Existence and uniqueness of solutions for
Non–Linear Stochastic Partial Differential Equations

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ABSTRACT:

We state some results on existence and uniqueness for the solution of non linear stochastic PDEs with deviating arguments. In fact, we consider the equation

$$dx(t) + (A(t,x(t)) + B(t,x(\tau(t))) + f(t))\,dt = (C(t,x(\rho(t))) + g(t))\,dw_t\,,$$

where $A(t,.)$, $B(t,.)$ *and* $C(t,.)$ *are suitable families of non linear operators in Hilbert spaces,* w_t *is a Hilbert valued Wiener process, and* τ , ρ *are functions of delay. If* A *satisfies a coercivity condition and a monotonicity hypothesis, and if* B , C *are Lipschitz continuous, we prove that there exists a unique solution of an initial value problem for the precedent equation. Some examples of interest for the applications are given to illustrate the results.*

1. Introduction

The study of stochastic PDE's has greatly developed over the last years. Stochastic PDE's are used in modelling physical phenomena [5], population biology [7], filtering [13], etc.

The main aim of this paper is to study this type of equation with delay terms. In fact, we prove existence and uniqueness of solution (in Itô's sense) for a rather general type of stochastic PDEs with non linear monotone operators and with delays. We deal with the following stochastic parabolic equation:

$$(1)\qquad \begin{cases} dx(t) + [A(t,x(t)) + B(t,x(\tau(t))) + f(t)]\; dt = [C(t,x(\ rho(t))) + g(t)]\,dw_t\,, \quad t>0 \\ x(0) = x_0\,, \end{cases}$$

where $A(t,.)$, $B(t,.)$, $C(t,.)$ are families of operators in Hilbert spaces, non linear eventually, and satisfying a monotonicity condition; w_t is a Hilbert valued Wiener process, and τ , ρ are delay functions.

When there are not delays ($\tau(t)=\rho(t)=t$), the equation (1) has been studied: in the case $B=C=0$, for A non linear, in Bensoussan [2] and Curtain [6], and for some type of non linear operators A, in Bensoussan–Temam [3,4] and Marcus [10]; in the case $C\neq 0$, $B=0$, for linear A and C, in Balakrishann [1], for linear A and non linear C in Dawson [6], and for non linear monotone A and Lipschitz continuous C in Pardoux [12].

In the case with deviating arguments, Real [14,15] studies a rather general case when all of the operators are linear and there exists a term which is a non continuous martingale. However, we have not found in the literature the case we are going to analyze here.

We will adapt to our problem one of the most important method for solving non linear PDEs (see Lions [9]): the monotonicity method. Pardoux [12] also used an adaptation of that method for another type of non linear monotone equations: when $B=0$ and without delays.

In Section 2 we shall state the problem and the notation we are going to use. Uniqueness of solution will be proved, in Section 3, using Itô's formula. In Section 4, we state the existence results. Some extensions of the results are given in Section 5. Finally, we illustrate our theory with several important examples appearing in the applications.

2. Statement of the problem

The theory of stochastic integrals in Hilbert spaces is well developed (see [8], [12], for example).

The variational method we are going to use, forces us to work with the classical pair of real separable Hilbert spaces V , H satisfying $V\hookrightarrow H$ (injection continuous and dense). We identify H with its dual space, and denote by V' the dual of V . Then, we have

$$V\hookrightarrow H\equiv H'\hookrightarrow V' .$$

We will denote by $\|.\|$, $|.|$ and $\|.\|_*$ the norms in V , H and V' respectively; by $\langle .,.\rangle$ the duality product between V',V , and by $(.,.)$ the scalar product in H .

Let us fix $T>0$ and, let w_t be a Wiener process defined on the complete probability space $(\Omega,\mathcal{F},P)$ and taking values in the separable Hilbert space K , with incremental covariance operator W . Let $(\mathcal{F}_t)_{t\geq 0}$ be the σ-algebra generated by $\{w_s, 0\leq s\leq t\}$, then w_t is a martingale relative to $(\mathcal{F}_t)_{t\geq 0}$ and we have the following representation of w_t :

$$w_t=\sum_{i=1}^{\infty}\beta_t^i e_i,$$

where (e_i) is an orthonormal set of eigenvectors of W, β_t^i are mutually independent real Wiener processes with incremental covariance $\lambda_i>0$, $We_i=\lambda_i e_i$ and $\operatorname{tr} W=\sum_{i=1}^{\infty}\lambda_i$ (tr denotes the trace of an operator, see [8], [12], [13]).

