Existence of invariant manifolds for coupled parabolic and hyperbolic stochastic partial differential equations

Tomás Caraballo†, Igor Chueshov‡, José A. Langa†

†Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, Spain.

‡Department of Mechanics and Mathematics, Kharkov University, 61077 Kharkov, Ukraine.

E-mail: caraball@us.es; chueshov@univer.kharkov.ua; langa@us.es

Abstract. An abstract system of coupled nonlinear parabolic-hyperbolic partial differential equations subjected to additive white noise is considered. The system models temperature dependent or heat generating wave phenomena in a continuum random medium. Under suitable conditions, the existence of an exponentially attracting random invariant manifold for the coupled system is proved, and as a consequence, the system can be reduced to a single stochastic hyperbolic equation with a modified nonlinear term. Finally it is also proved that this random manifold converges to its deterministic counterpart when the intensity of noise tends to zero.

AMS classification scheme numbers: 35B40, 37H10, 37L25; Secondary 35M10, 37L55.

Submitted to: *Nonlinearity*

1. Introduction and statement of the problem

A description of wave propagation phenomena in random media is usually based on the study of stochastically (or randomly) perturbed hyperbolic partial differential equations (see, e.g., Sobczyk (1984) and the references therein). If these wave phenomena are temperature dependent or heat generating, then the hyperbolic equations are coupled with a stochastic parabolic (heat) equation (see, e.g., Chow (1973) or Hori (1973)). To this respect, the question of how a thermal environment may influence on the long time dynamics of the system arises. In this paper we consider this question and show that, under some conditions, temperature field is a slave variable for wave (master) variables. In particular this means that the thermal effects at large time scale can be taken into account by modifying a forcing (nonlinear) term in the corresponding stochastic hyperbolic equation.

As a model to present our results we consider a system of stochastic differential equations consisting of the hyperbolic equation

$$v_{tt} + \gamma v_t + Lv = F(v, v_t, u) + \dot{W}_1, \quad \text{in } X_1, \tag{1}$$

and the parabolic one

$$u_t + \nu Au = G(v, v_t, u) + K(v, v_t) + \dot{W}_2, \quad \text{in } X_2, \tag{2}$$

where X_1 and X_2 are infinite dimensional separable Hilbert spaces, γ and ν are positive parameters, and the operators and the noises appearing in (1) and (2) satisfy the following assumptions:

(A1) L and A are positive linear self-adjoint operators in X_1 and X_2 respectively with domains $D(L)$ and $D(A)$.

(A2) F and G are nonlinear mappings,

$$F : D(L^{1/2}) \times X_1 \times D(A^\alpha) \mapsto X_1,$$

$$G : D(L^{1/2}) \times X_1 \times D(A^\alpha) \mapsto X_2,$$

where $\alpha \in [0, 1)$, and there exist constants M_F and M_G such that

$$\|F(v_0, v_1, u) - F(\hat{v}_0, \hat{v}_1, \hat{u})\|_{X_1} \le M_F \left(\|L^{1/2}(v_0 - \hat{v}_0)\|^2_{X_1} + \|v_1 - \hat{v}_1\|^2_{X_1} + \|A^\alpha(u - \hat{u})\|^2_{X_2} \right)^{1/2} \tag{3}$$

and

$$\|G(v_0, v_1, u) - G(\hat{v}_0, \hat{v}_1, \hat{u})\|_{X_2} \le M_G \left(\|L^{1/2}(v_0 - \hat{v}_0)\|^2_{X_1} + \|v_1 - \hat{v}_1\|^2_{X_1} + \|A^\alpha(u - \hat{u})\|^2_{X_2} \right)^{1/2}. \tag{4}$$

(A3) The mapping $K : D(L^{1/2}) \times X_1 \mapsto [D(A^\beta)]'$ possesses the property

$$\|A^{-\beta} \left(K(v_0, v_1) - K(\hat{v}_0, \hat{v}_1) \right) \|_{X_2} \le M_K \left(\|L^{1/2}(v_0 - \hat{v}_0)\|^2_{X_1} + \|v_1 - \hat{v}_1\|^2_{X_1} \right)^{1/2} \tag{5}$$

for some $0 \le \beta \le 1 - \alpha$, where M_K is a positive constant.

(A4) For every $i = 1, 2$, $W_i(t)$, $t \in \mathbb{R}$, is a two-sided X_i-valued Wiener process with covariance operator $K_i = K_i^* \geq 0$ such that $\operatorname{tr} K_1 < \infty$ and $\operatorname{tr} K_2 A^{2(\alpha+\varepsilon)-1} < \infty$ for some $\varepsilon > 0$. We assume for simplicity that W_1 and W_2 are independent, and denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the corresponding probability space, and by $\dot{W}_i$ the generalized derivative with respect to t in (1) and (2).

Although it is possible to consider other kinds of randomness to model stochastic wave phenomena, the main reason which justifies the use of additive noise is that it usually models background effects and small effects that have been omitted or neglected in a deterministic modeling procedure. To this respect, from the physical point of view, it is important to know whether qualitative properties of the simplified (deterministic) model are robust enough to perturbations by additive noises. Our result in Section 6 answers this question for system (1)–(2).

We also note that system (1)–(2) is an abstract model for a thermoelastic phenomenon in a random medium which can be described by the following equations (see, e.g., Chow (1973)):

$$v_{tt} + \gamma v_t - \mu\Delta v - (\mu+\lambda)\nabla \operatorname{div} v = -\kappa\nabla\theta + \widetilde{F}(v, \nabla v) + \dot{W}_1,\ t > 0, x \in \mathcal{O}, \tag{6}$$

$$\theta_t - \nu\Delta\theta = -\delta \cdot \operatorname{div} v_t + \widetilde{G}(\theta, \nabla\theta) + \dot{W}_2,\ t > 0, x \in \mathcal{O}, \tag{7}$$

where $\mathcal{O}$ is a domain in $\mathbb{R}^d$, $d = 2, 3$, $v = v(x,t) \in \mathbb{R}^d$ denotes the displacement vector, $\theta = \theta(x,t)$ the temperature, and $\mu, \lambda, \kappa, \nu, \delta$ are positive constants, where μ and λ are Lamé moduli. The parameter $\gamma > 0$ describes resistance forces, and the functions $\widetilde{F}$ and $\widetilde{G}$ satisfy suitable conditions. The white noise processes $\dot{W}_1$ and $\dot{W}_2$ (see below for more details) model random fluctuations in external loads ($\dot{W}_1$) and in thermal sources ($\dot{W}_2$).

System (6)-(7) can be easily set in our abstract formulation. To this end, we first need to equip these equations with suitable boundary conditions. For example, we can consider Dirichlet type boundary conditions

$$v = 0,\ \theta = 0 \ \text{ for } \ t > 0,\ x \in \partial\mathcal{O}. \tag{8}$$

If we assume that $\widetilde{F} : \mathbb{R}^{d+d^2} \mapsto \mathbb{R}^d$ and $\widetilde{G} : \mathbb{R}^{1+d} \mapsto \mathbb{R}$ are globally Lipschitz, then (A1)-(A3) hold for our problem (6), (7) and (8), by setting $X_1 = [L_2(\mathcal{O})]^d$, $X_2 = L_2(\mathcal{O})$, $\alpha = \beta = 1/2$, $K(v, v_t) = -\delta \operatorname{div} v_t$, $L = -\mu\Delta - (\mu+\lambda)\nabla \operatorname{div}$ and $A = -\Delta$ with Dirichlet boundary conditions, and finally, F and G are defined in the obvious way.

We note that the asymptotic behaviour of deterministic thermoelastic models has been receiving increasing attention over the last years (see, e.g., Chandrasekharaiah

(1998), Chueshov (2004), Jiang and Racke (2000), Munoz Rivera and Barreto (1998), Munoz Rivera and Racke (1995), and the references therein).

It is also worth mentioning that, as we do not assume any compactness properties concerning the resolvents of the operators L and A, our problem (6), (7) and (8) on *unbounded* domains can be also included within the scope of our theory, after an appropriate redetermination of the linear and nonlinear terms in the equations.

As far as we know, there are no publications on the dynamics of coupled parabolic-hyperbolic *stochastic* partial differential equations, although stochastic parabolic and wave equations have been widely studied by many authors (see, e.g, the monographs Cerrai (2001), Da Prato and Zabczyk (1996) and the references therein for the parabolic case and the papers Barbu and Da Prato (2002), Carmona and Nualart (1993), Dalang and Frangos (1998), Da Prato and Zabczyk (1992), Millet and Morien (2001), Millet and Sanz-Solé (2000), Peszat and Zabczyk (2000), Quer-Sardanyons and Sanz-Solé (2004) for the wave case).

Our main objective in this paper is to prove a reduction principle for the random dynamical system generated by problem (1)–(2) which will allow us to rewrite our coupled system as an equivalent problem for a single stochastic hyperbolic equation with a conveniently modified nonlinear term. To be more precise, we will prove that, for ν large enough, in the phase space

$$\mathcal{H}_0 = D(L^{1/2}) \times X_1 \times X_2$$

of the random dynamical system generated by (1) and (2), there exists an invariant exponentially attracting (random) surface of the form

$$\mathcal{M}(\omega) = \left\{ (v, \bar{v}, \Phi(\omega, v, \bar{v})) \ : \ (v, \bar{v}) \in D(L^{1/2}) \times X_1 \right\} \subset \mathcal{H}_0, \tag{9}$$

where $\Phi : \Omega \times D(L^{1/2}) \times X_1 \mapsto X_2$ is a Lipschitz mapping for each $\omega \in \Omega$ and a stationary process with respect to t (see Theorem 4.1 for more details). Under some additional conditions the existence of this surface $\mathcal{M}$ makes it possible to prove that the long-time behaviour of the system (1) and (2) can be described by the reduced problem

$$v_{tt} + \gamma v_t + Lv = F(v, v_t, \Phi(\theta_t \omega, v, v_t)) + \dot{W}_1, \quad \text{in } X_1. \tag{10}$$

For a similar result in the deterministic framework we refer to Leung (2003) and Chueshov (2004). We also mention that, in contrast with (1), the reduced system (10) contains a *random* nonlinear term of the form $F^*(v, v_t, \theta_t \omega)$ and, hence, cannot be considered as a perturbation of a deterministic system by an additive white noise process.

The approach which we adopt in this paper relies on some ideas from the theory of inertial manifolds started by Foias et al. (1988) and developed by many authors (see, e.g., the monographs by Chueshov (1999), Constantin et al. (1989), Temam (1988) for the deterministic case and the papers by Bensoussan and Flandoli (1995), Chueshov (1995), Chueshov and Girya (1995), Chueshov and Scheutzow (2001), Duan et al. (2003) for the stochastic case and also the references therein). To cover our main case $\alpha+\beta=1$ we invoke the idea of the Lyapunov-Perron method (see, e.g., Chow and Lu (1988) and Chow et al. (1992)) in the form presented in Miklavčič (1991) for the deterministic case. To the best of our knowledge, this idea has not been used earlier in the study of invariance properties of stochastic systems.

The paper is organized as follows. In the preliminary Section 2 we represent the problem as a first order stochastic differential equation, for the reader's convenience recall the basic definitions from the theory of random dynamical systems, and collect several results on stochastic convolutions in a form adapted to our situation. In Section 3 we prove the existence and uniqueness of mild solutions to problem (1) and (2) and show that this problem generates a filtered random dynamical system (RDS). Section 4 contains our main result which is a type of reduction principle (see Theorem 4.1). In Section 5 we establish some properties of the reduced system. In Section 6 we estimate the distance between $\mathcal{M}(\omega)$ and its deterministic counterpart $\mathcal{M}^{det}$ in terms of the covariance operators of W_1 and W_2 (see Theorem 6.1). In particular, we prove that $\mathcal{M}(\omega)$ converges to $\mathcal{M}^{det}$ when the intensity of the noise tends to zero. Some final comments and conclusions are presented in the last section.

