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PULLBACK ATTRACTORS FOR STOCHASTIC HEAT EQUATIONS IN MATERIALS WITH MEMORY

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Abstract. We study the asymptotic behaviour of a non-autonomous stochastic reaction-diffusion equation with memory. In fact, we prove the existence of a random pullback attractor for our stochastic parabolic PDE with memory. The randomness enters in our model as an additive Hilbert valued noise. We first prove that the equation generates a random dynamical system (RDS) in an appropriate phase space. Due to the fact that the memory term takes into account the whole past history of the phenomenon, we are not able to prove compactness of the generated RDS, but its asymptotic compactness, ensuring thus the existence of the random pullback attractor.

1. **Introduction and motivation of the problem.** The main aim of this paper is to analyse the long-time behaviour of stochastic diffferential systems with memory terms, expressed by convolution integrals, which represent the past history of one or more variables. In particular, we focus on a non-autonomous stochastic reaction-diffusion equation with memory.

Needless to say that many physical phenomena are better described if one considers in the equations of the model some terms which take into account the past history of the system. Although, in some situations, the contribution of the past history may not be so relevant to significantly affect the long time dynamics of the problem, in certain models, such as those describing high viscosity liquids at low temperatures, or the thermomechanical behaviour of polymers (see, [16], [25] and the references therein) the past history plays a nontrivial role.

On the other hand, it is sensible to assume that the models of certain phenomena from the real world are more realistic if some kind of uncertainty, for instance, some randomness or environmental noise, is also considered in the formulation. We will consider an additive noise in our model which we interpret as the environmental

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noisy effect produced on the system, and will exploit the recent theory of random dynamical systems (see [1], [7], [12], [27]) to obtain information on the asymptotic behaviour of our model, in particular we will be able to prove the existence of a pullback (random) attractor.

In the remaining part of this section, we first describe the deterministic problem, and later we consider the stochastic version to be analysed in this paper. In Section 2 we state the assumptions on the coefficients in our equation as well as the necessary background on the theory of pullback attractors. The existence and uniqueness of solution to our model and the construction of the associate random dynamical system is proved in Section 3, while the existence of a pullback attractor is shown in the last section.

1.1. **The deterministic non-autonomous model.** The starting point for our considerations is the following (deterministic) heat conduction model.

Let $\mathcal{O}$ be a regular enough bounded domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$. We denote by $v = v(x, t)$ the temperature at position $x \in \bar{\mathcal{O}}$ and time t. Following the theory developed by Coleman & Gurtin [10], Gurtin & Pipkin [21] and Nunziato [24] we assume that the density $e(x, t)$ of the internal energy and the heat flux $q(x, t)$ are related to the temperature and its gradient by the constitutive relations:

$$e(x,t) = b_0 v(x,t), \quad t \in \mathbb{R},\ x \in \bar{\mathcal{O}} \tag{1}$$

and

$$q(x,t) = -c_0 \nabla v(x,t) + \int_{-\infty}^{t} \gamma(t-s) \nabla v(x,s) ds, \quad t \in \mathbb{R},\ x \in \bar{\mathcal{O}}. \tag{2}$$

Here the constants $b_0 > 0$ and $c_0 > 0$ are called respectively the heat capacity and the thermal conduction, γ is the heat flux relaxation function (the standard example is $\gamma(s) = \gamma_0 e^{-d_0 t}$ with $d_0 > 0$ and $\gamma_0 < 0$).

The energy balance for the system has the form

$$\partial_t e(x,t) = -\operatorname{div} q(x,t) + f(v(x,t), x, t), \quad t \in \mathbb{R},\ x \in \bar{\mathcal{O}}, \tag{3}$$

where $f(v, x, t)$ is the energy supply which may depend on the temperature. Thus we arrive at the following non-autonomous heat equation with memory

$$b_0 \partial_t v(x,t) = c_0 \Delta v(x,t) - \int_{-\infty}^{t} \gamma(t-s) \Delta v(x,s) ds + f(v(x,t), x, t), \tag{4}$$

where $t > 0$, $x \in \mathcal{O}$. We also need to impose some (natural) boundary conditions for $v(x, t)$.

1.2. **The stochastic model.** We are interested in the case in which the function f describing the energy supply in (4) possesses a stochastic term representing an environmental (white) noise. More precisely, we assume that the energy supply function $f(v, x, t)$ has the form

$$f(v, x, t) = -f(v) + \partial_t W(x, t),$$

where $W(x, t)$ is a Wiener process in $L_2(\mathcal{O})$. Thus, we have the following non-autonomous SPDE with memory

$$v_t - \nu \Delta v + \int_{-\infty}^{t} \gamma(t-s) \Delta v(s) ds + f(v) = \partial_t W(t) \tag{5}$$

in the bounded domain $\mathcal{O}$, with the boundary condition

$$v(t, x) = 0 \quad \text{for} \quad x \in \partial\mathcal{O}. \tag{6}$$

We also need to equip (5) with the initial datum:

$$v(t,x) = v_0(t,x) \quad \text{for} \quad t \le 0,\ x \in \mathcal{O}. \tag{7}$$

We note that the *linear* version of this problem has been studied in [9]. In particular, it was shown in [9] that, in the case $f \equiv 0$, the solution to (5)-(7) is a Gaussian process which possesses some regularity properties and converges in law to a stationary process as $t \to \infty$.

Our goal in this paper is to study the long time dynamics of solutions to (5) in the nonlinear situation. Namely, we prove that solutions to (5) converge (in the pullback sense) to a compact random attractor. Since pullback convergence implies forward convergence in law, our Theorem 3 extends the result given in [9] to nonlinear models. **Our results also generalize deterministic (autonomous) results on the dynamical systems with memory which are perturbed by non-autonomous features (random, in fact, in our case). This generalization is not trivial due to the appearance of an additional source for the non-compactness, which is represented by the additional variable ω in a probability space Ω (see below for more details). We also emphasize that memory systems possesses infinite retarded time, which definitely enhance all the difficulties.**

As a consequence, our analysis comes to reinforce the power and suitability of the theory of pullback attractors in the study of dynamical systems generated by stochastic partial differential equations, now in the special case of stochastic systems with memory.

We also note that the long time behaviour of the deterministic version (4) of problem (5) was studied in [5, 11, 17, 18] for the autonomous case and in [19, 20] for the case of time-dependent coefficients.