As an abuse of notation, we also use $|.|$ for the norm in the linear continuous operator space $\mathcal{L}(K,H)$.

We denote by $I^p(0,T;V)$, for $p>1$, the space of $V-$valued processes $(x(t))_{t\in[0,T]}$ (we will write $x(t)$ for short) measurable (from $[0,T]\times\Omega$ in V), and satisfying:

i) $x(t)$ is $\mathcal{F}_t-$measurable a.e. in t (in the sequel, we will write a.e.t.)

ii) $E\int_0^T |x_t|^p\,dt<+\infty$.

It is easy to check that $I^p(0,T;V)$ is a closed subspace of $L^p(\Omega\times[0,T],\mathcal{F}\otimes\mathcal{B}([0,T]),dP\otimes dt;V)$, where by $\mathcal{B}([0,T])$ we denote the Borel σ–algebra.

For short, we shall write $L^2(\Omega;C(-h,T;H))$ instead of $L^2(\Omega,\mathcal{F},dP;C(-h,T;H))$, where $C(-h,T;H)$ denotes the space of continuous functions from $[-h,T]$ to H.

Let $A(t,.):V\to V'$ be a family of non linear operators defined a.e.t., and let $p>1$. We make the following hypotheses:

(a.1) Coercivity: $\exists\,\alpha>0$, $\lambda\in\mathbf{R}$ such that:

$$2\langle A(t,x),x\rangle+\lambda|x|^2\geq\alpha\|x\|^p\,,\ \ \forall x\in V\,,\ \text{a.e.t.}$$

(a.2) Monotonicity: $2\langle A(t,x)-A(t,y),x-y\rangle+\lambda|x-y|^2\geq 0\,,\ \forall x,y\in V$, a.e.t.

(a.3) Boundedness: $\exists\,\beta>0$: $\|A(t,x)\|_*\leq\beta\|x\|^{p-1}$, $\forall x\in V$, a.e.t.

(a.4) Hemicontinuity: $\theta\in\mathbf{R}\to\langle A(t,x+\theta y),z\rangle\in\mathbf{R}$ is continuous $\forall x,y,z\in V$, a.e.t.

(a.5) Measurability:

$$t\in(0,T)\to A(t,x)\in V' \ \text{ is Lebesgue – measurable } \ \forall x\in V\,,\ \ \text{a.e.t.}$$

Let $B(t,.):H\to H$ be a family of operators defined a.e.t., and satisfying:

(b.1) $B(t,0)=0$

(b.2) Lipschitz condition: $\exists\,k_1$ such that

$$|B(t,x)-B(t,y)|\leq k_1|x-y|\,,\quad\forall x,y\in H\,,\quad\text{a.e.t.}$$

(b.3) Measurability: $t\in(0,T)\to B(t,x)\in H$ is Lebesgue–measurable, $\forall x\in V$.

And let $C(t,.):H\to\mathcal{L}(K,H)$ be another family defined a.e.t. and verifying:

(c.1) $C(t,0)=0$

(c.2) Lipschitz condition: $\exists\,k_2$ such that

$$|C(t,x)-C(t,y)|\leq k_2|x-y|\,,\quad\forall x,y\in H\,,\quad\text{a.e.t.}$$

(c.3) Measurability: $t\in(0,T)\to C(t,x)\in\mathcal{L}(K,H)$ is Lebesgue–measurable $\forall x\in H$.

We also consider two measurable functions (of delay) $\rho,\tau:[0,T]\to[0,T]$, such that

$$(\rho.\tau)\qquad\qquad 0\leq\rho(t),\tau(t)\leq t\,,\ \ \forall t\in[0,T].$$

For f, g we suppose that

(f.g) $$f \in I^2(0,T;H)\,, g \in I^2(0,T;\mathcal{L}(K,H)).$$

And finally, we are given an initial value $x_0 \in L^2(\Omega,\mathcal{F}_0,P;H)$.