2. Basic definitions and auxiliary facts

First of all, we will rewrite system (1) and (2) as a first order stochastic partial differential system and will analyze the corresponding Cauchy problem; in other words, problem (1)–(2) is equivalent to

$$\frac{dV}{dt} + \mathcal{A}V = \mathcal{B}(V) + \dot{\mathcal{W}},\ t > s, \quad V\big|_{t=s} = V_0, \tag{11}$$

where $s \in \mathbb{R}$, $V = V(t) = (v(t), v_t(t), u(t))^T$, $\mathcal{W} = (0, W_1, W_2)^T$ and

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 \\ L & \gamma & 0 \\ 0 & 0 & \nu A \end{pmatrix}, \quad \mathcal{B}(V) = \begin{pmatrix} 0 \\ F(v, v_t, u) \\ G(v, v_t, u) + K(v, v_t) \end{pmatrix}. \tag{12}$$

We consider now problem (11) in the scale of spaces

$$\mathcal{H}_\sigma = D(L^{1/2}) \times X_1 \times D(A^\sigma), \quad \sigma \in \mathbb{R},$$

which are equipped with the norms

$$|V|_\sigma = \left(\|L^{1/2}v_0\|_{X_1}^2 + \|v_1\|_{X_1}^2 + \|A^\sigma u_0\|_{X_2}^2\right)^{1/2}, \quad V = (v_0, v_1, u_0).$$

Recall that if $\sigma < 0$, then $D(A^\sigma)$ is the completion of X_2 with respect to the norm $\|A^\sigma \cdot \|_{X_2}$.

It is straightforward to check that the operator $\mathcal{A}$ generates a strongly continuous semigroup $e^{-\mathcal{A}t}$ in each space $\mathcal{H}_\sigma$ and

$$e^{-\mathcal{A}t} = \begin{pmatrix} T_t & 0 \\ 0 & e^{-\nu A t} \end{pmatrix}, \tag{13}$$

where T_t is the strongly continuous *group* in $D(L^{1/2}) \times X_1$ generated by the equation

$$v_{tt} + \gamma v_t + Lv = 0, \ t > 0, \ \text{ in } X_1. \tag{14}$$

Let P denote the orthoprojector in $\mathcal{H}_\sigma$ onto the first two components, i.e.

$$P(v_0, v_1, u_0) = (v_0, v_1, 0) \quad \text{for} \quad (v_0, v_1, u_0) \in \mathcal{H}_\sigma, \tag{15}$$

and $Q = I - P$. One can easily establish by a direct calculation (see, e.g., Foias et al. (1998) and also Chueshov (1999) or Temam (1988)) the following *dichotomy* estimates

$$\left| e^{-\mathcal{A}t} PV \right|_\sigma \le e^{-\gamma t} |PV|_\sigma, \quad t \le 0, \ V \in \mathcal{H}_\sigma, \tag{16}$$

$$\left| e^{-\mathcal{A}t} PV \right|_\sigma \equiv |T_t PV|_{D(L^{1/2}) \times X_1} \le |PV|_\sigma, \quad t \ge 0, \ V \in \mathcal{H}_\sigma, \tag{17}$$

$$\left| e^{-\mathcal{A}t} QV \right|_\sigma \le \left[\left(\frac{\sigma}{\nu t} \right)^\sigma + \lambda_1^\sigma \right] e^{-\nu \lambda_1 t} |QV|_0, \quad t > 0, \ V \in \mathcal{H}_\sigma, \ \sigma > 0, \tag{18}$$

where $\lambda_1 > 0$ is the minimal point in the spectrum of A. We also note (see, e.g., Chueshov (1999, Lemma 5.7.1)) that there exist positive constants C_0 and γ_0 such that

$$|T_t y|_{D(L^{1/2}) \times X_1} \le C_0 e^{-\gamma_0 t} \, |y|_{D(L^{1/2}) \times X_1}, \quad t \ge 0, \ y \in D(L^{1/2}) \times X_1. \tag{19}$$

2.1. Random dynamical systems

We recall now some concepts from the theory of random dynamical systems (see, e.g. Arnold (1998) for more details). As usual, $\mathbb{R}_+$ denotes the set of all non-negative elements of $\mathbb{R}$.

Definition 2.1 Let X be a topological space. *A random dynamical system* (RDS) with time $\mathbb{R}_+$ and state space X is a pair (θ, ϕ) consisting of the following two objects:

(i) A metric dynamical system (MDS) $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$, i.e. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a family of measure preserving transformations $\{\theta_t : \Omega \mapsto \Omega, \, t \in \mathbb{R}\}$ such that

(a) $\theta_0 = \mathrm{id}, \quad \theta_t \circ \theta_s = \theta_{t+s} \quad$ for all $\quad t, s \in \mathbb{R}$;

(b) the map $(t, \omega) \mapsto \theta_t \omega$ is measurable and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

(ii) A (perfect) cocycle ϕ over θ of continuous mappings of X with one-sided time $\mathbb{R}_+$, i.e. a measurable mapping

$$\phi : \mathbb{R}_+ \times \Omega \times X \mapsto X, \quad (t, \omega, x) \mapsto \phi(t, \omega)x$$

such that the mapping $\phi(\cdot, \omega) : (t, x) \mapsto \phi(t, \omega)x$ is continuous for all $\omega \in \Omega$ and satisfies the cocycle property:

$$\phi(0, \omega) = \mathrm{id}, \quad \phi(t+s, \omega) = \phi(t, \theta_s \omega) \circ \phi(s, \omega) \quad \text{for all} \quad t, s \geq 0 \ \text{ and } \ \omega \in \Omega.$$

Definition 2.2 Let θ be an MDS, $\overline{\mathcal{F}}$ the $\mathbb{P}$-completion of $\mathcal{F}$, and $\mathbf{F} = \{\mathcal{F}_t, t \in \mathbb{R}\}$ a family of sub-σ-algebras of $\overline{\mathcal{F}}$ such that (i) $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s < t$; (ii) $\mathcal{F}_s = \bigcap_{h>0} \mathcal{F}_{s+h}$, $s \in \mathbb{R}$, i.e., the filtration $\mathbf{F}$ is right-continuous; (iii) $\mathcal{F}_s$ contains all $\mathbb{P}$-null sets in $\overline{\mathcal{F}}$, $s \in \mathbb{R}$; and (iv) θ_s is $(\mathcal{F}_{t+s}, \mathcal{F}_t)$-measurable for all $s, t \in \mathbb{R}$. Then $(\theta, \mathbf{F})$ is called a *filtered metric dynamical system* (FMDS). If, in addition, (θ, ϕ) is an RDS such that $\phi(t, \cdot)x$ is $(\mathcal{F}_t, \mathcal{B}(X))$-measurable for every $t \geq 0$ and $x \in X$, then $(\theta, \mathbf{F}, \phi)$ is called a *filtered random dynamical system* (FRDS).

We note that $(\theta, \mathbf{F}, \phi)$ is an FRDS if and only if (θ, ϕ) is an RDS, $(\theta, \mathbf{F})$ is an FMDS and $\phi(\cdot, \cdot)x$ is adapted to $\mathbf{F}$ for every $x \in X$. Recall that an X-valued stochastic process $Y(t), t \in T \subseteq \mathbb{R}$ is called *adapted* with respect to the filtration $\mathbf{F}$ if $Y(t)$ is $(\mathcal{F}_t, \mathcal{B}(X))$-measurable for every $t \in T$.

2.2. Stochastic convolution

We consider a pair of two-sided independent Wiener processes $W_1(t)$ and $W_2(t)$, $t \in \mathbb{R}$, with values in X_1 and X_2 respectively on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance operators K_1 and K_2 possessing the properties

$$K_i = K_i^* \geq 0, \quad \operatorname{tr} K_1 < \infty, \quad \operatorname{tr} K_2 A^{2(\alpha+\varepsilon)-1} < \infty,$$

for some $\varepsilon > 0$. For the definitions and properties of such processes see Da Prato and Zabczyk (1992). The property $\operatorname{tr} K_1 < \infty$ implies that W_1 has almost surely strongly continuous trajectories in X_1 (see Da Prato and Zabczyk (1992, p.119)). In the second case, there exists a Hilbert space $\overline{X}_2$ (containing X_2) such that W_2 has strongly continuous paths in $\overline{X}_2$. In addition to the distributional properties of $\mathcal{W} = (0, W_1, W_2)^T$, we will assume that there exists a filtered MDS $(\theta, \mathbf{F})$ such that

(a) $\mathcal{W}(t) - \mathcal{W}(s)$ is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}, s \le t$ and independent of $\mathcal{F}_s$ for $s < t$;

(b) $\mathcal{W}(t+s,\omega) - \mathcal{W}(s,\omega) = \mathcal{W}(t, \theta_s\omega)$, $s,t \in \mathbb{R}$, $\omega \in \Omega$ (*helix* property).

We refer to the monograph Arnold (1998) for the construction of this FMDS and the corresponding Wiener processes.

Now, let us consider the following stochastic integral

$$\eta(t,s) = \int_s^t e^{-(t-\tau)\mathcal{A}} d\mathcal{W}(\tau) \equiv \int_s^t e^{-(t-\tau)\mathcal{A}} \begin{pmatrix} 0 \\ dW_1(\tau) \\ dW_2(\tau) \end{pmatrix}, \quad t > s. \tag{20}$$

The integral in (20) exists as an operator stochastic integral (see, e.g., Da Prato and Zabczyk (1992)). The process $\eta(t,s)$ has the form $\eta(t,s) = (\eta_1(t,s), \eta_2(t,s))^T$, where $\eta_1(t,s)$ and $\eta_2(t,s)$ are centered (independent) Gaussian processes in $D(L^{1/2}) \times X_1$ and $D(A^\alpha) \subset X_2$ respectively, of the form

$$\eta_1(t,s) = \int_s^t T_{t-\tau} \begin{pmatrix} 0 \\ dW_1(\tau) \end{pmatrix}, \quad \eta_2(t,s) = \int_s^t e^{-\nu(t-\tau)A} dW_2(\tau). \tag{21}$$

One can also prove (see, e.g., Da Prato and Zabczyk (1992)) that

$$\mathbb{E}|\eta(t,s)|_\sigma^2 = \mathbb{E}|\eta_1(t,s)|^2_{D(L^{1/2})\times X_1} + \mathbb{E}\|A^\sigma \eta_2(t,s)\|^2_{X_2}$$

and

$$\mathbb{E}|\eta_1(t,s)|^2_{D(L^{1/2})\times X_1} = \int_s^t \operatorname{tr}\left\{ T_{t-\tau} \hat{K}_1 T^*_{t-\tau} \right\} d\tau, \tag{22}$$

where $\hat{K}_1 = \operatorname{diag}\{0, K_1\}$ is an operator in $D(L^{1/2}) \times X_1$, and

$$\mathbb{E}\|A^\sigma \eta_2(t,s)\|^2_{X_2} = \frac{1}{2\nu} \operatorname{tr}\left\{ K_2 A^{2\sigma-1}(1 - e^{-2\nu(t-s)A}) \right\}, \quad \sigma \le \alpha. \tag{23}$$

We note that (22) and (19) imply

$$\mathbb{E}|\eta_1(t,s)|^2_{D(L^{1/2})\times X_1} \le C_0^2 \cdot \operatorname{tr} K_1 \cdot \left(1 - e^{-2\gamma_0(t-s)}\right) \tag{24}$$

and also, since T_t is a strongly continuous contraction semigroup, by Da Prato and Zabczyk (1992, Theorem 6.10) there exists a constant $C > 0$ independent of K_1 such that

$$\mathbb{E} \sup_{t\in[0,1]} |\eta_1(t,0)|^2_{D(L^{1/2})\times X_1} \le C\mathbb{E}|\eta_1(1,0)|^2_{D(L^{1/2})\times X_1} \le C \cdot \operatorname{tr} K_1. \tag{25}$$

We will write $\Pi = \{(t,s) \; : \; -\infty \le s \le t < \infty, \; t > -\infty\}$. By Chueshov and Scheutzow (2001, Proposition 3.1), there exists a (perfect) modification of the processes $\eta_1(t,s)$ and $\eta_2(t,s)$ such that the following properties hold.

