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ON THE STRUCTURE OF THE GLOBAL ATTRACTOR FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS WITH WEAK CONVERGENCE

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Abstract. The aim of this paper is to describe the structure of global attractors for non-autonomous dynamical systems with recurrent coefficients (with both continuous and discrete time). We consider a special class of this type of systems (the so–called weak convergent systems). It is shown that, for weak convergent systems, the answer to Seifert's question (Does an almost periodic dissipative equation possess an almost periodic solution?) is affirmative, although, in general, even for scalar equations, the response is negative. We study this problem in the framework of general non-autonomous dynamical systems (cocycles). We apply the general results obtained in our paper to the study of almost periodic (almost automorphic, recurrent, pseudo recurrent) and asymptotically almost periodic (asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent) solutions of different classes of differential equations.

1. Introduction.

Denote by $\mathbb{R}^n$ the n–dimensional real Euclidian space with the norm $|\cdot|$, and by $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ the space of all continuous functions $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ equipped with the compact-open topology.

Consider the differential equation

$$x' = f(t,x), \tag{1}$$

where $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Assume that the right-hand side of (1) satisfies hypotheses ensuring the existence, uniqueness and extendability of solutions of (1), i.e., for all $(t_0,x_0)\in\mathbb{R}\times\mathbb{R}^n$ there exists a unique solution $x(t;t_0,x_0)$ of equation (1) with initial data t_0,x_0, and defined for all $t\geq t_0$.

Recall (see, for example, [18, 32]) that equation (1) is said to be uniformly dissipative (or uniformly ultimately bounded) if there exists a number $r_0>0$ such that,

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for every $r > 0$, there is $L(r) > 0$ such that, if $|x_0| \le r$, then $|x(t; t_0, x_0)| \le r_0$, for all $t \ge t_0 + L(r)$.

Equation (1) (respectively, the function f) is called *regular*, if for every $x_0 \in \mathbb{R}^n$ and $g \in H(f) := \overline{\{f_\tau : \tau \in \mathbb{R}\}}$ (where by bar we denote the closure in the space $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $f_\tau(t, x) := f(t + \tau, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$), the equation

$$x' = g(t, x) \tag{2}$$

possesses a unique solution $\varphi(t, x, g)$ passing through the point x_0 at the initial moment $t = 0$, and defined on $\mathbb{R}_+ := \{t \in \mathbb{R} | \ t \ge 0\}$.

Theorem 1.1. [9, Ch.II] *Suppose that $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $H(f)$ is a compact subset of $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Then, the following statements are equivalent:*

(i) *equation (1) is uniformly dissipative;*
(ii) *there exists a positive number R_0 such that*

$$\limsup_{t \to +\infty} |\varphi(t, x, g)| \le R_0, \tag{3}$$

for all $(x, g) \in \mathbb{R}^n \times H(f)$.

At light of Theorem 1.1, it is said that equation (1) is *dissipative* (in fact, the family of equations (2) is collectively dissipative, but we use this shorter terminology) if (3) holds.

Seifert's Problem (see [19] for more details): Suppose that equation (1) is dissipative and the function f is almost periodic (with respect to the time variable). Does equation (1) possess an almost periodic solution?

Fink and Fredericson [19] and Zhikov [33] established that, in general, even when equation (1) is scalar, the answer to Seifert's question is negative.

Related to this result, there are the following interesting questions:

a) To extract some classes of dissipative differential equations for which the response to Seifert's problem is positive;
b) To indicate the additional (it is desirable "optimal") conditions which, jointly with dissipativity, guarantee the existence of at least one almost periodic solution of equation (1).

Below we include a short survey on results concerning the questions a) and b).

a) For the following classes of dissipative equations of type (1), the response to Seifert's question is affirmative: linear equations (see [9, Ch.II]), quasi-linear equations (weak non-linear perturbations of linear equations) (see [7, 9]); holomorphic equations (see [5, 6, 8, 9]).

b) Zubov (see [36]) established that equation (1) admits a unique almost periodic solution if it is convergent, i.e., it admits a unique solution which is bounded on $\mathbb{R}$ and also uniformly globally asymptotically stable. This result was generalized for equations (1) with recurrent coefficients by Cheban [9, Ch.II] and with pseudo recurrent coefficients by Caraballo and Cheban [4].

Levitan and Zhikov (see, for example, [23]) proved that, for low-dimensional equations (namely, for $n \le 3$), equation (1) admits at least one almost periodic solution if (1) is uniformly positively stable, i.e., for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x_1 - x_2| < \delta$ implies $|\varphi(t, x_1, g) - \varphi(t, x_2, g)| < \varepsilon$, for all $t \ge 0$ and $g \in H(f)$.

The main result for ODEs (Theorem 4.2 and its generalizations) that we prove in this paper is the following: we show that if equation (1) is weak convergent (i.e., there exists a positive number L such that $\lim_{t\to+\infty}|\varphi(t,x_1,g)-\varphi(t,x_2,g)|=0$ for all $|x_i|\le L$ $(i=1,2)$ and $g\in H(f)$), and f is pseudo recurrent with respect to the time variable (in particular, f is recurrent, almost automorphic, Bohr almost periodic or quasi periodic), then, equation (1) admits a unique pseudo recurrent (respectively, recurrent, almost automorphic, Bohr almost periodic, quasi periodic) solution. If this solution is Lyapunov stable, then the Levinson center (the compact global attractor) is a minimal almost periodic set. If it is not Lyapunov stable, then the Levinson center contains a minimal almost periodic set, but it is not minimal (this means, in particular, that equation (1) admits a family (more than one) of solutions which are bounded on $\mathbb{R}$).

We present our results in the framework of general non-autonomous dynamical systems (cocycles) and we apply our abstract theory to several classes of differential equations.

The paper is organized as follows.

In Section 2, we collect some notions (global attractor, minimal set, point/compact dissipativity, non-autonomous dynamical systems with convergence, quasi periodicity, Levitan/Bohr almost periodicity, almost automorphy, recurrence, pseudo recurrence, Poisson stability, etc) and facts from the theory of dynamical systems which will be necessary in this paper.

Section 3 is devoted to the study of a special class of non-autonomous dynamical systems (NDS): the so-called NDS with weak convergence. We give a generalization of the notion of convergent NDS. On the one hand, this type of NDS is very close to NDS with convergence (because they conserve some properties of convergent systems) and larger than that of convergent systems. On the other hand, we analyze the class of compact dissipative NDS with nontrivial Levinson center. The main results of our paper are proved in this section, namely Theorem 3.5 and Theorem 3.8 (see also Corollary 3.9 and Corollary 3.10) which provide sufficient conditions for the existence of a unique minimal set in the Levinson center which is homeomorphic to the base dynamical system (driving system). This means, in particular, that if the base dynamical system is a compact minimal set consisting of recurrent (respectively, almost automorphic, Bohr almost periodic, quasi periodic, periodic, stationary) points, then, under the conditions of Theorem 3.5, the Levinson center of a non-autonomous dynamical system contains a unique minimal set which consists of recurrent (respectively, almost automorphic, Bohr almost periodic, quasi periodic, periodic, stationary) points.

In Section 4, we exhibit some applications of our abstract results to different classes of differential equations. Namely, almost periodic and asymptotically almost periodic solutions (Subsection 4.1), uniformly compatible (by the character of recurrence with the right hand side) solutions of strict dissipative equations (Subsection 4.2).

2. **Nonautonomous Dynamical Systems with Convergence.**

Let us start by recalling some concepts and notations about the theory of non-autonomous dynamical systems which will be necessary for our analysis.

2.1. **Compact Global Attractors of Dynamical Systems.**

Let (X,ρ) be a metric space, $\mathbb{R}$ $(\mathbb{Z})$ be the group of real (integer) numbers, $\mathbb{R}_+$ $(\mathbb{Z}_+)$ be the semigroup of nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T}\subseteq\mathbb{S}$ $(\mathbb{S}_+\subseteq\mathbb{T})$ be a sub-semigroup of the additive group $\mathbb{S}$.

A *dynamical system* is a triplet $(X,\mathbb{T},\pi)$, where $\pi:\mathbb{T}\times X\to X$ is a continuous mapping satisfying the following conditions:

$$\pi(0,x)=x\ (\forall x\in X);$$

$$\pi(s,\pi(t,x))=\pi(s+t,x)\ (\forall t,s\in\mathbb{T}\text{ and }x\in X).$$

If $\mathbb{T}=\mathbb{R}$ $(\mathbb{R}_+)$ or $\mathbb{Z}$ $(\mathbb{Z}_+)$, the dynamical system $(X,\mathbb{T},\pi)$ is called a *group (semi-group)*. When $\mathbb{T}=\mathbb{R}_+$ or $\mathbb{R}$, the dynamical system $(X,\mathbb{T},\pi)$ is called a *flow*, but if $\mathbb{T}\subseteq\mathbb{Z}$, then $(X,\mathbb{T},\pi)$ is called a *cascade* (*discrete flow*).