Now, we state the following problem:

$$(PC) \qquad \begin{cases} \textit{To find a process } x \in I^p(0,T;V) \cap L^2(\Omega;C(0,T;H)) \textit{ such that :} \\ x(t) + \int_0^t [A(s,x(s)) + B(s,x(\tau(s))) + f(s)]\ ds \\ \qquad = x_0 + \int_0^t [C(s,x(\rho(s))) + g(s)]\ dw_s\,, \quad P-\text{a.s.}, \quad \forall t \in [0,T]. \end{cases}$$

Remark 2.1. We observe that, if $x \in L^2(0,T;H)$ then, by (b.1)–(b.3), $B(x) \in L^2(0,T;H)$, where $B(x)(t) = B(t,x(t))$. Moreover, $x \in L^2(0,T;H) \to B(x) \in L^2(0,T;H)$ is continuous, and so, measurable. Since $x \in H \to B(t,x) \in H$ is continuous a.e.t., it follows that, if $x(t)$ is an H–valued stochastic process and $\mathcal{F}_t$–adapted, then $B(t,x(t))$ also is. In addition, if $x \in L^2(\Omega\times(0,T);H)$, then $B(x) \in L^2(\Omega\times(0,T);H)$ too. Finally, if x^n is a bounded sequence in $L^2(\Omega\times(0,T);H)$, $B(x^n)$ also is bounded.

Similar observations are deduced from (c.1)–(c.3) for $C : L^2(0,T;H) \to L^2(0,T;\mathcal{L}(K,H))$ defined by $C(x)(t) = C(t,x(t))$.

These remarks and the measurability of ρ and τ imply that the integrals appearing in (PC) are well defined.

3. Uniqueness of solution

In this Section we will prove that there exists at most one solution of (PC). We will obtain this result from (a.2) and Itô's formula (see [8], [13] for that formula).

Theorem 3.1

Assume the hypotheses in Section 2. Then, there exists at most one solution of (PC) *in* $I^p(0,T;V) \cap L^2(\Omega;C(0,T;H))$.

Proof. Suppose that $x, y \in I^p(0,T;V) \cap L^2(\Omega;C(0,T;H))$ are solutions of (PC). Then, applying Itô's formula, we obtain

$$\begin{aligned} (3.1) \quad & E|x(t)-y(t)|^2 \\ &= -2E\int_0^t \langle A(s,x(s)) - A(s,y(s)), x(s)-y(s)\rangle\, ds \\ &\quad -2E\int_0^t (B(s,x(\tau(s))) - B(s,y(\tau(s))), x(s)-y(s))\, ds \\ &\quad + E\int_0^t \operatorname{tr}\left[(C(s,x(\rho(s))) - C(s,y(\rho(s))))\, W\, (C(s,x(\rho(s))) - C(s,y(\rho(s))))^*\right]\, ds. \end{aligned}$$

Now, by putting $z(t) = x(t) - y(t)$ and using conditions (a.2), (b.2), (c.2), it follows

$$
\begin{aligned}
(3.2) \qquad E|z(t)|^2 \leq \lambda E \int_0^t |z(s)|^2 \, ds + 2k_1 E \int_0^t |z(\tau(s))||z(s)| \, ds \\
+ k_2^2 \mathrm{tr}(W) E \int_0^t |z(\rho(s))|^2 \, ds \, .
\end{aligned}
$$

We are going to estimate the terms on the right-hand side of (3.2). First,

$$
(3.3) \qquad \lambda E \int_0^t |z(s)|^2 \, ds \leq |\lambda| \int_0^t \sup_{r \in [0,s]} E|z(r)|^2 \, ds.
$$

Using $(\rho . \tau)$ we get

$$
\begin{aligned}
(3.4) \qquad 2E \int_0^t |z(\tau(s))||z(s)| \, ds &\leq E \int_0^t |z(\tau(s))|^2 \, ds + E \int_0^t |z(s)|^2 \, ds \\
&\leq 2 \int_0^t \sup_{r \in [0,s]} E|z(r)|^2 \, ds \, ,
\end{aligned}
$$

$$
(3.5) \qquad E \int_0^t |z(\rho(s))|^2 \, ds \leq \int_0^t \sup_{r \in [0,s]} E|z(r)|^2 \, ds \, .
$$

Consequently, $(3.2) - (3.5)$ yield

$$
(3.6) \qquad \sup_{r \in [0,t]} E|z(r)|^2 \leq \left[|\lambda| + 2k_1 + 2k_2^2 \mathrm{tr}\,(W) \right] \int_0^t \sup_{r \in [0,s]} E|z(r)|^2 \, ds \, .
$$

Finally, Gronwall's Lemma implies

$$
(3.7) \qquad \sup_{r \in [0,t]} E|z(r)|^2 = 0 \, , \ \ \forall t \in [0,T] \, .
$$

Obviously, uniqueness follows from (3.7). ∎

4. Existence of solution

First, we state a theorem on existence and uniqueness of solution for a stochastic evolution equation, and an energy equality. Next, we will prove the existence of solution of (PC) using this result.