2. **Hypotheses and preliminaries.** To start with let us establish the assumptions to be imposed on the terms in our problem (5):

- $f(\cdot) \in C^1(\mathbb{R})$ possesses the property $f'(v) \ge -c$ for all $v \in \mathbb{R}$ and also satisfies the relations
$$vf(v) \ge a_0|v|^{p+1} - c, \quad |f'(v)| \le a_1|v|^{p-1} + c, \quad v \in \mathbb{R}, \tag{8}$$
where a_i and c are positive constants and $p \ge 1$;
- the kernel $\gamma(s)$ belongs to $C^2(\mathbb{R}_+)$, $\lim_{s\to\infty}\gamma(s) = 0$, and the function $\mu(s) \equiv -\gamma'(s)$ possesses the properties
$$\mu(s) \ge 0, \quad \mu'(s) + \delta\mu(s) \le 0, \tag{9}$$
where δ is a positive constant;
- $W(t)$, $t \in \mathbb{R}$, is a two-sided $L_2(\mathcal{O})$-valued Wiener process with covariance operator $K = K^* \ge 0$ such that
$$\operatorname{tr} K(-\Delta)^{2\alpha-1} < \infty \quad \text{for some} \quad \alpha > \alpha_* \equiv 1 + \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p+1}\right), \tag{10}$$
where Δ is the Laplace operator with the Dirichlet boundary conditions. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the corresponding probability space, and by $\dot{W} \equiv \partial_t W$ the generalized derivative with respect to t.

We note that the hypotheses concerning $\gamma(s)$ implies that

$$0 \le \mu(s) \le \mu(0)e^{-\delta s}, \quad s \in \mathbb{R}_+, \tag{11}$$

and

$$0 \le \gamma(s) \le \frac{\mu(0)}{\delta} e^{-\delta s}, \quad s \in \mathbb{R}_+. \tag{12}$$

It also follows from (9) that either (i) $\mu(s) > 0$ for all $s \in \mathbb{R}_+$ or (ii) there exists $s_* > 0$ such that $\mu(s) > 0$ for $s \in [0, s_*)$ and $\mu(s) = 0$ for all $s \ge s_*$. In the latter case we have a retarded problem with *finite* delay and therefore we will concentrate mainly on the first case.

We suppose that we can also associate with the Wiener process W on $(\Omega, \mathcal{F}, \mathbb{P})$ a metric dynamical system $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$, i.e. we can define a family of measure preserving transformations $\{\theta_t : \Omega \mapsto \Omega, t \in \mathbb{R}\}$ such that

(i) $\theta_0 = id, \quad \theta_t \circ \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$;
(ii) the map $(t, \omega) \mapsto \theta_t\omega$ is measurable and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$;
(iii) the following *helix property* holds:

$$W(t+s, \omega) - W(s, \omega) = W(t, \theta_s\omega), \quad s, t \in \mathbb{R}, \quad \omega \in \Omega.$$

We now introduce the Ornstein-Uhlenbeck process $\eta(t)$ given by

$$\eta(t; \omega) = \left(\int_{-\infty}^{t} e^{-\nu(t-\tau)A} dW(\tau) \right)(\omega), \quad t \in \mathbb{R}, \tag{13}$$

where, from now on, A denotes the operator $-\Delta$ with Dirichlet boundary conditions on $\mathcal{O}$. The integral in (13) exists as an operator stochastic integral and gives a stationary Gaussian process (see, e.g., [22] or [14, 15]) which solves the stochastic equation

$$d\eta + \nu A\eta dt = dW.$$

We can also involve a perfection procedure to define $\eta(t; \omega) \equiv \overline{\eta}(\theta_t\omega)$ for *all* $\omega \in \Omega$ (for details see Proposition 3.1 in [8]). Moreover $t \mapsto \overline{\eta}(\theta_t\omega)$ is continuous from $\mathbb{R}$ into $D(A^{\alpha_*})$ for each $\omega \in \Omega$ and the following *temperedness* condition

$$\sup_{t\in\mathbb{R}} \{\| A^{\alpha_*}\overline{\eta}(\theta_t\omega) \| e^{-\beta|t|}\} < \infty \quad \text{for all} \quad \beta > 0, \ \omega \in \Omega, \tag{14}$$

is fulfilled. We refer to [8] for details and further references. We also note that for α_* given in (10) we have (see, e.g., [29, Chap.4]) that

$$D(A^{\alpha_*}) \subset W_2^{2\alpha_*}(\mathcal{O}) \subset W_{p+1}^2(\mathcal{O}),$$

where $W_q^s(\mathcal{O})$ is the L_q-based Sobolev space of order s. Therefore $\omega \mapsto \Delta\overline{\eta}(\omega)$ is a tempered random variable with values in $L_{p+1}(\mathcal{O})$. We will use this observation later.

Below we need the notion of a random closed set. We recall the following definitions (see [1] or [4]).

Definition 1 (Random Closed Set)**.** Let X be a Polish space with a metric d_X. The multifunction $\omega \mapsto D(\omega) \neq \emptyset$ is said to be a *random closed set* if the mapping $\omega \mapsto \mathrm{dist}_X(v, D(\omega))$ is measurable for any $v \in X$, where $\mathrm{dist}_X(v, B)$ is the distance in X between the element v and the set $B \subset X$, and $D(\omega)$ is closed for each $\omega \in \Omega$. For ease of notations, we denote the random closed set $\omega \mapsto D(\omega)$ by $\widehat{D}$ or $\{D(\omega)\}$. If $D(\omega)$ is a compact set for every $\omega \in \Omega$, then $\widehat{D}$ is called a *random compact set.* A random closed set $\{D(\omega)\}$ is said to be *tempered* if there exists $v_0 \in X$ such that $D(\omega) \subset \{v \in X : \mathrm{d}_X(v, v_0) \le r(\omega)\}$ for all $\omega \in \Omega$, where the random variable $r(\omega) > 0$ is tempered, i.e.

$$\sup_{t\in\mathbb{R}} \{r(\theta_t\omega)e^{-\beta|t|}\} < \infty \quad \text{for all} \quad \beta > 0, \ \omega \in \Omega.$$

Now we introduce the notion of a random dynamical system.

Definition 2. Let X be a Polish space. A pair (θ, ϕ) consisting of a metric dynamical system $\theta = (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ and a measurable mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \mapsto X$$

possessing the following cocycle property:

$$\phi(0, \omega, u) = u, \quad \phi(t, \theta_\tau \omega, \phi(\tau, \omega, u)) = \phi(t + \tau, \omega, u), \quad \forall u \in X,\ t, \tau \geq 0,$$

is called a *random dynamical system* (driven by the metric dynamical system θ) with phase space X and cocycle ϕ.

