^{\xi}_{1/2}(\mathbb{R},\mathrm{X})\ni\tilde{\omega}\mapsto(\hat{z}(\tilde{\omega})(t),\hat{z}(\tilde{\omega}))\in\mathrm{X}\times L^4(a,b;\mathrm{X}) \tag{2.18}$$

*is continuous. Moreover, the above result is valid with the space $C^{\xi}_{1/2}(\mathbb{R},\mathrm{X})$ being replaced by $\Omega(\xi,\mathrm{X})$.*

**Theorem 2.13.** *Under the assumptions of Proposition 2.11, for any $\omega\in C^{\xi}_{1/2}(\mathbb{R},\mathrm{X})$,*

$$\hat{z}(\vartheta_s\omega)(t)=\hat{z}(\omega)(t+s), \quad t,s\in\mathbb{R}. \tag{2.19}$$

*In particular, for any $\omega\in\Omega$ and all $t,s\in\mathbb{R}$, $\hat{z}(\vartheta_s\omega)(0)=\hat{z}(\omega)(s)$.*

For $\zeta\in C_{1/2}(\mathbb{R},\mathrm{X})$ we put

$$(\tau_s\zeta)=\zeta(t+s), \quad t,s\in\mathbb{R}.$$

Thus, $\tau_s$ is a a linear and bounded map from $C_{1/2}(\mathbb{R},\mathrm{X})$ into itself. Moreover, the family $(\tau_s)_{s\in\mathbb{R}}$ is a $C_0$ group on $C_{1/2}(\mathbb{R},\mathrm{X})$.

Using this notation Theorem 2.13 can be rewritten in the following way.

**Corollary 2.14.** *For $s\in\mathbb{R}$ $\tau_s\circ\hat{z}=\hat{z}\circ\vartheta_s$, i.e.*

$$\tau_s\big(\hat{z}(\omega)\big)=\hat{z}\big(\vartheta_s(\omega)\big),\ \omega\in C^{\xi}_{1/2}(\mathbb{R},\mathrm{X}).$$

Note that for any $\nu > 0$ and $\alpha \geq 0$, $(\nu A + \alpha I)^{\delta} : \mathrm{E} \to \mathrm{X}$ is a bounded linear map and so is the induced map $\Omega(\xi, \mathrm{E}) \ni \omega \mapsto (\nu A + \alpha I)^{\delta}\omega \in \Omega(\xi, \mathrm{X})$.

For $\delta$ as in the Assumption A.1, $\alpha \geq 0$, $\nu > 0$, $\xi \in (\delta, 1/2)$ and $\omega \in C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$ (so that $(\nu A + \alpha I)^{-\delta}\omega \in C^{\xi}_{1/2}(\mathbb{R}, \mathrm{X})$), we put $z_\alpha(\omega) := \hat{z}\big((\nu A + \alpha I)^{-\delta}\omega\big) \in C_{1/2}(\mathbb{R}, \mathrm{X})$. Hence, for any $t \geq 0$,

$$\begin{aligned} z_\alpha(\omega)(t) &:= \int_{-\infty}^{t} (\nu A + \alpha I)^{1+\delta} e^{-(t-r)(\nu A + \alpha I)} \\ &\qquad\qquad \big[(\nu A + \alpha I)^{-\delta}\omega(t) - (\nu A + \alpha I)^{-\delta}\omega(r)\big]\, dr \qquad (2.20) \\ &= \int_{-\infty}^{t} (\nu A + \alpha I)^{1+\delta} e^{-(t-r)(\nu A + \alpha I)} \big((\nu A + \alpha I)^{-\delta}\vartheta_r\omega\big)(t-r)\, dr. \end{aligned}$$

It follows from Theorem 2.13 that

$$z_\alpha(\vartheta_s\omega)(t) = z_\alpha(\omega)(t+s), \quad \omega \in C^{\xi}_{1/2}(\mathbb{R}, \mathrm{X}),\ t, s \in \mathbb{R}. \tag{2.21}$$

Let us make the following crucial observation, see Proposition 6.10 in [10]: the process $z_\alpha$ is an $X$-valued stationary and ergodic. Hence, by the Strong Law of Large Numbers, see [21] for a similar argument,

$$\lim_{t\to\infty} \frac{1}{t}\int_{-t}^{0} |z_\alpha(s)|^2_{\mathrm{X}}\, ds = \mathbb{E}|z_\alpha(0)|^2_{\mathrm{X}}, \quad \mathbb{P} - \text{a.s. on } C^{\xi}_{1/2}(\mathbb{R}, \mathrm{X}). \tag{2.22}$$

Therefore, it follows from [10, Proposition 6.10] that we find $\alpha_0$ such that for all $\alpha \geq \alpha_0$,

$$\mathbb{E}|z_\alpha(0)|^2_{\mathrm{X}} < \frac{\nu^2\lambda_1}{6C^2}, \tag{2.23}$$

where $\lambda_1$ is the constant appearing in the Poincaré inequality (2.1) and $C > 0$ is a certain universal constant.

By $\Omega_\alpha(\xi, \mathrm{E})$ we denote the set of those $\omega \in \Omega(\xi, \mathrm{E})$ for which the equality (2.22) holds true. As mentioned above, the set $\Omega_\alpha(\xi, \mathrm{E})$ is $\mathbb{P}$-conegligible. Moreover, in view of Corollary 2.14 that the set $\Omega_\alpha(\xi, \mathrm{E})$ is invariant with respect to the flow $\vartheta$, i.e. for all $\alpha \geq 0$ and all $t \in \mathbb{R}$, $\vartheta_t\big(\Omega_\alpha(\xi, \mathrm{E})\big) \subset \Omega_\alpha(\xi, \mathrm{E})$. Therefore, we fix

$$\xi \in (\delta, \frac{1}{2})$$

and set

$$\Omega := \hat{\Omega}(\xi, \mathrm{E}) = \bigcap_{n=1}^{\infty} \Omega_n(\xi, \mathrm{E}).$$

For reasons that will become clear later we take as a model of a metric DS the quadruple $\Big(\hat{\Omega}(\xi, \mathrm{E}), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta}\Big)$ where $\hat{\mathcal{F}}$, $\hat{\mathbb{P}}$ and $\hat{\vartheta}$ are respectively the natural restrictions of $\mathcal{F}$, $\mathbb{P}$ and $\vartheta$ to $\hat{\Omega}(\xi, \mathrm{E})$.