- Process $\eta_1(t,s)$:

(i) $(t,s) \mapsto \eta_1(t,s,\omega)$ is continuous from Π into $D(L^{1/2}) \times X_1$, $\omega \in \Omega$;

(ii) $(t,s,\omega) \mapsto \eta_1(t,s,\omega)$ is measurable from $\Pi \times \Omega$ into $D(L^{1/2}) \times X_1$;

(iii) quasi-stationarity:

$$\eta_1(t,s,\omega) = \eta_1(t+\tau, s+\tau, \theta_{-\tau}\omega), \quad (t,s) \in \Pi,\ \tau \in \mathbb{R},\ \omega \in \Omega; \tag{26}$$

(iv) evolution relation: for all $-\infty \le \tau < s \le t,\ \omega \in \Omega$,

$$\eta_1(t,s,\omega) = \eta_1(t,\tau,\omega) - T_{t-s}\eta_1(s,\tau,\omega); \tag{27}$$

(v) temperedness: for all $\beta > 0,\ \omega \in \Omega$,

$$\sup_{t\in\mathbb{R}} \left\{ |\eta_1(t,-\infty,\omega)|_{D(L^{1/2})\times X_1}\, e^{-\beta|t|} \right\} < \infty. \tag{28}$$

- Process $\eta_2(t,s)$:

(i) $(t,s) \mapsto \eta_2(t,s,\omega)$ is continuous from Π into $D(A^\alpha)$, $\quad \omega \in \Omega$;

(ii) $(t,s,\omega) \mapsto \eta_2(t,s,\omega)$ is measurable as a map from $\Pi \times \Omega$ into $D(A^\alpha)$;

(iii) quasi-stationarity:

$$\eta_2(t,s,\omega) = \eta_2(t+\tau, s+\tau, \theta_{-\tau}\omega), \quad (t,s) \in \Pi,\ \tau \in \mathbb{R},\ \omega \in \Omega; \tag{29}$$

(iv) evolution relation: for $-\infty \le \tau < s \le t,\ \omega \in \Omega$,

$$\eta_2(t,s,\omega) = \eta_2(t,\tau,\omega) - e^{-\nu(t-s)A}\eta_2(s,\tau,\omega); \tag{30}$$

(v) temperedness: for all $\beta > 0,\ \omega \in \Omega$,

$$\sup_{t\in\mathbb{R}} \left\{ \|A^\alpha \eta_2(t,-\infty,\omega)\|_{X_2} e^{-\beta|t|} \right\} < \infty. \tag{31}$$

We note that formally Proposition 3.1, as it is stated in Chueshov and Scheutzow (2001), cannot be applied to the process η_1. However the arguments given in the proof of this proposition rely only on the fact that the corresponding semigroup (this is T_t in our case) is strongly continuous and exponentially stable and therefore they cover the case of processes like $\eta_1(t,s)$.

We also recall (see, e.g., Arnold (1998)) that a random variable $v(\omega)$ with values in a Banach space X is said to be *tempered* iff

$$\sup_{t\in\mathbb{R}} \left\{ e^{-\beta|t|} \|v(\theta_t\omega)\|_X \right\} < \infty \quad \text{for all} \quad \beta > 0,\ \omega \in \Omega.$$

By (26) and (29) we have that $\eta_i(t,-\infty,\omega) = \eta_i(0,-\infty,\theta_t\omega) \equiv \tilde{\eta}_i(\theta_t\omega)$ for $t \in \mathbb{R},\ \omega \in \Omega$, and $i = 1,2$, where, due to (28) and (31), the Gaussian random variables $\tilde{\eta}_1(\omega)$ and $\tilde{\eta}_2(\omega)$ are tempered in $D(L^{1/2}) \times X_1$ and X_2 respectively.

3. Mild solutions and generation of an RDS

For a given $\sigma \in \mathbb{R}$, we denote by $C([a,b];\mathcal{H}_\sigma)$ the space of strongly continuous functions on the interval $[a,b]$ with values in $\mathcal{H}_\sigma$, and by $L_2([a,b];\mathcal{H}_\sigma)$ the space of measurable functions $h(\cdot)$ with values in $\mathcal{H}_\sigma$ such that

$$|h|^2_{L_2([a,b];\mathcal{H}_\sigma)} \equiv \int_a^b |h(t)|^2_\sigma dt < \infty.$$

Definition 3.1 Let $s \in \mathbb{R}$, $T > s$ and $V_0 \in \mathcal{H}_{\alpha-1/2}$. A process $V(t) \equiv V(t, s, \omega; V_0)$ which, for each $\omega \in \Omega$, belongs to the space $L_2([s,T];\mathcal{H}_\alpha) \cap C([s,T];\mathcal{H}_{\alpha-1/2})$ is said to be *a mild solution* to problem (11) on the interval $[s,T]$ if $V(s) = V_0$ and

$$V(t) = \mathcal{R}[V](t) \equiv e^{-\mathcal{A}(t-s)}V_0 + \int_s^t e^{-\mathcal{A}(t-\tau)}\mathcal{B}(V(\tau))d\tau + \eta(t,s) \tag{32}$$

for almost all $t \in [s,T]$ and $\omega \in \Omega$, where $\eta(t,s)$ is given by (20).

In this section we prove the existence and uniqueness of mild solutions to (11). One of the key ingredients in our proof is a fixed point argument in the space $L_2([s, s+T];\mathcal{H}_\alpha)$ which relies on the following assertion.

Lemma 3.2 *Let $e^{-\mathcal{A}t}$ be a strongly continuous semigroup in a Hilbert space H with a self-adjoint and positive generator $\mathcal{A} = \mathcal{A}^* > 0$ and $f \in L_2([s, s+T]; H)$ for some $T > 0$ and $s \in \mathbb{R}$. Then the function*

$$\mathcal{I}_s^\beta[f](t) = \int_s^t e^{-\mathcal{A}(t-\tau)}\mathcal{A}^\beta f(\tau)d\tau, \ \ t \in [s, s+T], \tag{33}$$

belongs to $L_2([s, s+T]; D(\mathcal{A}^{1-\beta}))$ and the estimate

$$\int_s^{s+T} \|\mathcal{A}^\alpha \mathcal{I}_s^\beta[f](t)\|_H^2 dt \le (2T)^{2-2(\alpha+\beta)} \int_s^{s+T} \|f(t)\|_H^2 dt \tag{34}$$

holds for any $0 \le \beta \le 1$, $-\beta \le \alpha \le 1-\beta$.

This lemma was proved in Chueshov (2004) for the case $s = 0$. For arbitrary s the argument is basically the same.

We can now prove our main result in this section.

Theorem 3.3 *For every $V_0 \in \mathcal{H}_{\alpha-1/2}$ and $T > 0$ problem (11) has a unique mild solution $V(t)$ on the interval $[s,T]$ provided either $\alpha+\beta = 1$ and $\nu > M_K$, where M_K is the constant in (A3), or $\alpha+\beta < 1$. Moreover, if $\alpha - 1/2 \le \sigma_0 \le \sigma < \min(1-\beta, 1/2)$, then*

$$V(t) \in C((s,T];\mathcal{H}_\sigma) \quad \text{and} \quad |V(t)|_\sigma \le C_T(\omega)\cdot|t-s|^{-\sigma+\sigma_0}, \ t \in (s,T], \tag{35}$$

for $V_0 \in \mathcal{H}_{\sigma_0}$ and $\omega \in \Omega$, where $C_T(\omega)$ is a positive random variable. If $\sigma_0 = \sigma$, then $V(t) \in C([s,T];\mathcal{H}_\sigma)$ for all $\omega \in \Omega$. Furthermore, the process $t \mapsto V(t,\omega)$ is adapted to the filtration $\mathbf{F}$.

Define the map $\phi : \mathbb{R}_+ \times \Omega \times \mathcal{H}_\sigma \mapsto \mathcal{H}_\sigma$, where $\alpha - 1/2 \le \sigma < \min(1-\beta, 1/2)$, by the formula $\phi(t,\omega)V_0 := V(t, 0, \omega; V_0)$. Then **(i)** *$(\theta, \mathbf{F}, \phi)$ is a FRDS, and* **(ii)**

$V(t,s,\omega;V_0)=\phi(t-s,\theta_s\omega,V_0)$ solves (32) for every $s\in\mathbb{R}$, every $t>s$, and every $\omega\in\Omega$.

Proof. We split the proof into several steps.

STEP 1. We first prove that, for each $\omega\in\Omega$, there exists a unique solution to Eq. (32) in the space $L_2([s,s+T];\mathcal{H}_\alpha)$ for a small enough $T>0$.

Since

$$2\nu\int_s^t \|A^{1/2}e^{-\nu A(\tau-s)}u_0\|_{X_2}^2 d\tau \le \|u_0\|_{X_2}^2, \quad u_0\in X_2,$$

it follows that $e^{-\mathcal{A}(t-s)}V_0\in L_2([s,s+T];\mathcal{H}_\alpha)$ provided that $V_0\in\mathcal{H}_{\alpha-1/2}$. Therefore, since $\eta(t,s)\in C(\Pi;\mathcal{H}_\alpha)$, the fact that $\mathcal{R}$ maps $L_2([s,s+T];\mathcal{H}_\alpha)$ into itself for every $\omega\in\Omega$ can be obtained from the contraction estimate for $\mathcal{R}$ given below.