Random dynamical systems are generated by differential equations with random coefficients or stochastic differential equations with a unique and global solution. We refer to [1] for more details on the general theory of random dynamical systems.

Below we also need the following concept of (random) pullback attractor for random dynamical systems, (see, e.g., [1, 12, 26, 27] and the references therein). The appearance of this concept is motivated by the corresponding definition of a global attractor (cf. [2, 6, 28], for example).

Definition 3. Let (θ, ϕ) be a random dynamical system with the phase space X. We denote by $\mathcal{D}$ the collection of all tempered random sets in X. A random closed set $\{\mathfrak{A}(\omega)\}$ from $\mathcal{D}$ is said to be a *random pullback attractor* for (θ, ϕ) in $\mathcal{D}$ if (i) $\widehat{\mathfrak{A}}$ is an invariant set, i.e. $\phi(t, \omega, \mathfrak{A}(\omega)) = \mathfrak{A}(\theta_t \omega)$ for $t \geq 0$ and $\omega \in \Omega$; and (ii) $\widehat{\mathfrak{A}}$ is pullback attracting in $\mathcal{D}$, i.e.

$$\lim_{t \to +\infty} \operatorname{dist}_X \{\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \,|\, \mathfrak{A}(\omega)\} = 0, \quad \omega \in \Omega, \tag{15}$$

for all $\widehat{D} \in \mathcal{D}$, where $\operatorname{dist}_X\{A|B\} = \sup_{a \in A} \inf_{b \in B} d_X(a, b)$.

Remark 1. It is worth mentioning that, although it is also possible to define a random pullback attractor as an invariant random compact set which attracts all the deterministic bounded sets (as was originally introduced in the paper by Crauel and Flandoli [12])), and then prove that it attracts the more general class of all tempered sets (which, in particular, contains all the deterministic bounded sets), from the practical point of view, it may be more convenient for our problem to deal with this more general class of tempered sets, since, in the end, the attractor will belong to this family.

We conclude this section with a theorem ensuring the existence of random pullback attractor under suitable assumptions (see, e.g., [7, Theorem 1.8.1] for more details).

Theorem 1. *Let (θ, ϕ) be an asymptotically compact RDS in $\mathcal{D}$, i.e. there exists a compact attracting set $\widehat{B}_0$ (in the sense of (15)) in $\mathcal{D}$. Then, this RDS possesses a unique random compact pullback attractor $\{\mathfrak{A}(\omega)\}$ in the universe $\mathcal{D}$, and $\mathfrak{A}(\omega) \subset B_0(\omega)$ for all $\omega \in \Omega$. This attractor has the form*

$$\mathfrak{A}(\omega) = \bigcap_{t > 0} \overline{\bigcup_{\tau \geq t} \phi(\tau, \theta_{-\tau}\omega, B_0(\theta_{-\tau}\omega))} \ \textit{ for every } \ \omega \in \Omega.$$

3. **Existence and uniqueness of solution. Generation of RDS on an appropriate phase space.** In this section we rewrite problem (5)–(7) as a random PDE with memory and show that this PDE possesses a unique solution which allows to construct a random dynamical system in an appropriate phase space.

Introducing a new variable $u = v - \eta$, our problem (5)–(7) can be rewritten as a random parabolic equation with memory of the form

$$u_t - \nu\Delta u + \int_{-\infty}^{t} \gamma(t-s)\Delta u(s)ds + f(u+\eta) = \eta_\gamma(t) \tag{16}$$

in the bounded domain $\mathcal{O}$ with the boundary condition

$$u(t,x) = 0 \quad \text{for} \quad x \in \partial\mathcal{O}, \tag{17}$$

where $\eta_\gamma(t)$ is a process defined by

$$\eta_\gamma(t,\omega) = -\int_{-\infty}^{t} \gamma(t-s)\Delta\eta(s,\omega)ds \equiv \int_{-\infty}^{t} \gamma(t-s)A\eta(s,\omega)ds. \tag{18}$$

We also need to equip (16) with an initial value:

$$u(t,x) = u_0(t,x) \equiv v_0(t,x) - \eta(t) \quad \text{for} \quad t \le 0,\ x \in \mathcal{O}. \tag{19}$$

It is straightforward to prove the following result.

Lemma 1. *The process $t \mapsto \eta_\gamma(t,\omega)$ given by* (18) *is continuous with values in $D(A^{\alpha_*-1})$. Moreover $\eta_\gamma(t,\omega) \equiv \overline{\eta}_\gamma(\theta_t\omega)$, where*

$$\overline{\eta}_\gamma(\omega) = \int_0^\infty \gamma(s)A\overline{\eta}(\theta_{-s}\omega)ds$$

is a tempered random variable with values in $D(A^{\alpha_*-1})$. Here above $D(A^\beta)$ denotes the domain of A^β for $\beta \ge 0$.

Now following the idea introduced by Dafermos [13] (see also [11, 17] and the survey [20]), we introduce the new variable

$$q(t;s,x) = \int_0^s u(t-\tau,x)d\tau \equiv \int_{t-s}^{t} u(\tau,x)d\tau, \quad s \ge 0.$$

Integrating by parts we can rewrite problem (16)–(19) as

$$u_t - \nu\Delta u - \int_0^\infty \mu(s)\Delta q(t;s)ds + f(u+\eta(t)) = \eta_\gamma(t), \quad x \in \mathcal{O},\ t > 0; \tag{20}$$

$$q_t + \partial_s q = u, \quad (s;x) \in \mathbb{R}_+ \times \mathcal{O},\ t > 0; \tag{21}$$

in the bounded domain $\mathcal{O}$, with the boundary condition

$$u(t,x) = 0,\ q(t,s,x) = 0,\ x \in \partial\mathcal{O},\ s \in \mathbb{R}_+,\ t > 0; \quad q(t,0,x) = 0,\ x \in \mathcal{O},\ t > 0. \tag{22}$$

We also need to equip (20) and (21) with the initial data

$$u(0,x) = u_0(x),\ q(0;s,x) = q_0(s,x) \quad \text{for} \quad s \in \mathbb{R}_+,\ x \in \mathcal{O}. \tag{23}$$

Now we introduce an appropriate phase space to handle our situation. We denote $H \equiv H^0 = L_2(\mathcal{O})$ and $H^\sigma = D(A^{\sigma/2})$, where $\sigma \ge 0$. We also consider the weighted Hilbert spaces

$$\mathcal{W}_\mu^\sigma \equiv L_2(\mathbb{R}_+, \mu(s)ds; H^\sigma)$$

which consists of H^σ-valued measurable functions $q(s)$ such that

$$\|q\|^2_{\mathcal{W}^\sigma_\mu} \equiv \int_{\mathbb{R}_+} \mu(s)\|A^{\sigma/2}q(s)\|^2_H ds$$

(in the case when $\mu(s) = 0$ for $s \geq s_*$ instead of $\mathbb{R}_+$ we take the interval $[0, s_*]$). We also denote

$$\mathcal{H} \equiv H \times \mathcal{W}^1_\mu.$$

We start with the following assertion.