**Proposition 2.15.** *The quadruple $\Big(\hat{\Omega}(\xi, \mathrm{E}), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta}\Big)$ is a metric DS. For each $\omega \in \hat{\Omega}(\xi, \mathrm{E})$ the limit in (2.22) exists.*

Let us now formulate an immediate and important consequence of the above result in which $C > 0$ is the constant appearing in (2.23).

**Corollary 2.16.** *For each $\omega \in \hat{\Omega}(\xi, \mathrm{E})$ there exits $t_0 = t_o(\omega) \geq 0$, such that*

$$\frac{3C^2}{\nu} \int_{-t}^{0} |z_\alpha(s)|_{\mathrm{X}}^2 \, ds < \frac{\nu \lambda_1 t}{2}, \ t \geq t_0, \tag{2.24}$$

Finally we define a map $\varphi = \varphi_\alpha : \mathbb{R}_+ \times \Omega \times \mathrm{H} \to \mathrm{H}$ by

$$(t, \omega, x) \mapsto v\big(t, z_\alpha(\omega)\big)\big(x - z(\omega)(0)\big) + z_\alpha(\omega)(t) \in \mathrm{H}, \tag{2.25}$$

where $v(t, v_0)$, $t \geq 0$, is a solution to the following problem

$$\frac{dv}{dt} = -\nu \mathrm{A} v - B(v) - B(v, z) - B(z, v) - B(z) + \alpha z + f, \tag{2.26}$$

$$v(0) = v_0. \tag{2.27}$$

Because of Theorem 4.5 from [10], which, for the completeness sake, we state below as Theorem 2.18, and because $z_\alpha(\omega) \in C_{1/2}(\mathbb{R}, \mathrm{X})$, $z_\alpha(\omega)(0)$ is a well defined element of H. Consequently, the map $\varphi_\alpha$ is well defined.

**Definition 2.17.** *Suppose that $z \in L^4_{\mathrm{loc}}\big([0, \infty); \mathbb{L}^4(\mathcal{O})\big) \cap L^4_{\mathrm{loc}}([0, \infty); \mathrm{V})$, $f \in \mathrm{V}'$ and $v_0 \in \mathrm{H}$. A function $v \in C([0, \infty); \mathrm{H}) \cap L^2_{\mathrm{loc}}([0, \infty); \mathrm{V}') \cap L^4_{\mathrm{loc}}\big([0, \infty); \mathbb{L}^4(\mathcal{O})\big)$ is a solution to problem (2.26)-(2.27) iff $v(0) = v_0$ and (2.26) holds in the weak sense, i.e. for any $\varphi \in \mathrm{V}$*

$$\frac{d}{dt}(v(t), \varphi) = -\nu\big((v(t), \varphi)\big) - b\big(v(t) + z(t), \varphi, v(t) + z(t)\big) + (\alpha z(t) + f, \varphi), \tag{2.28}$$

*in the distributions sense on $(0, \infty)$.*

**Theorem 2.18.** *Assume that $\alpha \geq 0$, $v_0 \in \mathrm{H}$, $f \in \mathrm{V}'$ and $z \in L^4_{\mathrm{loc}}\big([0, \infty); \mathbb{L}^4(\mathcal{O})\big) \cap L^2_{\mathrm{loc}}([0, \infty); \mathrm{V}')$.*

*(i) Then there exists a unique solution $v$ of problem (2.26)-(2.27).*

*(ii) If in addition, $v_0 \in \mathrm{V}$, $f \in \mathrm{H}$ and $z \in C(\mathbb{R}; V) \cap L^2_{\mathrm{loc}}\big(\mathbb{R}; D(\mathrm{A})\big)$, then $v \in C([0, \infty); \mathrm{V}) \cap L^2_{\mathrm{loc}}\big([0, \infty); D(\mathrm{A})\big)$.*

It was proved in [10, Proposition 6.16] that the map $\varphi_\alpha$ does not depend on $\alpha$ and hence, from now on, it will be denoted by $\varphi$. Furthermore, we have the following result, see [10, Theorems 6.15 and 8.8].

**Theorem 2.19.** *Suppose that $\mathcal{O} \subset \mathbb{R}^2$ is a Poincaré domain and that Assumption A.1 is satisfied. Then the map $\varphi$ is an asymptotically compact RDS over the metric DS $\Big(\hat{\Omega}(\xi, \mathrm{E}), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta}\Big)$.*

Our previous results yield the existence and the uniqueness of solutions to problem (2.12) as well as its continuous dependence on the data (in particular on the initial value $u_0$ and the force $f$). Moreover, if we define, for $x \in \mathrm{H}$, $\omega \in \Omega$, and $t \geq s$,

$$u(t, s; \omega, u_0) := \varphi(t - s; \vartheta_s \omega) u_0 = v\big(t, s; \omega, u_0 - z(s)\big) + z(t), \tag{2.29}$$

then for each $s \in \mathbb{R}$ and each $u_0 \in \mathrm{H}$, the process $u(t)$, $t \geq s$, is a solution to problem (2.12).