Let P be defined by (15) and $Q=I-P$. It follows from (17) and (3) that

$$|P\mathcal{R}[V_1](t)-P\mathcal{R}[V_2](t)|_\alpha \le M_F\int_s^t |V_1(\tau)-V_2(\tau)|_\alpha d\tau \tag{36}$$

for any $V_1,V_2\in L_2([s,s+T];\mathcal{H}_\alpha)$. Therefore

$$|P\mathcal{R}[V_1]-P\mathcal{R}[V_2]|_{L_2([s,s+T];\mathcal{H}_\alpha)} \le M_F\cdot T\cdot |V_1-V_2|_{L_2([s,s+T];\mathcal{H}_\alpha)}. \tag{37}$$

We also have that

$$Q\mathcal{R}[V_1]-Q\mathcal{R}[V_2]=(0,0,\mathcal{I}_0^s[G(V_1)-G(V_2)])+\frac{1}{\nu^\beta}\left(0,0,\mathcal{I}_s^\beta[K_\Delta^\beta]\right),$$

where $\mathcal{I}_s^\beta$ is given by (33) with $\mathcal{A}=\nu A$ and $H=X_2$, $V_i=(v_i,\bar{v}_i,u_i)$ are elements from the space $L_2([s,s+T];\mathcal{H}_\alpha)$, and $K_\Delta^\beta=A^{-\beta}\left(K(v_1,\bar{v}_1)-K(v_2,\bar{v}_2)\right)$. Consequently from (34) and hypotheses (A2) and (A3) we obtain that

$$|Q\mathcal{R}[V_1]-Q\mathcal{R}[V_2]|_{L_2([s,s+T];\mathcal{H}_\alpha)} \le q(T,\nu)\cdot |V_1-V_2|_{L_2([s,s+T];\mathcal{H}_\alpha)}, \tag{38}$$

where

$$q(T,\nu)=\frac{M_G(2T)^{1-\alpha}}{\nu^\alpha}+\frac{M_K(2T)^{1-(\alpha+\beta)}}{\nu^{\alpha+\beta}}.$$

Therefore, it follows from (37) and (38) that

$$|\mathcal{R}[V_1]-\mathcal{R}[V_2]|_{L_2([s,s+T];\mathcal{H}_\alpha)} \le (M_FT+q(T,\nu))\cdot |V_1-V_2|_{L_2([s,s+T];\mathcal{H}_\alpha)}.$$

Thus, in the case $\alpha+\beta<1$, for every $\nu>0$ we can choose T_0 independent of V_0 such that $M_FT_0+q(T_0,\nu)<1$. If $\alpha+\beta=1$, then we can make this choice only if $\nu>M_K$. In any case, equation (32) has a unique solution $V(t)$ defined in the interval $[s,s+T_0]$ and which belongs to $L_2([s,s+T_0];\mathcal{H}_\alpha)$.

STEP 2. Now we prove that $V(t)$ satisfies (35) with $T=s+T_0$. To this end, observe that it is sufficient to prove that

$$\mathcal{R}_0[V](t):=-e^{-\mathcal{A}t}V_0-\eta(t,s,\omega)+\mathcal{R}[V](t)\in C([s,s+T_0];\mathcal{H}_\sigma) \tag{39}$$

for $\sigma < \min(1-\beta, 1/2)$.

Indeed, for $t_1 > t_2 \geq s$, we have that

$$\mathcal{R}_0[V](t_1) - \mathcal{R}_0[V](t_2) = \int_{t_2}^{t_1} e^{-\mathcal{A}(t_1-\tau)}\mathcal{B}(V(\tau))d\tau + \int_{s}^{t_2} e^{-\mathcal{A}(t_2-\tau)}\left[e^{-\mathcal{A}(t_1-t_2)} - 1\right]\mathcal{B}(V(\tau))d\tau. \tag{40}$$

By (17) we obtain that

$$|P\mathcal{R}_0[V](t_1) - P\mathcal{R}_0[V](t_2)|_1 \leq C\int_{t_2}^{t_1}(1+|V(\tau)|_\alpha)\,d\tau + \int_{s}^{t_2}\left|\left[e^{-\mathcal{A}(t_1-t_2)} - 1\right]P\mathcal{B}(V(\tau))\right|_0 d\tau,$$

where, as before, P is defined by (15). Thus, the Lebesgue convergence theorem implies that

$$P\mathcal{R}_0[V](t) \in C([s, s+T_0]; \mathcal{H}_1) \quad \text{for every} \quad \omega \in \Omega, \tag{41}$$

and, consequently, using (13) and (32) we see that

$$PV(t) \in C([s, s+T_0]; \mathcal{H}_1) \quad \text{and} \quad \max_{t\in[s,s+T_0]} |PV(t)|_1 \leq C_T(\omega). \tag{42}$$

Thus, we only need to check the continuity of the functions

$$\mathcal{Q}_1(t) = \int_s^t e^{-\nu A(t-\tau)}G(V(\tau))d\tau, \quad \text{and} \quad \mathcal{Q}_2(t) = \int_s^t e^{-\nu A(t-\tau)}K(v(\tau), \bar{v}(\tau))d\tau.$$

Here we keep denoting $V = (v, \bar{v}, u)$. By a similar representation to that in (40) and by the relation

$$\| A^\sigma e^{-tA} \| \leq \max_{\lambda>0} |\lambda^\sigma e^{-t\lambda}| = \left(\frac{\sigma}{et}\right)^\sigma, \quad t > 0,\ \sigma > 0,$$

we obtain that

$$\|A^\sigma(\mathcal{Q}_1(t_1) - \mathcal{Q}_1(t_2))\|_{X_2} \leq C_1\int_{t_2}^{t_1}(1+|V(\tau)|_\alpha)\frac{d\tau}{|t_1-\tau|^\sigma} + C_2\left\|A^{-\varepsilon}\left[e^{-\nu A(t_1-t_2)} - 1\right]\right\| \cdot \int_s^{t_2}(1+|V(\tau)|_\alpha)\frac{d\tau}{|t_2-\tau|^{\sigma+\varepsilon}} \tag{43}$$

for any $\sigma < 1/2$ and $0 < \varepsilon < 1/2 - \sigma$. Therefore, since

$$\| A^{-\varepsilon}(1-e^{-\nu At}) \| \leq \sup_{\lambda>0}\left\{\frac{1-e^{-\nu\lambda t}}{\lambda^\varepsilon}\right\} \leq \sup_{\lambda>0}\left\{\frac{1\wedge(\nu\lambda t)}{\lambda^\varepsilon}\right\} = (\nu t)^\varepsilon, \quad t > 0,$$

from the Hölder inequality it follows that $\mathcal{Q}_1 \in C([s, s+T_0], D(A^\sigma))$. Similarly, thanks to (42) and (A3) we have

$$\|A^{-\beta}K(v(t), \bar{v}(t))\|_{X_2} \leq C(\omega) \quad \text{for all} \quad t \in [s, s+T_0],\ \omega \in \Omega,$$

and we then find that

$$\|A^\sigma(\mathcal{Q}_2(t_1) - \mathcal{Q}_2(t_2))\|_{X_2} \le C_1 \int_{t_2}^{t_1} \frac{d\tau}{|t_1-\tau|^{\sigma+\beta}} + C_2 \left\| A^{-\varepsilon}\left[e^{-\nu A(t_1-t_2)} - 1\right]\right\| \cdot \int_s^{t_2} \frac{d\tau}{|t_2-\tau|^{\sigma+\beta+\varepsilon}}.$$

Thus, as above, we can conclude that $\mathcal{Q}_2 \in C([s, s+T_0], D(A^\sigma))$ for any $\sigma < 1-\beta$. Since $Q\mathcal{R}_0[V](t) = \mathcal{Q}_1(t) + \mathcal{Q}_2(t)$, using (41) we obtain (39).

Consequently, our problem (11) possesses a unique mild solution on the interval $[s, s+T_0]$.

Step 3. Since T_0 does not depend on the initial datum V_0, we can repeat the same procedure on the interval $[s+T_0, s+2T_0]$ as many times as necessary. This implies the existence of a unique mild solution on any interval $[s, T]$.

Step 4. Finally, by the quasi-stationarity relations (26) and (29) we deduce from the uniqueness of the mild solutions that

$$V(t, s, \omega; V_0) = V(t+\tau, s+\tau, \theta_{-\tau}\omega; V_0), \ s \le t, \tau \in \mathbb{R}, \tag{44}$$

for all $V_0 \in \mathcal{H}_\sigma, \ \omega \in \Omega$, as well as

$$V(t, 0, \omega; V_0) = V(t, s, \omega; V(s, 0, \omega; V_0)), \ 0 \le s \le t, \ V_0 \in \mathcal{H}_\sigma, \ \omega \in \Omega.$$

Therefore

$$\phi(t+s, \omega)V_0 = V(t+s, 0, \omega; V_0) = V(t+s, s, \omega; V(s, 0, \omega; V_0)) = V(t, 0, \theta_s\omega, V(s, 0, \omega; V_0)) = \phi(t, \theta_s\omega)\phi(s, \omega)V_0,$$

for $t, s \ge 0$, i.e. ϕ satisfies the cocycle property. The continuity and measurability properties of ϕ follow from those of V. It also follows from (44) that

$$\phi(t-s, \theta_s\omega)V_0 = V(t-s, 0, \theta_s\omega; V_0) = V(t, s, \omega; V_0),$$

which completes the proof. □

Remark 3.4 If $\alpha < 1-\beta$, one can then prove that

$$V(t) \in C((s, T]; \mathcal{H}_\sigma) \quad \text{for any} \quad \sigma < 1-\beta. \tag{45}$$

The point is that in this case using a Gronwall type argument (see, e.g., Henry (1981, Sect.7.1)) one can prove that

$$|V(t)|_\alpha \le C_T \cdot (t-s)^{-1/2}, \quad V_0 \in \mathcal{H}_{\alpha-1/2}, \ t \in (s, T].$$

Using this relation we obtain from (43) that $\mathcal{Q}_1 \in C([s, T_0], D(A^\sigma))$ for $\sigma < 1$. Thus (45) holds.

4. Existence of an invariant manifold

Now we can prove our major result in this paper.

Theorem 4.1 *Assume that hypotheses (A1)–(A4) hold and $\nu > \nu_0$, where*

$$\nu_0 = \frac{\gamma}{\lambda_1} + \frac{2}{\lambda_1}\left[M_F + \lambda_1^{\alpha} M_G + \lambda_1^{\alpha+\beta} M_K\right] \tag{46}$$

and $\lambda_1 > 0$ is the minimal point in the spectrum of A. Then, there exists a random mapping $\Phi(\omega, \cdot) : D(L^{1/2}) \times X_1 \mapsto X_2$ such that

$$\|A^{\sigma}(\Phi(\omega, D_1) - \Phi(\omega, D_2))\|_{X_2} \le C_{\sigma}\|D_1 - D_2\|_{D(L^{1/2})\times X_1}, \tag{47}$$

for all $D_1, D_2 \in D(L^{1/2}) \times X_1$ and for any σ satisfying the inequality

$$\alpha - 1/2 < \sigma < \min(1-\beta, 1/2), \tag{48}$$

where $C_{\sigma} > 0$ is a (deterministic) constant. Moreover, the random surface

$$\mathcal{M}(\omega) = \left\{(v, \bar{v}, \Phi(\omega, v, \bar{v})) \ : \ (v, \bar{v}) \in D(L^{1/2}) \times X_1\right\} \subset \mathcal{H}_{\sigma}, \tag{49}$$

is positively invariant with respect to the cocycle $\phi(t, \omega)$, i.e., $\phi(t, \omega)\mathcal{M}(\omega) \subseteq \mathcal{M}(\theta_t\omega)$. This surface $\mathcal{M}$ is exponentially attracting in the following sense: for any mild solution $V(t)$ to Eq. (11) there exists $V^* \in \mathcal{M}(\omega)$ such that

$$\left[\int_0^{\infty} e^{2\mu t}\, |V(t) - \phi(t, \omega)V^*|_{\alpha}^2\, dt\right]^{1/2} < R_1(\omega) + C|V(0)|_{\sigma}, \tag{50}$$

and also

$$|V(t) - \phi(t, \omega)V^*|_{\sigma} < e^{-\mu t}\left(R_2(\omega) + C|V(0)|_{\sigma}\right), \quad t > 0, \tag{51}$$

where $\mu = (\gamma + \nu\lambda_1)/2$, $R_1(\omega)$ and $R_2(\omega)$ are scalar tempered random variables and C is a deterministic constant.