Theorem 4.1

Assume the hypotheses in Section 2, with $\lambda = 0$. *Then, there exists a unique process* $x \in I^p(0,T;V) \cap L^2(\Omega; C(0,T;H))$ *such that*

$$
x(t) + \int_0^t \left[A(s, x(s)) + f(s) \right] \, ds = x_0 + M_t \, , \; P - \text{a.s.} \, , \ \ \forall t \in [0,T] \, ,
$$

where M_t *is a* H*–valued continuous, square integrable* $\mathcal{F}_t$*–martingale. The solution also verifies the following energy equality:*

$$|x(t)|^2 + 2\int_0^t \langle A(s,x(s)), x(s)\rangle\, ds + 2\int_0^t (f(s), x(s))\, ds \tag{4.1}$$
$$= |x_0|^2 + 2\int_0^t (x(s), dM_s) + \mathrm{tr}\langle\langle M\rangle\rangle_t\,, \quad P-\text{a.s.}\,, \quad \forall t\in[0,T]\,,$$

where $\langle\langle M\rangle\rangle_t$ *denotes the quadratic variation of* M_t *(see Métivier and Pellaumail [11]).*

Proof. See [11], [15] and the references given there. ∎

Remark 4.1. We observe that, in our situation (see the proof of theorem 4.2), the martingale M_t will be $\int_0^t g(s)\, dw_s$ and hence, the energy equality yields

$$E|x(t)|^2 + 2E\int_0^t \langle A(s,x(s)), x(s)\rangle\, ds + 2E\int_0^t (f(s), x(s))\, ds \tag{4.2}$$
$$= E|x_0|^2 + E\int_0^t \mathrm{tr}(g(s)Wg(s)^*)\, ds\,, \quad P-\text{a.s.}\,, \quad \forall t\in[0,T]\,.$$

In Pardoux [12,13] and Ichikawa [8] we can find a rather general Itô's formula.

Now, using a Picard's scheme, we can prove the existence of solution for the problem (PC).

Theorem 4.2

Assume the conditions in Section 2. Then, there exists a unique solution of (PC) *in* $I^p(0,T;V)\cap L^2(\Omega; C(0,T;H))$.

Proof. Uniqueness holds from theorem 3.1. For the existence, we consider the equations

$$x^1(t) + \int_0^t \left[A(s,x^1(s)) + \frac{\lambda}{2}x^1(s)\right] ds + \int_0^t f(s)\, ds \tag{4.3}$$
$$= x_0 + \int_0^t g(s)\, dw_s$$

$$x^{n+1}(t) + \int_0^t \left[A(s,x^{n+1}(s)) + \frac{\lambda}{2}x^{n+1}(s)\right] ds \tag{4.4}$$
$$+ \int_0^t B(s,x^n(\tau(s)))\, ds + \int_0^t f(s)\, ds$$
$$= x_0 + \int_0^t \frac{\lambda}{2}x^n(s)\, ds + \int_0^t C(s,x^n(\rho(s)))\, dw_s$$
$$+ \int_0^t g(s)\, dw_s\,, \quad \forall n = 1,2,3,...$$

By (a.1)–(a.5), the family $A_1(t,.) : V \to V'$ defined by $A_1(t,x) = A(t,x) + (\lambda/2)x$, satisfies the assumptions in theorem 4.1. Consequently, (4.3) has a unique solution $x^1 \in I^p(0,T;V)\cap L^2(\Omega; C(0,T;H))$.

We note that, from (b.2),(c.2) and the measurability of the functions ρ , τ , it follows:

i) The map $(t,\omega)\in(0,T)\times\Omega\to B(t,x^1(\tau(t))\in H$ belongs to $I^2(0,T;H)$.

ii) The map $(t,\omega)\in(0,T)\times\Omega\to C(t,x^1(\rho(t))\in H$ belongs to $I^2(0,T;\mathcal{L}(K,H))$, and so, $\int_0^{\cdot} C(s,x^1(\rho(s)))\,dw_s$ is a continuous and square integrable $\mathcal{F}_t$–martingale.