Let us now recall Lemma 8.3 and Lemma 8.5 from [10] in which $\lambda_1$ is the constant appearing in the Poincaré inequality (2.1) and

$$[v]^2 := \nu \|v\|^2 - \frac{\lambda_1}{2} |v|^2, \ v \in V. \tag{2.30}$$

**Lemma 2.20.** *Suppose that $v$ is a solution to problem (2.26) on the time interval $[a, \infty)$ with $z \in L^4_{\mathrm{loc}}\big([a, \infty), \mathbb{L}^4(\mathcal{O})\big) \cap L^2_{\mathrm{loc}}([a, \infty), \mathrm{V}')$ and $\alpha \geq 0$. Denote $\beta = \frac{\nu \lambda_1}{2}$ and $g(t) = \alpha z(t) - B\big(z(t), z(t)\big)$, $t \in [a, \infty)$. Then, for any $t \geq \tau \geq a$*

$$\begin{aligned} |v(t)|^2 \leq & |v(\tau)|^2 e^{-\nu\lambda_1 (t-\tau) + \frac{3C^2}{\nu} \int_\tau^t |z(s)|^2_{\mathbb{L}^4} ds} \\ & + \frac{3}{\nu} \int_\tau^t \{ |g(s)|^2_{\mathrm{V}'} + |f|^2 \} e^{-\nu\lambda_1 (t-s) + \frac{3C^2}{\nu} \int_s^t (|z(\zeta)|^2_{\mathbb{L}^4}) d\zeta} ds. \end{aligned} \tag{2.31}$$

$$\begin{aligned} |v(t)|^2 = & |v(\tau)|^2 e^{-\nu\lambda_1(t-\tau)} + 2 \int_\tau^t e^{-\nu\lambda_1(t-s)} (\langle B(v(s), z(s)), v(t) \rangle \\ & + \langle g(s), v(s) \rangle + \langle f, v(s) \rangle - [v(s)]^2) \, ds. \end{aligned} \tag{2.32}$$

**Lemma 2.21.** *Under the above assumptions, for each $\omega \in \Omega(\xi, \mathrm{E})$,*

$$\lim_{t \to -\infty} |z(\omega)(t)|^2 e^{\nu\lambda_1 t + \int_t^0 \frac{3C^2}{\nu} |z(\omega)(s)|^2_{\mathbb{L}^4} ds} = 0.$$

Finally, let us recall a result containing in itself [10, Lemmas 8.6 and 8.7].

**Lemma 2.22.** *Under the above assumptions, for each $\omega \in \Omega(\xi, \mathrm{E})$,*

$$\int_{-\infty}^0 \big[1 + |z(\omega)(t)|^2_{\mathbb{L}^4} + |z(\omega)(t)|^4_{\mathbb{L}^4}\big] e^{\nu\lambda_1 t + \int_t^0 \frac{3C^2}{\nu} |z(\omega)(s)|^2_{\mathbb{L}^4} ds} \, dt < \infty.$$

Since the proof of Lemma 8.6 from [10] is miraculously missing from the final version of that paper, below we will present a detailed proof of Lemma 2.22. In fact, it is enough to consider the integral with the 4th moment of $z$ as the cases of 1 and of the 2nd moment follow analogously.

*Proof.* It is enough to consider the case of $|z(\omega)(t)|^4_{\mathbb{L}^4}$. Let us fix $\omega \in \Omega$. By Corollary 2.16 we can find $t_0 \geq 0$ such that for $t \geq t_0$,

$$\int_{-t}^{-t_0} (-\nu\lambda_1 + \frac{3C^2}{\nu} |z(s)|^2_{\mathbb{L}^4}) \, ds \leq -\frac{\nu\lambda_1 (t - t_0)}{2}.$$

By the continuity of all relevant functions, it is sufficient to prove that the integral $\int_{-\infty}^{-t_0} |z(\omega)(t)|^4_{\mathbb{L}^4} e^{\nu\lambda_1 t + \int_t^0 \frac{3C^2}{\nu} |z(\omega)(s)|^2_{\mathbb{L}^4}\, ds}\, dt$ is finite.

Because of inequality (2.17), we can find a constant $\rho_2 = \rho_2(\omega)$, such that

$$\frac{|z(t)|_{\mathbb{L}^4}}{1+|t|} \le \rho_2,\ t \in \mathbb{R}.$$

Therefore, with $\rho_3(\omega) := e^{\int_{-t_0}^0 (-\nu\lambda_1 + \frac{3C^2}{\nu} |z(r)|^2_{\mathbb{L}^4})\, dr} < \infty$, we have, for every $\omega \in \Omega$,

$$\begin{aligned}
&\int_{-\infty}^{-t_0} |z(s)|^4_{\mathbb{L}^4} e^{\int_s^0 (-\nu\lambda_1 + \frac{3C^2}{\nu} |z(r)|^2_{\mathbb{L}^4})\, dr}\, ds \\
&\quad = \rho_3 \int_{-\infty}^{-t_0} |z(s)|^4_{\mathbb{L}^4} e^{\int_s^{-t_0} (-\nu\lambda_1 + \frac{3C^2}{\nu} |z(r)|^2_{\mathbb{L}^4})\, dr}\, ds \\
&\qquad \le \rho_2^4 \rho_3 e^{\frac{\nu\lambda_1}{2} t_0} \int_{-\infty}^{-t_0} |s|^4 e^{\frac{\nu\lambda_1}{2} s}\, ds < \infty.
\end{aligned}$$

□

**Definition 2.23.** *A function $r : \Omega \to (0,\infty)$ belongs to the class $\mathfrak{R}$ if and only if*

$$\limsup_{t\to\infty} r(\vartheta_{-t}\omega)^2 e^{-\nu\lambda_1 t + \frac{3C^2}{\nu} \int_{-t}^0 |z(\omega)(s)|^2_{\mathbb{L}^4}\, ds} = 0, \tag{2.33}$$

*where $C > 0$ is the constant appearing in (2.23).*
*We denote by $\mathfrak{DR}$ the class of all closed and bounded random sets $D$ on $H$ such that the function radious $\Omega \ni \omega \mapsto r(D(\omega)) := \sup\{|x|_H : x \in B\}$ belongs to the class $\mathfrak{R}$.*

Observe that by Corollary 2.16, the constant functions belong to $\mathfrak{R}$. The following result lists a couple of other important examples of functions belonging to class $\mathfrak{R}$.