Remark 4.2 It follows from (51) and from the positive invariance property of $\mathcal{M}(\omega)$ with respect to the cocycle ϕ that

$$\sup\left\{\mathrm{dist}_{\mathcal{H}_{\sigma}}\left(\phi(t, \omega)V_0, \mathcal{M}(\theta_t\omega)\right) \ : V_0 \in B\right\} \le C_B(\omega)e^{-\mu t}, \quad \omega \in \Omega,$$

for any bounded set B from H_{σ}, where σ satisfies (48). Since $R_2(\omega)$ is tempered, relation (51) also implies that

$$\lim_{t\to\infty} \sup\left\{e^{\tilde{\mu}t}\mathrm{dist}_{\mathcal{H}_{\sigma}}\left(\phi(t, \theta_{-t}\omega)V_0, \mathcal{M}(\omega)\right) \ : V_0 \in B\right\} = 0, \quad \omega \in \Omega,$$

for any $\tilde{\mu} < \mu$. Thus the manifold $\mathcal{M}(\omega)$ is uniformly exponentially attracting in the both *forward* and *pullback* sense.

The rest of this section is devoted to the proof of Theorem 4.1. We will proceed in several steps which have been structured in subsections.

4.1. Construction of the manifold $\mathcal{M}$

According to the standard Lyapunov-Perron procedure (see, e.g., Chow and Lu (1988), Chow et al. (1992), Chueshov (1999), Miklavčič (1991)) but modified for stochastic systems (see Chueshov (1995), Chueshov and Girya (1995), Chueshov and Scheutzow (2001)), in order to construct an invariant manifold we should first solve the integral equation

$$V(t,s;\omega) = \mathfrak{B}_D[V;\omega](t,s), \quad t \le s, \tag{52}$$

for every $s \in \mathbb{R}$, where $D \in P\mathcal{H}_\alpha$, $\mathfrak{B}_D[V;\omega] = \mathfrak{I}_D[\mathcal{B}(V);\omega]$, $\mathcal{B}(V)$ is defined in (12), and $\mathfrak{I}_D[V;\omega]$ is given by

$$\begin{aligned}\mathfrak{I}_D[V;\omega](t,s) &= e^{-\mathcal{A}(t-s)}D - \int_t^s e^{-\mathcal{A}(t-\tau)}PV(\tau,s)d\tau \\ &+ \int_{-\infty}^t e^{-\mathcal{A}(t-\tau)}QV(\tau,s)d\tau - e^{-\mathcal{A}(t-s)}P\eta(s,t,\omega) + Q\eta(t,-\infty,\omega) \\ &\equiv \mathfrak{I}_D^{det}[V](t,s) - e^{-\mathcal{A}(t-s)}P\eta(s,t,\omega) + Q\eta(t,-\infty,\omega).\end{aligned} \tag{53}$$

Here as above $Q = I - P$ and P is defined by (15).

For each fixed $s \in \mathbb{R}$, we consider Eq. (52) and the operators $\mathfrak{B}_D$ and $\mathfrak{I}_D$ in the spaces

$$Y_{\alpha,s} = \left\{ V(\cdot) \,:\, e^{\mu(\cdot - s)}V(\cdot) \in L^2(-\infty,s;\mathcal{H}_\alpha) \right\}, \tag{54}$$

where $\mu \in (\gamma, \nu\lambda_1)$ will be chosen later, with the norm

$$|V|_{Y_{\alpha,s}} = \left(\int_{-\infty}^s e^{2\mu(t-s)}|V(t)|_\alpha^2 dt \right)^{1/2}.$$

We first point out some properties of the stochastic term in Eq. (52), which is useful in our considerations.

From relations (28), (31) and (16) we have that the random function

$$\begin{aligned} t \mapsto \Sigma(s,t,\omega) &\equiv -e^{-\mathcal{A}(t-s)}P\eta(s,t,\omega) + Q\eta(t,-\infty,\omega) \\ &= (-T_{t-s}\eta_1(s,t,\omega), \eta_2(t,-\infty,\omega))^T \end{aligned} \tag{55}$$

belongs to the space $Y_{\alpha,s}$ for every $\omega \in \Omega$ and $s \in \mathbb{R}$. It is easy to see from (26) and (29) that

$$\Sigma(s,t+s,\omega) = \Sigma(0,t,\theta_s\omega) \quad \text{for all} \quad t \le 0,\ s \in \mathbb{R},\ \omega \in \Omega.$$

Therefore a simple calculation gives us the following relation between the solutions to the problem (52) for different values of s:

$$V(t+s,s;\omega) = V(t,0;\theta_s\omega) \quad \text{for all} \quad t \le 0,\ s \in \mathbb{R},\ \omega \in \Omega. \tag{56}$$

Consequently, it is sufficient to prove the existence and uniqueness of solutions to (52) only for the case $s = 0$. This observation and also the deterministic argument given in Chueshov (2004) make it possible to prove the following assertion.

Proposition 4.3 *Let $s \in \mathbb{R}$ and $\gamma < \mu < \nu\lambda_1$. Then, for every $D \in P\mathcal{H}_\alpha$ and $\omega \in \Omega$ the operator $\mathfrak{B}_D[\cdot;\omega]$ is continuous from $Y_{\alpha,s}$ into itself and*

$$|\mathfrak{B}_{D_1}[V_1;\omega] - \mathfrak{B}_{D_2}[V_2;\omega]|_{Y_{\alpha,s}} \le |D_1 - D_2|_0 + \kappa_\alpha(\nu,\mu)\cdot|V_1 - V_2|_{Y_{\alpha,s}},\ \omega \in \Omega, \quad (57)$$

for every $D_1, D_2 \in P\mathcal{H}_\alpha$ and $V_1, V_2 \in Y_{\alpha,s}$, where

$$\kappa_\alpha(\nu,\mu) = \frac{M_F}{\mu-\gamma} + \frac{\lambda_1^\alpha M_G + \lambda_1^{\alpha+\beta} M_K}{\nu\lambda_1 - \mu}. \quad (58)$$

Now we take $\mu = (\gamma + \nu\lambda_1)/2$. In this case $\kappa_\alpha(\nu,\mu) < 1$ under the condition $\nu > \nu_0$, where ν_0 is given by (46). Thus $\mathfrak{B}_D[\cdot;\omega]$ is a contraction in $Y_{\alpha,s}$ and hence Eq. (52) has a unique solution $V(\cdot,s) \equiv V(\cdot,s;\omega,D)$ in the space $Y_{\alpha,s}$ for each $\omega \in \Omega$. Using the same (standard) argument as in the deterministic case (see Chueshov (2004)) one can show that this solution $V(\cdot,s)$ possesses the properties

$$V(\cdot) \equiv V(\cdot,s) \in C((-\infty,s],\mathcal{H}_\sigma), \quad \sigma < \min(1-\beta, 1/2), \quad (59)$$

and

$$\sup_{t\le s}\left\{e^{\mu(t-s)}|V(t,s;\omega,D_1) - V(t,s;\omega,D_2)|_\sigma\right\} \le C_\sigma|D_1 - D_2|_0 \quad (60)$$

for any $D_1, D_2 \in P\mathcal{H}_\alpha$ and $\omega \in \Omega$, where C_σ is a positive constant. Moreover, it follows directly from (52) that for every $r \in (-\infty, s)$ and for almost all $t \in [r,s]$ the function $V(\cdot,s)$ satisfies the relation

$$V(t,s) = e^{-\mathcal{A}(t-r)}V(r,s) + \int_r^t e^{-\mathcal{A}(t-\tau)}\mathcal{B}(V(\tau,s))d\tau + \eta(t,r). \quad (61)$$

Now for every $s \in \mathbb{R}$ we define $\Phi_s : \Omega \times D(L^{1/2}) \times X_1 \to X_2$ as

$$\Phi_s(\omega,D) = \int_{-\inf ty}^s e^{-\nu A(s-\tau)}(G(V(\tau,s)) + K(v(\tau,s),\bar{v}(\tau,s)))d\tau + \eta_2(s,-\infty), \quad (62)$$

where $V(t,s) = (v(t,s),\bar{v}(t,s),u(t,s))$ solves the integral equation (52).

It is easy to see from (56) that $\Phi_s(\omega,D) = \Phi_0(\theta_s\omega,D) \equiv \Phi(\theta_s\omega,D)$, i.e. $s \mapsto \Phi_s(\omega,D)$ is a stationary process. Therefore, it follows from (52) and (61) that the random surface $\mathcal{M}(\omega)$ given by (49) is positively invariant with respect to the cocycle ϕ. Moreover, the relation (60) implies the Lipschitz property (47).

4.2. Tracking properties

We will use the method developed in Miklavčič (1991) for the proof of a tracking property for inertial manifolds in the deterministic case.

Let $V_0 = (v_0, v_1, u_0) \in \mathcal{H}_\sigma$, where σ satisfies (48), and let $V(t) \equiv V(t, 0, \omega; V_0)$ be a mild solution to (11) for $s = 0$. We extend $V(t)$ on the semi-axis $(-\infty, 0]$ by the formula $V(t) = (v_0, v_1, (1 + |t|A)^{-1}u_0)$. It is easy to see that

$$V(\cdot) \in Y_{\alpha,0} \cap C((-\infty, 0], \mathcal{H}_\sigma)$$

and

$$|V|^2_{Y_{\alpha,0}} \equiv \int_{-\infty}^{0} e^{2\mu t}|V(t)|^2_\alpha dt \leq C|V_0|^2_\sigma. \tag{63}$$

Now we consider the following space

$$\mathcal{Z} = \left\{ Z(\cdot) \,:\, |Z|^2_{\mathcal{Z}} \equiv \int_{-\infty}^{\infty} e^{2\mu t}|Z(t)|^2_\alpha dt < \infty \right\}$$

and define the random function

$$Z_0(t, \omega) = \begin{cases} -V(t) + \mathfrak{B}_{PV_0}[V; \omega](t, 0), & \text{for } t \leq 0; \\ \\ e^{-\mathcal{A}t}\left[-V_0 + \mathfrak{B}_{PV_0}[V; \omega](0, 0)\right], & \text{for } t > 0, \end{cases} \tag{64}$$

where $\mathfrak{B}$ is the same as in (52). Below we need the following properties of the random function $Z_0(t, \omega)$.

