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EXISTENCE OF EXPONENTIALLY ATTRACTING STATIONARY SOLUTIONS FOR DELAY EVOLUTION EQUATIONS

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Abstract. We consider the exponential stability of semilinear stochastic evolution equations with delays when zero is not a solution for these equations. We prove the existence of a non-trivial stationary solution exponentially stable, for which we use a general random fixed point theorem for general cocycles. We also construct stationary solutions with the stronger property of attracting bounded sets uniformly, by means of the theory of random dynamical systems and their conjugation properties.

1. **Introduction.** The asymptotic behaviour of stochastic partial differential equations is an important task which has been receiving much attention during the last decades. In particular, stochastic evolution equations containing some sort of delay or retarded argument have also been extensively studied due to their importance in applications (see, for example, [2], [4], [5], [11], [15], [24] and [25]).
However, even in the non-delay framework, most results in the literature are concerned with the exponential stability of *constant* stationary solutions, mainly the trivial one (see [12] and [18] in the finite dimensional context, and [7], [16], [13], [14] among others in the infinite dimensional framework).
In the recent work [6], the asymptotic behaviour of semilinear stochastic partial differential equations has been analysed, focusing on the exponential stability of their non-constant stationary solutions in mean square and pathwise.
Our aim in this work is to prove analogous results in the case in which the non-linear term can eventually contain some hereditary features. Although it may be possible to develop a parallel analysis to that one in [6], but with necessary modifications due to the different nature of the problem, we will use a different technique in this paper. In fact, our results will be deduced as consequences of a general fixed point

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theorem for general cocycles. In this way, by simply choosing appropriate phase spaces we can deduce both types of stability (in mean square and pathwise) under a unified treatment.
These generalized fixed points will provide stationary solutions of our problem having different properties, depending on the chosen setup. In the first case, the stationary solution will be exponentially attracting in mean square, while in the second it will be pathwise exponentially attracting.
The content of the paper is as follows. In Section 2 we state a general theorem ensuring the existence of generalized fixed points for cocycles. In Section 3, a semilinear stochastic evolution equation with finite delay is considered. First, we construct a cocycle associated to our model and prove the existence of a mean square exponentially attracting solution. In addition, this stationary solution is proved to attract with probability one. However, the exceptional set does depend on the initial value. To overcome this disadvantage we exploit the tools from the theory of random dynamical systems and construct another cocycle which possesses a random fixed point exponentially attracting for every path. To do this, we consider a more specific noisy term in our equation since it is not always possible to generate a random dynamical system from general stochastic PDEs.

2. **A generalized fixed point theorem for cocycles.** In this section we will establish a theorem which ensures the existence and uniqueness of generalized fixed points for nonautonomous and random dynamical systems. The main property describing the dynamics of such a system is termed *cocycle property,* which is a generalization of the semigroup property for autonomous systems.
Let $X = (X(r), \|\cdot\|_r)$ for $r \in \mathbb{R}$ be a family of Banach spaces.
We say that $\psi = \psi(\cdot,\cdot,\cdot)$ is a *cocycle* on X if, for any $t \in \mathbb{R}^+$ and $r \in \mathbb{R}$, the mapping $\psi(t, r, \cdot) : X(r) \to X(r + t)$ satisfies

$$\begin{aligned} \psi(0, r, \cdot) &= \mathrm{id}_{X(r)}, \\ \psi(t + s, r, \cdot) &= \psi(t, r + s, \psi(s, r, \cdot)). \end{aligned} \tag{1}$$

Indeed, deleting all the r's in the above formula one gets the well known semigroup property. A family $\psi^* = (\psi^*(r))_{r\in\mathbb{R}}$, where $\psi^*(r) \in X(r)$ for all $r \in \mathbb{R}$, is called a *generalized fixed point* for the cocycle ψ if

$$\psi(t, r, \psi^*(r)) = \psi^*(t + r), \quad \text{for all } t \geq 0 \text{ and } r \in \mathbb{R}.$$

Let κ be a positive constant. A family $\xi = (\xi(r))_{r\in\mathbb{R}}$, $\xi(r) \in X(r)$ is called κ-growing if

$$\lim_{r\to-\infty} \|\xi(r)\|_r e^{(\kappa-\varepsilon)r} = 0 \quad \text{for every } \varepsilon > 0.$$

The following theorem formulates sufficient conditions for the existence of a *pullback* attracting generalized fixed point for a cocycle.

Theorem 1. *We consider the family of Banach spaces* $X = (X(r), \|\cdot\|_r)_{r\in\mathbb{R}}$. *Let* ψ *be a cocycle defined on* X *and* $\mathcal{P}$ *be some non-empty subset of* κ*-growing families on* X *for a* $\kappa > 0$. *In addition,*

$(A1)$ *Suppose that there exists a mapping*

$$K : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$$

such that

$$\|\psi(t, r, x) - \psi(t, r, y)\|^2_{r+t} \leq K(t, r)\|x - y\|^2_r \quad \textit{for all} \quad x, y \in X(r),$$

and for every $\varepsilon > 0$

$$\lim_{t\to\infty} K(t,r)e^{(\kappa-\varepsilon)t} = 0, \quad \lim_{t\to\infty} K(t,r-t)e^{(\kappa-\varepsilon)t} = 0.$$

$(A2)$ *Suppose that* $(\psi(N, r-N, \xi(r-N)))_{N\in\mathbb{N}}$ *is a Cauchy sequence in* $X(r)$ *for every* $r \in \mathbb{R}$ *and* ξ *in* $\mathcal{P}$. *Then the limit family is contained in* $\mathcal{P}$.

$(A3)$ *We have that for* $x \in \mathcal{P}$, $r \in \mathbb{R}$ *and for every* $\varepsilon > 0$

$$\sup_{q\in[0,1]} \|\psi(q, r-q-t, x(r-q-t)) - x(r-t)\|_{r-t} = o(e^{(\kappa-\varepsilon)t})$$

for $t \to \infty$.

Then there exists a generalized fixed point $\psi^* \in \mathcal{P}$ *for* ψ *which is the unique one (in* $\mathcal{P}$*). Moreover, for all families* $\xi \in \mathcal{P}$ *the following convergence*

$$\lim_{t\to\infty} \|\psi(t, r-t, \xi(r-t)) - \psi^*(r)\|_r = 0 \qquad \text{(pullback convergence)}$$

holds exponentially fast for all $r \in \mathbb{R}$. *In addition, we have*

$$\lim_{t\to\infty} \|\psi(t, r, \xi(r)) - \psi^*(r+t)\|_{r+t} = 0$$

for all families $\xi \in \mathcal{P}$ *exponentially fast.*

Proof. This theorem is a version of similar results in Duan *et al.* [10], Schmalfuß [21], [23]. Although we do not prefer to include the complete proof, we would like to mention that the idea is to show that, for every $\xi \in \mathcal{P}$, the Cauchy sequence $\{\psi(N, r-N, \xi(r-N))\}_{N\in\mathbb{N}}$ has a limit in $X(r)$ where its limit does not depend on the family ξ. This follows from $(A1)$ and $(A2)$. According to $(A3)$ $(\psi(t, r-t, \xi(r-t)))$ has the same limit for $t \to \infty$ as the above sequence which causes the existence of a generalized fixed point. □

3. **Mean square exponentially attracting stationary solutions for a delay model.** In this section we deal with the concept of exponentially attracting stationary solutions for a kind of delay stochastic non-linear evolution equation generated by random fixed points.

3.1. **Preliminaries on a semilinear stochastic evolution equation with delays.** We start to describe the noise driving the differential equation.
Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a filtered probability space such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \qquad \text{for } s \le t.$$

In what follows, we consider a two-sided Wiener process W taking values in some separable Hilbert space U with covariance Q being a trace class symmetric operator on U. For instance, as it is usual in the literature, the previous probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is to be chosen as the set of continuous paths $C_0(\mathbb{R}, U)$ which are zero at zero equipped with the compact open topology. $\mathcal{F}$ is supposed to be the associated Borel σ-algebra and $\mathbb{P}$ is the Wiener measure with respect to the covariance Q. We also set $\mathcal{F}_t = \sigma\{\omega(u) - \omega(v) : v,\, u \le t\}$.
Note that the above probability space is not complete. Its completion is denoted by $(\Omega, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ where $\{\bar{\mathcal{F}}_t\}_{t\in\mathbb{R}}$ is a normal filtration, see Da Prato and Zabczyk [8], Page 75. For the extension of $\mathbb{P}$ to $\bar{\mathcal{F}}$ we choose the same notation $\mathbb{P}$.
We now introduce a measurable flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ on the above non-completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\theta : (\mathbb{R} \times \Omega, \mathcal{F} \otimes \mathcal{B}(\mathbb{R})) \to (\Omega, \mathcal{F}), \quad \theta_{t+\tau} = \theta_t \circ \theta_\tau, \quad \theta_0 = \mathrm{id}_\Omega, \tag{2}$$

where by means of $\mathcal{B}(\mathbb{S})$ we denote the Borel σ-algebra of open subsets in the topological space $\mathbb{S}$.
The *Wiener shift* operators which form the flow θ

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \, \omega \in \Omega$$

leave the Wiener measure $\mathbb{P}$ invariant. More precisely, $\mathbb{P}$ is ergodic with respect to θ and, in addition, we have that

$$\theta_u^{-1}\mathcal{F}_t = \mathcal{F}_{t+u} \tag{3}$$

for any t, $u \in \mathbb{R}$, see Arnold [1], Page 72. In particular, we mention that the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system.
As this probability space is canonical, we have for a Wiener process W and its shift operators

$$W(t,\omega) = \omega(t), \quad W(t,\theta_s\omega) = \omega(t+s) - \omega(s) = W(t+s,\omega) - W(s,\omega).$$