**Proposition 2.24.** *Define functions $r_i : \Omega \to (0,\infty)$, $i = 1,2,3$ by the following formulae, for $\omega \in \Omega$,*

$$\begin{aligned}
r_1^2(\omega) &:= |z(\omega)(0)|^2_H, \\
r_2^2(\omega) &:= \sup_{s\le 0} |z(\omega)(s)|^2_H e^{\nu\lambda_1 s + \frac{3C^2}{\nu} \int_s^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr} \\
r_3^2(\omega) &:= \int_{-\infty}^0 |z(\omega)(s)|^2_H e^{\nu\lambda_1 s + \frac{3C^2}{\nu} \int_s^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}\, ds \\
r_4^2(\omega) &:= \int_{-\infty}^0 |z(\omega)(s)|^4_{\mathbb{L}^4} e^{\nu\lambda_1 s + \frac{3C^2}{\nu} \int_s^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}\, ds \\
r_5^2(\omega) &:= \int_{-\infty}^0 e^{\nu\lambda_1 s + \frac{3C^2}{\nu} \int_s^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}\, ds.
\end{aligned}$$

*Then all these functions belong to class $\mathfrak{R}$.*
*The class $\mathfrak{R}$ is closed with respect to sum, multiplication by a constant and if $r \in \mathfrak{R}$, $0 \le \bar{r} \le r$, then $\bar{r} \in \mathfrak{R}$.*

*Proof.* Since by Theorem 2.13, $z(\vartheta_{-t}\omega)(s) = z(\omega)(s-t)$, we have

$$\begin{aligned} r_2^2(\vartheta_{-t}\omega) &= \sup_{s\le 0} |z(\vartheta_{-t}\omega)(s)|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 |z(\vartheta_{-t}\omega)(r)|^2_{\mathbb{L}^4}\, dr} \\ &= \sup_{s\le 0} |z(\omega)(s-t)|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 |z(\omega)(r-t)|^2_{\mathbb{L}^4}\, dr} \\ &= \sup_{s\le 0} |z(\omega)(s-t)|^2 e^{\nu\lambda_1 (s-t) + \frac{3C^2}{\nu}\int_{s-t}^{-t} |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr} e^{\nu\lambda_1 t} \\ &= \sup_{\sigma\le -t} |z(\omega)(\sigma)|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{-t} |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr} e^{\nu\lambda_1 t} \end{aligned}$$

Hence, multiplying the above by $e^{-\nu\lambda_1 t} e^{\frac{3C^2}{\nu}\int_{-t}^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}$ we get

$$r_2^2(\vartheta_{-t}\omega) e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr} \le \sup_{\sigma\le -t} |z(\omega)(\sigma)|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{0} |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}.$$

This, together with Lemma 2.21 concludes the proof in the case of function $r_2$. In the case of $r_1$, we have

$$r_1^2(\vartheta_{-t}\omega) e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr} = |z(\omega)(-t)|^2 e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^{0} |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}.$$

Thus, by Lemma 2.21 we infer that $r_1$ also belongs to the class $\mathfrak{R}$. The argument in the case of function $r_3$ is similar but the completness sake we include it here. From the first part of the proof we infer that

$$r_3^2(\vartheta_{-t}\omega) e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr} \le \int_{-\infty}^{-t} |z(\omega)(\sigma)|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{0} |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}\, d\sigma.$$

Since by Lemma 2.22 $\int_{-\infty}^0 |z(\omega)(\sigma)|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{0} |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}\, d\sigma$ is finite, by the Lebesgue monotone Theorem we conclude that

$$\int_{-\infty}^{-t} |z(\omega)(\sigma)|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{0} |z(\omega)(r)|^2_{\mathbb{L}^4}\, dr}\, d\sigma \to 0 \text{ as } t \to \infty.$$

The proof in the other cases is analogous. The proof of the second part of the Proposition is trivial. This concludes the proof. $\square$

Now we are ready to state and prove the main result of this paper.

**Theorem 2.25.** *Suppose that the domain $\mathcal{O} \subset \mathbb{R}^2$ is a Poincaré domain and that the Assumption A.1 is satisfied. Consider the metric DS $\mathfrak{T} = \left(\hat{\Omega}(\xi, \mathrm{E}), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta}\right)$ from Proposition 2.15, and the RDS $\varphi$ on $H$ over $\mathfrak{T}$ generated by the 2D stochastic Navier-Stokes equations with additive noise (2.12) satisfying Assumption A1. Then the following properties hold.*

*(i) there exists a $\mathfrak{DR}$-absorbing set $B \in \mathfrak{DR}$;*

*(ii) the RDS $\varphi$ is $\mathfrak{DR}$-asymptotically compact;*

*(iii) the family $A$ of sets defined by $A(\omega) = \Omega_B(\omega)$ for all $\omega \in \Omega$, is the minimal $\mathfrak{DR}$-attractor for $\varphi$, is $\hat{\mathcal{F}}$-measurable, and*

$$A(\omega) = \overline{\bigcup_{C \subset H} \Omega_C(\omega)} \quad \hat{\mathbb{P}} - a.s., \tag{2.34}$$

*where the union in (2.34) is made for all bounded and closed nonempty deterministic subsets $C$ of $H$.*

*Proof.* In view of Theorem 2.8 and Remark 2.10, it is enough to show points (i)-(ii). We prove now point (i). The proof of point (ii) will be done in the next section.

Let $D$ be a random set from the class $\mathfrak{DR}$. Let $r_D(\omega)$ be the radius of $D(\omega)$, i.e. $r_D(\omega) := \sup\{|x|_H : x \in D(\omega)\}$, $\omega \in \Omega$.