Proposition 1. *For any initial data $(u_0; q_0) \in \mathcal{H}$, problem (20)–(23) possesses a unique solution $(u(t); q(t)) = (u(t; \omega, u_0, q_0); q(t; \omega, u_0, q_0))$, such that for each $\omega \in \Omega$,*

$$u(\cdot; \omega, u_0, q_0) \in C([0,T], H) \cap L_2(0,T; D(A^{1/2})) \cap L_{p+1}((0,T) \times \mathcal{O}) \text{ for every } T > 0,$$

and $q(\cdot; \omega, u_0, q_0) \in C(\mathbb{R}_+, \mathcal{W}^1_\mu)$. Moreover, for each $\omega \in \Omega$ the mapping

$$\{t; u_0; q_0\} \mapsto (u(t; \omega, u_0, q_0); q(t; \omega, u_0, q_0))$$

is continuous from $\mathbb{R}_+ \times \mathcal{H}$ into $\mathcal{H}$, and for each $\{t; u_0; q_0\} \in \mathbb{R}_+ \times \mathcal{H}$, the mapping

$$\omega \in \Omega \mapsto (u(t; \omega, u_0, q_0); q(t; \omega, u_0, q_0)) \in \mathcal{H} \tag{24}$$

is $\mathcal{F}$-measurable.

Proof. The existence, uniqueness and continuity properties of solutions for every fixed $\omega \in \Omega$ can be established by the standard deterministic approach involving the compactness method based on Galerkin approximations and the theory of monotone operators (see, e.g. [5] or [11] for the corresponding considerations in the autonomous deterministic case). The $\mathcal{F}$-measurability of the mapping (24) follows from the same property for the corresponding Galerkin approximations which are finite dimensional differential-integral equations. □

Remark 2. To obtain the existence of a unique solution to the problem considered we can also apply the result which was established in [3] for a problem like (5)–(7) in the case of a much more general (multiplicative) diffusion term. However this result provides us with solutions which are defined *almost surely* in $\omega \in \Omega$. In contrast, Proposition 1 gives us a *perfect* solution, i.e., a solution defined for all $\omega \in \Omega$. This is a key fact which allows us to construct the corresponding cocycle and to show that problem (20)–(23) generates an RDS. We refer to [1] for a discussion of the role played by the perfection procedure in the theory of random dynamical systems.

Remark 3. Suppose that $u_0 \in H^1_0(\mathcal{O})$ and $q_0 \in \mathcal{W}^2_\mu$. Then, as $\overline{\eta}_\gamma$ is a tempered random variable with values in $L_2(\mathcal{O})$, using the finite dimensional approximations of the solution $(u(t); q(t))$ of problem (20)–(23), one can prove that in fact, for every $\omega \in \Omega$, $u \in C([0,T], H^1_0(\mathcal{O})) \cap L_2(0,T; D(A)) \cap L_{p+1}((0,T) \times \mathcal{O})$ for every $T > 0$, and $q \in C(\mathbb{R}_+, \mathcal{W}^2_\mu)$. The corresponding arguments are almost the same as in Subsection 4.2. However we do not give details because we do not use these smoothness properties in our further considerations.

Our main result in this section is the following assertion.

Theorem 2 (Generation of RDS)**.** *Problem (20)–(23) generates an RDS in the phase space $\mathcal{H}$ with the cocycle ϕ given by the formula*

$$\phi(t, \omega, U_0) = U(t) \equiv (u(t); q(t)), \quad U_0 = (u_0; q_0), \tag{25}$$

where $(u(t); q(t))$ is the solution to (20)–(23).

Proof. We only need to establish the cocycle property for the mapping $\phi(\cdot,\cdot,\cdot)$ given by (25). This follows from the uniqueness of solution and the fact that problem (20)–(23) can be written as an evolution equation in $\mathcal{H}$ of the form

$$U_t + \mathcal{A}U + F(\theta_t\omega, U) = 0, \quad U(t) = (u(t); q(t))^T.$$

□

4. **Existence of the pullback attractor.** Our main result is the following assertion.

Theorem 3 (Pullback Attractor)**.** *The RDS (θ,ϕ) generated in $\mathcal{H}$ by problem (20)–(23) possesses a compact random pullback attractor $\hat{\mathfrak{A}} = \{\mathfrak{A}(\omega)\}$ in the collection $\mathcal{D}$ of all tempered sets in $\mathcal{H}$. This attractor $\hat{\mathfrak{A}}$ is a tempered set in the space*

$$D(A^{1/2}) \times \left\{ q \in \mathcal{W}_\mu^2 \;:\; \partial_s q \in \mathcal{W}_\mu^1 \right\}. \tag{26}$$

If $\nu\lambda_1 + \inf_{s\in\mathbb{R}} f'(s) > 0$, where λ_1 is the smallest eigenvalue of A, then the attractor $\hat{\mathfrak{A}}$ is a singleton, i.e. there exists a random variable $V(\omega)$ with values in $\mathcal{H}$ such that $\mathfrak{A}(\omega) = \{V(\omega)\}$ for every $\omega \in \Omega$.

To prove Theorem 3 we show that conditions in Theorem 1 are fulfilled. We split the proof into two steps.

4.1. **Pullback dissipativity.** The first step in our argument is the following assertion.