Given two real numbers $a < b$ and a separable Banach space $\mathcal{H}$, we denote by $I^2(a,b;\mathcal{H})$ the closed subspace of $L^2(\Omega \times [a,b], \bar{\mathcal{F}} \otimes \mathcal{B}([a,b]), \mathrm{d}\mathbb{P}\otimes\mathrm{d}t; \mathcal{H})$ of all stochastic processes $(t,\omega) \to X(t,\omega) \in \mathcal{H}$ such that $X(t)$ is $\bar{\mathcal{F}}_t$-adapted.
We denote by $L_s^2(\Omega; C(a,b;\mathcal{H}))$ the space of processes $u \in L^2(\Omega, \bar{\mathcal{F}}, \mathrm{d}\mathbb{P}; C(a,b;\mathcal{H}))$ such that $u(t)$ is $\bar{\mathcal{F}}_{t+s}$-measurable for each t in $[a,b]$, where $s \in \mathbb{R}$ and $C(a,b;\mathcal{H})$ denotes the space of all continuous functions from $[a,b]$ into $\mathcal{H}$ equipped with supremum norm, which will be denoted by $||\cdot||_{C(a,b;\mathcal{H})}$. $||\cdot||_{L^2}$ is defined to be the norm in the spaces $L_s^2(\Omega; C(a,b;\mathcal{H}))$ for all $s \in \mathbb{R}$. We also write $L^2(\Omega; C(a,b;\mathcal{H}))$ instead of $L_0^2(\Omega; C(a,b;\mathcal{H}))$.
Let us fix a $h > 0$ and consider $T > 0$. For brevity we denote $C_{\mathcal{H}} = C(-h,0;\mathcal{H})$. If we have a function $u \in C(-h,T;\mathcal{H})$, for each $t \in [0,T]$ we denote by $u_t \in C_{\mathcal{H}}$ the function defined by $u_t(s) = u(t+s)$, $-h \le s \le 0$. Moreover, if $y \in L^2(-h,T;\mathcal{H})$ we also denote by $y_t \in L^2(-h,0;\mathcal{H})$, for almost every (in the sequel, a.e.) $t \in (0,T)$, the function defined by $y_t(s) = y(t+s)$, a.e. $s \in (-h,0)$.

For the following let W be the Wiener process on the probability space $(\Omega, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ introduced above. In addition, assume that there exists a Gelfand triplet $V \subset H \subset V'$ of separable Hilbert spaces, where V' denotes the dual of V. We denote by $|\cdot|$, $\|\cdot\|_V$ the norms in H and V respectively. The inner product in H will be denoted by $(\cdot,\cdot)$, and the duality mapping between V' and V by $\langle\cdot,\cdot\rangle$.

We will study the qualitative behaviour of the following delay stochastic evolution equations

$$\begin{cases} du = Audt + f(u_t)dt + B(u)dW, \ t \ge 0, \\ u(t) = \xi(t), \ t \in [-h,0], \\ u \in I^2(0,T;V) \cap L^2(\Omega; C(-h,T;H)), \text{ for all } T > 0. \end{cases} \tag{P}$$

We mention that the equation in (P) has to be satisfied in V' so that we consider our solutions from a variational point of view. The random variable ξ is supposed to be $(\bar{\mathcal{F}}_0, \mathcal{B}(C_H))$–measurable. Let us denote by $a_1 > 0$ the constant of the injection $V \subset H$, i.e.

$$a_1 |u|^2 \le \|u\|_V^2, \qquad \text{for } v \in V,$$

and let $-A : V \to V'$ be a positive, linear and continuous operator for which there exists a constant $a_2 < 0$ such that

$$\langle -Au, u\rangle \ge -a_2\|u\|_V^2, \quad \text{for all } u \in V.$$

Therefore, $-A$ generates a norm in V which is equivalent to the previous one. In the sequel, we suppose that $\|\cdot\|_V$ denotes this norm in V, which is then defined by $\|u\|_V^2 = \langle -Au, u\rangle$, for all $u \in V$. Moreover, it is also well-known (see, for instance, Dautray and Lions [9]) that A is the generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ in H satisfying

$$\|S(t)\|_{\mathcal{L}(H)} \leq e^{at},$$

where $a = a_1 a_2 < 0$.

Also, suppose that $f : C_H \to H$ is a mapping satisfying the following properties:

(f.1) there exists a constant $C_f > 0$ such that for all $\xi, \widetilde{\xi} \in C_H$

$$|f(\xi) - f(\widetilde{\xi})| \leq C_f \|\xi - \widetilde{\xi}\|_{C_H},$$

(f.2) for any positive continuous function ρ over $(-h, +\infty)$ there exists a constant $K_f = K_f(\rho, h) \geq 0$ such that for all $u, \widetilde{u} \in C(-h, +\infty; H)$, and all $t \geq 0$

$$\int_0^t \rho(s)\, |f(u_s) - f(\widetilde{u}_s)|^2 \,\mathrm{d}s \leq K_f^2 \int_{-h}^t \rho(s)\, |u(s) - \widetilde{u}(s)|^2 \,\mathrm{d}s.$$

Assumption (f.2) is often assumed in the context of delay partial differential equations. See [6], [3], [4], [17] for some particular examples.

Finally, $B : H \to \mathcal{L}_2^Q(U, H)$ is supposed to be Lipschitz continuous with respect to the Hilbert-Schmidt norm $\mathcal{L}_2^Q(U, H)$ of linear operators from U to H (see Da Prato and Zabczyk [8] Chapter 4):

$$\mathrm{tr}_H((B(u) - B(v))Q(B(u) - B(v))^*) := \|B(u) - B(v)\|_{\mathcal{L}_2^Q}^2 \leq L_B^2 |u - v|^2$$

for $u,\, v \in H$.

We now recall the following theorem on the existence, uniqueness and regularity of solutions to (P) (cf. Márquez-Durán [17] and Caraballo *et al.* [6]).

Theorem 2. *For any $s \in \mathbb{R}$ and $\xi \in L_s^2 := L_s^2(\Omega; C_H)$ there exists a unique (variational and mild) solution to problem (P) with continuous trajectories in C_H, denoted by $u(t, s, \xi)$, defined for $t \geq -h$, and such that $u(t, s, \xi) = \xi(t)$ for $t \in [-h, 0]$, where the Wiener process is $\theta_s W$ instead of W. Moreover, for any $\tau \geq 0, s \in \mathbb{R}$ it holds that*

$$u(t, s + \tau, u(\tau, s, \xi)) = u(t + \tau, s, \xi),\ \mathbb{P} - a.s. \ \ \textit{for all} \ \ t \geq 0. \tag{4}$$

Notice that the process $\theta_s W(\cdot, \omega) = W(\cdot, \theta_s \omega) = W(\cdot + s, \omega) - W(s, \omega)$, for $s \in \mathbb{R}$, is also a Wiener process with covariance Q. For $t \geq 0$, this process is adapted to the filtration $\{\bar{\mathcal{F}}_{s+t}\}_{t\geq 0}$ which follows from (3).

Moreover, observe that the following equality holds for $\xi \in L_0^2$:

$$u(\cdot, 0, \xi)(\theta_s \cdot) = u(\cdot, s, \xi^s), \quad \mathbb{P} - \text{a.s.}, \tag{5}$$

where $\xi^s(\cdot, \cdot) := \xi(\cdot, \theta_s \cdot)$. Indeed, by (3) the random variable $\xi(\cdot, \theta_s \omega)$ is $\bar{\mathcal{F}}_s$-measurable, and both sides of (5) are driven by the same Wiener process $(\theta_s W)$ with the same initial condition. Then, the uniqueness of solutions of (P) proves (5).

As we have already mentioned, the intention of this article is to find mean square and/or omega-wise exponentially attracting stationary solutions to problem (P), which means to find a solution process for which the finite dimensional distributions do not depend on time shifts. Suppose, we can find an $(\bar{\mathcal{F}}_0, \mathcal{B}(C_H))$-measurable random variable ξ^* with values in C_H (i.e. a measurable mapping $\xi^* : \Omega \to C_H$)

such that, if we choose the initial condition $\xi^*(\cdot,\omega)$, then the mapping $u_t(\cdot,\omega) := \xi^*(\cdot,\theta_t\omega)$ solves problem (P). More precisely, the process given by

$$v(t,\omega) = \begin{cases} \xi^*(t,\omega), & \text{if } t \in [-h,0] \\ \xi^*(0,\theta_t\omega), & \text{if } t > 0 \end{cases}$$

is a solution of (P). Then we have for $t_1 < t_2 < \cdots < t_n$

$$\begin{aligned} &\mathbb{P}\left(\xi^*(\cdot,\theta_{t_1}\omega) \in B_1, \cdots, \xi^*(\cdot,\theta_{t_n}\omega) \in B_n\right) \\ &\qquad = \mathbb{P}(\xi^*(\cdot,\theta_{t_1+t}\omega) \in B_1, \cdots, \xi^*(\cdot,\theta_{t_n+t}\omega) \in B_n) \end{aligned}$$

for any $t \in \mathbb{R}$ and Borel sets $B_1, \cdots, B_n$ from $\mathcal{B}(C_H)$, which follows directly from the θ_t-invariance of $\mathbb{P}$. Hence ξ^* generates a stationary solution.

3.2. **Mean square exponentially attracting stationary solutions.** In this paragraph we establish the mean square exponential attractivity for the stationary solutions to problem (P).
Now, we define an appropriate cocycle and will prove that it possesses a unique generalized fixed point which generates a mean square exponentially attracting stationary solution to our problem (P).

Lemma 1. *Consider the Banach space $X(r) = L^2(\Omega, \bar{\mathcal{F}}_r, \mathbb{P}; C_H)$, for $r \in \mathbb{R}$, with its usual norm $||\cdot||_{L^2} = \sqrt{\mathbb{E}\|\cdot\|_{C_H}^2}$ which is independent of r. For $t \geq 0, r \in \mathbb{R}$, define $\psi(t,r,\cdot) : X(r) \to X(t+r)$ by*

$$\psi(t,r,\xi) = u_t(\cdot,r,\xi), \quad \textit{for } \xi \in X(r). \tag{6}$$

Then, ψ satisfies the cocycle property (1).

Proof. The proof follows from Theorem 2, in particular from the equality (4). ☐

Our next aim is to prove the existence of a generalized fixed point for this cocycle ψ. Let $\mathcal{P}$ introduced in Section 2 be given by the families of C_H-valued random variables $\{\xi(r)\}_{r\in\mathbb{R}}$ such that $\xi(r)$ is $\bar{\mathcal{F}}_r-$measurable and $\|\xi(r)\|_{L^2}$ is uniformly bounded in r. To check that assumptions in Theorem 1 are fulfilled we need some preliminary results concerning the solutions of problem (P).
From now on, we denote $\mu_0 = 2a + L_B^2 + 2K_f$ and suppose it is a negative constant. We first establish a result concerning the mean square attractivity for the solution to problem (P).