Lemma 4.4 *For every $\omega \in \Omega$ the random function $Z_0(t, \omega)$ belongs to $\mathcal{Z}$. Moreover for every σ satisfying (48) there exist a deterministic constant C and scalar tempered random variables $\widetilde{R}_1(\omega)$ and $\widetilde{R}_2(\omega)$ such that*

$$|Z_0|_{\mathcal{Z}} \leq \widetilde{R}_1(\omega) + C|V_0|_\sigma \quad \text{and} \quad \sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|Z_0(t)|_\sigma \right\} \leq \widetilde{R}_2(\omega) + C|V_0|_\sigma. \tag{65}$$

Proof. We split $Z_0(t, \omega)$ into deterministic and stochastic parts, $Z_0(t, \omega) = Z_0^{det}(t) + Z_0^{st}(t, \omega)$, where

$$Z_0^{det}(t) = \begin{cases} -V(t) + \mathfrak{I}_{PV_0}[\mathcal{B}(V)](t, 0), & \text{for } t \leq 0; \\ \\ e^{-\mathcal{A}t}\left[-V_0 + \mathfrak{I}_{PV_0}[\mathcal{B}(V)](0, 0)\right], & \text{for } t > 0, \end{cases}$$

and

$$Z_0^{st}(t, \omega) = \begin{cases} \left(-T_t\eta_1(0, t), \eta_2(t, 0)\right)^T, & \text{for } t \leq 0; \\ \\ \left(0, 0, e^{-\nu A t}\eta_2(0, -\infty)\right)^T, & \text{for } t > 0, \end{cases} \tag{66}$$

Since, by (16) and (18)

$$R_1^*(\omega) \equiv |Z_0^{st}(\omega)|^2_{\mathcal{Z}} \le \int_{-\infty}^0 e^{2(\mu-\gamma)t}\left[|\eta_1(0,t)|^2_{D(L^{1/2})\times X_1} + \|A^\alpha \eta_2(t,-\infty)\|^2_{X_2}\right] dt$$
$$+\|A^\alpha \eta_2(0,-\infty)\|^2_{X_2}$$

and

$$R_2^*(\omega) \equiv \sup_{t\in\mathbb{R}} \left\{ e^{\mu t} |Z_0^{st}(t,\omega)|_\sigma \right\}$$
$$\le c_0 \sup_{t\in\mathbb{R}} \left\{ e^{(\mu-\gamma)t}\left[|\eta_1(0,t)|_{D(L^{1/2})\times X_1} + \|A^\sigma \eta_2(t,-\infty)\|_{X_2}\right]\right\},$$

it follows from (27), (28) and (31) that $R_1^*(\omega)$ and $R_2^*(\omega)$ are tempered random variables. Therefore, estimating the deterministic part $Z_0^{det}(t)$ by the standard method we arrive at the estimates (65) with $\widetilde{R}_i(\omega) = C_1 + C_2 R_i^*(\omega)$, where C_1 and C_2 are deterministic constants. □

Now we define an integral operator $\mathfrak{R} : \mathcal{Z} \mapsto \mathcal{Z}$ by the formula

$$\mathfrak{R}[Z](t) = Z_0(t) + \int_{-\infty}^t e^{-\mathcal{A}(t-\tau)} Q\left[\mathcal{B}(Z(\tau)+V(\tau)) - \mathcal{B}(V(\tau))\right] d\tau$$
$$- \int_t^\infty e^{-\mathcal{A}(t-\tau)} P\left[\mathcal{B}(Z(\tau)+V(\tau)) - \mathcal{B}(V(\tau))\right] d\tau. \quad (67)$$

Let us prove that $\mathfrak{R}$ is a contraction in $\mathcal{Z}$.

By (3) and (16) we have that

$$e^{\mu t}|P\left(\mathfrak{R}[Z_1](t) - \mathfrak{R}[Z_2](t)\right)|_\alpha \le M_F \int_t^\infty e^{(\mu-\gamma)(t-\tau)} \cdot e^{\mu\tau}\left|Z_1(\tau) - Z_2(\tau)\right|_\alpha d\tau$$
$$\equiv \int_{-\infty}^\infty e(t-\tau) f(\tau) d\tau, \quad (68)$$

where $e(t) = 0$ for $t > 0$, $e(t) = M_F e^{(\mu-\gamma)t}$ for $t \le 0$ and $f(t) = e^{\mu t}\left|Z_1(t) - Z_2(t)\right|_\alpha$, $t \in \mathbb{R}$. Thus, using the Fourier transformation and the Plancherel formula (cf. Lemma 2.2 in Chueshov (2004)) we obtain that

$$|P\left(\mathfrak{R}[Z_1] - \mathfrak{R}[Z_2]\right)|_{\mathcal{Z}} \le \frac{M_F}{\mu-\gamma} \cdot |Z_1 - Z_2|_{\mathcal{Z}}.$$

Similarly,

$$Q\left(\mathfrak{R}[Z_1](t) - \mathfrak{R}[Z_2](t)\right) = (0; 0; \mathcal{Q}_1(t) + \mathcal{Q}_2(t)), \quad (69)$$

where

$$\mathcal{Q}_1(t) = \int_{-\infty}^t e^{-\nu A(t-\tau)}\left[G(Z_1(\tau)+V(\tau)) - G(Z_2(\tau)+V(\tau))\right] d\tau,$$
$$\mathcal{Q}_2(t) = \int_{-\infty}^t (e^{-\nu A(t-\tau)} A^\beta Q\left[A^{-\beta}\left(K(P[Z_1(\tau)+V(\tau)]) - K(P[Z_2(\tau)+V(\tau)])\right)\right]) d\tau.$$

To estimate $\mathcal{Q}_1$ and $\mathcal{Q}_2$ we use the following result from Chueshov (2004).

Lemma 4.5 *Let $e^{-\mathcal{A}t}$ be a strongly continuous semigroup in a Hilbert space H with a self-adjoint and positive generator $\mathcal{A} = \mathcal{A}^* > 0$. Let $\lambda_{min} > 0$ be the minimal point in the spectrum of $\mathcal{A}$. The following assertions hold: For any $0 \le \beta \le 1$ and $\mu \ge 0$ the mapping $f \in L_2(\mathbb{R}; H) \mapsto \mathcal{I}^\beta[f] \in L_2(\mathbb{R}; D(\mathcal{A}^{1-\beta}))$, where*

$$\mathcal{I}^\beta[f](t) = \int_{-\infty}^t e^{-\mathcal{A}(t-\tau)}(\mathcal{A}+\mu)^\beta f(\tau)d\tau, \quad t \in \mathbb{R},$$

is continuous, and the estimate

$$\int_{\mathbb{R}} \|(\mathcal{A}+\mu)^\alpha \mathcal{I}^\beta[f](t)\|_H^2 dt \le \frac{(\lambda_{min}+\mu)^{2(\alpha+\beta)}}{\lambda_{min}^2} \int_{\mathbb{R}} \|f(t)\|_H^2 dt$$

holds for any $0 \le \beta \le 1$, $-\beta \le \alpha \le 1-\beta$ and $\mu \ge 0$.

Applying Lemma 4.5 with $\mathcal{A} = \nu A - \mu$ we obtain that

$$|\mathcal{Q}_1|_{\mathcal{Z}} \le \frac{\lambda_1^\alpha M_G}{\nu\lambda_1 - \mu} |Z_1 - Z_2|_{\mathcal{Z}} \quad \text{and} \quad |\mathcal{Q}_2|_{\mathcal{Z}} \le \frac{\lambda_1^{\alpha+\beta} M_K}{\nu\lambda_1 - \mu} |Z_1 - Z_2|_{\mathcal{Z}}\,.$$

Since $\mu = (\gamma + \nu\lambda_1)/2$, we have that

$$|\mathfrak{R}[Z_1] - \mathfrak{R}[Z_2]|_{\mathcal{Z}} \le q \cdot |Z_1 - Z_2|_{\mathcal{Z}} \quad \text{for every} \quad Z_1, Z_2 \in \mathcal{Z}. \tag{70}$$

Here $q = \kappa_\alpha(\nu, (\gamma+\nu\lambda_1)/2) < 1$ under the condition $\nu > \nu_0$, where κ_α and ν_0 are given by (58) and (46). Thus by the contraction principle there exists a unique solution $Z \in \mathcal{Z}$ to the equation $Z = \mathfrak{R}[Z]$ in $\mathcal{Z}$.

Now using the same calculation as in Chueshov (2004) and Miklavčič (1991) we can conclude that the function $\widetilde{V}(t) = Z(t) + V(t)$, where $Z \in \mathcal{Z}$ solves the equation $Z = \mathfrak{R}[Z]$, satisfies the relation

$$\widetilde{V}(t) = \begin{cases} \mathfrak{B}_{P\widetilde{V}(0)}[\widetilde{V}, \omega](t, 0), & \text{if } t \le 0; \\ \\ \phi(t, 0, \omega)\widetilde{V}(0), & \text{if } t > 0. \end{cases} \tag{71}$$

In particular, $\widetilde{V}(0) = \mathfrak{B}_{P\widetilde{V}(0)}[\widetilde{V}, \omega](0,0)$ and, therefore, by the definition of the operator $\mathfrak{B}$ we obtain that

$$\widetilde{V}(0) = P\widetilde{V}(0) + \int_{-\infty}^0 e^{\mathcal{A}\tau} Q\mathcal{B}(\widetilde{V}(\tau))d\tau + Q\eta(0, -\infty).$$

By (62) this implies that $\widetilde{V}(0) = \left(P\widetilde{V}(0), \Phi(\omega, P\widetilde{V}(0))\right)$. Therefore

$$\widetilde{V}(t) = \phi(t, 0, \omega)\widetilde{V}(0) \in \mathcal{M}(\theta_t\omega) \quad \text{for} \quad t \ge 0.$$

Thus to complete the proof we only need to establish (50) and (51).

Since $\widetilde{V}(t) = Z(t) + V(t)$ and

$$Z(t) = \mathfrak{R}[Z](t) = Z_0(t) + \mathfrak{R}[Z](t) - \mathfrak{R}[0](t), \tag{72}$$

from (65) and (70) we obtain the relation

$$|Z|_{\mathcal{Z}} \leq (1-q)^{-1} \cdot |Z_0|_{\mathcal{Z}} \leq (1-q)^{-1} \cdot \left(\widetilde{R}_1(\omega) + C|V_0|_\sigma \right), \tag{73}$$

which implies (50).

Now we prove (51). Since $|PU|_\sigma = |PU|_\alpha$, from (68) we have that

$$\begin{aligned} e^{\mu t}|P\left(\mathfrak{R}[Z](t) - \mathfrak{R}[0](t)\right)|_\sigma &\leq M_F \int_t^\infty e^{(\mu-\gamma)(t-\tau)} \cdot e^{\mu\tau}\, |Z(\tau)|_\alpha \, d\tau \\ &\leq M_F \left[\int_t^\infty e^{2(\mu-\gamma)(t-\tau)} d\tau \right]^{1/2} \cdot |Z|_{\mathcal{Z}} = \frac{M_F}{\sqrt{2(\mu-\gamma)}} \cdot |Z|_{\mathcal{Z}}. \end{aligned}$$

Thus

$$\sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|P\left(\mathfrak{R}[Z](t) - \mathfrak{R}[0](t)\right)|_\sigma \right\} \leq \frac{M_F}{\sqrt{2(\mu-\gamma)}}\ cdot\, |Z|_{\mathcal{Z}}. \tag{74}$$

Similarly, using (69) we have that

$$\begin{aligned} &|Q\left(\mathfrak{R}[Z](t) - \mathfrak{R}[0](t)\right)|_\sigma \\ &\leq M_G \int_{-\infty}^t \|A^\sigma e^{-\nu A(t-\tau)}\| \cdot |Z(\tau)|_\alpha d\tau \\ &+ M_K \int_{-\infty}^t \|A^{\sigma+\beta} e^{-\nu A(t-\tau)}\| \cdot |PZ(\tau)|_0 d\tau \\ &\leq a_1 \cdot e^{-\mu t} \cdot |Z|_{\mathcal{Z}} + a_2 \cdot e^{-\mu t} \cdot \sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|PZ(t)|_\sigma \right\}, \end{aligned}$$

where

$$\begin{aligned} a_1 &= M_G \sup_{t\in\mathbb{R}} \left[\int_{-\infty}^t \|A^\sigma e^{-(\nu A-\mu)(t-\tau)}\|^2 d\tau \right]^{1/2} < \infty, \\ a_2 &= M_K \sup_{t\in\mathbb{R}} \int_{-\infty}^t \|A^{\sigma+\beta} e^{-(\nu A-\mu)(t-\tau)}\| d\tau < \infty, \end{aligned}$$

(the finiteness of a_1 and a_2 follows from (18) by a straightforward computation). From (72) and (74) we have that

$$\sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|PZ(t)|_\sigma \right\} \leq \sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|Z_0(t)|_\sigma \right\} + \frac{M_F}{\sqrt{2(\mu-\gamma)}} \cdot |Z|_{\mathcal{Z}}.$$

Therefore

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|Q\left(\mathfrak{R}[Z](t) - \mathfrak{R}[0](t)\right)|_\sigma \right\} \\ &\leq a_2 \sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|Z_0(t)|_\sigma \right\} + \left[a_1 + \frac{a_2 M_F}{\sqrt{2(\mu-\gamma)}} \right] \cdot |Z|_{\mathcal{Z}}. \end{aligned} \tag{75}$$

Consequently, using relations (72), (74) and (65) we obtain that

$$\sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|Z(t)|_\sigma \right\} \leq (1+a_2)\widetilde{R}_2(\omega) + \left[a_1 + \frac{(1+a_2)M_F}{\sqrt{2(\mu-\gamma)}} \right] \cdot |Z|_{\mathcal{Z}}.$$

Thus by (73) we have

$$\sup_{t\in\mathbb{R}} \left\{ e^{\mu t}|Z(t)|_\sigma \right\} \le c_0(\widetilde{R}_1(\omega)+\widetilde{R}_2(\omega)) + c_1|V_0|_\sigma .$$

with appropriate (deterministic) constants c_0 and c_1. This implies (51) and completes the proof of Theorem 4.1.