Proposition 2. *The RDS (θ,ϕ) is pullback dissipative in $\mathcal{D}$, i.e. there exists a tempered random variable $R(\omega) > 0$ such that for any random closed set $\widehat{D}$ from $\mathcal{D}$ we can find $t_0(\omega,\widehat{D}) > 0$ such that*

$$\|\phi(t,\theta_{-t}\omega,U)\|_{\mathcal{H}} \le R(\omega) \quad \text{for all} \quad U \in D(\theta_{-t}\omega),\ t \ge t_0(\omega,\widehat{D}).$$

Thus the random ball $B_0(\omega) = \{U \in \mathcal{H} : \|U\|_{\mathcal{H}} \le R(\omega)\}$ is absorbing. This ball is forward invariant and absorbing if we take

$$R^2(\omega) = c_0 \int_{-\infty}^0 e^{\nu_0\tau} \left(1 + \|\overline{\eta}(\theta_\tau\omega)\|_{L_{p+1}(\mathcal{O})}^{p+1} + \|A^{-1/2}\overline{\eta}_\gamma(\theta_\tau\omega)\|^2 \right) d\tau, \tag{27}$$

where $\nu_0 = \min\{\nu\lambda_1, \delta\}$, and the constant c_0 does not depend on the properties of the function $\mu(s)$.

Proof. Although the calculations below seem to be formal, they can be justified by using the corresponding Galerkin approximations.

Multiplying (20) by u in H, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|A^{1/2}u\|^2 + \int_0^\infty \mu(s)(Aq(t;s),u(t))ds + (f(u+\eta),u) = (\eta_\gamma,u). \tag{28}$$

Using (21) and (9) we have that

$$\begin{aligned} &\int_0^\infty \mu(s)(Aq(t;s),u(t))ds = \int_0^\infty \mu(s)(Aq, q_t + q_s)ds \\ &= \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)\|A^{1/2}q(t;s)\|^2 ds - \frac{1}{2}\int_0^\infty \mu'(s)\|A^{1/2}q(t;s)\|^2 ds \\ &\ge \frac{1}{2}\frac{d}{dt}\|q(t)\|_{W_\mu^1}^2 + \frac{\delta}{2}\|q(t)\|_{W_\mu^1}^2. \end{aligned} \tag{29}$$

From (8) we also have that

$$
\begin{aligned}
(f(u+\eta),u) &= \int_{\mathcal{O}} f(u)udx + \int_{\mathcal{O}}\int_0^1 f'(u+\lambda\eta)\eta u dx \\
&\geq a_0\|u\|^{p+1}_{L_{p+1}(\mathcal{O})} - c_1\int_{\mathcal{O}}\left(1+|u|^{p-1}+|\eta|^{p-1}\right)|\eta||u|dx - c_2 \\
&\geq \frac{a_0}{2}\|u\|^{p+1}_{L_{p+1}(\mathcal{O})} - b_0\left(1+\|\eta\|^{p+1}_{L_{p+1}(\mathcal{O})}\right). \qquad (30)
\end{aligned}
$$

Now from (28)–(30) we obtain that

$$
\frac{d}{dt}\left(\|u\|^2+\|q\|^2_{W^1_\mu}\right)+\nu\|A^{1/2}u\|^2+\delta\|q\|^2_{W^1_\mu}+a_0\|u\|^{p+1}_{L_{p+1}(\mathcal{O})}\leq R_0^2(\theta_t\omega), \qquad (31)
$$

where

$$
R_0^2(\omega)=c\left(1+\|\overline{\eta}(\omega)\|^{p+1}_{L_{p+1}(\mathcal{O})}+\|A^{-1/2}\overline{\eta}_\gamma(\omega)\|^2\right). \qquad (32)
$$

This implies that

$$
\|\phi(t,\omega,U)\|^2_{\mathcal{H}}\leq e^{-\nu_0 t}\|U\|^2_{\mathcal{H}}+\int_0^t e^{-\nu_0(t-\tau)}R_0^2(\theta_\tau\omega)d\tau, \quad t\geq 0,
$$

which allows to complete the proof of Proposition 2. □

Remark 4. It follows from (31) that

$$
\frac{d}{dt}\|U\|^2_{\mathcal{H}}+\nu_*\|U\|^2_{\mathcal{H}}+\frac{\nu}{2}\|A^{1/2}u\|^2+a_0\|u\|^{p+1}_{L_{p+1}(\mathcal{O})}\leq R_0^2(\theta_t\omega),
$$

for any $0<\nu_*\leq\nu_1\equiv\frac{1}{2}\min\{\nu\lambda_1,\delta\}$, where $U=(u;q)$. Multiplying by $e^{\nu_* t}$ after integration we obtain

$$
\begin{aligned}
&\int_0^t e^{-\nu_*(t-\tau)}\left[\frac{\nu}{2}\|A^{1/2}u(\tau)\|^2+a_0\|u(\tau)\|^{p+1}_{L_{p+1}(\mathcal{O})}\right]d\tau \\
&\qquad\leq \|U_0\|^2_{\mathcal{H}}e^{-\nu_* t}+\int_0^t e^{-\nu_*(t-\tau)}R_0^2(\theta_\tau\omega)d\tau, \qquad (33)
\end{aligned}
$$

for any $0<\nu_*\leq\nu_1\equiv\frac{1}{2}\min\{\nu\lambda_1,\delta\}$, where $R_0(\omega)$ is given by (32).

4.2. **Asymptotic compactness.** To prove asymptotic compactness we use the splitting method.

Let $U(t)=(u(t);q(t))$ be the solution to problem (20)–(23). We represent it in the form

$$
U(t)=(u(t);q(t))=(u^{st}(t);q^{st}(t))+(u^c(t);q^c(t))\equiv U^{st}(t)+U^c(t).
$$

Here $U^{st}(t)$ (stable part) and $U^c(t)$ (compact part) satisfy the boundary condition (22) and solve the following problems

$$
\begin{aligned}
&u^{st}_t-\nu\Delta u^{st}-\int_0^\infty\mu(s)\Delta q^{st}(t;s)ds \\
&\qquad +f(u^{st}+u^c+\eta)-f(u^c+\eta)+Mu^{st}=0, \quad x\in\mathcal{O},\ t>0; \qquad (34)\\
&q^{st}_t+\partial_s q^{st}=u^{st}, \quad x\in\mathcal{O},\ s\in\mathbb{R}_+,\ t>0;\\
&u^{st}(0,x)=u_0(x),\ q^{st}(0;s,x)=q_0(s,x), \quad s\in\mathbb{R}_+,\ x\in\mathcal{O};
\end{aligned}
$$

where the constant $M>0$ is chosen such that $f'(s)+M\ge 0$ for all $s\in\mathbb{R}$, and

$$\begin{aligned} &u^c_t-\nu\Delta u^c-\int_0^\infty \mu(s)\Delta q^c(t;s)ds\\ &\qquad\quad + f(u^c+\eta)+Mu^c = Mu+\eta_\gamma(t), \quad x\in\mathcal{O},\ t>0; \\ &q^c_t+\partial_s q^c = u^c, \quad x\in\mathcal{O},\ s\in\mathbb{R}_+,\ t>0;\\ &u^c(0,x)=0,\ q^c(0;s,x)=0, \quad s\in\mathbb{R}_+,\ x\in\mathcal{O}. \end{aligned} \tag{35}$$

In the deterministic case a similar splitting was used in [11]. We also note that the well-posedness of problems (34) and (35) can be established similarly to Proposition 1.