The function $\pi(\cdot,x):\mathbb{T}\to X$ is called the *motion* passing through the point x at the initial moment $t=0$, and the set $\Sigma_x:=\pi(\mathbb{T},x)$ is called the *trajectory* of this motion.

A nonempty set $M\subseteq X$ is called *positively invariant* (*negatively invariant*, *invariant*) with respect to the dynamical system $(X,\mathbb{T},\pi)$ or, simply, *positively invariant (negatively invariant, invariant)*, if $\pi(t,M)\subseteq M$ $(M\subseteq\pi(t,M),\pi(t,M)=M)$ for every $t\in\mathbb{T}$.

A closed positively invariant set is said to be *minimal* if it does not contain any own closed positively invariant subset.

It is easy to see that every positively invariant minimal set is invariant.

Let $M\subseteq X$. The set

$$\omega(M):=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}\pi(\tau,M)}$$

is called the *ω-limit* of M.

The set $W^s(\Lambda)$, defined by the equality

$$W^s(\Lambda):=\{x\in X|\lim_{t\to+\infty}\rho(\pi(t,x),\Lambda)=0\}$$

is called the *stable manifold* of the set $\Lambda\subseteq X$.

For $p\in X$, $M\subset X$ and $\delta>0$, let us denote by $B(M,\delta)=\{x\in X|\ \rho(x,M)<\delta\}$.

The set M is called:

- *orbitally stable* if for every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that $\rho(x,M)<\delta$ implies $\rho(\pi(t,x),M)<\varepsilon$, for all $t\geq 0$;
- *attracting* if there exists $\gamma>0$ such that $B(M,\gamma)\subset W^s(M)$;
- *asymptotically stable* if it is orbitally stable and attracting;
- *globally asymptotically stable* if it is asymptotically stable and $W^s(M)=X$.

The dynamical system $(X,\mathbb{T},\pi)$ is called:

- *point dissipative* if there exists a nonempty compact subset $K\subseteq X$ such that, for every $x\in X$,
$$\lim_{t\to+\infty}\rho(\pi(t,x),K)=0; \tag{4}$$
- *compact dissipative* if the equality (4) takes place uniformly w.r.t. x on the compact subsets of X;
- *locally complete (compact)* if for any point $p\in X$, there exist $\delta_p>0$ and $l_p>0$ such that the set $\pi(l_p,B(p,\delta_p))$ is relatively compact.

Let $(X,\mathbb{T},\pi)$ be compact dissipative, and K be a compact set attracting every compact subset of X. Let us set

$$J:=\omega(K):=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}\pi(\tau,K)}. \tag{5}$$

It can be shown (see [9, Ch.I]) that the set J defined by equality (5) does not depend on the choice of the attractor K, but is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set J is called the *Levinson center* of the compact dissipative dynamical system $(X, \mathbb{T}, \pi)$.

Some properties of this set can be found in [9, 20].

2.2. **Non-Autonomous Dynamical Systems with Convergence.** Given two dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$, a triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$, where h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, is called a *non-autonomous dynamical system* (see [2, 9]). Recall (see, for example, [9, 12]) that the non-autonomous dynamical system $\langle (X,\mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is said to be *convergent* if the following conditions are satisfied:

- (i) the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compact dissipative;
- (ii) the set $J_X \bigcap X_y$ contains no more than one point for all $y \in J_Y$, where $X_y := h^{-1}(y) := \{x \in X \mid h(x) = y\}$ and J_X (respectively, J_Y) is the Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$).

Some sufficient conditions and criteria ensuring the convergence of a dynamical system can be found in [9, Ch.II].

Thus, a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is convergent, if the systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compact dissipative with Levinson centers J_X and J_Y respectively, and J_X has "trivial" sections, i.e., $J_X \bigcap X_y$ consists of a single point for all $y \in J_Y$. In this case, the Levinson center J_X of the dynamical system $(X, \mathbb{T}_1, \pi)$ is a copy (a homeomorphic image) of the Levinson center J_Y of the dynamical system $(Y, \mathbb{T}_2, \sigma)$. Thus, the dynamics on J_X is the same as on J_Y.

Remark 2.1. 1. We note that convergent systems are, in some sense, the simplest dissipative dynamical systems. If Y is compact and invariant, $\mathbb{T}_2 = \mathbb{S}$, $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a convergent non-autonomous dynamical system, and J_X is the Levinson center of $(X, \mathbb{T}_1, \pi)$, then $(J_X, \mathbb{T}_2, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are homeomorphic. Although the Levinson center of a convergent system can be completely described, it may be sufficiently complicated.

2. The concept of convergent system of differential equations is well developed (see, for example, B. P. Demidovich [16, 17, 18], Pavlov *et al.* [24], V. A. Pliss [25, 26], V. I. Zubov [35], and many others). The non-autonomous system of differential equations

$$x' = f(t, x) \tag{6}$$

is called *convergent*, if it admits a unique solution defined and bounded on $\mathbb{R}$, which is uniformly globally asymptotically stable. It is possible to show that the non-autonomous dynamical system generated by the convergent equation (6) is convergent. However, the concept of convergent non-autonomous dynamical system is much more general (see [9, 12] and the bibliography therein).

Given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of x, if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$, for all $t \in \mathbb{T}$).

A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$, there exists a positive number l such that, in any segment of length l, there is an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent, and the set $H(x) := \overline{\{\pi(t, x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*, where the bar denotes the closure in X.

Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} \ : \ \text{such that } \{\pi(t_n,x)\} \to x \text{ and } \{t_n\} \to \infty\}$ and $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{T} \ : \ \text{such that } \{\pi(t_n,x)\} \text{ is convergent and } \{t_n\} \to \infty\}$.

A point $x \in X$ is called *Poisson stable in the positive direction* if there exists a sequence $\{t_n\} \in \mathfrak{N}_x$ such that $t_n \to +\infty$ as $n \to \infty$.

Let $(X, \mathbb{T}, \pi)$ be a two-sided dynamical system (i.e., $\mathbb{T} = \mathbb{S}$). A point $x \in X$ is called *Poisson stable in the negative direction* if there exists a sequence $\{t_n\} \in \mathfrak{N}_x$ such that $t_n \to -\infty$ as $n \to \infty$. The point $x \in X$ is called *Poisson stable* if it is Poisson stable in both directions.

The dynamical system $(X, \mathbb{T}, \pi)$ is said to be

(i) *transitive* if there exists a point $x_0 \in X$ such that $H(x_0) = X$;
(ii) *pseudo recurrent* if X is compact, transitive, and every point $x \in X$ is Poisson stable.

A point $x \in X$ is called *pseudo recurrent* (see [28, 30]) if the dynamical system $(H(x), \mathbb{T}, \pi)$ is pseudo recurrent.

Remark 2.2. *Every recurrent point is pseudo recurrent, but there exist pseudo recurrent points which are not recurrent (see* [28, 30]*).*

An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}^m$. Let $(\mathcal{T}^m, \mathbb{T}, \sigma)$ be an irrational winding of $\mathcal{T}^m$, i.e., $\sigma(t,\nu) := (\nu_1 t, \nu_2 t, \dots, \nu_m t)$ for all $t \in \mathbb{S}$ and $\nu \in \mathcal{T}^m$.

A point $x \in X$ is called *quasi-periodic*, with frequency $\nu := (\nu_1, \nu_2, \dots, \nu_m) \in \mathcal{T}^m$, if there exists a continuous function $\Phi : \mathcal{T}^m \to X$ such that $\pi(t,x) := \Phi(\sigma(t,\omega))$ for all $t \in \mathbb{T}$, where $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is an irrational winding of the torus $\mathcal{T}^m$ and $\omega \in \mathcal{T}^m$.

A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called *Levitan almost periodic* (see [2, 23]) if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$, and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Remark 2.3. *Let $x_i \in X_i$ $(i = 1, 2, \dots, m)$ be a Levitan almost periodic point of the dynamical system $(X_i, \mathbb{T}, \pi_i)$. Then, the point $x := (x_1, x_2, \dots, x_m) \in X := X_1 \times X_2 \times \ldots \times X_m$ is also Levitan almost periodic in the product dynamical system $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is defined by the equality $\pi(t,x) := (\pi_1(t,x_1), \pi_2(t,x_2), \dots, \pi_m(t,x_m))$, for all $t \in \mathbb{T}$ and $x := (x_1, x_2, \dots, x_m) \in X$.*

Recall (see [11]) that the point $x \in X$ is called *asymptotically τ–periodic* (respectively, *asymptotically quasi periodic*, *asymptotically Bohr almost periodic*, *asymptotically almost automorphic*, *asymptotically recurrent*, *asymptotically pseudo recurrent*) if there exists a τ-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent) point $p \in X$ such that

$$\lim_{t \to +\infty} \rho(\pi(t,x), \pi(t,p)) = 0.$$

Thus, we can prove the following result.