Theorem 3. *Suppose all the assumptions on A, B and f hold. Then, for each $\mu \in [\mu_0, 0]$ there exists a $K_1 = K_1(\mu,h) > 0$ such that*

$$\mathbb{E}\left|u(t,s,\xi) - u(t,s,\eta)\right|^2 \leq K_1||\xi - \eta||_{L^2}^2 e^{\mu t}, \tag{7}$$

for all $t \geq 0$, $s \in \mathbb{R}$, and $\xi, \eta \in L_s^2$.
Moreover, for each $\mu \in (\mu_0, 0]$ there exists a constant $K_2 = K_2(\mu,h) > 0$ such that for any solution u of problem (P) we have that

$$\mathbb{E}\int_0^t e^{-\mu r}\left|u(r,s,\xi) - u(r,s,\eta)\right|^2 dr \leq K_2||\xi - \eta||_{L^2}^2, \tag{8}$$

for all $t \geq 0$, $s \in \mathbb{R}$, and $\xi, \eta \in L_s^2$.

Proof. Let $\mu \in [\mu_0, 0]$, $s \in \mathbb{R}$ be fixed and denote $u(t) := u(t,s,\xi), v(t) := u(t,s,\eta)$. Then, for any $t \geq 0$ it follows from $(f.2)$ that

$$\begin{aligned}
&2\int_0^t \mathrm{e}^{-\mu r}(f(u_r)-f(v_r),u(r)-v(r))\mathrm{d}r \\
&\leq 2\left(\int_0^t \mathrm{e}^{-\mu r}\left|f(u_r)-f(v_r)\right|^2\mathrm{d}r\right)^{1/2}\left(\int_0^t \mathrm{e}^{-\mu r}\left|u(r)-v(r)\right|^2\mathrm{d}r\right)^{1/2} \\
&\leq 2K_f\int_{-h}^t \mathrm{e}^{-\mu r}\left|u(r)-v(r)\right|^2\mathrm{d}r.
\end{aligned}$$

Applying now Itô's formula to the process $\mathrm{e}^{-\mu t}\left|u(t)-v(t)\right|^2$, and taking into account the assumptions on A and B, we obtain for every $t \geq 0$

$$\begin{aligned}
\mathrm{e}^{-\mu t}\left|u(t)-v(t)\right|^2 &= \left|\xi(0)-\eta(0)\right|^2 - \mu\int_0^t \mathrm{e}^{-\mu r}\left|u(r)-v(r)\right|^2\mathrm{d}r \\
&\quad + 2\int_0^t \mathrm{e}^{-\mu r}\left\langle A(u(r)-v(r)),u(r)-v(r)\right\rangle\mathrm{d}r \\
&\quad + 2\int_0^t \mathrm{e}^{-\mu r}(f(u_r)-f(v_r),u(r)-v(r))\mathrm{d}r \\
&\quad + \int_0^t \mathrm{e}^{-\mu r}||B(u(r))-B(v(r))||^2_{\mathcal{L}_2^Q}\mathrm{d}r \\
&\quad + 2\int_0^t \mathrm{e}^{-\mu r}(u(r)-v(r),(B(u(r))-B(v(r)))\mathrm{d}W(r,\theta_s\omega)) \\
&\leq \left|\xi(0)-\eta(0)\right|^2 + 2K_f\int_{-h}^0 \mathrm{e}^{-\mu r}\left|u(r)-v(r)\right|^2\mathrm{d}r \\
&\quad + (-\mu+2a+L_B^2+2K_f)\int_0^t \mathrm{e}^{-\mu r}\left|u(r)-v(r)\right|^2\mathrm{d}r \\
&\quad + 2\int_0^t \mathrm{e}^{-\mu r}(u(r)-v(r),(B(u(r))-B(v(r)))\mathrm{d}W(r,\theta_s\omega)) \\
&\leq \left|\xi(0)-\eta(0)\right|^2 + 2K_f||\xi-\eta||^2_{L^2}\int_{-h}^0 \mathrm{e}^{-\mu r}\mathrm{d}r \\
&\quad + 2\int_0^t \mathrm{e}^{-\mu r}(u(r)-v(r),(B(u(r))-B(v(r)))\mathrm{d}W(r,\theta_s\omega)) \quad \mathbb{P}-\text{a.s.}
\end{aligned} \tag{9}$$

Then (7) follows easily by taking the expectation after having replaced t by $t \wedge T_N$, where T_N is the family of stopping times

$$T_N(\omega) = \inf\{t \geq 0 : \left|u(t)\right|^2 + \left|v(t)\right|^2 \geq N\}$$

such that

$$\lim_{N\to\infty}(t \wedge T_N) = t,\ \mathbb{P}-\text{a.s.},$$

since u, v have continuous paths.

Next, we prove (8). To this end, we consider $\mu \in (\mu_0, 0]$. Thanks to (9) we know that

$$\begin{aligned}
&\mathrm{e}^{-\mu t}\mathbb{E}\,|u(t,s,\xi)-u(t,s,\eta)|^2\\
&\le \mathbb{E}\,|\xi(0,\cdot)-\eta(0,\cdot)|^2 + 2K_f h||\xi-\eta||^2_{L^2}\\
&\quad + (-\mu+2a+L_B^2+2K_f)\mathbb{E}\int_0^t e^{-\mu r}\,|u(r,s,\xi)-u(r,s,\eta)|^2\,\mathrm{d}r.
\end{aligned}$$

Taking into account that $-\mu+2a+L_B^2+2K_f<0$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^t e^{-\mu r}\,|u(r,s,\xi)-u(r,s,\eta)|^2\,\mathrm{d}r\\
&\le -(1+2K_f h)(-\mu+2a+L_B^2+2K_f)^{-1}||\xi-\eta||^2_{L^2},
\end{aligned}$$

which proves (8). □

Now, using this lemma, we can prove the following result.

Lemma 2. *For each $\mu \in (\mu_0, 0)$ there exists a $K_3 = K_3(\mu, h) > 0$ such that*

$$||\psi(t,s,\xi)-\psi(t,s,\eta)||^2_{L^2}\ \big(=||u_t(\cdot,s,\xi)-u_t(\cdot,s,\eta)||^2_{L^2}\big) \le K_3||\xi-\eta||^2_{L^2}\,e^{\mu t}, \quad (10)$$

for all $t \ge 0$, $s \in \mathbb{R}$, and $\xi, \eta \in L^2_s$.

Proof. We use the same notation as in the previous proof. Let us assume that $t \ge h$. Applying Itô's formula on the intervals $[0, t+\sigma]$, for $\sigma \in [-h, 0]$, and $[0, t-h]$, and then subtracting the obtained equalities, we can deduce that

$$\begin{aligned}
&\mathrm{e}^{-\mu(t+\sigma)}\,|u_t(\sigma)-v_t(\sigma)|^2\\
&= \mathrm{e}^{-\mu(t-h)}\,|u(t-h)-v(t-h)|^2 - \mu\int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}\,|u(r)-v(r)|^2\,\mathrm{d}r\\
&\quad + 2\int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}\,\langle A(u(r)-v(r)), u(r)-v(r)\rangle\,\mathrm{d}r\\
&\quad + 2\int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}(f(u_r)-f(v_r), u(r)-v(r))\mathrm{d}r\\
&\quad + \int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}||B(u(r))-B(v(r))||^2_{\mathcal{L}_2^Q}\mathrm{d}r\\
&\quad + 2\int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}(u(r)-v(r), (B(u(r))-B(v(r)))\mathrm{d}W(r,\theta_s\omega)).
\end{aligned}$$

Thanks to hypothesis $(f.2)$ and using the assumptions on A, B and the fact that $-\mu+\mu_0<0$, we obtain

$$\begin{aligned}
&\mathrm{e}^{-\mu(t+\sigma)}\,|u_t(\sigma)-v_t(\sigma)|^2\\
&\le \mathrm{e}^{-\mu(t-h)}\,|u(t-h)-v(t-h)|^2\\
&\quad + 2K_f\int_{-h}^{0}\mathrm{e}^{-\mu r}\,|u(r)-v(r)|^2\,\mathrm{d}r + 2K_f\int_0^{t-h}\mathrm{e}^{-\mu r}\,|u(r)-v(r)|^2\,\mathrm{d}r\\
&\quad + 2\int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}(u(r)-v(r), (B(u(r))-B(v(r)))\mathrm{d}W(r,\theta_s\omega)).
\end{aligned}$$

Taking into account (8), it follows

$$
\begin{aligned}
&\mathrm{e}^{-\mu t}\mathbb{E}\left[\sup_{-h\le\sigma\le 0}|u_t(\sigma)-v_t(\sigma)|^2\right]\\
&\le \mathrm{e}^{-\mu t}\mathbb{E}\,|u(t-h)-v(t-h)|^2+2\mathrm{e}^{-\mu h}K_f(h+K_2)||\xi-\eta||_{L^2}^2\\
&\quad+2\mathrm{e}^{-\mu h}\mathbb{E}\left[\sup_{-h\le\sigma\le 0}\int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}(u(r)-v(r),(B(u(r))-B(v(r)))\mathrm{d}W(r,\theta_s\omega))\right].
\end{aligned}
$$

On the other hand, Burkholder-Davis-Gundy's inequality yields that

$$
\begin{aligned}
&2\mathrm{e}^{-\mu h}\mathbb{E}\left[\sup_{-h\le\sigma\le 0}\int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}(u(r)-v(r),(B(u(r))-B(v(r)))\mathrm{d}W(r,\theta_s\omega))\right]\\
&\le 6\mathrm{e}^{-\mu h}\mathbb{E}\left[\sup_{-h\le\sigma\le 0}\left(\mathrm{e}^{\frac{-\mu(t+\sigma)}{2}}|u(t+\sigma)-v(t+\sigma)|\right)\right.\\
&\qquad\qquad\left.\left(\int_{t-h}^{t}\mathrm{e}^{-\mu r}||B(u(r))-B(v(r))||_{\mathcal{L}_2^Q}^2\mathrm{d}r\right)^{1/2}\right]\\
&\le \frac{1}{2}\mathrm{e}^{-\mu t}\mathbb{E}\left[\sup_{-h\le\sigma\le 0}|u_t(\sigma)-v_t(\sigma)|^2\right]\\
&\quad+18\mathrm{e}^{-2\mu h}L_B^2\mathbb{E}\int_{t-h}^{t}\mathrm{e}^{-\mu r}|u(r)-v(r)|^2\,\mathrm{d}r.
\end{aligned}
$$