Remark 4.6 The existence of a positively invariant manifold of the form (49) can be also established in the case $\gamma = 0$ under the the condition $\nu > \nu_0$, where ν_0 is given by (46) with $\gamma = 0$. The point is that we can introduce artificial small damping in problem (1) with $\gamma = 0$ by the considering the equation

$$v_{tt} + \varepsilon v_t + Lv = F_\varepsilon(v, v_t, u) + \dot{W}_1, \quad \text{in } X_1, \tag{76}$$

where $F_\varepsilon(v, v_t, u) = F(v, v_t, u) + \varepsilon v_t$. We choose ε such that the relation $\nu > \nu_0$ remains true, where ν_0 is defined by (46) with $\gamma = \varepsilon$ and with the term $M_F + \varepsilon$ instead of M_F. Now we can apply Theorem 4.1 to problem (76) and (2).

5. The reduced system

Assume the hypotheses of Theorem 4.1 hold and let $\Phi \equiv \Phi_0$ be given by (62) with $s = 0$. Consider the problem

$$\begin{cases} v_{tt} + \gamma v_t + Lv = F(v, v_t, \Phi(\theta_t\omega, v, v_t)) + \dot{W}_1, \ t > s, \quad \text{in } X_1, \\ v|_{t=s} = v_0, \ v_t|_{t=s} = v_1, \end{cases} \tag{77}$$

and define its *mild solution* on the interval $[s, T]$ as a random function

$$D(t,\omega) \equiv D(t,s;\omega,v_0,v_1) = (v(t,\omega), v_t(t,\omega)) \in C([s,T], D(L^{1/2}) \times X_1) \tag{78}$$

such that

$$\int_s^T \|A^\alpha \Phi(\theta_t, v(t,\omega), v_t(t,\omega))\|_{X_2}^2 dt < \infty, \quad \omega \in \Omega, \tag{79}$$

and

$$\begin{bmatrix} v(t,\omega) \\ v_t(t,\omega) \end{bmatrix} = T_{t-s} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} + \int_s^t T_{t-\tau} \begin{bmatrix} 0 \\ F(v(\tau), v_t(\tau), \Phi(\omega, v(\tau), v_t(\tau))) \end{bmatrix} d\tau + \eta_1(t,s)$$

for almost all $t \in [s, T]$ and $\omega \in \Omega$, where T_t is the evolution group generated by (14) and $\eta_1(t, s)$ is given by (21).

Proposition 5.1 *Let $v_0 \in D(L^{1/2})$ and $v_1 \in X_1$. Then under the conditions of Theorem 4.1 problem (77) has a mild solution on any interval $[s, T]$. If $\alpha < \min(1 - \beta, 1/2)$, then this solution is unique and any mild solution $(\hat{v}(t), \hat{v}_t(t))$ to problem (77) generates a mild solution to problem (1) and (2) by the formula*

$$(v(t), v_t(t), u(t)) = (\hat{v}(t), \hat{v}_t(t), \Phi(\theta_t\omega, \hat{v}(t), \hat{v}_t(t))). \tag{80}$$

Moreover, in this case the manifold $\mathcal{M}$ is invariant with respect to the cocycle ϕ generated by (1) and (2).

Proof. Let $V(t) = (v(t), v_t(t), u(t))$ be a mild solution to problem (11) with the initial data $V_0 = (v_0, v_1, \Phi(\omega, v_0, v_1))$. Since $\mathcal{M}$ given by (49) is positively invariant, we have that

$$PV(t) = (v(t), v_t(t), 0) \equiv (D(t), 0) \quad \text{and} \quad QV(t) = (0, 0, \Phi(\theta_t\omega, D(t))).$$

Thanks to Theorem 3.3 $D(t)$ possesses the property (78). We also have that

$$\int_s^T |QV(t)|_\alpha^2 dt \leq \int_s^T |V(t)|_\alpha^2 dt < \infty.$$

Thus (79) holds. Consequently $D(t)$ is a mild solution to (77).

If $\alpha < \min(1-\beta, 1/2)$, then by Theorem 4.1 with $\sigma = \alpha$ we have that

$$\|A^\alpha(\Phi(\omega, D_1) - \Phi(\omega, D_2))\|_{X_2} \leq C\|D_1 - D_2\|_{D(L^{1/2})\times X_1},$$

for $D_i \in D(L^{1/2}) \times X_1$. This implies that the function

$$D \mapsto F_\Phi(\omega, D) := F(v_0, v_1, \Phi(\omega, v_0, v_1)), \quad D = (v_0, v_1) \in D(L^{1/2}) \times X_1,$$

is globally Lipschitz, i.e.

$$\|F_\Phi(\omega, D_1) - F_\Phi(\omega, D_2))\|_{X_1} \leq C\|D_1 - D_2\|_{D(L^{1/2})\times X_1}, \tag{81}$$

with $D_i \in D(L^{1/2}) \times X_1$. Therefore a Gronwall type argument gives us the uniqueness of solutions to (77). Relation (80) easily follows from the uniqueness theorem for (77).

Property (81) makes it also possible to solve (77) backwards in time and, hence, one can prove that $\mathcal{M}$ is invariant with respect to the cocycle $\phi(t, \omega)$. □

Observe now that Theorem 4.1 implies that for any mild solution $V(t) = (v(t), v_t(t), u(t))$ to problem (1) and (2) with initial data $V_0 \in \mathcal{H}_\sigma$, where σ satisfies (48), there exists a mild solution $D(t) = (\hat{v}(t), \hat{v}_t(t))$ to reduced problem (77) such that

$$\|L^{1/2}(v(t) - \hat{v}(t)\|_{X_1}^2 + \|v_t(t) - \hat{v}_t(t)\|_{X_1}^2 + \|A^\sigma\left[u(t) - \Phi(\omega, \hat{v}(t), \hat{v}_t(t))\right]\|_{X_2}^2 \leq Ce^{-\mu(t-s)}$$

for any $t \geq s$ with positive constants C and μ. Thus under the conditions of Theorem 4.1, the long-time behaviour of solutions to (1) and (2) can be described completely by solutions to problem (77). Moreover, under the condition $\alpha < \min(1-\beta, 1/2)$, due to relation (80), every limiting regime of the reduced system (77) is realized in the coupled system (1) and (2).

6. Distance between random and deterministic manifolds

Theorem 4.1 can be also applied to the deterministic version of problem (1) and (2):

$$v_{tt} + \gamma v_t + Lv = F(v, v_t, u), \quad \text{in } X_1, \tag{82}$$

$$u_t + \nu A u = G(v, v_t, u) + K(v, v_t), \quad \text{in } X_2. \tag{83}$$

In this case Theorem 4.1 give us the existence of a (deterministic) invariant exponentially attracting manifold $\mathcal{M}^{det}$ in the space $\mathcal{H}_\sigma$ of the form

$$\mathcal{M}^{det} = \left\{ (v, \bar{v}, \Phi^{det}(v, \bar{v})) \ : \ (v, \bar{v}) \in D(L^{1/2}) \times X_1 \right\}, \tag{84}$$

where $\Phi^{det} \ : \ D(L^{1/2}) \times X_1 \mapsto D(A^\sigma) \subset X_2$ is a globally Lipschitz mapping and σ satisfies (48).

Our goal in this section is to estimate the mean value distance between the deterministic ($\mathcal{M}^{det}$) and random ($\mathcal{M}(\omega)$) manifolds.

Theorem 6.1 *There exist a positive constant C such that*

$$\mathbb{E}\left\{ \sup_{D \in D(L^{1/2}) \times X_1} \| A^\sigma (\Phi(\cdot, D) - \Phi^{det}(D)) \|^2_{X_2} \right\} \le C \left(\operatorname{tr} K_1 + \operatorname{tr} K_2 A^{2\alpha - 1} \right), \tag{85}$$

where σ satisfies (48). Thus, the random manifold $\mathcal{M}(\omega)$ is close to its deterministic counterpart when $\operatorname{tr} K_1 + \operatorname{tr} K_2 A^{2\alpha-1}$ becomes small.

Proof. It follows from the definition (see (62)) of the functions Φ and Φ^{det} that

$$\Phi(\omega, D) - \Phi^{det}(D) = \int_{-\infty}^{0} e^{\nu A \tau} \left[G(V^{st}(\tau)) - G(V^{det}(\tau)) \right] d\tau$$
$$+ \int_{-\infty}^{0} e^{\nu A \tau} \left[K(v^{st}(\tau), \bar{v}^{st}(\tau)) - K(v^{det}(\tau), \bar{v}^{det}(\tau)) \right] d\tau + \eta_2(0, -\infty),$$

where $V^{st}(t) \equiv (v^{st}(t), \bar{v}^{st}(t), u^{st}(t))$ and $V^{det}(t) \equiv (v^{det}(t), \bar{v}^{det}(t), u^{det}(t))$ are defined on the semi-axis $(-\infty, 0]$ and solve the equations

$$V^{st}(t) = \mathfrak{I}_D[\mathcal{B}(V^{st}); \omega](t, 0) \text{ and } V^{det}(t) = \mathfrak{I}^{det}_D[\mathcal{B}(V^{det})](t, 0), \tag{86}$$

where $\mathfrak{I}_D$ and $\mathfrak{I}^{det}_D$ are defined as in (53).

Using the same method as in the proof of relation (75) we can conclude that

$$\| A^\sigma (\Phi(\cdot, D) - \Phi^{det}(D)) \|_{X_2} \le \| A^\sigma \eta_2(0, -\infty) \|_{X_2} + a_1 |V^{st} - V^{det}|_{Y_{\alpha,0}}$$
$$+ a_2 \sup_{t \le 0} \left\{ e^{\mu t} |P(V^{st}(t) - V^{det}(t))|_0 \right\}, \tag{87}$$

where a_1 and a_2 are deterministic constants.