Lemma 2.4. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be a convergent non-autonomous dynamical system, and J_X (respectively, J_Y) be the Levinson center of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$), and $y_0 \in J_Y$ be a τ–periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent) point. Then, the following statements hold:*

(i) *the point $x_0 \in J_X \bigcap X_{y_0}$ is also τ–periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent);*
(ii) *every point $x \in X_{y_0}$ is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically recurrent, asymptotically pseudo recurrent).*

Proof. The first statement follows directly from the corresponding definition. Let $x \in X_{y_0}$ be an arbitrary point. We will show that $\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,x_0)) = 0$. Indeed, if we suppose that it is not true, then there are $\varepsilon_0 > 0$ and $t_n \to +\infty$ ($\{t_n\} \subseteq \mathbb{T}_1$) such that

$$\rho(\pi(t_n,x),\pi(t_n,x_0)) \geq \varepsilon_0. \tag{7}$$

Since the dynamical system $(X,\mathbb{T}_1,\pi)$ is compact dissipative, we may suppose that the sequences $\{\pi(t_n,x)\}$ and $\{\pi(t_n,x_0)\}$ are convergent. Denote by $p := \lim_{n\to\infty} \pi(t_n,x)$, and $p_0 := \lim_{n\to\infty} \pi(t_n,x_0)$. Then, $p,p_0 \in X_q \subseteq J_X$, where $q := \lim_{n\to\infty} \sigma(t_n,y_0)$. Since the non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is convergent, then $p = p_0$. On the other hand, passing to the limit in (7), we obtain $\rho(p,p_0) \geq \varepsilon_0 > 0$. This contradiction proves our statement. □

3. **Non-Autonomous Dynamical Systems with Weak Convergence.** In this section we will study a class of non-autonomous dynamical systems which is very close to convergent systems, but possessing a non-trivial global attractor. This means that this class of non-autonomous systems will conserve almost all properties of convergent systems, but will have a "non-trivial" global attractor J_X, i.e., there exists at least one point $y \in J_Y$ such that the set $J_X \bigcap X_y$ contains more than one point.

A non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be *weak convergent*, if the following conditions hold:

(i) the dynamical systems $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ are compact dissipative with Levinson centers J_X and J_Y respectively;

(ii) it follows that
$$\lim_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0,$$
for all $x_1,x_2 \in J_X$ with $h(x_1) = h(x_2)$.

Remark 3.1. It is clear that every convergent non-autonomous dynamical system is weak convergent. The opposite statement is not true in general.

Indeed, the last statement can be confirmed by the following example. Let $(X,\mathbb{T},\pi)$ be an autonomous dynamical system with compact global attractor J, which possesses a unique stationary attracting point p (i.e., $\lim_{|t|\to+\infty} \rho(\pi(t,x),p) = 0$ for all $x \in J$) and $J \neq \{p\}$). For example, consider the dynamical system $(X,\mathbb{R},\pi)$ on the space $X = \mathbb{R}^2$, generated by following system of differential equations

$$\begin{cases} x' = \dfrac{x^2(2y-x)+2y^5}{(x^2+y^2)[1+(x^2+y^2)^2]} \\[2ex] y' = \dfrac{8y^2(y-x)}{(x^2+y^2)[1+(x^2+y^2)^2]} \end{cases}. \tag{8}$$

The phase plane of this dynamical system (8) is described in Figure 1. For more details see [31] and also [3, Ch.II, pp.94-98] and [21, Ch.V, pp.191-194]. For a similar example the reader is referred to [1, Ch.V, p.59] (Example 1.7.8).

We can now prove the following result which will be crucial in the proof of one of our main results (namely, Theorem 3.5).

Lemma 3.2. *Let $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a non-autonomous dynamical system, and assume that the following conditions hold:*

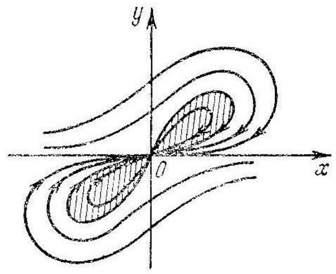

FIGURE 1.

(i) *Y is a compact minimal set;*
(ii) *there exists a point $x_0 \in X$ with relatively compact positive semi-trajectory $\Sigma^+_{x_0} = \{\pi(t, x_0) \mid t \geq 0\}$;*
(iii)
$$\lim_{t\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0,$$
for all $x_1, x_2 \in X$ with $h(x_1) = h(x_2)$.

Then, there exists a unique compact minimal set $M \subseteq X$ such that:

(i) *the section $M \bigcap X_y$ of the set M consists of a single point m_y for all $y \in Y$;*
(ii) *every positive semi-trajectory Σ^+_x is relatively compact;*
(iii)
$$\lim_{t\to+\infty} \rho(\pi(t, x), m_{\sigma(t,h(x))}) = 0, \tag{9}$$
for all $x \in X$.

Proof. Since the positive semi-trajectory $\Sigma^+_{x_0}$ of x_0 is relatively compact, the ω–limit set ω_{x_0} of the point x_0 is nonempty, compact, invariant and contains at least one minimal subset $M \subseteq \omega_{x_0}$. We will prove that the dynamical system $(X, \mathbb{T}_1, \pi)$ has at most one minimal set. Indeed, if we suppose that M_1 and M_2 are two different minimal sets of $(X, \mathbb{T}_1, \pi)$, then $M_1 \bigcap M_2 = \emptyset$ and, in particular, $M_{1y} \bigcap M_{2y} = \emptyset$ for all $y \in Y$, where $M_{iy} := h^{-1}(y) \bigcap M_i$ $(i = 1, 2)$. Let $x_i \in M_{iy}$ and $t_n \to +\infty$ such that

$\sigma(t_n, y) \to y$ and $\pi(t_n, x_i) \to \bar{x}_i \in M_{iy}$ $(i = 1, 2)$ as $n \to \infty$;

$$\lim_{n\to\infty} \rho(\pi(t_n, x_1), \pi(t_n, x_2)) = 0. \tag{10}$$

It is now easy to see that there exists such a sequence. From the equality (10) we have $\bar{x}_1 = \bar{x}_2 \in M_{1y} \bigcap M_{2y}$. This is a contradiction and, therefore, M is the unique compact minimal set of the dynamical system $(X, \mathbb{T}_1, \pi)$.

Let $y \in Y$ be an arbitrary point, then it is recurrent. By Lemma 6.5.19 in [12, Ch.VI, p.226], there exists a unique recurrent point $m_y \in M_y$ such that the equality (9) holds for all $x \in M_y$. Now, we will prove that $M_y = \{m_y\}$. Indeed, if $x \in M_y$ then, there exists a sequence $t_n \to +\infty$ such that $\pi(t_n, m_y) \to x$ because M is minimal. On the other hand, $\sigma(t_n, y) \to y$ and, since the point m_y is uniformly compatible by the character of recurrence with the point y, then $\pi(t_n, m_y) \to m_y$. Thus, we have $x = m_y$ and, consequently, $M_y = \{m_y\}$ for all $y \in Y$.

To finish the proof it is sufficient to note that $\lim_{t\to+\infty}\rho(\pi(t,x),m_{\sigma(t,h(x))})=0$, for all $x\in X$. □

Let $(X,\mathbb{T},\pi)$ be a dynamical system. Denote by $\Omega_X := \overline{\cup\{\omega_x \mid x\in X\}}$ and by $D^+(M) := \bigcap_{\varepsilon>0}\overline{\bigcup_{t\geq 0}\pi(t,B(M,\varepsilon))}$, where $M\subseteq X$.

Corollary 3.3. *Under the assumptions in Lemma 3.2, the dynamical system $(X,\mathbb{T}_1,\pi)$ is point dissipative and Ω_X is a compact minimal set.*

Below we give an example of point dissipative, but not compact dissipative, dynamical system with weak convergence.

Example 3.4. Let $\varphi\in C(\mathbb{R},\mathbb{R})$ be a function possessing the following properties:

1. $\varphi(0)=0$;
2. $supp(\varphi)=[0,2]$;
3. $\varphi\in C^\infty(\mathbb{R},\mathbb{R})$;
4. $\varphi(1)=1$;
5. the function φ is monotone increasing from 0 to 1 and it is decreasing from 1 to 2;
6. $x\varphi(x^{-1})\to 0$ as $x\to+\infty$.