Therefore, applying Theorem 3 we obtain

$$
\begin{aligned}
&\frac{\mathrm{e}^{-\mu t}}{2}\mathbb{E}\left[\sup_{-h\le\sigma\le 0}|u_t(\sigma)-v_t(\sigma)|^2\right]\\
&\le \mathrm{e}^{-\mu t}\mathbb{E}\,|u(t-h)-v(t-h)|^2+2\mathrm{e}^{-\mu h}K_f(h+K_2)||\xi-\eta||_{L^2}^2\\
&\quad+18\mathrm{e}^{-2\mu h}L_B^2\mathbb{E}\int_{t-h}^{t+\sigma}\mathrm{e}^{-\mu r}|u(r)-v(r)|^2\,\mathrm{d}r\\
&\le K_1\mathrm{e}^{-\mu h}||\xi-\eta||_{L^2}^2+2K_f\mathrm{e}^{-\mu h}(h+K_2)||\xi-\eta||_{L^2}^2\\
&\quad+18K_2\mathrm{e}^{-2\mu h}L_B^2||\xi-\eta||_{L^2}^2.
\end{aligned}
$$

The case $0\le t<h$ can be easily deduced from the previous analysis by noticing that

$$
\begin{aligned}
||u_t(\cdot,s,\xi)-u_t(\cdot,s,\eta)||_{C_H}^2\le&\sup_{\sigma\in[-h,-t]}|u(t+\sigma,s,\xi)-u(t+\sigma,s,\eta)|^2\\
&+\sup_{\sigma\in[-t,0]}|u(t+\sigma,s,\xi)-u(t+\sigma,s,\eta)|^2.
\end{aligned}
$$

□

Recall that the family $\mathcal{P}$ is formed by the families of C_H-valued random variables $\{\xi(r)\}_{r\in\mathbb{R}}$ such that $\xi(r)$ is $\bar{\mathcal{F}}_r-$ measurable and that $||\xi(r)||_{L^2}$ is uniformly bounded in $r\in\mathbb{R}$. Then a similar analysis to the one carried out in the previous lemma allows to prove the following consequence.

Corollary 1. *For any given $\{\xi(r)\}_{r\in\mathbb{R}}\in\mathcal{P}$ it holds that $||\psi(t,r,\xi(r))||_{L^2}^2$ is uniformly bounded for $t\ge 0$, $r\in\mathbb{R}$, and thus $\{\psi(t,r,\xi(r))\}_{r\in\mathbb{R}}\in\mathcal{P}$, for $t\ge 0$ and $\{\xi(r)\}_{r\in\mathbb{R}}\in\mathcal{P}$.*

Theorem 4. *Under the previous assumptions, there exists a generalized fixed point $\{\xi^*(r)\}_{r\in\mathbb{R}} \in \mathcal{P}$ for the cocycle ψ defined by (6). This fixed point is the only one which belongs to $\mathcal{P}$ and generates a mean square exponentially attracting stationary solution of (P).*

Proof. We will apply Theorem 1 to our cocycle. Let us consider a fixed $\mu \in (\mu_0, 0)$ and set $K(t,r) := K_3 e^{\mu t}$, for $t \geq 0, r \in \mathbb{R}$, where K_3 is the constant appearing in (10). Then, conditions in $(A1)$ from Theorem 1 are satisfied by taking $\kappa = -\mu$. Condition $(A3)$ follows for any $(\xi(r))_{r\in\mathbb{R}} \in \mathcal{P}$ thanks to Corollary 1. To complete the proof, consider a random variable $\xi \in L^2_0$. Then $(\xi(r))_{r\in\mathbb{R}} = (\xi(\cdot, \theta_r \cdot))_{r\in\mathbb{R}} \in \mathcal{P}$ thanks to the $\{\theta_t\}_{t\in\mathbb{R}}$-invariance of $\mathbb{P}$. Let us assume that $\{\psi(N, r-N, \xi(r-N))\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $X(r)$ for every $r \in \mathbb{R}$, and let us prove that

$$\xi^*(r) := \left(\lim_{N\to\infty} \psi(N, r-N, \xi(r-N))\right)_{r\in\mathbb{R}} \in \mathcal{P}.$$

Then, as $\xi^*(r)$ is obtained as an L^2_r limit, it is therefore $\bar{\mathcal{F}}_r-$ measurable. Moreover, thanks to Corollary 1, $\xi^*(r)$ is uniformly bounded for $r \in \mathbb{R}$ and thus $(\xi^*(r))_{r\in\mathbb{R}} \in \mathcal{P}$.

Notice that this fixed point $\xi^* \in \mathcal{P}$ satisfies

$$\psi(t, r, \xi^*(r)) = \xi^*(t+r), \quad \mathbb{P}\text{-a.s. for all } t \geq 0 \text{ and } r \in \mathbb{R},$$

which means that

$$u_t(\cdot, r, \xi^*(r)) = \xi^*(t+r), \text{ for all } t \geq 0 \text{ and } r \in \mathbb{R}.$$

Let us finally prove that ξ^* generates a mean square exponentially attracting stationary solution to our problem (P). Denote by $u^*(s,\omega) := \xi^*(0,\omega)(s)$, $s \in [-h, 0], \omega \in \Omega$, and let us now prove that

$$u_t(\cdot, 0, u^*(\cdot,\cdot)) = u^*(\cdot, \theta_t \cdot), \ \mathbb{P}\text{-a.s. in } [-h, 0], \text{ for any } t \geq 0.$$

On account of Lemma 2 we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{-h\leq\sigma\leq 0} |u_t(\sigma, 0, u^*(\cdot,\cdot)) - u^*(\sigma, \theta_t \cdot)|^2\right] \\
&= \mathbb{E}\left[\sup_{-h\leq\sigma\leq 0} \Big| u_t(\sigma, 0, (L^2)\lim_{N\to\infty} u_N(\cdot, -N, \xi(\cdot, \theta_{-N}\cdot))) \right. \\
&\left. -(L^2)\lim_{N\to\infty} u_N(\sigma, t-N, \xi(\cdot, \theta_{t-N}\cdot))\Big|^2\right] \\
&= \lim_{N\to\infty} \mathbb{E}\left[\sup_{-h\leq\sigma\leq 0} |u_N(\sigma, t-N, u_t(\cdot, -N, \xi(\cdot, \theta_{-N}\cdot))) - u_N(\sigma, t-N, \xi(\cdot, \theta_{t-N}\cdot))|^2\right] \\
&\leq K_3 \lim_{N\to\infty} e^{\mu N} ||u_t(\cdot, -N, \xi(\cdot, \theta_{-N}\cdot)) - \xi(\cdot, \theta_{t-N}\cdot)||^2_{L^2} \\
&\leq K_3 \lim_{N\to\infty} e^{\mu N} ||u_t(\cdot, 0, \xi(\cdot,\cdot)) - \xi(\cdot, \theta_t \cdot)||^2_{L^2} = 0
\end{aligned}$$

since $\mu < 0$ and t is fixed.

The exponential attracting property of the generalized fixed point implies immediately the mean square exponential attractivity for the stationary solution. Indeed, from (10) it follows

$$\mathbb{E}||u_t(\cdot, 0, \xi(\cdot,\cdot)) - u_t(\cdot, 0, u^*(\cdot,\cdot))||^2_{C_H} \leq K_1 e^{\mu t} ||\xi - u^*||^2_{L^2}$$

for all $t \geq 0$, and $\xi \in L^2_0$, and, in particular,

$$\mathbb{E}\, |u_t(\sigma, 0, \xi(\cdot,\cdot)) - u^*(\sigma, \theta_t \cdot)|^2 \leq K_1 e^{-\mu h} ||\xi - u^*||^2_{L^2} e^{\mu t}$$

for all $t \geq h$, $\sigma \in [-h, 0]$. $\square$

3.3. **Additional almost sure exponential stability of the stationary solution.** In addition to the mean square exponential stability of the stationary solution to problem (P) proved in the previous subsection, we can now show the almost sure exponential convergence. We point out that the exceptional set depends on each initial datum.

Theorem 5. *Under the assumptions in Theorem 4, for any random variable $\xi \in L_0^2$ we have that*

$$\lim_{t\to\infty} ||u_t(\cdot, 0, \xi(\cdot,\cdot)) - u^*(\cdot, \theta_t \cdot))||_{C_H}^2 = 0,$$

$\mathbb{P}-$almost surely exponentially fast, where u^* is the stationary solution to (P) given by the unique generalized fixed point in $\mathcal{P}$.