Lemma 2. *We have the following estimate*

$$\|U^{st}(t)\|^2_{\mathcal{H}}\le e^{-\nu_0 t}\|U_0\|^2_{\mathcal{H}}, \quad U_0=(u_0;q_0)\in\mathcal{H},\ t\ge 0,$$

where $\nu_0>0$ *is the same as in Proposition 2.*

Proof. The argument is the same as in Proposition 2. Indeed we clearly have relation (28) with $\overline{\eta}_\gamma\equiv 0$ and *non-negative* term

$$(f(u+u^c+\eta)-f(u^c+\eta)+Mu,u)$$

instead of $(f(u+\eta),u)$. Therefore using (29) we obtain

$$\frac{d}{dt}\left(\|u^{st}\|^2+\|q^{st}\|^2_{W^1_\mu}\right)+\nu\|A^{1/2}u^{st}\|^2+\delta\|q^{st}\|^2_{W^1_\mu}\le 0,$$

which implies the conclusion of the lemma. □

Now we deal with the compact part $U^c(t)$.

Lemma 3. *We have the following estimate*

$$\begin{aligned} \|U^c(t)\|^2_{\mathcal{H}}+\int_0^t e^{-\nu_*(t-\tau)}\left[\frac{\nu}{2}\|A^{1/2}u^c(\tau)\|^2+a_0\|u^c(\tau)\|^{p+1}_{L_{p+1}(\mathcal{O})}\right]d\tau \\ \le \int_0^t e^{-\nu_*(t-\tau)}\left[C\|u(\tau)\|^2+R_0^2(\theta_\tau\omega)\right]d\tau, \end{aligned}\tag{36}$$

for any $0<\nu_*\le\nu_1\equiv\frac{1}{2}\min\{\nu\lambda_1,\delta\}$, *where* $R_0(\omega)$ *is given by* (32).

Proof. In the same way as in Proposition 2 we can obtain the following analogue to relation (31):

$$\begin{aligned} \frac{d}{dt}\left(\|u^c\|^2+\|q^c\|^2_{W^1_\mu}\right)+\nu\|A^{1/2}u^c\|^2+\delta\|q^c\|^2_{W^1_\mu}+a_0\|u^c\|^{p+1}_{L_{p+1}(\mathcal{O})} \\ \le C\|u(t)\|^2+R_0^2(\theta_t\omega), \end{aligned}\tag{37}$$

where $R_0(\omega)$ is given by (32). Thus we have

$$\frac{d}{dt}\|U^c\|^2_{\mathcal{H}}+\nu_*\|U^c\|^2_{\mathcal{H}}+\frac{\nu}{2}\|A^{1/2}u^c\|^2+a_0\|u^c\|^{p+1}_{L_{p+1}(\mathcal{O})}\le C\|u(t)\|^2+R_0^2(\theta_t\omega),$$

for any $0<\nu_*\le\nu_1$. Multiplying by $e^{\nu_* t}$ after integration we obtain (36). □

Lemma 4. *We have the following estimate*

$$\begin{aligned} &\|A^{1/2}u^c(t)\|^2+\|q^c(t)\|^2_{W^2_\mu}+\frac{\nu}{2}\int_0^t e^{-\nu_*(t-\tau)}\|Au^c(\tau)\|^2d\tau \\ &\le C\int_0^t e^{-\nu_*(t-\tau)}\left[\|u(\tau)\|^2+R_1^2(\theta_\tau\omega)\right]d\tau \end{aligned}\tag{38}$$

for any $0 < \nu_* \le \nu_1 \equiv \frac{1}{2}\min\{\nu\lambda_1, \delta\}$, *where*

$$R_1^2(\omega) = \left(1 + \|\overline{\eta}(\omega)\|^{p+1}_{L_{p+1}(\mathcal{O})} + \|A\overline{\eta}(\omega)\|^{p+1}_{L_{p+1}(\mathcal{O})} + \|\overline{\eta}_\gamma(\omega)\|^2\right). \tag{39}$$

Proof. We multiply the first equation in (35) by $Au^c(t)$ in H and find that

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}u^c\|^2 + \nu\|Au^c\|^2 + \int_0^\infty \mu(s)(Aq^c(t;s), Au^c(t))ds$$
$$+(f(u^c+\eta) + Mu^c, Au^c) + M\|A^{1/2}u^c\|^2 = (Mu + \eta_\gamma, Au^c). \tag{40}$$

As in the proof of relation (29) one can see that

$$\int_0^\infty \mu(s)(Aq^s(t;s), Au^s(t))ds \ge \frac{1}{2}\frac{d}{dt}\|q^c(t)\|^2_{W^2_\mu} + \frac{\delta}{2}\|q^c(t)\|^2_{W^2_\mu}. \tag{41}$$

We also have that

$$(f(u^c+\eta), Au^c) = (\nabla[f(u^c+\eta)], \nabla(u^c+\eta)) - (f(u^c+\eta), A\eta) + (f(0), Au^c)$$
$$\ge -M\|\nabla(u^c+\eta)\|^2 - \left[\int_{\mathcal{O}} |f(u^c+\eta)|^{1+1/p}\, dx\right]^{\frac{p}{p+1}} \cdot \|A\eta\|_{L_{p+1}(\mathcal{O})} + (f(0), Au^c)$$
$$\ge -M\|A^{1/2}(u^c+\eta)\|^2 - C\left(1 + \|u^c\|^{p+1}_{L_{p+1}(\mathcal{O})} + \|A\eta\|^{p+1}_{L_{p+1}(\mathcal{O})}\right) - \frac{\nu}{4}\|Au^c\|^2$$
$$\ge -2M\|A^{1/2}u^c\|^2 - C\left(1 + \|u^c\|^{p+1}_{L_{p+1}} + \|\eta\|^{p+1}_{L_{p+1}} + \|A\eta\|^{p+1}_{L_{p+1}}\right) - \frac{\nu}{4}\|Au^c\|^2.$$

Therefore (40) and (41) imply that

$$\frac{d}{dt}\left(\|A^{1/2}u^c\|^2 + \|q^c(t)\|^2_{W^2_\mu}\right) + \nu\|Au^c\|^2 + \delta\|q^c(t)\|^2_{W^2_\mu}$$
$$\le C\left(\|A^{1/2}u^c\|^2 + \|u^c\|^{p+1}_{L_{p+1}(\mathcal{O})} + \|u\|^2 + R_1^2(\theta_t\omega)\right).$$

Thus using Lemma 3 we obtain (38). □

Now we derive further estimates which are necessary for the compactness.