Again, by these remarks, we can apply theorem 4.1 and we get that there exists a unique process $x^2\in I^p(0,T;V)\cap L^2(\Omega;C(0,T;H))$, which is solution of (4.4) for $n=1$. By recurrence, we obtain a sequence of solutions for $(4.3)-(4.4)$, $\{x^n\}_{n\geq 1}\subset I^p(0,T;V)\cap L^2(\Omega;C(0,T;H))$.

In the sequel, we shall prove that the sequence $\{x^n\}$ is convergent to a process in $I^p(0,T;V)\cap L^2(\Omega;C(0,T;H))$, and this process is the solution of (PC). We shall split this proof in four steps.

STEP 1.– $\{x^n\}$ *is a Cauchy sequence in* $L^2(\Omega;C(0,T;H))$.

Indeed, for $n>1$, it follows from Itô's formula for the process $x^{n+1}(t)-x^n(t)$,

$$\begin{aligned}(4.5)\quad & |x^{n+1}(t)-x^n(t)|^2+2\int_0^t\langle A(x^{n+1})-A(x^n),x^{n+1}-x^n\rangle\,ds\\ &+\lambda\int_0^t|x^{n+1}-x^n|^2\,ds+2\int_0^t\left(B(x^n_\tau)-B(x^{n-1}_\tau),x^{n+1}-x^n\right)\,ds\\ &=\lambda\int_0^t\left(x^{n+1}-x^n,x^n-x^{n-1}\right)\,ds+2\int_0^t\left(x^{n+1}-x^n,(C(x^n_\rho)-C(x^{n-1}_\rho))\,dw_s\right)\\ &+\int_0^t\mathrm{tr}\left[\left(C(x^n_\rho)-C(x^{n-1}_\rho)\right)W\left(C(x^n_\rho)-C(x^{n-1}_\rho)\right)^*\right]\,ds,\end{aligned}$$

where, by definition: $x^n:=x^n(s)$, $A(x^n):=A(s,x^n(s))$, $B(x^n_\tau):=B(s,x^n(\tau(s)))$ and $C(x^n_\rho):=C(s,x^n(\rho(s)))$.

¿From (a.2),

$$\begin{aligned}(4.6)\qquad |x^{n+1}(t)-x^n(t)|^2\leq&|\lambda|\int_0^t|x^{n+1}-x^n||x^n-x^{n-1}|\,ds\\ &+2\left|\int_0^t\left(x^{n+1}-x^n,(C(x^n_\rho)-C(x^{n-1}_\rho))\,dw_s\right)\right|\\ &+\int_0^t\left|\mathrm{tr}\left[\left(C(x^n_\rho)-C(x^{n-1}_\rho)\right)W\left(C(x^n_\rho)-C(x^{n-1}_\rho)\right)^*\right]\right|\,ds\\ &+2\int_0^t|B(x^n_\tau)-B(x^{n-1}_\tau)||x^{n+1}-x^n|\,ds.\end{aligned}$$

Consequently, (4.6) yields

$$\begin{aligned}(4.7)\qquad & E\left[\sup_{0\leq\theta\leq t}|x^{n+1}(\theta)-x^n(\theta)|^2\right]\leq|\lambda|E\int_0^t|x^{n+1}-x^n||x^n-x^{n-1}|\,ds\\ &+\mathrm{tr}(W)E\int_0^t|C(x^n_\rho)-C(x^{n-1}_\rho)|^2\,ds\\ &+2E\int_0^t|B(x^n_\tau)-B(x^{n-1}_\tau)||x^{n+1}-x^n|\,ds\\ &+2E\left[\sup_{0\leq\theta\leq t}\left|\int_0^t\left(x^{n+1}-x^n,(C(x^n_\rho)-C(x^{n-1}_\rho))\,dw_s\right)\right|\right].\end{aligned}$$

Now, we estimate the terms on the right-hand side of (4.7), and we apply the inequality

$$2ab\leq\frac{a^2}{l^2}+l^2b^2\,,\quad a,b\in\mathbf{R}\,,\ \ l>0\,,$$

for suitable l.