Proof. Take $\xi \in L_0^2$, and fix a constant $\mu \in (\mu_0, 0)$. Let us denote

$$u(t, 0, \xi(\cdot,\cdot)) = u^1(t), \qquad u(t, 0, u^*(\cdot,\cdot)) = u^2(t).$$

We first prove that there exists a positive constant C such that for any $N \in \mathbb{N}$

$$\mathbb{E}\left[\sup_{N\leq t\leq N+1} \left|u^1(t) - u^2(t)\right|^2\right] \leq C||\xi - u^*||_{L^2}^2 e^{\mu N}. \tag{11}$$

Indeed, Itô's formula, Lemma 3 and the assumptions on A and B imply

$$\begin{aligned}
&\mathbb{E}\left[\sup_{N\leq t\leq N+1} \left|u^1(t) - u^2(t)\right|^2\right] \\
=&K_1 \mathrm{e}^{\mu N}||\xi - u^*||_{L^2}^2 + (2a + L_B^2)K_1 \frac{\mathrm{e}^{\mu N}}{-\mu}||\xi - u^*||_{L^2}^2 \\
&+ 2\mathbb{E}\int_N^{N+1} (f(u_r^1) - f(u_r^2), u^1(r) - u^2(r))\mathrm{d}r \\
&+ 2\mathbb{E}\left[\sup_{N\leq t\leq N+1} \int_N^t (u^1(r) - u^2(r), (B(u^1(r)) - B(u^2(r)))\mathrm{d}W(r,\omega))\right].
\end{aligned}$$

From $(f.2)$ and (8) it follows

$$\begin{aligned}
&2\mathbb{E}\int_N^{N+1} (f(u_r^1) - f(u_r^2), u^1(r) - u^2(r))\mathrm{d}r \\
&\leq 2\mathbb{E}\int_N^{N+1} \mathrm{e}^{-\mu(r-N)}(f(u_r^1) - f(u_r^2), u^1(r) - u^2(r))\mathrm{d}r \\
&\leq 2\mathrm{e}^{\mu N}\mathbb{E}\int_0^{N+1} \mathrm{e}^{-\mu r} \left|f(u_r^1) - f(u_r^2)\right| \left|u^1(r) - u^2(r)\right| \mathrm{d}r \\
&\leq 2K_f \mathrm{e}^{\mu N}\mathbb{E}\int_{-h}^{N+1} \mathrm{e}^{-\mu r} \left|u^1(r) - u^2(r)\right|^2 \mathrm{d}r \\
&\leq 2K_f \mathrm{e}^{\mu N} h||\xi - u^*||_{L^2}^2 + 2K_f K_2 \mathrm{e}^{\mu N}||\xi - u^*||_{L^2}^2.
\end{aligned}$$

On the other hand, Burkholder-Davis-Gundy's inequality yields that

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{N\leq t\leq N+1}\int_N^t (u^1(r)-u^2(r),(B(u^1(r))-B(u^2(r)))\mathrm{d}W(r,\omega))\right]\\
&\leq 6\mathbb{E}\left[\sup_{N\leq t\leq N+1}\left|u^1(t)-u^2(t)\right|\left(\int_N^{N+1}||B(u^1(r))-B(u^2(r))||^2_{\mathcal{L}_2^Q}\mathrm{d}r\right)^{1/2}\right]\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{N\leq t\leq N+1}\left|u^1(t)-u^2(t)\right|^2\right]+18K_1L_B^2\frac{\mathrm{e}^{\mu N}}{-\mu}||\xi-u^*||^2_{L^2}.
\end{aligned}$$

Then (11) follows from the previous estimates.
Now, given $\varepsilon>0$, the Doob-Chebyshev inequality implies

$$\begin{aligned}
&\mathbb{P}\left[\sup_{N\leq t\leq N+1}\left|u^1(t)-u^2(t)\right|^2\geq \mathrm{e}^{(\mu+\varepsilon)N}\right]\\
&\leq \mathrm{e}^{-(\mu+\varepsilon)N}\mathbb{E}\left[\sup_{N\leq t\leq N+1}\left|u^1(t)-u^2(t)\right|^2\right]\\
&\leq C\mathrm{e}^{-\varepsilon N}||\xi-u^*||^2_{L^2},
\end{aligned}$$

and therefore, the Borel-Cantelli lemma can now be applied to assure that there exist a $k=k(\mu,h)$ and a $\gamma>0$, and a subset $\Omega'\subset\Omega$ with $\mathbb{P}(\Omega')=0$ such that, for each $\omega\notin\Omega'$, there exists a positive random variable $T(\omega)$ such that

$$|u_t(0,0,\xi(\cdot,\cdot))-u_t(0,0,u^*(\cdot,\cdot))|^2\leq k||\xi-u^*||^2_{L^2}\mathrm{e}^{-\gamma t},\ \forall t\geq T(\omega),$$

and then

$$|u_t(\sigma,0,\xi(\cdot,\cdot))-u_t(\sigma,0,u^*(\cdot,\cdot))|^2\leq k\mathrm{e}^{\gamma h}||\xi-u^*||^2_{L^2}\mathrm{e}^{-\gamma t},\ \forall t\geq T(\omega)+h,$$

for all $\sigma\in[-h,0]$, which finishes this proof. □

4. **Pathwise exponentially attracting stationary solutions.** A disadvantage of the method used in the last section to prove the pathwise asymptotic stability of the stationary solution u^* is that the exceptional set depends on each initial condition ξ. However, it is a very interesting task to try to prove the existence of such a stationary solution which attracts exponentially any other solution for all sample paths, or at least, with a *common* exceptional (null) set to all the initial data. We will overcome this disadvantage in this section by exploiting the methods of the theory of *random dynamical systems* (a comprehensive presentation can be found in Arnold [1]). However, we will need to consider a more particular form for our stochastic partial differential equations. Indeed, although it is a well-known fact that finite-dimensional stochastic differential equations generate random dynamical systems (see Arnold [1] Chapter 1), this is not true in general for infinite dimensional equations. However, for some especial classes of stochastic partial differential equations the existence of random dynamical systems is established (see, for instance, Mohammed *et al.* [19] and [20]).

4.1. **Preliminaries on random dynamical systems.** A random dynamical system on a separable Banach space X is given by a measurable mapping

$$\chi:(\mathbb{R}^+\times\Omega\times X,\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathcal{B}(X))\to(X,\mathcal{B}(X)),$$

satisfying the cocycle property:

$$\begin{aligned}&\chi(t+\tau,\omega,\xi)=\chi(t,\theta_\tau\omega,\chi(\tau,\omega,\xi)),\quad t,\,\tau\in\mathbb{R}^+,\,\omega\in\Omega,\,\xi\in X,\\&\chi(0,\omega,\xi)=\xi,\end{aligned}\tag{12}$$

where θ is the flow of shift operators (for instance the Wiener shift) previously introduced. We emphasize that (12) has to be fulfilled for *any* $\omega\in\Omega$. However, it is sufficient to replace Ω by a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set of full measure. Outside this invariant set we can re-define χ by the identity mapping on X. This we will use later. The measurability of (2) remains true if we replace $\mathcal{F}$ by its trace σ-algebra, see [6].

We will show that the mapping χ is related to the solution of a delay stochastic or random differential equation. From now on, we will always suppose that the mapping

$$X\ni\xi\mapsto\chi(t,\omega,\xi)\in X$$

is continuous for any t, ω.

The next result will be used to obtain a conjugated random dynamical system to a given random dynamical system. The proof can be found in [1].

Lemma 3. *Let χ be a random dynamical system. Suppose that the mapping $T:\Omega\times X\to X$ has the following properties: For fixed $\omega\in\Omega$ the mapping $T(\omega,\cdot)$ is a homeomorphism on X. For fixed $x\in X$ the mappings $T(\cdot,x)$, $T^{-1}(\cdot,x)$ are measurable. Then the mapping*

$$(t,\omega,x)\in\mathbb{R}^+\times\Omega\times X\to T(\theta_t\omega,\chi(t,\omega,T^{-1}(\omega,x)))=:\phi(t,\omega,x)\in X\tag{13}$$

satisfies (12). Hence ϕ is a random dynamical system.

The measurability of ϕ follows from the properties of T. Later on, we will transform our stochastic delay evolution equation containing a noisy term into an evolution equation without noise but with random coefficients, both of them generating a random dynamical system on $\mathbb{R}^+\times\Omega\times C_H$ satisfying the property (13), which means that ϕ and χ are conjugated random dynamical systems.

A random variable Y on $(\Omega,\mathcal{F},\mathbb{P})$ with values in X is called *tempered* if

$$\lim_{t\to\pm\infty}\frac{\log^+||Y(\theta_t\omega)||_X}{|t|}=0,\tag{14}$$

or equivalently if $t\to||Y(\theta_t\omega)||_X$ has a subexponential growth for $t\to\pm\infty$, in other words, for $\varepsilon>0$ and $\omega\in\Omega$ there exists a $t_0(\varepsilon,\omega)\ge0$ such that for $|t|\ge t_0(\varepsilon,\omega)$ it holds

$$||Y(\theta_t\omega)||_X\le e^{\varepsilon|t|}.$$

Similar to the definition of a generalized fixed point an $(\mathcal{F},\mathcal{B}(X))$-measurable random variable χ^* is called a *random fixed point* if

$$\chi(t,\omega,\chi^*(\omega))=\chi^*(\theta_t\omega)\tag{15}$$

for $t\ge0$, $\omega\in\Omega$. χ^* is an *exponentially attracting* random fixed point if for any random variable ξ we have

$$\lim_{t\to\infty}||\chi(t,\omega,\xi(\omega))-\chi^*(\theta_t\omega)||_X=0\quad\text{for }\omega\in\Omega\tag{16}$$

with exponential speed. If χ is defined by the solution mapping of a delay stochastic/random differential equation then the process $(t,\omega)\mapsto\chi^*(\theta_t\omega)$ is a *stationary* solution of a delay stochastic/random differential equation.

4.2. **Construction of the cocycle for a delay stochastic evolution equation.** In this paragraph we construct the cocycle generated by a delay stochastic partial differential equation and set the appropriate framework in which we can apply Theorem 1 to prove the existence of a unique stationary solution exponentially stable with probability one. For the reasons given above we will consider our delay stochastic evolution equations in the Stratonovich form. We also suppose that the diffusion part of this equation is given by

$$B(u) \circ dW = B_1 u \circ dw_1 + \cdots + B_N u \circ dw_N,$$

where $w_1, \cdots, w_N$ are one-dimensional mutually independent standard Wiener processes and $W = (w_1, \cdots, w_N)$ so that the phase space U for the Wiener process is given by $\mathbb{R}^N$, and $B_i \in \mathcal{L}(H)$ for $i = 1, \cdots N$. We will denote $b_i = \|B_i\|_{\mathcal{L}(H)}$. This means that we will consider the equation

$$\begin{cases} du = (Au + f(u_t))dt + \sum_{i=1}^{N} B_i u \circ dw_i, \\ u(t) = \xi(t), \ t \in [-h, 0]. \end{cases} \tag{17}$$

In the sequel we consider a deterministic initial condition $\xi \in C_H$. The operators B_i generate C_0-groups which will be denoted by S_{B_i}. In addition, we suppose the operators A, $B_1, \cdots, B_N$ mutually commute, what implies that these groups and the semigroup generated by A are also mutually commuting. A has been introduced in Section 3.

Now we want to transform (17) into a partial differential equation whose right-hand side contains ω as a parameter. For fixed ω this equation can be solved as a deterministic nonautonomous differential equation. We will be able to construct a cocycle mapping.