A function φ with properties 1. – 6. can be constructed as follows. Let

$$\varphi_0(t) := \begin{cases} \exp\left([t^2-1]^{-1}+1\right) & , \ |t|<1 \\ 0 & , \ |t|\geq 1. \end{cases}$$

Then, the function $\varphi(t) := \varphi_0(t-1)$ is as desired. We set $X := \{a\varphi\left(\frac{t}{a}+h\right) \mid h\in\mathbb{R},\ a>0\}\cup\{\theta\}$, where θ is the function from $C(\mathbb{R},\mathbb{R})$ identically equal to 0. It is possible to show that the set X is closed in $C(\mathbb{R},\mathbb{R})$, and it is invariant with respect to shifts. Thus, on the set X is induced a dynamical system (on $C(\mathbb{R},\mathbb{R})$ is defined the dynamical system of translations or Bebutov's dynamical system), which we denote by $(X,\mathbb{R},\sigma)$. We will indicate some properties of this system:

(i) for every function $\psi\in X$ the set $\{\sigma(t,\psi) : t\in\mathbb{R}\}$ is relatively compact and $\omega_\psi=\alpha_\psi=\{\theta\}$, where α_ψ denotes the α-limit associated to ψ;
(ii) the dynamical system $(X,\mathbb{R},\sigma)$ is pointwise dissipative, and $\Omega_X=\{\theta\}$;
(iii) $D^+(\Omega_X)=X$ and, consequently, the dynamical system $(X,\mathbb{R},\sigma)$ is not compact dissipative because the set X, evidently, is not compact;
(iv) the dynamical system $(X,\mathbb{R},\sigma)$ does not admit a maximal compact invariant set.

The necessary example is therefore constructed.

The subset $A\subseteq X$ is said to be *chain transitive* (see [15, 22]) if for any $a,b\in A$, and any $\varepsilon>0$ and $L>0$, there are finite sequences $x_1,x_2,\dots,x_m\in A$ with $a=x_1, b=x_m$, and $t_1,t_2,\dots,t_m\geq L$ such that $\rho(\pi(t_i,x_i),x_{i+1})<\varepsilon$ $(1\leq i\leq m-1)$. The sequence $\{x_1,x_2,\dots,x_m\}$ is called an ε–chain in A connecting a and b.

We can now establish and prove the main results of our paper.

Theorem 3.5. *Let $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be a non-autonomous dynamical system satisfying the following conditions:*

(i) *Y is a compact minimal set;*
(ii) *the dynamical system $(X,\mathbb{T}_1,\pi)$ is compact dissipative with Levinson center J;*

(iii) *it holds that*

$$\lim_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0,$$

for all $x_1, x_2 \in J$ *with* $h(x_1) = h(x_2)$.

Then, the following statements hold:

(i) *there exists a unique compact minimal set* $M \subseteq J$ *such that*
   (a) $M \subseteq \partial J$, *where* ∂J *is the boundary of the set* J;
   (b) *the section* $M \bigcap X_y$ *of the set* M *consists of a single point* m_y *for all* $y \in Y$;
   (c) *it holds that*

$$\lim_{t\to+\infty} \rho(\pi(t,x), m_{\sigma(t,h(x))}) = 0,$$

   for all $x \in X$;

(ii) *the Levinson center* J *of the system* $(X, \mathbb{T}_1, \pi)$ *is chain transitive.*

Proof. According to Lemma 3.2, there exists a unique minimal set $M \subset X$ for the dynamical system $(X, \mathbb{T}_1, \pi)$ with properties (b) and (c). To finish the first statement of our theorem it is sufficient to show that $M \subseteq \partial J$. Let $x \in \partial J$, then $M \subseteq \omega_x \subseteq \partial J$ because J is the maximal compact invariant set of $(X, \mathbb{T}_1, \pi)$, and the set ∂J is also compact and invariant.

Let $a, b \in J$, $\varepsilon > 0$, $L > 0$ and $m \in M$. Then, there exist $t_1 \geq L$, $t_2 \geq L$, and an entire trajectory γ of the dynamical system $(X, \mathbb{T}_1, \pi)$, passing through the point b at the initial moment ($\gamma(0) = b$), such that

$$\rho(\pi(t_1, a), m) < \frac{\varepsilon}{3}, \tag{11}$$

and

$$\rho(\gamma(-t_2), m) < \frac{\varepsilon}{3}. \tag{12}$$

We set $x_1 = a$, $x_2 = \gamma(-t_2)$ and $x_3 = b$. Then, from the inequalities (11) and (12) we obtain

$$\rho(\pi(t_i, x_i), x_{i+1}) < \varepsilon \tag{13}$$

($i = 1, 2$). Inequality (13) means that $\{x_1, x_2, x_3\}$ is an ε–chain in J connecting points a and b. □

Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system. A subset $M \subseteq X$ is called

- *stable* if for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, m) < \delta$ ($m \in M, h(x) = h(m)$) implies $\rho(\pi(t, x), \pi(t, m)) < \varepsilon$ for all $t \geq 0$;
- *globally attracting* if $\lim\limits_{t\to+\infty} \rho(\pi(t,x), M_{\sigma(t,h(x))}) = 0$, for all $x \in X$;
- *globally asymptotically stable* if it is stable and globally attracting.

Then, we can prove the following theorems concerning the structure of compact positively invariant sets.

Theorem 3.6. *Let* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ *be a non-autonomous compact dissipative dynamical system,* $M \neq \emptyset$ *be a compact positively invariant set. Suppose that the following conditions are fulfilled:*

(i) $h(M) = Y$;
(ii) $M \bigcap X_y$ *contains a single point for all* $y \in Y$;
(iii) M *is globally asymptotically stable.*

Then, M *is orbitally stable.*

Proof. We will show that set M is orbitally stable in $(X, \mathbb{T}_1, \pi)$. Suppose that it is not true. Then, there exist $\varepsilon_0 > 0, \delta_n \to 0, x_n \in B(M, \delta_n)$ and $t_n \to +\infty$ such that

$$\rho(\pi(t_n, x_n), M) \geq \varepsilon_0. \tag{14}$$

Since M is compact, then we may suppose that the sequence $\{x_n\}$ is convergent. Let $x_0 := \lim_{n\to+\infty} x_n$, with $x_{y_n} \in M_{y_n}, \rho(x_n, M) = \rho(x_n, x_{y_n})$ and $y_0 = h(x_0)$. Then, $x_0 = \lim_{n\to+\infty} x_{y_n}$ and $x_0 \in M_{y_0}$. Let $q_n = h(x_n)$, and note that

$$\rho(x_n, x_{q_n}) \leq \rho(x_n, x_{y_n}) + \rho(x_{y_n}, x_{q_n}) \to 0 \tag{15}$$

as $n \to +\infty$, because $q_n \to y_0$ and $x_{q_n} \to x_0$. Taking into account (15), and the asymptotic stability of the set M, we have

$$\rho(\pi(t_n, x_n), \pi(t_n, x_{q_n})) \to 0. \tag{16}$$

But the equalities (14) and (16) are contradictory. Hence, the set M is orbitally stable in $(X, \mathbb{T}_1, \pi)$. □

Theorem 3.7. *Suppose the conditions of Theorem 3.5 are fulfilled, and M is the unique compact minimal set of the dynamical system $(X, \mathbb{T}_1, \pi)$.*

The following statements hold:

(i) *if M is stable, then $J = M$;*
(ii) *if M is unstable, then $J \neq M$ and $D^+(M) = J$.*

Proof. Note that, under the conditions of the theorem, $\Omega_X = M$. Let M be stable. By Theorem 3.6 the set M is orbitally stable and, consequently, $D^+(M) = M$. On the other hand, the dynamical system $(X, \mathbb{T}_1, \pi)$ is compact dissipative and, consequently, $J = D^+(\Omega_X)$. Thus, we have $J = D^+(M) = M$.

By Theorem 3.5 we have $\Omega_X = M$ and, consequently, according to Theorem 1.11 in [9, Ch.I, p.21], $J = D^+(M)$. On the other hand, since the dynamical system $(X, \mathbb{T}_1, \pi)$ is compact dissipative, then $D^+(M) = M$ only for orbitally stable sets. Thus, $M \subseteq J$ and $M \neq J$. Finally, we note that $M \subseteq \alpha_\gamma$ for every entire trajectory γ (with $\gamma(\mathbb{S}) \subseteq J$) of the dynamical system $(X, \mathbb{T}_1, \pi)$ and, by Theorem 1.35 in [9, Ch.I, p.45], $J = D^+(M)$. □

Theorem 3.8. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be a non-autonomous dynamical system satisfying the following conditions:*

(i) *the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are point dissipative;*
(ii) *Ω_Y is a compact minimal set;*
(iii)
$$\lim_{t\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0,$$
for all $x_1, x_2 \in X$ with $h(x_1) = h(x_2)$;
(iv) *for every $y \in \Omega_Y$, the set $L_{\tilde{X}} \bigcap \tilde{X}_y$ contains at most one point, where $L_{\tilde{X}} := \{x \in \tilde{X} :$ there exists at least one entire motion $\gamma \in \Phi_x$ such that $\gamma(\mathbb{S}) \subseteq \tilde{X}$ and $\gamma(\mathbb{S})$ is relatively compact $\}$, where $\tilde{X} := h^{-1}(\Omega_Y)$.*

Then, there exists a unique compact minimal set $M \subseteq X$ such that

(i) *the section $M \bigcap X_y$ of the set M consists of a single point m_y for all $y \in Y$;*
(ii) *$\Omega_X = M$.*