Let $\omega \in \Omega$ be fixed. For given $s \le 0$ and $x \in \mathrm{H}$, let $v$ be the solution of (2.26) on time interval $[s, \infty)$ with the initial condition $v(s) = x - z(s)$. Applying (2.31) with $t = 0, \tau = s \le 0$, we get

$$\begin{aligned} |v(0)|^2 \quad &\le \quad 2|x|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 |z(r)|^2_{\mathbb{L}^4}\,dr} + 2|z(s)|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 |z(r)|^2_{\mathbb{L}^4}\,dr} \\ &+ \quad \frac{3}{\nu}\int_s^0 \{\|g(t)\|^2_{\mathrm{V}'} + \|f\|^2_{\mathrm{V}'}\} e^{\nu\lambda_1 t + \frac{3C^2}{\nu}\int_t^0 |z(r)|^2_{\mathbb{L}^4}\,dr}\,dt. \end{aligned} \tag{2.35}$$

Set, for $\omega \in \Omega$,

$$\begin{aligned} r_{11}(\omega)^2 = 2 + \sup_{s\le 0}\Big\{ & 2|z(s)|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 |z(r)|^2_{\mathbb{L}^4}\,dr} \\ + \frac{3}{\nu}\int_s^0 & \{\|g(t)\|^2_{\mathrm{V}'} + \|f\|^2_{\mathrm{V}'}\} e^{\nu\lambda_1 t + \frac{3C^2}{\nu}\int_t^0 |z(r)|^2_{\mathbb{L}^4}\,dr}\,dt \Big\}, \end{aligned} \tag{2.36}$$

$$r_{12}(\omega) = |z(0)(\omega)|_{\mathrm{H}}. \tag{2.37}$$

By Lemma 2.22 and Proposition 2.24 we infer that both $r_{11}$ and $r_{12}$ belong to $\mathfrak{R}$ and also that $r_{13} := r_{11} + r_{12}$ belongs to $\mathfrak{R}$ as well. Therefore the random set $B$ defined by $B(\omega) := \{u \in \mathrm{H} : |u| \le r_{13}(\omega)\}$ belongs to the family $\mathfrak{DR}$.

We will show now that $B$ absorbs $D$. Let $\omega \in \Omega$ be fixed. Since $r_D \in \mathfrak{R}$ there exists $t_D(\omega) \ge 0$, such that

$$r_0(\vartheta_{-t}\omega)^2 e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 |z(\omega)(s)|^2_{L^4}\,ds} \le 1, \text{ for } t \ge t_D(\omega).$$

Thus, if $x \in D(\vartheta_{-t}\omega)$ and $s \ge t_D(\omega)$, then by (2.35), $|v(0, \omega; s, x - z(s))| \le r_{11}(\omega)$. Thus we infer that

$$|u(0, s; \omega, x)| \le |v(0, s; \omega, x - z(s))| + |z(0)(\omega)| \le r_{13}(\omega).$$

In other words, $u(0, s; \omega, x) \in B(\omega)$, for all $s \ge t_D(\omega)$. This proves that $B$ absorbs $D$. $\square$

## 3. Proof of the $\mathfrak{DR}$-asymptotical compactness property of the RDS $\varphi$ generated by stochastic NSEs

We consider here the RDS $\varphi$ over the metric DS $\left(\hat{\Omega}(\xi, \mathrm{E}), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta}\right)$, from Proposition 2.15 and the family $\mathfrak{DR}$ defined in Definition 2.23. The main result in this section is the following result.

**Proposition 3.1.** *Assume that for each random set $D$ belonging to $\mathfrak{DR}$, there exists a random set $B$ belonging to $\mathfrak{DR}$ such that $B$ absorbs $D$. Then, the RDS $\varphi$ is $\mathfrak{DR}$-asymptotically compact.*

Let us recall that the RDS $\varphi$ is independent of the auxiliary parameter $\alpha \in \mathbb{N}$. For reasons that will become clear in the course of the proof we choose $\alpha$ such that $\mathbb{E}|z_\alpha(0)|^2_{\mathbb{L}^4} \leq \frac{\nu^2\lambda_1}{6C^2}$, where $z_\alpha(t)$, $t \in \mathbb{R}$ is the Ornstein-Uhlenbeck process from section 2, $C > 0$ is a certain universal constant. Such a choice is possible because of Proposition 2.15. Let us choose $\alpha \in \mathbb{N}$ such that the condition (2.23) is satisfied.

For simplicity of notation we will denote the space $\hat{\Omega}(\xi, \mathrm{E})$ simply by $\Omega$ and the process $z_\alpha(t)$, $t \in \mathbb{R}$ by $z(t)$, $t \in \mathbb{R}$.

*Proof.* Suppose that $D$ is a closed random set from the class $\mathfrak{DR}$ and $B \in \mathfrak{DR}$ is a closed random set which absorbs $D$. We fix $\omega \in \Omega$. Let us take an increasing sequence of positive numbers $(t_n)_{n=1}^\infty$ such that $t_n \to \infty$ and an H-valued sequence $(x_n)_n$ such that $x_n \in D(\vartheta_{-t_n}\omega)$, for all $n \in \mathbb{N}$.
**Step I.** Reduction. Since $B$ absorbs $D$, $\varphi(t_n, \vartheta_{-t_n}\omega, D(\vartheta_{-t_n}\omega)) \subset B(\omega)$ for $n \in \mathbb{N}$ sufficiently large. Since $B(\omega)$ as a bounded set in $H$ is weakly precompact in $H$, without loss of generality we may assume that

$$\varphi(t_n, \vartheta_{-t_n}\omega), D(\vartheta_{-t_n}\omega)) \subset B(\omega)$$

for all $n \in \mathbb{N}$ and, for some $y_0 \in H$,

$$\varphi(t_n, \vartheta_{-t_n}\omega, x_n) \rightharpoonup y_0 \quad \text{weakly in H.} \tag{3.1}$$

Our aim is to prove that for some subsequence

$$\varphi(t_{n'}, \vartheta_{-t_{n'}}\omega, x_{n'}) \to y_0 \text{ strongly in H.} \tag{3.2}$$

Since $z(0) \in \mathrm{H}$, then

$$\varphi(t_n, \vartheta_{-t_n}\omega, x_n - z(0)) \to y_0 - z(0) \quad \text{weakly in H.} \tag{3.3}$$

In particular,

$$|y_0 - z(0)| \leq \liminf_{n\to\infty} |\varphi(t_n, \vartheta_{-t_n}\omega, x_n - z(0))|. \tag{3.4}$$