For what follows, we will fix a one-dimensional Wiener process with $U = \mathbb{R}$ and $\mathbb{E}|w(1)|^2 = 1$. We consider the one-dimensional stochastic differential equation

$$dz = -\lambda\, z\, dt + dw(t) \tag{18}$$

for some $\lambda > 0$. This equation has a random fixed point z^* in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein-Uhlenbeck process.

The proof of the following result can be found, for instance, in Caraballo *et al.* [6]:

Lemma 4. *Let λ be a positive number and consider the probability space as in Section 3 with $U = \mathbb{R}$. There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\Omega' \in \mathcal{F}$ of full measure such that, for $w(t, \omega) = \omega(t)$ in the canonical interpretation*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} = 0,$$

and, for such ω, the random variable given by

$$z^*(\omega) := -\lambda \int_{-\infty}^{0} e^{\lambda\tau} \omega(\tau) \mathrm{d}\tau$$

is well defined. Moreover, for $\omega \in \Omega'$, the mapping

$$\begin{aligned}(t, \omega) \to z^*(\theta_t\omega) &= -\lambda \int_{-\infty}^{0} e^{\lambda\tau} \theta_t\omega(\tau) \mathrm{d}\tau \\ &= -\lambda \int_{-\infty}^{0} e^{\lambda\tau} \omega(t + \tau) \mathrm{d}\tau + \omega(t)\end{aligned}$$

is a stationary solution of (18) with continuous trajectories. In addition, for $\omega \in \Omega'$

$$\lim_{t\to\pm\infty}\frac{|z^*(\theta_t\omega)|}{|t|}=0,\qquad \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z^*(\theta_\tau\omega)\mathrm{d}\tau=0,$$
$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t |z^*(\theta_\tau\omega)|\mathrm{d}\tau=\mathbb{E}|z^*|<\infty. \tag{19}$$

Let $\lambda_1,\cdots,\lambda_N$ be a set of positive numbers. For any pair λ_j, w_j we have a stationary Ornstein-Uhlenbeck process generated by a random variable $z_j^*(\omega)$ on Ω_j', with properties formulated in Lemma 4, defined on the metric dynamical system $(\Omega_j',\mathcal{F}_j,\mathbb{P}_j,\theta)$. We set

$$(\tilde\Omega,\mathcal{F},\mathbb{P},\theta), \tag{20}$$

where

$$\tilde\Omega=\Omega'_1\times\cdots\times\Omega'_N,\quad \mathcal{F}=\bigotimes_{i=1}^N\mathcal{F}_i,\quad \mathbb{P}=\mathbb{P}_1\times\mathbb{P}_2\times\cdots\times\mathbb{P}_N,$$

and θ is the flow of Wiener shifts.

To find random fixed points for (17) we will transform this equation into an evolution equation with random coefficients but without white noise by means of the operators T and $\widetilde{T}$ defined in the following way. For every $\omega\in\tilde\Omega$ let

$$T(\omega):=S_{B_1}(z_1^*(\omega))\circ\cdots\circ S_{B_N}(z_N^*(\omega))$$

be a family of random linear homeomorphisms on H. The inverse operator is well defined by

$$T^{-1}(\omega):=S_{B_N}(-z_N^*(\omega))\circ\cdots\circ S_{B_1}(-z_1^*(\omega)).$$

These operators can be easily extended to linear homeomorphisms $\widetilde{T}(\omega)$ and $\widetilde{T}^{-1}(\omega)$ on C_H. Indeed, for any $\xi\in C_H$, let us define

$$\left(\widetilde{T}(\omega)\xi\right)(s):=T(\omega)\xi(s),\quad \text{for}\ \ s\in[-h,0].$$

Because of the estimate

$$\|T^{-1}(\omega)\|\le e^{\|B_1\||z_1^*(\omega)|}\cdot\ldots\cdot e^{\|B_N\||z_N^*(\omega)|}$$

and (19), it follows that $\|T^{-1}(\theta_t\omega)\|$ has a subexponential growth as $t\to\pm\infty$ for any $\omega\in\tilde\Omega$ (we will also denote $||\cdot||$ for the norm $||\cdot||_{\mathcal{L}(H)}$ when no confusion is possible). Hence $\|T^{-1}\|$ is tempered. Analogously, $\|T\|$ is also tempered. On the other hand, since z_j^*, $j=1,\cdots,N$, are independent Gaussian random variables, by the ergodic theorem we still have a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\bar\Omega\in\mathcal{F}$ of full measure such that

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t\|T(\theta_\tau\omega)\|^2\mathrm{d}\tau=\mathbb{E}\|T\|^2\le\prod_{j=1}^N\mathbb{E}(\|S_{B_j}(z_j^*)\|^2)<\infty, \tag{21}$$
$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t\|T^{-1}(\theta_\tau\omega)\|^2\mathrm{d}\tau=\mathbb{E}\|T^{-1}\|^2\le\prod_{j=1}^N\mathbb{E}(\|S_{Bj}(-z_j^*)^2\|)<\infty.$$

Remark 1. We now consider θ defined in (2) on $\tilde\Omega\cap\bar\Omega$ instead of Ω. This mapping has the same properties as the original one if we choose for $\mathcal{F}$ the trace σ-algebra with respect to $\tilde\Omega\cap\bar\Omega$ denoted also by $\mathcal{F}$ for a metric dynamical system defined on $\tilde\Omega\cap\bar\Omega$ for which we use the old notation $(\Omega,\mathcal{F},\mathbb{P},\theta)$.

Sometimes, in other publications (14), (15), (16) only have to be satisfied for ω

contained in a $\{\theta_t\}_{t\in\mathbb{R}}$–invariant set of full measure. However, by our definition of the metric dynamical system we can state that these properties will hold for every $\omega \in \Omega$.

We now formulate an evolution equation with random coefficients but without white noise

$$\begin{cases} \dfrac{dv}{dt} = \left(A + \sum\limits_{i=1}^{N} \lambda_i z_i^*(\theta_t\omega)B_i\right) v + T^{-1}(\theta_t\omega) f(\widetilde{T}(\theta_t\omega)v_t), \\ v(t) = \xi(t),\ t \in [-h, 0], \end{cases} \tag{22}$$

where ξ belongs to C_H, which will be our phase space.

Lemma 5. *Suppose that A, $B_1, \cdots, B_N$, $\lambda_1, \ldots, \lambda_N$ satisfy the preceding assumptions. Then*
i) the random evolution equation (22) possesses a unique solution, and this solution generates a random dynamical system $\chi : \mathbb{R}^+ \times \Omega \times C_H \to C_H$ defined for $t \geq 0$, $\omega \in \Omega$ and $\xi \in C_H$ by

$$\chi(t, \omega, \xi) = v_t(\cdot, \omega, \xi).$$

ii) the process $\phi : \mathbb{R}^+ \times \Omega \times C_H \to C_H$ defined for $t \geq 0$, $\omega \in \Omega$ and $\xi \in C_H$ by

$$\phi(t, \omega, \xi) = \widetilde{T}(\theta_t\omega)\chi(t, \omega, \widetilde{T}^{-1}(\omega)\xi)$$

is another random dynamical system such that

$$\phi(t, \omega, \xi) = u_t(\cdot, \omega, \xi),$$

being u a solution version to problem (17), unique modulo $\mathbb{P}$.

Proof. i) We mention only that by the continuity and linearity of T and $\widetilde{T}$ the mapping $\xi \in C_H \to T^{-1}(\theta_t\omega)f(\widetilde{T}(\theta_t\omega)\xi) \in H$ is Lipschitz continuous, where the Lipschitz constant is uniformly bounded on any interval $[0, T]$. Hence we can prove the existence of a mild solution of (22) for every ω. The proof of measurability is straightforward.
ii) This part follows in a standard way by applying the chain rule to the function y^t defined as

$$y^t(s) := \left(\widetilde{T}(\theta_t\omega)v_t(\cdot, \omega, \widetilde{T}^{-1}(\omega)\xi)\right)(s) = T(\theta_{t+s}\omega)v(t + s, \omega, \widetilde{T}^{-1}(\omega)\xi),\ s \in [-h, 0],$$

and taking into account the commutativity of the operators involved (see [6] for a similar situation in the non-delay case).
On account of Lemma 3, ϕ and χ are conjugated random dynamical systems. □

4.3. **Existence of exponentially stationary solutions.** In the following let us apply the general method from Section 2 to find an exponentially attracting random fixed point ξ^*.

Theorem 6. *Suppose A, $B_1, \cdots, B_N$ mutually commute and $W = (w_1, \cdots, w_N)$ satisfies the assumptions at the beginning of this section. In addition, there are positive constants $\lambda_1, \cdots, \lambda_N$ such that*

$$a + \sum_{i=1}^{N} b_i\lambda_i\mathbb{E}|z_i^*| + \frac{K_f}{2}\left(\prod_{i=1}^{N}\mathbb{E}||S_{B_i}(z_i^*)||^2 + \prod_{i=1}^{N}\mathbb{E}||S_{B_i}(-z_i^*)||^2\right) < 0 \tag{23}$$

holds.
Then the random dynamical systems χ and ϕ possess, respectively, a tempered random fixed point χ^* and ϕ^*, which are unique under all tempered random variables in C_H and which attract exponentially fast every random variable in C_H.

Proof. First of all, we point out that to prove the theorem it is sufficient to show that (22) has a unique exponentially attracting generalized fixed point. The conjugation technique then gives the existence of a fixed point for (17).