$$(4.8)\qquad |\lambda| E\int_0^t |x^{n+1}-x^n||x^n-x^{n-1}|\,ds \le \frac{1}{4}E\left[\sup_{0\le\theta\le t}|x^{n+1}(\theta)-x^n(\theta)|^2\right]+\lambda^2 T\int_0^t E\left[\sup_{0\le\theta\le s}|x^n(\theta)-x^{n-1}(\theta)|^2\right]ds\,.$$

$$(4.9)\qquad \mathrm{tr}(W)E\int_0^t |C(x^n_\rho)-C(x^{n-1}_\rho)|^2\,ds \le \mathrm{tr}(W)k_2^2\int_0^t E\left[\sup_{0\le\theta\le s}|x^n(\theta)-x^{n-1}(\theta)|^2\right]ds\,.$$

$$(4.10)\qquad 2E\int_0^t |B(x^n_\tau)-B(x^{n-1}_\tau)||x^{n+1}-x^n|\,ds \le \frac{1}{4T}E\int_0^t |x^{n+1}-x^n|^2\,ds+4k_1^2TE\int_0^t |x^n(\tau(s))-x^{n-1}(\tau(s))|^2\,ds \le \frac{1}{4}E\left[\sup_{0\le\theta\le t}|x^{n+1}(\theta)-x^n(\theta)|^2\right]+4k_1^2T\int_0^t E\left[\sup_{0\le\theta\le s}|x^n(\theta)-x^{n-1}(\theta)|^2\right]ds\,.$$

Burkholder–Davis–Gundy's inequality implies

$$(4.11)\qquad 2E\left[\sup_{0\le\theta\le t}\left|\int_0^t \left(x^{n+1}-x^n,(C(x^n_\rho)-C(x^{n-1}_\rho))\,dw_s\right)\right|\right] \le 6\mathrm{tr}\,(W)E\left[\left(\sup_{0\le\theta\le t}|x^{n+1}(\theta)-x^n(\theta)|^2\right)\int_0^t |C(x^n_\rho)-C(x^{n-1}_\rho)|^2\,ds\right]^{1/2} \le \frac{1}{4}E\left[\sup_{0\le\theta\le t}|x^{n+1}(\theta)-x^n(\theta)|^2\right]+36k_2^2\mathrm{tr}\,(W)\int_0^t E\left[\sup_{0\le\theta\le s}|x^n(\theta)-x^{n-1}(\theta)|^2\right]ds\,.$$

If we set

$$(4.12)\qquad \varphi^n(t)=E\left[\sup_{0\le\theta\le t}|x^{n+1}(\theta)-x^n(\theta)|^2\right],$$

then, (4.7) – (4.11) yield

$$(4.13)\qquad \varphi^n(t)\le\frac{3}{4}\varphi^n(t)+(\lambda^2T+k_2^2\mathrm{tr}(W)+4k_1^2T+36k_2^2)\int_0^t \varphi^{n-1}(s)\,ds\,,$$

and so, there exists $k>0$ such that

$$(4.14)\qquad \varphi^n(t)\le k\int_0^t \varphi^{n-1}(s)\,ds\,.$$

By iteration from (4.14), we get

$$(4.15)\qquad \varphi^n(t)\le\frac{k^{n-1}T^{n-1}}{(n-1)!}\varphi^1(T)\,,\quad \forall n>1\,,\quad \forall t\in[0,T]\,.$$

Therefore,

$$(4.16)\qquad E\left[\sup_{0\le\theta\le T}|x^{n+1}(\theta)-x^n(\theta)|^2\right]\le\frac{k^{n-1}T^{n-1}}{(n-1)!}\varphi^1(T)\,,\quad \forall n>1\,.$$

Obviously, (4.16) implies that $\{x^n\}$ is a Cauchy sequence in $L^2(\Omega; C(0,T;H))$.

STEP 2.– *The sequence* $\{x^n\}$ *is bounded in* $I^p(0,T;V)$.

Indeed, Itô's formula for $|x^n|^2$, with $n \geq 2$, yields

$$
\begin{aligned}
(4.17) \qquad & E|x^n(T)|^2 + 2E\int_0^T \langle A(x^n), x^n\rangle\, ds + \lambda E\int_0^T |x^n|^2\, ds \\
&= E|x_0|^2 - 2E\int_0^T (B(x_\tau^{n-1}), x^n)\, ds - 2E\int_0^T (f, x^n)\, ds \\
&+ \lambda E\int_0^T (x^n, x^{n-1})\, ds + E\int_0^T \operatorname{tr}\left[\left(C(x_\rho^{n-1}) + g\right) W \left(C(x_\rho^{n-1}) + g\right)^*\right] ds\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(4.18) \qquad & 2E\int_0^T \langle A(x^n), x^n\rangle\, ds + \lambda E\int_0^T |x^n|^2\, ds \\
&\leq E|x_0|^2 + 2E\int_0^T |B(x_\tau^{n-1})||x^n|\, ds + 2E\int_0^T |f||x^n|\, ds \\
&+ |\lambda| E\int_0^T |x^n||x^{n-1}|\, ds + \operatorname{tr}(W) E\int_0^T |C(x_\rho^{n-1}) + g|^2\, ds\,.
\end{aligned}
$$