Proof. Let $N := \Omega_Y$. Under our assumptions, N is a compact minimal set of the dynamical system $(Y, \mathbb{T}, \sigma)$. Consider the non-autonomous dynamical system $\langle(\tilde{X}, \mathbb{T}_1, \tilde{\pi}), (N, \mathbb{T}_2, \tilde{\sigma}), \tilde{h}\rangle$, where $\tilde{X} := h^{-1}(N)$, $(\tilde{X}, \mathbb{T}_1, \tilde{\pi})$ (respectively, $(N, \mathbb{T}_2, \tilde{\sigma})$) is the restriction of $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y, \mathbb{T}_2, \sigma)$) on $\tilde{X}$ (respectively, on N) and $\tilde{h} := h\big|_{\tilde{X}}$. According to Lemma 3.2, the dynamical system $(\tilde{X}, \mathbb{T}_1, \tilde{\pi})$ contains a unique compact minimal set M such that $h(M) = N$, and for all $y \in N$, the set $M_y = \tilde{h}^{-1}(y)$ consists of a single point. It is clear that the set M is also minimal with respect to the dynamical system $(X, \mathbb{T}_1, \pi)$. Repeating the arguments in the proof of Lemma 3.2, we can prove that M is the unique compact minimal set of $(X, \mathbb{T}_1, \pi)$. Let now $x \in X$ be an arbitrary point. Then, its ω–limit set ω_x is a non-empty, compact and invariant set. Since $h(\omega_x) = \omega_{h(x)} \subseteq N$, then $\omega_x \subseteq L_{\tilde{X}}$. Note that ω_x contains at least one compact minimal set. Taking into consideration that M is the unique compact minimal set in $(\tilde{X}, \mathbb{T}_1, \tilde{\pi})$, we then have $M \subseteq \omega_x$ and, consequently, $\omega_x \bigcap \tilde{X}_y = \{m_y\}$ for all $y \in N$. Thus, we conclude that $\omega_x = M$ and, consequently, $\Omega_X = M$. ☐

Denote by $\mathfrak{L}_x := \{\{t_n\} \in \mathfrak{M}_x : \ t_n \to +\infty\}$. Recall (see [11]) that the point $x \in X$ is called *comparable with $y \in Y$ by the character of recurrence in infinity* if $\mathfrak{L}_x \subseteq \mathfrak{L}_y$.

Corollary 3.9. *Let $\langle(X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system such that the following conditions hold:*

- (i) *the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are point dissipative;*
- (ii) *$N := \Omega_Y$ is a compact minimal set;*
- (iii)
$$\lim_{t\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0,$$
for all $x_1, x_2 \in X$ with $h(x_1) = h(x_2)$;
- (iv) *for every $y \in N$, the set $L_{\tilde{X}} \bigcap \tilde{X}_y$ contains at most one point.*

Then, there exists a unique compact minimal set $M \subseteq X$ such that

- (i) *the section $M \bigcap X_y$ of the set M consists of a single point m_y for all $y \in Y$;*
- (ii) *$\Omega_X = M$;*
- (iii) *every point $x \in X$ is comparable with $h(x)$ by the character of recurrence in infinity.*

Proof. The first and second statements follow from Theorem 3.8. To complete the proof we need to establish the third statement. Let $x \in X$, $y := h(x) \in H^+(y_0) := \overline{\{\sigma(t, y_0) : \ t \in \mathbb{T}_+\}}$ and $\{t_n\} \in \mathfrak{L}_y$. We will show that $\{t_n\} \in \mathfrak{L}_x$. Indeed, the sequence $\{\pi(t_n, x)\}$ is relatively compact because $(X, \mathbb{T}, \pi)$ is point dissipative. Let p be a limit point for the sequence $\{\pi(t_n, x)\}$, then there exists a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ such that $p = \lim_{k\to\infty} \pi(t_{n_k}, x)$. Denote by $q = \lim_{k\to\infty} \sigma(t_n, y)$, then $h(p) = q$. It is clear that $q \in \Omega_Y = N$ and $p \in L_X$. Since $L_X \bigcap \tilde{X}_q$ contains at most one point, we conclude that p is the unique limit point of the sequence $\{\pi(t_n, x)\}$ and, consequently, it is convergent. ☐

Corollary 3.10. *Let $\langle(X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system such that the following conditions hold:*

- (i) *the dynamical system $(X, \mathbb{T}, \pi)$ is point dissipative;*
- (ii) *there exists a point $y_0 \in Y$ such that $Y = H^+(y_0) := \overline{\{\sigma(t, y_0) : \ t \in \mathbb{T}_+\}}$;*

(iii) *the point* y_0 *is asymptotically stationary (respectively, asymptotically* τ*–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent);*

(iv)
$$\lim_{t\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0,$$
for all $x_1, x_2 \in X$ *with* $h(x_1) = h(x_2)$*;*

(v) *for every* $y \in N$*, the set* $L_{\tilde{X}} \bigcap \tilde{X}_y$ *contains at most one point.*

Then, there exists a unique compact minimal set $M \subseteq X$ *such that:*

(i) *the section* $M \bigcap X_y$ *of the set* M *consists of a single point* m_y *for all* $y \in Y$*;*

(ii) $\Omega_X = M$*;*

(iii) *every point* $x \in X$ *is asymptotically stationary (respectively, asymptotically* τ*–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent).*

Proof. The statements follow from Theorem 2.2.2 in [11, Ch.II, p.31] and Corollary 3.9. □

3.1. **Pseudo Recurrent Dynamical Systems with Convergence.** A non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be *uniformly stable in the positive direction on compact subsets of* X if, for arbitrary $\varepsilon > 0$ and every compact $K \subset X$, there is $\delta = \delta(\varepsilon, K) > 0$ such that the inequality $\rho(x_1, x_2) < \delta$ $(h(x_1) = h(x_2))$ implies that $\rho(\pi(t,x_1),\pi(t,x_2)) < \varepsilon$, for $t \in \mathbb{T}_1^+$, where $\mathbb{T}_1^+ := \{t \in \mathbb{T}_1 :\ t \geq 0\}$.

Let $X\dot{\times}X := \{(x_1,x_2) \mid x_1,x_2 \in X, h(x_1) = h(x_2)\}$. The function $V : X\dot{\times}X \to \mathbb{R}_+$ is said to be continuous, if $x_n^i \to x^i$ $(i = 1,2$ and $h(x_n^1) = h(x_n^2))$ implies $V(x_n^1,x_n^2) \to V(x^1,x^2)$.

If there exists a function $V : X\dot{\times}X \to \mathbb{R}_+$ with the following properties:

(i) V is continuous,

(ii) V is positive defined, i.e., $V(x_1,x_2) = 0$ if and only if $x_1 = x_2$,

(iii) $V(\pi(t,x_1),\pi(t,x_2)) \leq V(x_1,x_2)$ for all $(x_1,x_2) \in X\dot{\times}X$ and $t \in \mathbb{T}_1^+$,

then, the non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is called V*–monotone* (see [9], [10], [14], [23], and [34]).

Denote by $\mathcal{K} := \{a \in C(\mathbb{R}_+,\mathbb{R}_+) \mid a(0) = 0,\ a$ is strictly increasing$\}$.

Recall that the dynamical system $(X,\mathbb{T}_1,\pi)$ is called *asymptotically compact* if for every positively invariant bounded subset $M \subseteq X$, there exists a nonempty compact subset $K \subseteq X$ such that,
$$\lim_{t\to+\infty}\beta(\pi(t,M),K)=0,$$
where $\beta(A,B) := \sup_{a\in A}\rho(a,B)$ and $\rho(a,B) := \inf_{b\in B}\rho(a,b)$.

The next result will be crucial for our applications (particularly, Theorem 4.6).