Lemma 5. *We have the following relations*

$$\|\partial_s q^c(t)\|^2_{W^1_\mu} \le C\int_0^t e^{-\nu_*(t-\tau)}\left[\|u(\tau)\|^2 + R_0^2(\theta_\tau\omega)\right]d\tau, \tag{42}$$

and

$$\int_0^\infty s\mu(s)\|A^{1/2}q^c(t;s)\|^2 ds \le C\int_0^t e^{-\nu_*(t-\tau)}\left[\|u(\tau)\|^2 + R_0^2(\theta_\tau\omega)\right]d\tau, \tag{43}$$

for any $0 < \nu_* \le \nu_1 \equiv \frac{1}{2}\min\{\nu\lambda_1, \delta\}$, *where* $R_0(\omega)$ *is given by* (32).

Proof. It follows from the second equation in (35) and from the zero boundary and initial data for q^c that

$$q^c(t;s,x) = \begin{cases} \int_0^s u^c(t-\tau)d\tau, & \text{for } s \le t, \\ \\ \int_0^t u^c(t-\tau)d\tau, & \text{for } s > t. \end{cases} \tag{44}$$

Therefore

$$\partial_s q^c(t;s,x) = \begin{cases} u^c(t-s), & \text{for } s \le t, \\ 0, & \text{for } s > t. \end{cases}$$

Thus, by (11) we have that

$$\|\partial_s q^c(t)\|^2_{W^1_\mu} = \int_0^t \mu(t-s)\|A^{1/2}u^c(s)\|^2 ds \le \mu(0)\int_0^t e^{-\delta(t-s)}\|A^{1/2}u^c(s)\|^2 ds$$

Consequently, by Lemma 3 we obtain (42).

To prove (43) we note that (44) implies that

$$\|A^{1/2}q^c(t;s)\|^2 \le s\cdot \begin{cases} \int_{t-s}^t \|A^{1/2}u^c(\tau)\|^2 d\tau, & \text{for } s\le t,\\ \int_0^t \|A^{1/2}u^c(\tau)\|^2 d\tau, & \text{for } s>t.\end{cases}$$

Thus

$$\|A^{1/2}q^c(t;s)\|^2 \le s\cdot \begin{cases} e^{\nu_* s}\int_{t-s}^t e^{-\nu_*(t-\tau)}\|A^{1/2}u^c(\tau)\|^2 d\tau, & \text{for } s\le t,\\ e^{\nu_* t}\int_0^t e^{-\nu_*(t-\tau)}\|A^{1/2}u^c(\tau)\|^2 d\tau, & \text{for } s>t,\end{cases}$$

and hence

$$\|A^{1/2}q^c(t;s)\|^2 \le s e^{\nu_* s}\int_0^t e^{-\nu_*(t-\tau)}\|A^{1/2}u^c(\tau)\|^2 d\tau, \tag{45}$$

Consequently,

$$\int_0^\infty s\mu(s)\|A^{1/2}q^c(t;s)\|^2 \le C_\mu \int_0^t e^{-\nu_*(t-\tau)}\|A^{1/2}u^c(\tau)\|^2 d\tau,$$

where

$$C_\mu = \int_0^\infty s^2\mu(s)e^{\nu_* s}ds < \infty.$$

As above Lemma 3 implies (43). □

Remark 5. It is clear from the argument given in the proof of Lemma 5 that estimates (42) and (43) remain true if we replace the function $\mu(s)$ by the function $\tilde\mu(s) = \mu(s) + e^{-\beta s}$ with $\beta > 0$ large enough. Moreover, using (45) with A instead of $A^{1/2}$ and also (38) one can prove that

$$\|q^c(t)\|^2_{W^2_{\tilde\mu}} \le C\int_0^t e^{-\nu_*(t-\tau)}\left[\|u(\tau)\|^2 + R_1^2(\theta_\tau\omega)\right]d\tau.$$

This observation is important in the case when $\mu(s)\equiv 0$ for $s\ge s_*$ for some $s_*>0$.

Now we are in a position to construct a pullback attracting compact set.

It follows from Lemma 4 and (33) that

$$\|A^{1/2}u^c(t,\theta_{-t}\omega)\|^2 + \|q^c(t,\theta_{-t}\omega)\|^2_{W^2_\mu} \le C\|U_0(\theta_{-t}\omega)\|^2_{\mathcal H}e^{-\nu_* t} + R_*^2(\omega) \tag{46}$$

where the tempered random variable $R_*^2(\omega)$ has the form

$$R_*^2(\omega) = C\int_{-\infty}^0 e^{\nu_*\tau}R_1^2(\theta_\tau\omega)d\tau \tag{47}$$

with $R_1^2(\omega)$ given by (39). In a similar way Lemma 5 implies that

$$\|\partial_s q^c(t,\theta_{-t}\omega)\|^2_{W^1_\mu} + \int_0^\infty s\mu(s)\|A^{1/2}q^c(t;s,\theta_{-t}\omega)\|^2 ds \le C\|U_0(\theta_{-t}\omega)\|^2_{\mathcal H}e^{-\nu_* t} + R_*^2(\omega). \tag{48}$$

Now we introduce the random closed set $\mathfrak B$ in $\mathcal H$ by the formula

$$\mathfrak B(\omega) = \mathfrak B_1(\omega)\times\mathfrak B_2(\omega),$$

where

$$\mathfrak{B}_1(\omega) = \left\{ u \in H \;:\; \|A^{1/2}u\|^2 \le 1 + R_*^2(\omega) \right\}$$

and

$$\mathfrak{B}_2(\omega) = \left\{ q \in W_\mu^2 : \|q\|_{W_\mu^2}^2 + \|\partial_s q\|_{W_\mu^1}^2 + \int_0^\infty s\mu(s)\|A^{1/2}q(s)\|^2 ds \le 1 + 2R_*^2(\omega) \right\}.$$

Proposition 3. *The set $\mathfrak{B}(\omega)$ is uniformly pullback attracting in $\mathcal{D}$, i.e.*

$$\lim_{t\to\infty} \sup \left\{ \mathrm{dist}_{\mathcal{H}}(\phi(t, \theta_{-t}\omega, y), \mathfrak{B}(\omega)) \;:\; y \in D(\theta_{-t}\omega) \right\} = 0 \tag{49}$$

for any $\widehat{D} \in \mathcal{D}$ and any $\omega \in \Omega$. The same is true when we replace μ by $\tilde{\mu}$ (see Remark 5).