Since $\{x^n\}$ is convergent in $L^2(\Omega; C(0,T;H)$, it will be bounded in this space. Now, it is not difficult to check that there exists a positive constant, k', such that the right-hand side of (4.18) is bounded by this constant. As an example, we will estimate one of those terms:

$$
\begin{aligned}
2E\int_0^T |B(x_\tau^{n-1})||x^n|\, ds &\leq 2k_1 E\int_0^T |x^{n-1}(\tau(s))||x^n(s)|\, ds \\
&\leq k_1 E\int_0^T \left[|x^{n-1}(\tau(s))|^2 + |x^n(s)|^2\right] ds \\
&\leq k_1 E\int_0^T \left[\sup_{0\leq\theta\leq T} |x^{n-1}(\theta)|^2 + \sup_{0\leq\theta\leq T} |x^n(\theta)|^2\right] ds \\
&\leq Tk_1 \left[E\left(\sup_{0\leq\theta\leq T} |x^{n-1}(\theta)|^2\right) + E\left(\sup_{0\leq\theta\leq T} |x^n(\theta)|^2\right)\right] \\
&\leq Tk_1 \left(\|x^n\|^2_{L^2(\Omega;C(0,T;H))} + \|x^{n-1}\|^2_{L^2(\Omega;C(0,T;H))}\right).
\end{aligned}
$$

This fact, (4.18) and (a.1) lead to the following inequalities:

$$
(4.19) \qquad \alpha\int_0^T E\|x^n(s)\|^p\, ds \leq 2E\int_0^T \langle A(x^n), x^n\rangle\, ds + \lambda E\int_0^T |x^n|^2\, ds \leq k'\,,
$$

and Step 2 is proved.

STEP 3.– *We can take limits in* (4.4).

Indeed, from Step 1, $x^n \to x$, for some x in $L^2(\Omega; C(0,T;H))$. Since (b.2) and (c.2) hold, we also have $B(x_\tau^n) \to B(x_\tau)$ (in $L^2(\Omega; L^\infty(0,T;H))$), and $C(x_\rho^n) \to C(x_\rho)$ (in $L^2(\Omega; L^\infty(0,T;\mathcal{L}(K,H)))$).

On the other hand, by Step 2, $\{x^n\}$ has a subsequence which is weakly convergent in $I^p(0,T;V)$. But, since $x^n \to x$ in $L^2(\Omega; C(0,T;H))$, we can assure that $x^n \to x$ weakly in

$I^p(0,T;V)$ (in the sequel, we will denote this by $x^n \rightharpoonup x$ in $I^p(0,T;V)$). Nevertheless, it follows from (a.3) that $\{A(x^n)\}$ is bounded in $L^{p'}(\Omega\times(0,T);V')$ (with p' such that $(1/p)+(1/p')=1$), since

$$\int_0^T E\|A(t,x^n(t))\|_*^{p/(p-1)}\,dt \le \beta\int_0^T E\|x^n(t)\|^p\,dt \le k'/\alpha\,.$$

Therefore, from each subsequence of $\{A(x^n)\}$, we can get another subsequence weakly convergent in $L^{p'}(\Omega\times(0,T);V')$. Now, we will see that all the limits of different subsequences coincide. Indeed, let v_1, v_2 be two limits of two different subsequences. Since $x^n \to x$ in $L^2(\Omega;C(0,T;H))$, $B(x^n_\tau)\to B(x_\tau)$ in $L^2(\Omega;L^\infty(0,T;H))$ and $C(x^n_\rho)\to C(x_\rho)$ in $L^2(\Omega;L^\infty(0,T;\mathcal{L}(K,H)))$ then, (4.4) implies that all the sequence $\displaystyle\int_{t_1}^{t_2} A(s,x^n(s))\,ds$ converges in $L^1(\Omega;V')$ for all $t_1,t_2\in[0,T]$, and hence,

$$\int_{t_1}^{t_2} v_1(s)\,ds = \int_{t_1}^{t_2} v_2(s)\,ds \quad \forall t_1,t_2\in[0,T] \quad (\text{equality in } L^{p'}(\Omega;V')).$$