First step: Let us consider, for $\delta > 0$ small enough, the mapping

$$\gamma(\omega) = -2a - 2\sum_{i=1}^{N} b_i\lambda_i|z_i^*(\omega)| - (1+\delta)^{\frac{1}{2}} K_f \left(||T(\omega)||^2 + ||T^{-1}(\omega)||^2\right),$$

which satisfies, by (23) and (21), $\mathbb{E}\gamma =: \bar{\gamma} > 0$. According to Remark 1 where we introduced the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ we have that

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \gamma(\theta_\tau\omega)\mathrm{d}\tau = \bar{\gamma}, \qquad \text{for all } \omega \in \Omega.$$

We are interested in calculating a priori estimate for the solution of (22). Denote $v(t) = v_t(0, \omega, \xi)$, $t \in \mathbb{R}^+$. On account of the properties of the coefficients of (22) and the chain rule the following estimate holds:

$$\begin{aligned} e^{\int_0^t \gamma(\theta_l\omega)\mathrm{d}l}|v(t)|^2 \le |v(0)|^2 & \\ - \int_0^t (1+\delta)^{\frac{1}{2}} K_f \left(||T(\theta_s\omega)||^2 + ||T^{-1}(\theta_s\omega)||^2\right) & e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}|v(s)|^2\mathrm{d}s \\ + 2\int_0^t e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}(T^{-1}(\theta_s\omega)f(\widetilde{T}(\theta_s\omega)v_s), v(s))\mathrm{d}s. & \end{aligned} \tag{24}$$

The terms in the above integrals are locally integrable in s, l such that these integrals exist. Now we evaluate the last term in (24) by means of condition $(f.2)$.

$$\begin{aligned} & 2\int_0^t e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}(T^{-1}(\theta_s\omega)f(\widetilde{T}(\theta_s\omega)v_s), v(s))\mathrm{d}s \\ & \le \varepsilon \int_0^t ||T^{-1}(\theta_s\omega)||^2 e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}|v(s)|^2\mathrm{d}s \\ & + \varepsilon^{-1}(1+\delta)K_f^2 \int_{-h}^t ||T(\theta_s\omega)||^2 e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}|v(s)|^2\mathrm{d}s \\ & + \varepsilon^{-1}(1+\delta^{-1})|f(0)|^2 \int_0^t e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}s, \end{aligned} \tag{25}$$

where $\varepsilon > 0$. Choosing $\varepsilon = (1+\delta)^{\frac{1}{2}} K_f$ in the last inequality, if we suppose $t \ge \sigma$ we obtain from (24), (25)

$$\begin{aligned} e^{\int_0^{t+\sigma} \gamma(\theta_l\omega)\mathrm{d}l}|v(t+\sigma)|^2 \le |v(0)|^2 & \\ + (1+\delta)^{\frac{1}{2}} K_f ||\xi||^2_{C_H} & \int_{-h}^0 ||T(\theta_s\omega)||^2 e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}s \\ + (1+\delta^{-1})(1+\delta)^{-\frac{1}{2}} K_f^{-1}|f(0)|^2 & \int_0^{t+\sigma} e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}s. \end{aligned}$$

Now, if we take the supremum in the last inequality, we have

$$\begin{aligned}
&\sup_{\sigma\in[-h,0]}|v(t+\sigma)|^2 \qquad (26)\\
&\quad\le e^{-\int_0^t\gamma(\theta_l\omega)\mathrm{d}l}\left[\|\xi\|^2_{C_H}\left(e^{-2ah}+c_\delta\int_{-h}^0||T(\theta_s\omega)||^2e^{\int_0^s\gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}s\right)\right.\\
&\qquad\left.+\tilde{c}_\delta\int_0^t e^{\int_0^s\gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}s\right],
\end{aligned}$$

for all $t\ge h$, where we have denoted $c_\delta=e^{-2ah}(1+\delta)^{\frac12}K_f$ and $\tilde{c}_\delta=e^{-2ah}(1+\delta^{-1})(1+\delta)^{-\frac12}K_f^{-1}|f(0)|^2$. A similar estimate is true for $t\in[0,h)$.

Second step: We check the contraction condition which we need to get $(A1)$ from Theorem 1. Consider $\xi_1,\xi_2\in C_H$. Let $\omega\in\Omega$ and denote $v^i(t)=v_t(0,\omega,\xi_i)$, for $i=1,2$, and $\overline{v}(t)=v^1(t)-v^2(t)$, $t\in\mathbb{R}^+$.

Taking into account that $v^i(t)$ is a solution to the equation

$$\frac{dv^i(t)}{dt}=\left(A+\sum_{i=1}^N\lambda_i z_i^*(\theta_t\omega)B_i\right)v^i(t)+T^{-1}(\theta_t\omega)f(\widetilde{T}(\theta_t\omega)v_t^i),$$

we have that

$$\begin{aligned}
\frac{d|\overline{v}(t)|^2}{dt}=&2\left\langle\left(A+\sum_{i=1}^N\lambda_i z_i^*(\theta_t\omega)B_i\right)\overline{v}(t),\overline{v}(t)\right\rangle\\
&+2(T^{-1}(\theta_t\omega)(f(\widetilde{T}(\theta_t\omega)v_t^1)-f(\widetilde{T}(\theta_t\omega)v_t^2)),\overline{v}(t))\\
\le&\left(2a+2\sum_{i=1}^N\lambda_i b_i|z_i^*(\theta_t\omega)|+K_f||T^{-1}(\theta_t\omega)||^2\right)|\overline{v}(t)|^2\\
&+\frac{1}{K_f}|f(\widetilde{T}(\theta_t\omega)v_t^1)-f(\widetilde{T}(\theta_t\omega)v_t^2)|^2.
\end{aligned}$$

Therefore, using condition $(f.2)$,

$$\begin{aligned}
&e^{-2at}|\overline{v}(t)|^2\\
&\quad\le|\overline{v}(0)|^2+K_f\int_{-h}^0||T(\theta_s\omega)||^2e^{-2as}|\overline{v}(s)|^2\mathrm{d}s\\
&\qquad+\int_0^t\left(2\sum_{i=1}^N\lambda_i b_i|z_i^*(\theta_s\omega)|+K_f(||T^{-1}(\theta_s\omega)||^2+||T(\theta_s\omega)||^2)\right)e^{-2as}|\overline{v}(s)|^2\mathrm{d}s,
\end{aligned}$$

and by Gronwall's lemma,

$$\begin{aligned}
|\overline{v}(t)|^2\le&\left(1+K_f\int_{-h}^0||T(\theta_s\omega)||^2\mathrm{d}s\right)||\xi_1-\xi_2||^2_{C_H}\\
&\times e^{2at}\exp\left(\int_0^t 2\sum_{i=1}^N\lambda_i b_i|z_i^*(\theta_s\omega)|+K_f\left(||T^{-1}(\theta_s\omega)||^2+||T(\theta_s\omega)||^2\right)\mathrm{d}s.\right)
\end{aligned}$$

Now, setting $t \geq h$ and $\sigma \in [-h, 0]$, taking supremum,

$$\begin{aligned}\sup_{\sigma\in[-h,0]} &|\overline{v}(t+\sigma)|^2 \\ &\leq \left(1 + K_f \int_{-h}^0 ||T(\theta_s\omega)||^2 \mathrm{d}s\right) ||\xi_1 - \xi_2||^2_{C_H} \\ &\quad\times e^{-2ah} \exp\left(\int_0^t 2a + 2\sum_{i=1}^N \lambda_i b_i |z_i^*(\theta_s\omega)| + K_f \left(||T^{-1}(\theta_s\omega)||^2 + ||T(\theta_s\omega)||^2\right) \mathrm{d}s\right),\end{aligned}$$

since $\sup_{\sigma\in[-h,0]} e^{2a(t+\sigma)} = e^{-2ah}e^{2at}$. Therefore, if we set

$$\begin{aligned}K(t,\omega) := e^{-2ah} &\left(1 + K_f \int_{-h}^0 ||T(\theta_s\omega)||^2 \mathrm{d}s\right) \\ &\times \exp\left(\int_0^t 2a + 2\sum_{i=1}^N \lambda_i b_i |z_i^*(\theta_s\omega)| + K_f \left(||T^{-1}(\theta_s\omega)||^2 + ||T(\theta_s\omega)||^2\right) \mathrm{d}s\right),\end{aligned} \tag{27}$$

then, for $t \geq h$ it holds

$$||v_t(\cdot,\omega,\xi_1) - v_t(\cdot,\omega,\xi_2)||^2_{C_H} \leq K(t,\omega)||\xi_1 - \xi_2||^2_{C_H}.$$

Third step:
Now we apply Theorem 1. For each fixed $\omega \in \Omega$, we consider the family of Banach spaces given by $X(r) = C_H$ for $r \in \mathbb{R}$. In particular, we fix some $\bar{\omega} \in \Omega$ and define

$$\psi(t, r, \xi(r)) := \chi(t, \theta_r\bar{\omega}, \xi(\theta_r\bar{\omega})).$$

Then $(A1)$ from Theorem 1 follows from the second step, since by the definition of Ω (see (19) and (21)) we can set

$$\kappa = -a - \sum_{i=1}^N b_i\lambda_i\mathbb{E}|z_i^*| - \frac{K_f}{2}\left(\prod_{i=1}^N \mathbb{E}||S_{B_i}(z_i^*)||^2 + \prod_{i=1}^N \mathbb{E}||S_{B_i}(-z_i^*)||^2\right)$$

and $K(t, r) := K(t, \theta_r\bar{\omega})$ from (27). To see that $K(t, r - t) := K(t, \theta_{r-t}\bar{\omega})$ has the growth condition from $(A1)$ we need that $\int_{-h}^0 ||T(\theta_s\omega)||^2\mathrm{d}s$ is tempered what follows from the methods in the proof of Lemma 6.

For the fixed $\bar{\omega}$ we choose $\mathcal{P}$ to be the families $\xi = (\xi(r))_{r\in\mathbb{R}}$ with values in C_H such that $\|\xi(r)\|_{C_H}$ has a subexponential growth for $r \to \pm\infty$. These families are κ-growing. In particular, let Ξ be the set of random variables with values in C_H such that $\|\xi\|_{C_H}$ is tempered for $\xi \in \Xi$. Then the orbits $\xi = (\xi(\theta_r\bar{\omega}))_{r\in\mathbb{R}}$ are elements from $\mathcal{P}$.

Define the random variable

$$R(\omega) := 2\tilde{c}_\delta \int_{-\infty}^0 e^{-\int_s^0 \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}s. \tag{28}$$

From (26) we obtain

$$\begin{aligned}\|\psi(t,r-t,\xi(r-t))\|_{C_H}^2 \leq & e^{-\int_{-t}^0 \gamma(\theta_{r+l}\bar\omega)\mathrm{d}l}\|\xi(\theta_{r-t}\bar\omega)\|_{C_H}^2 \times \\ & \times\left(e^{-2ah}+c_\delta\int_{-\infty}^0\|T(\theta_{s+r-t}\bar\omega)\|^2 e^{\int_0^s\gamma(\theta_{l+r-t}\bar\omega)\mathrm{d}l}\mathrm{d}s\right) \\ & +\frac{1}{2}R(\theta_r\bar\omega)\leq R(\theta_r\bar\omega)\end{aligned}$$

for $|t|$ sufficiently large and $r\geq 0$ and $\xi\in\Xi$ because

$$t\to e^{-\int_{-t}^0\gamma(\theta_{l+r}\bar\omega)\mathrm{d}l}$$

tends to zero exponentially fast for $t\to\infty$. R and the infinite integral are tempered what follows from Lemma 6 below. The Cauchy sequence has a limit by the completeness of the phase space. Its norm is bounded by $\sqrt{R(\theta_r\bar\omega)}$. Hence the limit family is in $\mathcal{P}$ which gives $(A2)$.