Arguing as in [10], we can show that in order to prove (3.2) it is enough to prove that for some subsequence $\{n'\} \subset \mathbb{N}$

$$|y_0 - z(0)| \geq \limsup_{n'\to\infty} |\varphi(t_{n'}, \vartheta_{-t_{n'}}\omega, x_{n'}) - z(0)|. \tag{3.5}$$

**Step II.** Construction of a negative trajectory, i.e. a sequence $(y_n)_{n=-\infty}^0$ such that $y_n \in B(\theta_n \omega)$, $n \in \mathbb{Z}^-$, and

$$y_k = \varphi(k-n, \theta_n\omega, y_n), \; n < k \leq 0.$$

Since $B$ absorbs $D$, there exists a constant $N_1(\omega) \in \mathbb{N}$, such that

$$\{\varphi(-1+t_n, \vartheta_{1-t_n}\vartheta_{-1}\omega, x_n) : n \geq N_1(\omega)\} \subset B(\vartheta_{-1}\omega).$$

Hence we can find a subsequence $\{n'\} \subset \mathbb{N}$ and $y_{-1} \in B(\vartheta_{-1}\omega)$ such that

$$\varphi(-1+t_{n'}, \vartheta_{-t_{n'}}\omega, x_{n'}) \rightharpoonup y_1 \text{ weakly in H.} \tag{3.6}$$

Let us observe that the cocycle property, with $t = 1, s = t_{n'} - 1$, and $\omega$ being replaced by $\vartheta_{-t_{n'}}\omega$, reads as follows:

$$\varphi(t_{n'}, \vartheta_{-t_{n'}}\omega) = \varphi(1, \vartheta_{-1}\omega)\varphi(-1+t_{n'}, \vartheta_{-t_{n'}}\omega).$$

Hence, by Lemma [10, Lemma 7.2], from (3.1) and (3.6) we infer that $\varphi(1, \vartheta_{-1}\omega, y_1) = y_0$. By induction, for each $k = 1, 2, \ldots$, we can construct a subsequence $\{n^{(k)}\} \subset \{n^{(k-1)}\}$ and $y_{-k} \in B(\vartheta_{-k}\omega)$, such that $\varphi(1, \vartheta_{-k}\omega, y_{-k}) = y_{-k+1}$ and

$$\varphi(-k + t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega, x_{n^{(k)}}) \rightharpoonup y_{-k} \quad \text{weakly in H,} \quad \text{as } n^{(k)} \to \infty. \tag{3.7}$$

As above, the cocycle property with $t = k$, $s = t_{n^{(k)}}$ and $\omega$ being replaced by $\vartheta_{-t_{n^{(k)}}}\omega$, yields

$$\varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega) = \varphi(k, \vartheta_{-k}\omega)\varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega), \; k \in \mathbb{N}. \tag{3.8}$$

Hence, from (3.7) by applying [10, Lemma 7.1], we get

$$\begin{aligned} y_0 &= \text{w-}\lim_{n^{(k)}\to\infty} \varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega, x_{n^{(k)}}) \\ &= \text{w-}\lim_{n^{(k)}\to\infty} \varphi(k, \vartheta_{-k}\omega, \varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega, x_{n^{(k)}})) \\ &= \varphi(k, \vartheta_{-k}\omega, (\text{w-}\lim_{n^{(k)}\to\infty} \varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}})) = \varphi(k, \vartheta_{-k}\omega, y_{-k}), \end{aligned} \tag{3.9}$$

where w-lim denotes the limit in the weak topology on H. The same proof yields a more general property:

$$\varphi(j, \vartheta_{-k}\omega, y_{-k}) = y_{-k+j}, \quad \text{if } 0 \leq j \leq k.$$

Before we continue our proof, let us point out that, (3.9) means precisely that $y_0 = u(0, -k; \omega, y_{-k})$, where $u$ is defined by (2.29).

**Step III.** Proof of (3.5). From now on, until explicitly stated, we fix $k \in \mathbb{N}$, and we will consider problem (2.12) on the time interval $[-k, 0]$. From (2.29) and (3.8), with $t = 0$ and $s = -k$, we have

$$\begin{aligned} &|\varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega, x_{n^{(k)}} - z(0))|^2 \\ &\quad = \quad |\varphi(k, \vartheta_{-k}\omega, \varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega, x_{n^{(k)}} - z(0)))|^2 \\ &\quad = \quad |v(0, \omega; -k, \varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega, x_{n^{(k)}}) - z(-k))|^2. \end{aligned} \tag{3.10}$$

Let $v$ be the solution to (2.26) on $[-k,\infty)$ with $z=z_\alpha(\cdot,\omega)$ and the initial condition at time $-k$: $v(-k)=\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega,x_{n^{(k)}})-z(-k)$. In other words,

$$v(s)=v\Big(s,-k;\omega,\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega,x_{n^{(k)}})-z(-k)\Big),\quad s\ge -k.$$

From (2.32) with $t=0$ and $\tau=-k$ we infer that

$$\begin{aligned}
&|\varphi(t_{n^{(k)}},\vartheta_{-t_{n^{(k)}}}\omega,x_{n^{(k)}})-z(0)|^2 \qquad (3.11)\\
=\ & e^{-\nu\lambda_1 k}|\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega,x_{n^{(k)}})-z(-k)|^2\\
+\ & 2\int_{-k}^{0} e^{\nu\lambda_1 s}\left(b(v(s),z(s)),v(s)\right)+\langle g(s),v(s)\rangle+\langle f,v(s)\rangle-[v(s)]^2\big)\,ds.
\end{aligned}$$

To finish the proof it is enough to find a non-negative function $h\in L^1(-\infty,0)$ such that

$$\limsup_{n^{(k)}\to\infty}|\varphi(t_{n^{(k)}},\vartheta_{-t_{n^{(k)}}}\omega,x_{n^{(k)}})-z(0)|^2\le\int_{-\infty}^{-k}h(s)\,ds+|y_0-z(0)|^2. \quad (3.12)$$

For, if we define the diagonal process $(m_j)_{j=1}^\infty$ by $m_j=j^{(j)}$, $j\in\mathbb{N}$, then for each $k\in\mathbb{N}$, the sequence $(m_j)_{j=k}^\infty$ is a subsequence of the sequence $(n^{(k)})$ and hence by (3.12), $\limsup_j|\varphi(t_{m_j},\vartheta_{-t_{m_j}}\omega,x_{m_j})-z(0)|^2\le\int_{-\infty}^{-k}h(s)\,ds+|y_0-z(0)|^2$. Taking the $k\to\infty$ limit in the last inequality we infer that $\limsup_j|\varphi(t_{m_j},\vartheta_{-t_{m_j}}\omega,x_{m_j})-z(0)|^2\le|y_0-z(0)|^2$ which proves claim (3.5).