Now using (86) and the structure of the operators $\mathfrak{I}_D$ and $\mathfrak{I}_D^{det}$ we can write the following estimate

$$
\begin{aligned}
&|P(V^{st}(t) - V^{det}(t))|_0 \\
&\le M_F \int_t^0 e^{-\gamma(t-\tau)} \left|V^{st}(\tau) - V^{det}(\tau)\right|_\alpha d\tau + e^{\gamma t} \left|\eta_1(0,t)\right|_{D(L^{1/2})\times X_1}
\end{aligned}
$$

for $t \le 0$, which implies that

$$
\begin{aligned}
&\sup_{t\le 0} \left\{ e^{\mu t} |P(V^{st}(t) - V^{det}(t))|_0 \right\} \\
&\le C|V^{st} - V^{det}|_{Y_{\alpha,0}} + \sup_{t\le 0} \left\{ e^{(\mu-\gamma)t} \left|\eta_1(0,t)\right|_{D(L^{1/2})\times X_1} \right\}.
\end{aligned}
$$

Therefore from (87) we have that

$$
\begin{aligned}
&\|A^\sigma(\Phi(\cdot, D) - \Phi^{det}(D))\|_{X_2}^2 \\
&\le b_1 |V^{st} - V^{det}|_{Y_{\alpha,0}}^2 + b_2 \Delta_1(\omega; \eta_1, \eta_2) + 2\|A^\sigma \eta_2(0, -\infty)\|_{X_2}^2, \qquad (88)
\end{aligned}
$$

where b_1 and b_2 are deterministic constants and

$$
\Delta_1(\omega; \eta_1, \eta_2) = \sup_{t\le 0} \left\{ e^{2(\mu-\gamma)t} \left|\eta_1(0,t)\right|_{D(L^{1/2})\times X_1}^2 \right\}. \qquad (89)
$$

By (86) we have that

$$
|V^{st} - V^{det}|_{Y_{\alpha,0}} \le |\mathfrak{B}_D[V^{st}; \omega](\cdot, 0) - \mathfrak{B}_D[V^{det}; \omega](\cdot, 0)|_{Y_{\alpha,0}} + |\Sigma(0, \cdot)|_{Y_{\alpha,0}},
$$

where $\mathfrak{B}_D[V; \omega](t, 0)$ is the same as in (52) and $\Sigma(s,t)$ is given by (55). Thus by Proposition 4.3 we have that

$$
|V^{st} - V^{det}|_{Y_{\alpha,0}} \le (1-q)^{-1} |\Sigma(0, \cdot)|_{Y_{\alpha,0}},
$$

where $q = \kappa_\alpha(\nu, (\gamma + \nu\lambda_1)/2) < 1$. Therefore, using (88) we obtain the estimate

$$
\begin{aligned}
&\|A^\sigma(\Phi(\cdot, D) - \Phi^{det}(D))\|_{X_2}^2 \\
&\le 2\|A^\sigma \eta_2(0, -\infty)\|_{X_2}^2 + \frac{b_1}{(1-q)^2} |\Sigma(0, \cdot)|_{Y_{\alpha,0}}^2 + b_2 \Delta_1(\omega; \eta_1, \eta_2). \qquad (90)
\end{aligned}
$$

It easily follows from relations (23) and (24) and from the definition of $\Sigma(s,t)$ (see (55)) that

$$
\mathbb{E}\|A^\sigma \eta_2(0, -\infty)\|_{X_2}^2 \le C_1 \cdot \operatorname{tr} K_2 A^{2\alpha-1} \qquad (91)
$$

and

$$
\mathbb{E}|\Sigma(0, \cdot)|_{Y_{\alpha,0}}^2 \le C_2 \left(\operatorname{tr} K_1 + \operatorname{tr} K_2 A^{2\alpha-1} \right). \qquad (92)
$$

Now we calculate $\mathbb{E}\Delta_1(\omega;\eta_1,\eta_2)$. From (27) and (17) we obtain that

$$|\eta_1(0,t)|^2_{D(L^{1/2})\times X_1} \le 2\,|\eta_1(0,-\infty)|^2_{D(L^{1/2})\times X_1} + 2\,|\eta_1(t,-\infty)|^2_{D(L^{1/2})\times X_1}.$$

Hence by (24) we have that

$$\mathbb{E}\Delta_1(\cdot;\eta_1,\eta_2) \le C\mathrm{tr}\,K_1 + 2\mathbb{E}\sup_{t\le 0}\left\{e^{2(\mu-\gamma)t}\,|\eta_1(t,-\infty)|^2_{D(L^{1/2})\times X_1}\right\}.$$

Since

$$\begin{aligned}&\mathbb{E}\sup_{t\le 0}\left\{e^{2(\mu-\gamma)t}\,|\eta_1(t,-\infty)|^2_{D(L^{1/2})\times X_1}\right\}\\ &\quad\le \sum_{n=1}^{\infty} e^{-2(\mu-\gamma)(n-1)}\mathbb{E}\sup_{0\le t\le 1}\left\{|\eta_1(-n+t,-\infty)|^2_{D(L^{1/2})\times X_1}\right\}\end{aligned}$$

and, by (26), $\eta_1(-n+t,-\infty,\omega)=\eta_1(t,-\infty.\theta_{-n}\omega)$, we obtain from the invariance of the probability measure with respect to θ_t that

$$\mathbb{E}\sup_{t\le 0}\left\{e^{2(\mu-\gamma)t}\,|\eta_1(t,-\infty)|^2_{D(L^{1/2})\times X_1}\right\} \le C\mathbb{E}\sup_{0\le t\le 1}\left\{|\eta_1(t,-\infty)|^2_{D(L^{1/2})\times X_1}\right\}.$$

Using (27) we have that $\eta_1(t,-\infty)=\eta_1(t,0)+T_t\eta_1(0,-\infty)$. Consequently, by (24) and (25) we obtain that

$$\mathbb{E}\sup_{t\le 0}\left\{e^{2(\mu-\gamma)t}\,|\eta_1(t,-\infty)|^2_{D(L^{1/2})\times X_1}\right\} \le C\mathrm{tr}\,K_1$$

and hence $\mathbb{E}\Delta_1(\cdot;\eta_1,\eta_2)\le C\mathrm{tr}\,K_1$. Therefore (85) follows from (90) and (91). □

Conclusions

We have proved in this paper that the long time behaviour of coupled non-linear parabolic-hyperbolic partial differential equations perturbed by additive noise can be reduced to the analysis of a corresponding hyperbolic random equation with a modified nonlinear term. The main tool is the construction of an invariant random manifold for the coupled system which is given by the graph of a suitable Lipschitz mapping. Of course, one could consider other different expressions for the noisy terms in the equations (multiplicative, cylindrical, etc). We plan to investigate the possibility of doing a similar reduction in the future.

Acknowledgments

We would like to thank the referees for their useful suggestions and comments.
T. Caraballo and J.A. Langa have been partially supported by M.C.Y.T. (Spain) and FEDER (European Community) Project BFM 2002-03068. I. Chueshov has been partially supported by the INTAS Grant 2000-899.

References

L. Arnold, *Random Dynamical Systems.* Springer, Berlin-Heidelberg-New York, 1998.

V. Barbu and G. Da Prato, The stochastic nonlinear damped wave equation, *Appl. Math. Optim.* **46** (2002), 125–141.

A. Bensoussan and F. Flandoli, Stochastic inertial manifolds, *Stochastics and Stoch. Reports* **53** (1995), 13–39.

R. Carmona and D. Nualart, Random non-linear wave equation: smoothness of solutions, *Probab. Theory Rel. Fields*, **95** (1993), 87-102.

S. Cerrai, *Second order PDE's in finite and infinite dimension*, Lect. Notes Math. 1762, Springer, Berlin-Heidelberg, 2001.

D. S. Chandrasekharaiah, Hyperbolic thermoelastisity: a review of recent literature, *Appl. Mech. Rev.* **51** (1998), 705–729.

P. L. Chow, Thermoelastic wave propagation in a random medium and some related problems, *Int. J. Engng. Sci.* **11** (1973), 253–971.

S.-N. Chow and K. Lu, Invariant manifolds for flows in Banach spaces, *J. Diff. Eq.* **74** (1988), 285–317.

——, ——, and G.R. Sell, Smoothness of inertial manifolds, *J. Math. Anal. Appl.* **169** (1992), 283–312.

I. Chueshov, Approximate inertial manifolds of exponential order for semilinear parabolic equations subjected to additive white noise, *J. Dynam. Diff. Eqs.* **7** (1995), 549–566.

——, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems*, Acta, Kharkov, 1999, in Russian; English translation: Acta, Kharkov, 2002; see also http://www.emis.de/monographs/Chueshov/

——, A reduction principle for coupled nonlinear parabolic-hyperbolic PDE, *J. Evolution Equations* (2004), in press.

——and T. Girya, Inertial manifolds and forms for semilinear parabolic equations subjected to additive white noise, *Letters in Math. Phys.* **34** (1995), 69–76.

——and M. Scheutzow, Inertial manifolds and forms for stochastically perturbed retarded semilinear parabolic equations, *J. Dynam. Diff. Eqs.* **13** (2001), 355–380.

P. Constantin, C. Foias, B.Nicolaenko, R. Temam, *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations*, Springer, Berlin-Heidelberg-New York, 1989.

R. Dalang and N. Frangos, The stochastic wave equation in two spatial dimensions, *Ann. Probab.* **26** (1998), 187-212.

G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, 1992.

——, *Ergodicity for Infinite Dimensional Systems*, Cambridge Univ. Press,1996.

J. Duan, K. Lu, and B. Schmalfuß, Invariant manifolds for stochastic partial differential equations, *Ann. Probab.* **31** (2003), 2109-2135.

C. Foias, G.R. Sell, and R. Temam, Inertial manifolds for nonlinear evolution equations, *J. Diff. Eq.* **73** (1988), 309–353.

D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer, Berlin-Heidelberg-New York, 1981.

M. Hori, Statistical theory of effective electrical, thermal and magnetic properties of random heteregeneous materials, *J. Math. Phys.* **14** (1973), 514–523.

S. Jiang and R. Racke, *Evolution Equations in Thermoelasticity*, Chapman & Hall/CRC, Boca Raton, 2000.

A. W. Leung, Asymptotically stable invariant manifold for coupled nonlinear parabolic-hyperbolic partial differential equations, *J. Diff. Eq.* **187** (2003), 184–200.

M. Miklavčič, A sharp condition for existence of an inertial manifold, *J. Dyn. Diff. Eq.* **3** (1991), 437–456.

A. Millet and P. L. Morien, On a nonlinear stochastic wave equation in the plane: existence and uniqueness of the solution, *Ann. Appl. Probab.*, **11** (2001), 922-951.

——and M. Sanz-Solé, Approximation and support theorem for a wave equation in two space dimensions, *Bernoulli* **6** (2000), 887-915.

J.E. Munoz Rivera and R.K.Barreto, Existence and exponential decay in nonlinear thermoelasticity, *Nonlin. Anal.* **31** (1998), 149–162.

——and R.Racke, Smoothing properties, decay, and global existence of solutions to nonlinear coupled system of thermoelasticity type, *SIAM J. Math. Anal.* **26** (1995), 1547–1563.

S. Peszat and J. Zabczyk, Nonlinear stochastic wave and heat equations, *Probab. Theory Rel. Fields* **116** (2000), 421-443.

L. Quer-Sardanyons and M. Sanz-Solé, A stochastic wave equation in dimension 3: smoothness of the law, *Bernoulli* **10** (2004), 165-186.

K. Sobczyk, *Stochastic Wave Propagation*, Elsevier, Amsterdam, 1984.

R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer, Berlin-Heidelberg-New York, 1988.