Theorem 3.11. *Let* $\langle (X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ *be a non-autonomous dynamical system satisfying the following conditions:*

1. *the dynamical system* $(Y,\mathbb{S},\sigma)$ *is pseudo recurrent;*
2. *the dynamical system* $(X,\mathbb{T},\pi)$ *is asymptotically compact;*
3. *there exists a point* $x_0 \in X_{y_0}$ *with relatively compact positive semi-trajectory* $\Sigma_{x_0}^+ := \{\pi(t,x_0) :\ t \geq 0\}$*;*
4. *the non-autonomous dynamical system* $\langle (X,\ \mathbb{T},\ \pi),(Y,\ \mathbb{S},\ \sigma),\ h\rangle$ *is* V*–monotone;*

5. *for all* $(x_1, x_2) \in X\dot{\times}X \setminus \Delta_X$ *(where* $\Delta_X := \{(x,x) : x \in X\}$*) there exists a positive number* $t_0 = t_0(x_1, x_2) \in \mathbb{T}$ *such that* $V(\pi(t_0, x_1), \pi(t_0, x_2)) < V(x_1, x_2)$*;*
6. *there are functions* $a, b \in \mathcal{K}$ *such that* $Im(a) = Im(b)$*, and* $a(\rho(x_1, x_2)) \leq V(x_1, x_2) \leq b(\rho(x_1, x_2))$ *for all* $(x_1, x_2) \in X\dot{\times}X$*.*

Then, the following statements hold:

(i) *the NDS* $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ *is convergent;*
(ii) $J_X = \omega_{x_0}$*;*
(iii) $h(J_X) = Y$*.*

Proof. The proof follows the same lines than those of Theorem 3.10 in [4], and we omit them here. □

Corollary 3.12. *Let* $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ *be a non-autonomous dynamical system such that:*

(i) *the dynamical system* $(Y, \mathbb{S}, \sigma)$ *is transitive, i.e., there exists a point* $y_0 \in Y$ *such that* $H(y_0) = Y$*;*
(ii) *the point* y_0 *is* τ*–periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent);*
(iii) *the dynamical system* $(X, \mathbb{T}, \pi)$ *is asymptotically compact;*
(iv) *there exists a point* $x_0 \in X_{y_0}$ *with relatively compact positive semi-trajectory* $\Sigma^+_{x_0} := \{\pi(t, x_0) : t \geq 0\}$*;*
(v) *the non-autonomous dynamical system* $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ *is* V*–monotone;*
(vi) *for all* $(x_1, x_2) \in X\dot{\times}X \setminus \Delta_X$ *(where* $\Delta_X := \{(x,x) : x \in X\}$*), there exists a positive number* $t_0 = t_0(x_1, x_2) \in \mathbb{T}$ *such that* $V(\pi(t_0, x_1), \pi(t_0, x_2)) < V(x_1, x_2)$*;*
(vii) *there are functions* $a, b \in \mathcal{K}$ *such that* $Im(a) = Im(b)$*, and* $a(\rho(x_1, x_2) \leq V(x_1, x_2) \leq b(\rho(x_1, x_2))$ *for all* $(x_1, x_2) \in X\dot{\times}X$*.*

Then,

(i) *there exists a unique* τ*–periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent) point* $x_0 \in X_{y_0} := \{x \in X : h(x) = y_0\}$*;*
(ii) *every point* $x \in X$ *is asymptotically* τ*–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically recurrent, asymptotically pseudo recurrent).*

Proof. This statement directly follows from Theorem 3.11 and Lemma 2.4. □

Remark 3.13. 1. Note that in [4] we established an analogous result to Theorem 3.11 but under a stronger assumption. Namely, instead of Condition 5. of Theorem 3.11, we used in [4] the following one:

$$V(\pi(t, x_1), \pi(t, x_2)) < V(x_1, x_2) \tag{17}$$

for all $(x_1, x_2) \in X\dot{\times}X \setminus \Delta_X$ and $t > 0$.

2. It is clear that (17) implies Condition 5. of Theorem 3.11. The converse is not true. Below we give the corresponding counterexample.

Example 3.14. Consider the dynamical system $(X, \mathbb{T}, \pi)$ with discrete time $\mathbb{T} = \mathbb{Z}_+$. We set $X = C[0, 1]$, where $C[0, 1]$ is the Banach space of all continuous functions $\phi : [0, 1] \to \mathbb{R}$ endowed with the norm $||\phi||_{C[0,1]} := \max\limits_{t\in[0,1]} |\phi(t)|$ (or simply $||\phi||$). Consider the operator A, acting on the space $C[0, 1]$, defined by the equality $(A\phi)(t) := \int_0^t \phi(s)ds$. Now, consider the dynamical system on $C[0, 1]$ defined by the positive powers of the operator A, i.e., $\pi(n,\phi) := A^n\phi$ for all $n\in\mathbb{Z}_+$ and $\phi\in C[0,1]$. It is easy to check that the norm of the operator A is equal to 1 and, consequently, $||A^n(\phi_1-\phi_2)||\le||\phi_1-\phi_2||$. Thus, the dynamical system $(X,\mathbb{Z}_+,\pi)$ is $V-$monotone, if we take the function $V: X\times X\to\mathbb{R}_+$ defined by $V(\phi_1,\phi_2) := ||\phi_1-\phi_2||$. In this way, condition (17) does not hold for the dynamical system $(X,\mathbb{Z}_+,\pi)$ constructed above. On the other hand, it is also easy to check that the norm of the operator A^n is equal to $\frac{1}{n!}$. Thus, for all $\phi_i\in C[0,1]$ $(i=1,2)$ with $\phi_1\neq\phi_2$, we have $||A^n(\phi_1-\phi_2)||\le\frac{1}{n!}||\phi_1-\phi_2||<||\phi_1-\phi_2||$ for all $n\ge 2$. The necessary example is therefore constructed.

4. **Applications.** Let X and Y be two complete metric spaces. Denote by $C(X,Y)$ the space of all continuous functions $f: X\to Y$ equipped with the compact-open topology.

4.1. **Almost periodic solutions of almost periodic dissipative systems.** Let $(Y,\mathbb{R},\sigma)$ be a dynamical system on the metric space Y. In this subsection we suppose that Y is a compact space. Consider the differential equation

$$u' = f(\sigma(t,y),u) \quad (y\in Y), \tag{18}$$

where $f\in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$.

The function $f\in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$ (respectively, equation (18)) is said to be *regular* (see [27]) if for all $u\in\mathbb{R}^n$ and $y\in Y$, equation (18) admits a unique solution $\varphi(t,u,y)$ passing through the point $u\in\mathbb{R}^n$ at the initial moment $t=0$, and defined on $\mathbb{R}_+$.

It is well known (see [27]) that the mapping $\varphi:\mathbb{R}_+\times\mathbb{R}^n\times Y\to\mathbb{R}^n$ possesses the following properties:

(i) $\varphi(0,u,y)=u$ for all $u\in\mathbb{R}^n$ and $y\in Y$;
(ii) $\varphi(t+\tau,u,y)=\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for all $t,\tau\in\mathbb{R}_+$, $u\in\mathbb{R}^n$ and $y\in Y$;
(iii) the mapping φ is continuous.

Thus, the triplet $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{R},\sigma)\rangle$ is a cocycle (non-autonomous dynamical system) which is associated to (generated by) equation (18). In this case, the dynamical system $(Y,\mathbb{R},\sigma)$ is called the base dynamical system (or driving system).

Example 4.1. Let us consider the equation

$$u' = f(t,u), \tag{19}$$

where $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Along with equation (19), consider the family of equations

$$u' = g(t,u), \tag{20}$$

where $g\in H(f) := \overline{\{f_\tau:\ \tau\in\mathbb{R}\}}$ and f_τ is the τ-shift of f with respect to the time variable t, i.e., $f_\tau(t,u) := f(t+\tau,u)$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^n$. Suppose that the function f is regular (see [27]), i.e., for all $g\in H(f)$ and $u\in\mathbb{R}^n$ there exists a unique solution $\varphi(t,u,g)$ of equation (20). Denote by $Y=H(f)$, and $(Y,\mathbb{R},\sigma)$ the shift dynamical system on Y induced by the Bebutov dynamical system $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$. Now, the family of equations (20) can be written as

$$u' = F(\sigma(t,y),u), \quad (y\in Y),$$

if we define $F\in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$ by the equality $F(g,u) := g(0,u)$, for all $g\in H(f)$, and $u\in\mathbb{R}^n$.

In this section we suppose that equation (18) is regular. Equation (18) is called *dissipative* (see [9]), if there exists a positive number r such that

$$\limsup_{t\to+\infty} |\varphi(t,u,y)| < r$$

for all $u \in \mathbb{R}^n$ and $y \in Y$, where $|\cdot|$ is a norm in $\mathbb{R}^n$.

It is well known (see [19, 33]) that a dissipative equation with almost periodic coefficients (Y is an almost periodic minimal set) does not have, in general, an almost periodic solution. For certain classes of dissipative equations of the form (18), in the works [5]–[8] one can find sufficient conditions for the existence of at least one almost periodic solution. In this subsection we give a simple geometric condition which guarantees existence of a unique almost periodic solution, and this solution, in general, is not the unique solution of equation (18) which is bounded on $\mathbb{R}$.

We can now establish the following interesting result.

Theorem 4.2. *Suppose that the following conditions are fulfilled:*

(i) *equation (18) is regular and dissipative;*
(ii) *the space Y is compact, and the dynamical system $(Y,\mathbb{R},\sigma)$ is minimal;*
(iii) *for all $y \in Y$*

$$\lim_{t\to+\infty} |\varphi(t,u_1,y) - \varphi(t,u_2,y)| = 0, \tag{21}$$

where $\varphi(t,u_i,y)$ $(i = 1,2)$ is the solution of equation (18) passing through u_i at the initial moment $t = 0$, which is bounded on $\mathbb{R}$.