**Step IV.** Proof of (3.12). We begin with estimating the first term on the RHS of (3.11). If $-t_{n^{(k)}}<-k$, then by (2.29) and (2.31) we infer that

$$\begin{aligned}
&|\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega,x_{n^{(k)}})-z(-k)|^2e^{-\nu\lambda_1 k}\\
&\qquad\qquad=|v(-k,-t_{n^{(k)}};\vartheta_{-k}\omega,x_{n^{(k)}}-z(-t_{n^{(k)}}))|^2e^{-\nu\lambda_1 k}\\
\le\ & e^{-\nu\lambda_1 k}\Big\{|x_{n^{(k)}}-z(-t_{n^{(k)}})|^2e^{-\nu\lambda_1(t_{n^{(k)}}-k)+\frac{3C^2}{\nu}\int_{-t_{n^{(k)}}}^{-k}|z(s)|_{\mathbb{L}^4}^2\,ds}\\
&\frac{3}{\nu}\int_{-t_{n^{(k)}}}^{-k}\Big[\|g(s)\|_{\mathrm{V}'}^2+\|f\|_{\mathrm{V}'}^2\Big]e^{-\nu\lambda_1(-k-s)+\frac{3C^2}{\nu}\int_s^{-k}(|z(\zeta)|_{\mathbb{L}^4}^2)d\zeta}ds\Big\} \qquad (3.13)\\
\le\ & 2I_{n^{(k)}}+2II_{n^{(k)}}+\frac{3}{\nu}III_{n^{(k)}}+\frac{3}{\nu}\|f\|_{\mathrm{V}'}^2\,IV_{n^{(k)}},
\end{aligned}$$

where

$$\begin{aligned}
I_{n^{(k)}} &= |x_{n^{(k)}}|^2e^{-\nu\lambda_1 t_{n^{(k)}}+\frac{3C^2}{\nu}\int_{-t_{n^{(k)}}}^{-k}|z(s)|_{\mathbb{L}^4}^2\,ds},\\
II_{n^{(k)}} &= |z(-t_{n^{(k)}})|^2e^{-\nu\lambda_1 t_{n^{(k)}}+\frac{3C^2}{\nu}\int_{-t_{n^{(k)}}}^{-k}|z(s)|_{\mathbb{L}^4}^2\,ds},\\
III_{n^{(k)}} &= \int_{-\infty}^{-k}\|g(s)\|_{\mathrm{V}'}^2e^{\nu\lambda_1 s+\frac{3C^2}{\nu}\int_s^{-k}|z(\zeta)|_{\mathbb{L}^4}^2\,d\zeta}\,ds,\\
IV_{n^{(k)}} &= \int_{-\infty}^{-k}e^{\nu\lambda_1 s+\frac{3C^2}{\nu}\int_s^{-k}|z(\zeta)|_{\mathbb{L}^4}^2\,d\zeta}\,ds.
\end{aligned}$$

First, we will find a non-negative function $h \in L^1(-\infty, 0)$ such that

$$
\begin{aligned}
\limsup_{n^{(k)}\to\infty} |\varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega, x_{n^{(k)}}) - z(-k)|^2 e^{-\nu\lambda_1 k} \\
\le \int_{-\infty}^{-k} h(s)\, ds, \quad k \in \mathbb{N}.
\end{aligned} \tag{3.14}
$$

For this we will need one more auxiliary result.

**Lemma 3.2.** $\limsup_{n^{(k)}\to\infty} I_{n^{(k)}} = 0$.

*Proof.* Let us recall that, $\alpha \in \mathbb{N}$, $z(t) = z_\alpha(t)$, $t \in \mathbb{R}$, is the Ornstein-Uhlenbeck process from section 2, and $\mathbb{E}|z(0)|^2_{\mathrm{X}} = \mathbb{E}|z_\alpha(0)|^2_{\mathrm{X}} < \frac{\nu^2\lambda_1}{6C^2}$. Let us also recall that the space $\hat{\Omega}(\xi, \mathrm{E})$ was constructed in such a way that

$$
\lim_{n^{(k)}\to\infty} \frac{1}{-k - (-t_{n^{(k)}})} \int_{-t_{n^{(k)}}}^{-k} |z_\alpha(s)|^2_{\mathrm{X}} ds = \mathbb{E}|z(0)|^2_{\mathrm{X}} < \infty.
$$

Therefore, since the embedding $\mathrm{X} \hookrightarrow \mathbb{L}^4(\mathcal{O})$ is a contraction, we have for $n^{(k)}$ sufficiently large,

$$
\frac{3C^2}{\nu} \int_{-t_{n^{(k)}}}^{-k} |z(s)|^2_{\mathbb{L}^4}\, ds < \frac{\nu\lambda_1}{2}(t_{n^{(k)}} - k). \tag{3.15}
$$

Since the set $D(\omega)$ is bounded in H, there exists $\rho_1 > 0$ such that $|x_{n^{(k)}}| \le \rho_1$ for every $n^{(k)}$. Hence,

$$
\begin{aligned}
\limsup_{n^{(k)}\to\infty} |x_{n^{(k)}}|^2 e^{-\nu\lambda_1 t_{n^{(k)}} + \frac{3C^2}{\nu}\int_{-t_{n^{(k)}}}^{-k} |z(s)|^2_{\mathbb{L}^4} ds} \\
\le \limsup_{n^{(k)}\to\infty} \rho_1^2 e^{-\frac{\nu\lambda_1}{2}(t_{n^{(k)}} - k)} = 0.
\end{aligned} \tag{3.16}
$$

□

Therefore, by (3.13), and lemmas 2.21, 2.22 and 3.2, the proof of (3.14) is concluded, and it only remains to finish the proof of inequality (3.12), which we are going to do right now.