Proof. This follows from Lemma 2 and relations (46) and (48). □

Lemma 6. *If $\mu(s) > 0$ for all $s \ge 0$, then $\mathfrak{B}(\omega)$ is a random compact set in $\mathcal{H}$. In the case when $\mu(s) \equiv 0$ for $s \ge s_0$ for some $s_0 > 0$, the same assertion is true if we replace μ by $\tilde{\mu}$ in the definition of the set $\mathfrak{B}_2(\omega)$.*

Proof. Let $\mu(s) > 0$ for all $s \ge 0$. Since $D(A^{1/2})$ is compactly embedded in H, we only need to prove that the set

$$\mathfrak{B}_R = \left\{ q \in W_\mu^2 \;:\; \|q\|_{W_\mu^2}^2 + \|\partial_s q\|_{W_\mu^1}^2 + \int_0^\infty s\mu(s)\|A^{1/2}q(s)\|^2 ds \le R \right\}$$

is compact in W_μ^1 for every $R > 0$.

Let $\{q_N\}$ be a sequence in $\mathfrak{B}_R$. Then for each $T > 0$ this sequence is bounded in the space

$$\mathcal{L}_T = \left\{ q \in L_2(0, T; D(A)) \;:\; \partial_s q \in L_2(0, T; D(A^{1/2})) \right\}.$$

Therefore, by Aubin's compactness theorem (see, e.g., [23]) and a diagonal procedure, we can choose a subsequence $\{q_{N_k}\}$ which is Cauchy in $L_2(0, T; D(A^{1/2}))$ for every $T > 0$. Since

$$\begin{aligned} \|q_{N_k} - q_{N_m}\|_{W_\mu^1}^2 &\le \mu(0) \int_0^T \|A^{1/2}q_{N_k}(s) - A^{1/2}q_{N_m}(s))\|^2 ds \\ &\quad + \frac{2}{T} \int_T^\infty s\mu(s) \left[\|A^{1/2}q_{N_k}(s)\|^2 + \|A^{1/2}q_{N_m}(s)\|^2 \right] ds \\ &\le \mu(0) \int_0^T \|A^{1/2}q_{N_k}(s) - A^{1/2}q_{N_m}(s)\|^2 ds + \frac{4R}{T}, \end{aligned}$$

we obtain that

$$\limsup_{k,m\to\infty} \|q_{N_k} - q_{N_m}\|_{W_\mu^1}^2 \le 2RT^{-1} \quad \text{for every} \quad T > 0.$$

This implies that $\{q_{N_k}\}$ is a Cauchy sequence in W_μ^1. □

4.3. **Proof of Theorem 3.** By Proposition 3 and Lemma 6 the RDS (θ, ϕ) generated by problem (20)–(23) is asymptotically compact in $\mathcal{D}$. Therefore by Theorem 1 there exists a random compact pullback attractor $\hat{\mathfrak{A}}$. It follows from Proposition 3 that the attractor $\hat{\mathfrak{A}}$ is a tempered set in the space given by (26).

Now we prove the final statement of Theorem 3.

Let $U^1(t) = (u^1(t); q^1(t))$ and $U^2(t) = (u^2(t); q^2(t))$ be two solutions to problem (20)–(23) with initial data $U_0^1 = (u_0^1; q_0^1)$ and $U_0^2 = (u_0^2; q_0^2)$. Then the same calculations as in the proof of Proposition 2 give us the relation

$$\frac{1}{2}\frac{d}{dt}\left(\|u^1 - u^2\|^2 + \|q^1 - q^2\|^2_{W^1_\mu}\right) + \nu\|A^{1/2}(u^1 - u^2)\|^2 + \frac{\delta}{2}\|q^1 - q^2\|^2_{W^1_\mu} \\ +(f(u^1 + \eta) - f(u^2 + \eta), u^1 - u^2) \le 0.$$

This implies that

$$\frac{1}{2}\frac{d}{dt}\left(\|u^1 - u^2\|^2 + \|q^1 - q^2\|^2_{W^1_\mu}\right) + \beta\|u^1 - u^2\|^2 + \frac{\delta}{2}\|q^1 - q^2\|^2_{W^1_\mu} \le 0,$$

where $\beta = \nu\lambda_1 + \inf_{s\in\mathbb{R}} f'(s)$. Since in the case considered $\beta > 0$ we have that

$$\|U^1(t) - U^2(t)\|_{\mathcal{H}} \le e^{-\beta_* t}\|U_0^1 - U_0^2\|_{\mathcal{H}}, \quad t \ge 0, \tag{50}$$

where $\beta_* = \min\{\beta; \delta/2\} > 0$.

Now we consider two elements V_1 and V_2 from $\mathfrak{A}(\omega)$ for some fixed $\omega \in \Omega$. By the invariance property of pullback attractor for any $t > 0$ there exist V_1^t and V_2^t from $\mathfrak{A}(\theta_{-t}\omega)$ such that $V_i = \phi(t, \theta_{-t}\omega, V_i^t)$, $i = 1, 2$. Therefore (50) implies that

$$\|V_1 - V_2\|_{\mathcal{H}} \le e^{-\beta_* t}\|V_1^t - V_2^t\|_{\mathcal{H}}, \quad t \ge 0.$$

Since $\hat{\mathfrak{A}}$ is tempered in $\mathcal{H}$, after the limit transition $t \to +\infty$ we can conclude that $\|V_1 - V_2\|_{\mathcal{H}} = 0$. Thus $\mathfrak{A}(\omega)$ is a singleton. This completes the proof of Theorem 3.

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