To see $(A3)$ we note that from (26) by some straightforward integral transforms

$$\begin{aligned}&\sup_{q\in[0,1]}\|\psi(q,r-q,\xi(r-q))\|_{C_H}^2 \\ &\quad\leq e^{-2ah}\sup_{q\in[0,1]}e^{-\int_0^q\gamma(\theta_{r+l-q}\bar\omega)\mathrm{d}l}\sup_{q\in[0,1]}\|\xi(\theta_{r-q}\bar\omega)\|_{C_H}^2 \qquad (29)\\ &\quad+c_\delta\sup_{q\in[0,1]}e^{-\int_0^q\gamma(\theta_{r+l-q}\bar\omega)\mathrm{d}l}\int_{-\infty}^0\|T(\theta_{r+s-q}\bar\omega)\|^2 e^{\int_0^s\gamma(\theta_{r+l-q}\bar\omega)\mathrm{d}l}\mathrm{d}s+R(\theta_r\bar\omega)\\ &\leq e^{-2ah}e^{\int_{-1}^0 2\sum_{i=1}^N b_i\lambda_i|z_i^*(\theta_{l+r}\bar\omega)|+(1+\delta)^{\frac12}K_f\left(||T(\theta_{r+l}\bar\omega)||^2+||T^{-1}(\theta_{r+l}\bar\omega)||^2\right)\mathrm{d}l}\\ &\quad\times\sup_{q\in[0,1]}\|\xi(\theta_{r-q}\bar\omega)\|_{C_H}^2\\ &\quad+c_\delta\int_{-\infty}^0\|T(\theta_{r+s}\bar\omega)\|^2 e^{\int_0^s\gamma(\theta_{r+l}\bar\omega)\mathrm{d}l}\mathrm{d}s+R(\theta_r\bar\omega).\end{aligned}$$

The sum/product of tempered random variables is tempered. In addition, it is easy to see that $\sup_{p\in[-1,0]}\|\xi(\theta_p\omega)\|_{C_H}$ is tempered if $\|\xi(\omega)\|_{C_H}$ is.
Moreover, the first term on the right hand side of (29) is tempered by the Birkhoff ergodic theorem:

$$\lim_{t\to\infty}\frac{1}{t}\int_{-2}^0 2\sum_{i=1}^N b_i\lambda_i|z_i^*(\theta_{l-[t]}\omega)|+(1+\delta)^{\frac12}K_f\left(||T(\theta_{l-[t]}\omega)||^2+||T^{-1}(\theta_{l-[t]}\omega)||^2\right)\mathrm{d}l=0$$

for $\omega\in\Omega$, being $[t]$ the integer part of t.
The integral and the last term on the right hand side of (29) is defined by a tempered random variable what follows by Lemma 6 below.
Hence we can apply Theorem 1 which gives us a generalized fixed point ψ^*.

Fourth step: We set

$$\chi^*(\bar\omega):=\psi^*(0)=\psi^*(0,\bar\omega)\in B_{C_H}(0,R(\bar\omega))$$

for every $\bar\omega\in\Omega$. Note that by the cocycle property, this definition is correct. Hence χ^* satisfies the fixed point relation

$$\chi(t,\omega,\chi^*(\omega))=\chi^*(\theta_t\omega)\quad\text{for } t\geq 0 \text{ and } \omega\in\Omega.$$

According to the contraction condition given in the Second step we obtain (16) for every random variable $\xi \in C_H$. In particular, χ^* can be generated as a pointwise limit of random variables such that χ^* is a random variable.

If we define $\phi^*(\omega) := \widetilde{T}(\omega)\chi^*(\omega)$, as ϕ and χ are conjugated random dynamical systems, it follows

$$\begin{aligned}\phi(t,\omega,\phi^*(\omega)) &= \widetilde{T}(\theta_t\omega)\chi(t,\omega,\widetilde{T}^{-1}(\omega)\phi^*(\omega))\\ &= \widetilde{T}(\theta_t\omega)\chi(t,\omega,\chi^*(\omega))\\ &= \widetilde{T}(\theta_t\omega)\chi^*(\theta_t\omega) = \phi^*(\theta_t\omega).\end{aligned}$$

Since $\|\widetilde{T}\|_{\mathcal{L}(C_H)}$ and $\|\widetilde{T}^{-1}\|_{\mathcal{L}(C_H)}$ are tempered we have for ϕ^* the same uniqueness and convergence conclusion as for χ^*. □

Now we prove the temperedness statements formulated in the proof of the last theorem.

Lemma 6. *Under the assumptions of Theorem 6 the random variable R defined in (28) and*

$$\int_{-\infty}^0 \|T(\theta_s\omega)\|^2 e^{\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}s$$

are tempered.

Proof. We note that by the definition of Ω in Remark 1 we have

$$\left|\int_0^t (\gamma(\theta_{\tau+r}\omega)-\bar{\gamma})\mathrm{d}\tau\right| \le \varepsilon|t|, \qquad \log^+(\|T(\theta_t\omega)\|^2+1)\le\varepsilon|t|$$

for every $\omega\in\Omega$, $r\in\mathbb{R}$, $\varepsilon>0$ if $|t|\ge t_0(\omega,r,\varepsilon)$. It is clear that it is sufficient to prove that

$$\int_{-\infty}^0 (\|T(\theta_\tau\omega)\|^2+1)e^{-\int_{-\tau}^0\gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}s$$

is tempered. For every $\omega\in\Omega$ and $c<0$, consider $\varepsilon\in(0,\min(-\frac{c}{6},\frac{\bar{\gamma}}{4}))$. Then,

$$\begin{aligned}&\lim_{s\to-\infty} e^{-cs}\int_{-\infty}^0 (\|T(\theta_{\tau+s}\omega)\|^2+1)e^{-\int_\tau^0\gamma(\theta_{l+s}\omega)\mathrm{d}l}\mathrm{d}\tau\\ &=\lim_{s\to-\infty} e^{-cs}\int_{-\infty}^0 (\|T(\theta_{\tau+s}\omega)\|^2+1)e^{-\int_{\tau+s}^s\gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}\tau\\ &=\lim_{s\to-\infty} e^{-cs}\int_{-\infty}^0 e^{\int_{\tau+s}^0(-\gamma(\theta_l\omega)+\bar{\gamma})\mathrm{d}l+\bar{\gamma}(\tau+s)-\int_s^0(-\gamma(\theta_l\omega)+\bar{\gamma})\mathrm{d}l-\bar{\gamma}s+\log^+(\|T(\theta_{\tau+s}\omega)\|^2+1)}\mathrm{d}\tau\\ &\le\lim_{s\to-\infty} e^{-\frac{c}{2}s}\int_{-\infty}^0 e^{-\frac{c}{2}s-\epsilon(\tau+s)-\epsilon s+\bar{\gamma}\tau-\varepsilon(\tau+s)}\mathrm{d}\tau\\ &\le\lim_{s\to-\infty} e^{-\frac{c}{2}s}\int_{-\infty}^0 e^{\frac{\bar{\gamma}}{2}\tau}\mathrm{d}\tau=0.\end{aligned}$$

We consider now the case $s\to+\infty$. First of all, note that

$$e^{c|s|}\int_{-\infty}^0 (\|T(\theta_{\tau+s}\omega)\|^2+1)e^{-\int_\tau^0\gamma(\theta_{l+s}\omega)\mathrm{d}l}\mathrm{d}\tau$$

can be written as

$$e^{cs}\bigg(e^{-\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}\bigg(\int_0^s(\|T(\theta_\tau\omega)\|^2+1)e^{\int_0^\tau \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}\tau+\int_{-\infty}^0(\|T(\theta_\tau\omega)\|^2+1)e^{\int_0^\tau \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}\tau\bigg)\bigg).$$

We obtain that

$$\lim_{s\to+\infty} e^{cs}e^{-\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}\int_{-\infty}^0(\|T(\theta_\tau\omega)\|^2+1)e^{\int_0^\tau \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}\tau=0.$$

On the other hand, for every $\omega\in\Omega$ and $c<0$ there exists a $K(\omega,c)>0$ such that for $q>0$

$$\begin{aligned}&(\bar\gamma+\frac{c}{6})q-K(\omega,c)\le\int_0^q\gamma(\theta_\tau\omega)\mathrm{d}\tau\le(\bar\gamma-\frac{c}{6})q+K(\omega,c),\\&(\|T(\theta_t\omega)\|^2+1)\le e^{-\frac{c}{6}t},\qquad t\ge t_0(\omega,c)\end{aligned}$$

and then the term

$$e^{cs}e^{-\int_0^s \gamma(\theta_l\omega)\mathrm{d}l}\bigg(\int_0^{t_0}(\|T(\theta_\tau\omega)\|^2+1)e^{\int_0^\tau \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}\tau+e^{-\frac{c}{6}s}\int_{t_0}^s e^{\int_0^\tau \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}\tau\bigg)$$

can be estimated by

$$e^{cs}e^{K(\omega,c)}e^{(-\bar\gamma-\frac{c}{6})s}\bigg(\int_0^{t_0}(\|T(\theta_\tau\omega)\|^2+1)e^{\int_0^\tau \gamma(\theta_l\omega)\mathrm{d}l}\mathrm{d}\tau+\frac{e^{\frac{-c}{6}s}e^{K(\omega,c)}}{\bar\gamma-\frac{c}{6}}(e^{(\bar\gamma-\frac{c}{6})s}-e^{(\bar\gamma-\frac{c}{6})t_0})\bigg),$$

if s is sufficiently large, which gives the asserted convergence for $s\to+\infty$. We also note that $t\to\|T(\theta_t\omega)\|^2$ is locally integrable. □

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