**The end of the proof of inequality (3.12).**

Let us denote $\tilde{y}_k = y_k - z(-k)$ and

$$
\begin{aligned}
v^{n^{(k)}}(s) &= v(s, -k; \omega, \varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}} - z(-k)),\ s \in (-k, 0), \\
v_k(s) &= v(s, -k; \omega, y_{-k} - z(-k)),\ s \in (-k, 0).
\end{aligned}
$$

From property (3.7) and [10, Lemma 7.1] we infer that

$$
v^{n^{(k)}}(\cdot) \to v_k \text{ weakly in } L^2(-k, 0; \mathrm{V}). \tag{3.17}
$$

Since $e^{\nu\lambda_1\cdot}g(\cdot), e^{\nu\lambda_1\cdot}f \in L^2(-k,0;\mathrm{V}')$, we get

$$\lim_{n^{(k)}\to\infty}\int_{-k}^{0} e^{\nu\lambda_1 s}\langle g(s), v^{n^{(k)}}(s)\rangle\, ds = \int_{-k}^{0} e^{\nu\lambda_1 s}\langle g(s), v_k(s)\rangle\, ds, \tag{3.18}$$

and

$$\lim_{n^{(k)}\to\infty}\int_{-k}^{0} e^{\nu\lambda_1 s}\langle f, v^{n^{(k)}}(s)\rangle\, ds = \int_{-k}^{0} e^{\nu\lambda_1 s}\langle f, v_k(s)\rangle\, ds. \tag{3.19}$$

On the other hand, using the same methods as those in the proof of Theorem 2.18, there exists a subsequence of $\{v^{n^{(k)}}\}$, which, for the sake of simplicity of notation, is denoted as the old one and which satisfies

$$v^{n^{(k)}} \to v_k \quad \text{strongly in } L^2(-k,0;\mathbb{L}^2_{loc}(D)). \tag{3.20}$$

Next, since $e^{\nu\lambda_1 t}z(t)$, $t\in\mathbb{R}$, is an $\mathbb{L}^4$-valued process, Thus by [10, Corollary 5.3], (3.17) and (3.20), we infer that

$$\begin{aligned}\lim_{n^{(k)}\to\infty}&\int_{-k}^{0} e^{\nu\lambda_1 s} b(v^{n^{(k)}}(s), z(s)), v^{n^{(k)}}(s))\, ds\\ &= \int_{-k}^{0} e^{\nu\lambda_1 s} b(v_k(s), z(s), v_k(s))\, ds.\end{aligned} \tag{3.21}$$

Moreover, since the norms $[\cdot]$ and $\|\cdot\|$ are equivalent on V, and since for any $s\in(-k,0]$, $e^{-\nu k\lambda_1} \le e^{\nu\lambda_1 s} \le 1$, $(\int_{-k}^{0} e^{\nu\lambda_1 s}[\cdot]^2\, ds)^{1/2}$ is a norm in $L^2(-k,0;\mathrm{V})$ equivalent to the standard one. Hence, from (3.17) we obtain,

$$\int_{-k}^{0} e^{\nu\lambda_1 s}[v_k(s)]^2\, ds \le \liminf_{n^{(k)}\to\infty}\int_{-k}^{0} e^{\nu\lambda_1 s}[v^{n^{(k)}}(s)]^2\, ds.$$

In other words,

$$\limsup_{n^{(k)}\to\infty}\Big\{-\int_{-k}^{0}[v^{n^{(k)}}(s)]^2\, ds\Big\} \le -\int_{-k}^{0} e^{\nu\lambda_1 s}[v_k(s)]^2\, ds. \tag{3.22}$$

From (3.11), eqrefeqn:c6 and (3.21), and inequality (3.22) we conclude that

$$\begin{aligned}\limsup_{n^{(k)}\to\infty}&|\varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}} - z(0)|^2\\ &\le \int_{-\infty}^{-k} h(s)\, ds + 2\int_{-k}^{0} e^{\nu\lambda_1 s}\Big\{\langle B(v_k(s), z(s)), v_k(s)\rangle\\ &\qquad + \langle g(s), v_k(s)\rangle + \langle f, v_k(s)\rangle - [v_k(s)]^2\Big\}\, ds.\end{aligned} \tag{3.23}$$

On the other hand, from (3.9) and (2.32), we have

$$\begin{aligned} |y_0 - z(0)|^2 &= |\varphi(k, \vartheta_{-k}\omega)y_k - z(0)|^2 = |v(0, -k; \omega, y_k - z(-k))|^2 \\ &= |y_k - z(-k)|^2 e^{-\nu\lambda_1 k} + 2\int_{-k}^{0} e^{\nu\lambda_1 s}\Big\{\langle g(s), v_k(s)\rangle \qquad (3.24) \\ &+ \langle B(v_k(s), z(s)), v_k(s)\rangle + \langle f, v_k(s)\rangle - [v_k(s)]^2\Big\}\, ds. \end{aligned}$$

Hence, by combining (3.23) with (3.24), we get

$$\begin{aligned} &\limsup_{n^{(k)}\to\infty} |\varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}} - z(0)|^2 \\ &\le \int_{-\infty}^{-k} h(s)\, ds + |y_0 - z(0)|^2 - |y_k - z(-k)|^2 e^{-\nu\lambda_1 k} \\ &\le \int_{-\infty}^{-k} h(s)\, ds + |y_0 - z(0)|^2. \end{aligned}$$

which proves (3.12) and hence the proof of Proposition 3.1 is finished. □


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