Then,

(i) *if the point y is τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then equation (18) admits a unique τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent) solution $\varphi(t,u_y,y)$ $(u_y \in \mathbb{R}^n)$;*
(ii) *every solution $\varphi(t,x,y)$ is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent).*

Proof. Let $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{R},\sigma)\rangle$ be the cocycle associated to equation (18). Denote by $(X,\mathbb{R}_+,\pi)$ the skew-product dynamical system, where $X := \mathbb{R}^n \times Y$ and $\pi := (\varphi,\sigma)$ (i.e., $\pi(t,(u,y)) := (\varphi(t,u,y),\sigma(t,y))$ for all $x := (u,y) \in \mathbb{R}^n \times Y$ and $t \in \mathbb{R}_+$). Consider the non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by the cocycle φ (respectively, by equation (18)), where $h := pr_2 : X \to Y$ is the projection with respect to the second variable, i.e., $pr_2(\alpha,y) = y$. Since Y is compact, it is evident that the dynamical system $(Y,\mathbb{R},\sigma)$ is compact dissipative and its Levinson center J_Y coincides with Y. By Theorem 2.23 in [9], the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ is compact dissipative. Denote by J_X its Levinson center and by $I_y := pr_1(J_X \bigcap X_y)$ for all $y \in Y$, where $X_y := \{x \in X : h(x) = y\}$, and pr_1 is the projection function with respect to the first variable, i.e. $pr_1(\alpha,y) = \alpha$. According to the definition of the set $I_y \subseteq \mathbb{R}^n$, and by Theorem 2.24 in [9], $u \in I_y$ if and only if the solution $\varphi(t,u,y)$ is defined on $\mathbb{R}$ and bounded (i.e., the set $\overline{\varphi(\mathbb{R},u,y)} \subseteq \mathbb{R}^n$ is compact). Thus, $I_y = \{u \in \mathbb{R}^n \mid (u,y) \in J_X\}$. It is easy to see the condition (21) means that the non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is weak convergent. To finish the proof, it is sufficient to apply Lemma 6.5.19 in [12, Ch.VI, p.226] and Corollary 3.10 for the non-autonomous system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (18). □

Remark 4.3. Under the conditions of Theorem 4.2 there exists a unique almost periodic solution of equation (18), but equation (18) has, generally speaking, more than one solution defined and bounded on $\mathbb{R}$. Below, we will give an example which confirms this statement.

Example 4.4. Consider the following almost periodic system of two differential equations

$$\begin{cases} u' = \frac{(u-\sin t)^2(2v-u+2\sin\sqrt{2}t-\sin t)+2(v-\sin\sqrt{2}t)^5}{((u-\sin t)^2+(v-\sin\sqrt{2}t)^2)[1+((u-\sin t)^2+(v-\sin\sqrt{2}t)^2)^2]} + \cos t \\ v' = \frac{8(v-\sin\sqrt{2}t)^2(v-u+\sin\sqrt{2}t-\sin t)}{((u-\sin t)^2+(v-\sin\sqrt{2}t)^2)[1+((u-\sin t)^2+(v-\sin\sqrt{2}t)^2)^2]} + \sqrt{2}\cos\sqrt{2}t\ . \end{cases} \tag{22}$$

It is easy to check that the almost periodic function $\varphi : \mathbb{R} \to \mathbb{R}^2$ defined by the equality $\varphi(t) := (\sin t, \sin\sqrt{2}t)$ is a solution of system (22). Let now $x := u - \sin t$ and $y := v - \sin\sqrt{2}t$. Then, the system (22) reduces to (8). Thus, every solution ϕ of the system (22) possesses the form $\phi = \varphi + \psi$, where ψ is some solution of the system (8). Since (8) is weak convergent and admits more than one solution which is bounded on $\mathbb{R}$, the system (22) possesses the same property.

4.2. **Uniform compatible solutions of strict dissipative equations.** In this section we consider equation (18) when the driving system $(Y, \mathbb{R}, \sigma)$ is pseudo recurrent, and the function $f \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is strict dissipative with respect to its second variable $x \in \mathbb{R}^n$, i.e.,

$$\langle f(y, x_1) - f(y, x_2), x_1 - x_2\rangle < 0 \tag{23}$$

for all $x_1, x_2 \in \mathbb{R}^n$ $(x_1 \neq x_2)$ and $y \in Y$.

Recall (see [28, 29, 30]) that the point $x \in X$ is called *comparable (respectively, uniformly comparable) by the character of recurrence* with the point $y \in Y$ if $\mathfrak{N}_y \subseteq \mathfrak{N}_x$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_x$).

Let us now recall a result which plays an important role in the proof of our main result in this subsection.

Theorem 4.5. *(See [28, 30]) Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two dynamical systems, $x \in X$ and $y \in Y$. Then, the following statements hold:*

(i) *If x is comparable by the character of recurrence with y, and y is τ–periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then so is the point x.*
(ii) *If x is uniformly comparable by the character of recurrence with y, and y is τ–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then so is the point x.*

Following B. A. Shcherbakov [28, 30], a solution $\varphi(t, u, y)$ of equation (18) is said to be *compatible (respectively, uniformly compatible) by the character of recurrence with the right hand-side* if $\mathfrak{N}_y \subseteq \mathfrak{N}_{\varphi(\cdot,u,y)}$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi(\cdot,u,y)}$).

Theorem 4.6. *Let $(Y, \mathbb{R}, \sigma)$ be pseudo recurrent, $f \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ be strict dissipative with respect to the variable x, and assume that there exists at least one solution $\varphi(t, x_0, y)$ of equation (18) which is bounded on $\mathbb{R}_+$.*

Then,

(i) *equation (18) is convergent, i.e., the cocycle φ associated to equation (18) is convergent;*

(ii) *for all* $y \in Y$*, equation (18) admits a unique solution* $\varphi(t, x_y, y)$ *which is bounded on* $\mathbb{R}$ *and uniformly compatible, i.e.,* $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi(\cdot, x_y, y)}$*;*

(iii) *if the point* y *is* τ*–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent), then*

   (a) *equation (18) has a unique* τ*–periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent) solution;*

   (b) *every solution* $\varphi(t, x, y)$ *is asymptotically* τ*–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent);*

   (c) $\lim_{t\to\infty} |\varphi(t, x, y) - \varphi(t, x_y, y)| = 0$ *for all* $x \in \mathbb{R}^n$ *and* $y \in Y$*.*

Proof. Let $V : X\dot{\times}X \to \mathbb{R}_+$ be the mapping defined by the equality $V((x_1, y), (x_1, y)) := |x_1 - x_2|^2$ for all $x_1, x_2 \in \mathbb{R}^n$ and $y \in Y$, where $|\cdot|^2 := \langle \cdot, \cdot \rangle$ and $\langle \xi^1, \xi^2 \rangle := \sum_{i=1}^{n} \xi^1_i \xi^2_i$ ($\xi^j := (\xi^j_1, \xi^j_2, \dots, \xi^j_n) \in \mathbb{R}^n$ $(j = 1, 2)$). Let $(X, \mathbb{R}_+, \pi)$ be the skew-product dynamical system associated to the cocycle φ. Then,

$$\frac{dV(\pi(t,(u_1,y)), \pi(t,(u_2,y)))}{dt}\Big|_{t=0} = \langle f(y,u_1) - f(y,u_2), u_1 - u_2 \rangle < 0 \qquad (24)$$

for all $u_1, u_2 \in \mathbb{R}^n$ ($u_1 \neq u_2$) and $y \in Y$. From (24) it follows that $V(\pi(t,(u_1,y)), \pi(t(u_2,y)) < V((u_1,y),(u_2,y))$ for all $u_1, u_2 \in \mathbb{R}^n$ ($u_1 \neq u_2$) and $y \in Y$. Now to prove the first statement it is sufficient to apply Theorem 3.11.

According to the first statement of the theorem, the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is compact dissipative and, if J_X is its Levinson center, then $J_X \bigcap X_y$ consists of a single point $x_y = (u_y, y)$ and $\varphi(t, u_y, y)$ is the unique solution of equation (18) defined and bounded on $\mathbb{R}$. Now, we will prove that the solution $\varphi(\cdot, u_y, y)$ is uniformly compatible by the character of recurrence, i.e., $\mathfrak{M}_y \subseteq \mathfrak{M}_{\varphi(\cdot,u_y,y)}$. It is easy to see that the last statement is equivalent to the inclusion $\mathfrak{M}_y \subseteq \mathfrak{M}_{x_y}$. Let $\{t_k\} \in \mathfrak{M}_y$. Then, there exists a point $q \in Y$ such that $\sigma(t_n, y) \to q$ as $n \to \infty$. Consider the sequence $\{\pi(t_k, x_y)\}$. Since $x_y \in J_X$, the sequence $\{\pi(t_k, x_y)\}$ is relatively compact. Let p_1 and p_2 be two points of accumulation of this sequence. Then, there exist two subsequences $\{t^{(i)}_k\} \subseteq \{t_k\}$ $(i = 1, 2)$ such that $p_i = \lim_{k\to\infty} \pi(t^i_k, x_y)$ $(i = 1, 2)$. Since J_X is a compact invariant set, then $p_i \in J_X$ $(i = 1, 2)$. On the other hand, $\pi(t^i_k, x_y) = (\varphi(t^i_k, u_y, y), \sigma(t^i_k, y)) \to (\bar{u}_i, q) = p_i$ and, consequently, $p_i \in X_q$. Thus $p_i \in J_X \bigcap X_q$ $(i = 1, 2)$ and, consequently, $p_1 = p_2$. This means that the sequence $\{\pi(t_k, x_y)\}$ is convergent. The second statement is therefore proved.

Taking into account the first and second statements, to finish the proof of the third statement it is sufficient to apply Theorem 4.5. The theorem is completely proved. □

Remark 4.7. 1. Theorem 4.6 remains true if we replace the standard scalar product $\langle \cdot, \cdot \rangle$ on the space $\mathbb{R}^n$ by an arbitrary scalar product $\langle u, u \rangle_W := \langle Wu, u \rangle$, where $W = (w_{ij})^n_{i,j=1}$ ($w_{ij} \in \mathbb{R}$) is a symmetric and positive defined $n \times n$–matrix.

2. If we replace condition (23) by a stronger condition, then Theorem 4.6 is also true without the requirement that there exists at least one solution which is bounded on $\mathbb{R}_+$. Namely, if there exists a function $\zeta \in \mathcal{K}$ such that

$$\langle f(y,u_1) - f(y,u_2), u_1 - u_2 \rangle \leq -\zeta(|u_1 - u_2|^2)$$

for all $u_1, u_2 \in \mathbb{R}^n$ and $y \in Y$, where ζ possesses some additional properties (see, for example, [14]).

3. It is easy to see that Theorem 4.6 remains true also for equation (18) in an arbitrary Hilbert space H, if we suppose that the cocycle φ, generated by equation (18), is asymptotically compact (i.e., the corresponding skew-product dynamical system is asymptotically compact), and we replace the condition about the existence of at least one solution which is bounded on $\mathbb{R}_+$, by the existence of a relatively compact solution φ_0 on $\mathbb{R}_+$ (this means that $\varphi(\mathbb{R}_+)$ is a relatively compact subset from H).

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REFERENCES

[1] N. P. Bhatia and G. P. Szegö, "Stability Theory of Dynamical Systems", Lecture Notes in Mathematics, Springer, Berlin–Heidelberg–New York, 1970.

[2] I. U. Bronsteyn, "Extensions of Minimal Transformation Group", Noordhoff, 1979.

[3] B. F. Bylov, R. E. Vinograd, D. M. Grobman and V. V. Nemytskii, "Lyapunov Exponents Theory and Its Applications to Problems of Stabity", Moscow, Nauka, 1966, 576 pp. (in Russian)

[4] T. Caraballo and D. N. Cheban, *Levitan Almost Periodic and Almost Automorphic Solutions of Second-Order Monotone Differential Equations*, Submitted, (2010).

[5] D. N. Cheban, *Quasiperiodic solutions of the dissipative systems with quasiperiodic coefficients*, Differential Equations **22** (1986), no. 2, 267–278.

[6] D. N. Cheban, $\mathbb{C}$-*analytic dissipative dynamical systems*, Differential Equations **22** (1986), no. 11, 1915–1922.

[7] D. N. Cheban, *Boundedness, dissipativity and almost periodicity of the solutions of linear and weakly nonlinear systems of differential equations*, Dynamical systems and boundary value problems Kishinev, "Shtiintsa", (1987), 143–159.

[8] D. N. Cheban, *Global Pullback Atttactors of C-Analytic Nonautonomous Dynamical Systems*, Stochastics and Dynamics **1** (2001), no. 4, 511–535.

[9] D.N. Cheban, "Global Attractors of Non-Autonomous Dissipative Dynamical Systems", Interdisciplinary Mathematical Sciences 1. River Edge, NJ: World Scientific, 2004, 528pp.

[10] D.N. Cheban, *Levitan Almost Periodic and Almost Automorphic Solutions of V-monotone Differential Equations*, J. Dynamics and Differential Equations **20** (2008), no. 3, 669–697.

[11] D.N. Cheban, "Asymptotically Almost Periodic Solutions of Differential Equations", Hindawi Publishing Corporation, New York, 2009, 203 pp.

[12] D. N. Cheban, "Global Attractors of Set-Valued Dynamical and Control Systems", Nova Science Publishers, New York, 2010.

[13] D.N. Cheban and C. Mammana, *Invariant manifolds, global attractors and almost periodic solutions of non-autonomous difference equations*, Nonlinear Analysis TMA **56** (2004), no. 4, 465–484.

[14] D.N. Cheban and B. Schmalfuß, I*nvariant Manifolds, Global Attractors, Almost Automorphic and Almost Periodic Solutions of Non-Autonomous Differential Equations*, J. Math. Anal. Appl. **340** (2008), no. 1, 374–393.

[15] C. Conley, "Isolated Invariant Sets and the Morse Index", Region. Conf. Ser. Math., No.38, 1978. Am. Math. Soc., Providence, RI.

[16] B. P. Demidovich, *On Dissipativity of Certain Nonlinear Systems of Differential Equations, I,* Vestnik MGU **6** (1961), 19–27.

[17] B. P. Demidovich, *On Dissipativity of Certain Nonlinear Systems of Differential Equations, II,* Vestnik MGU **1** (1962), 3–8.
[18] B. P. Demidovich, "Lectures on Mathematical Theory of Stability", Moscow, Nauka, 1967. (in Russian)
[19] A. M. Fink and P. O. Fredericson, *Ultimate Boundedness Does not Imply Almost Periodicity,* Journal of Differential Equations **9** (1971), 280–284.
[20] J. K. Hale, "Asymptotic Behaviour of Dissipative Systems", Amer. Math. Soc., Providence, RI, 1988.
[21] W. Hahn, "Stability of motion", Springer-Verlag New York, Inc., New York 1967 xi+446 pp. (Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften, Band 138)
[22] M. W. Hirsch, H. L. Smith and X.-Q. Zhao, *Chain Transitivity, Attractivity, and Strong Repellers for Semidynamical Systems*, J. Dyn. Diff. Eqns **13** (2001), no. 1, 107–131.
[23] B.M. Levitan, V.V. Zhikov, "Almost Periodic Functions and Differential Equations", Cambridge Univ. Press, London, 1982.
[24] A. Pavlov, A. Pogrowsky, N. van de Wouw and N. Nijmeijer, *Convergent dynamics, a tribute to Boris Pavlovich Demidovich*, Systems and Control Letters, **52** (2007), no. 6, 257–261.
[25] V. A. Pliss, "Nonlocal Problems in the Theory of Oscillations", Nauka, Moscow, 1964 (in Russian). [English translation: Nonlocal Problems in the Theory of Oscillations, Academic Press, 1966.]
[26] V. A. Pliss, "Integral Sets of Periodic Systems of Differential Equations", Nauka, Moscow, 1977 (in Russian).
[27] G. R. Sell, "Topological Dynamics and Ordinary Differential Equations", Van Nostrand-Reinhold, London, 1971.
[28] B.A. Shcherbakov, "Topologic Dynamics and Poisson Stability of Solutions of Differential Equations", Ştiinţa, Chişinău, 1972. (In Russian)
[29] B.A. Shcherbakov, *The comparability of the motions of dynamical systems with regard to the nature of their recurrence*, Differential Equations **11** (1975), no. 7, 1246–1255.
[30] B.A. Shcherbakov, "Poisson Stability of Motions of Dynamical Systems and Solutions of Differential Equations", Ştiinţa, Chişinău, 1985. (In Russian)
[31] R. E. Vinograd, *Inapplicability of the method of characteristic exponents to the study of non-linear differential equations*, Mat. Sb. N.S. **41** (1957), no. 83, 431–438. (in Russian)
[32] Yoshizawa T., "Stability theory and the existence of periodic solutions and almost periodic solutions.", Applied Mathematical Sciences, Vol. 14, Springer-Verlag, New York-Heidelberg, 1975. vii+233 pp.
[33] V. V. Zhikov, *On Stability and Unstability of Levinson's centre*, Differentsial'nye Uravneniya **8** (1972), no. 12, 2167–2170.
[34] V.V. Zhikov, *Monotonicity in the Theory of Almost Periodic Solutions of Non-Linear operator Equations*, Mat. Sbornik **90** (1973), 214–228; English transl., Math. USSR-Sb. **19** (1974), 209-223.
[35] V. I. Zubov, "The Methods of A. M. Lyapunov and Their Application", Noordhoof, Groningen, 1964.
[36] V. I. Zubov, "Theory of Oscillations", Nauka, Moscow, 1979. (in Russian)

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