¿From this, it holds that $v_1=v_2$ in $L^{p'}(\Omega\times(0,T);V')$ and, finally, $A(x^n)\rightharpoonup v$ weakly in $L^{p'}(\Omega\times(0,T);V')$. In conclusion, we have proved:

$$x^n \to x \text{ in } L^2(\Omega;C(0,T;H)) \tag{4.20}$$

$$B(x^n_\tau)\to B(x_\tau) \text{ in } L^2(\Omega;L^\infty(0,T;H)) \tag{4.21}$$

$$C(x^n_\rho)\to C(x_\rho) \text{ in } L^2(\Omega;L^\infty(0,T;\mathcal{L}(K,H))) \tag{4.22}$$

$$x^n \rightharpoonup x \text{ weakly in } I^p(0,T;V) \tag{4.23}$$

$$A(x^n)\rightharpoonup v \text{ weakly in } L^{p'}(\Omega\times(0,T);V') \tag{4.24}$$

STEP 4.– *We take limits in* (4.4).

Since (4.20) – (4.24) hold, we can take limits in (4.4), and we obtain

$$\begin{aligned} x(t)+\int_0^t v(s)\,ds+\int_0^t B(s,x(\tau(s)))\,ds+\int_0^t f(s)\,ds \\ = x_0+\int_0^t C(s,x(\rho(s)))\,dw_s+\int_0^t g(s)\,dw_s. \end{aligned} \tag{4.25}$$

Thus, to finish the proof, it is sufficient to see that $A(s,x(s))=v(s)$ in $L^{p'}(\Omega\times(0,T);V')$.

¿From (4.17), it follows

$$\begin{aligned} 2E\int_0^T\langle A(x^n),x^n\rangle\,ds &= -\lambda E\int_0^T|x^n|^2\,ds+E|x_0|^2-E|x^n(T)|^2 \\ &-2E\int_0^T(B(x^{n-1}_\tau),x^n)\,ds-2E\int_0^T(f,x^n)\,ds \\ &+\lambda E\int_0^T(x^n,x^{n-1})\,ds+E\int_0^T \mathrm{tr}\,\Big[\big(C(x^{n-1}_\rho)+g\big)\,W\,\big(C(x^{n-1}_\rho)+g\big)^*\Big]\,ds\,. \end{aligned} \tag{4.26}$$

$(4.20) - (4.22)$ imply the existence of $\lim_{n\to\infty} 2E\int_0^T \langle A(x^n), x^n\rangle\, ds$, and

$$
\begin{aligned}
(4.27) \qquad \lim_{n\to\infty} 2E\int_0^T \langle A(x^n), x^n\rangle\, ds =& E|x_0|^2 - E|x(T)|^2 - 2E\int_0^T (f,x)\, ds \\
& - 2E\int_0^T (B(x_\tau), x)\, ds \\
& + E\int_0^T \mathrm{tr}\, \left[(C(x_\rho)+g)\, W\, (C(x_\rho)+g)^*\right]\, ds\,.
\end{aligned}
$$

But, (4.25) and Theorem 4.1 yield

$$
(4.28) \qquad \lim_{n\to\infty} E\int_0^T \langle A(x^n), x^n\rangle\, ds = E\int_0^T \langle v, x\rangle\, ds\,.
$$

¿From (a.2), we get

$$
(4.29) \qquad E\int_0^T \langle A(x^n) - A(z), x^n - z\rangle\, ds + \lambda E\int_0^T |x^n - z|^2\, ds \geq 0
$$

for all $z \in L^p(\Omega\times(0,T);V)\cap L^2(\Omega\times(0,T);H)$. Nevertheless, (4.20), (4.23), (4.24) allow us to take limits in (4.29) and, it holds

$$
(4.30) \qquad E\int_0^T \langle v - A(z), x - z\rangle\, ds + \lambda E\int_0^T |x - z|^2\, ds \geq 0\,.
$$

Now, if we set $z = x - \theta z_2$ (for $\theta > 0$, $z_2 \in L^p(\Omega\times(0,T);V)\cap L^2(\Omega\times(0,T);H)$), we get

$$
(4.31) \qquad E\int_0^T \langle v - A(x - \theta z_2), \theta z_2\rangle\, ds + \lambda\theta^2 E\int_0^T |z_2|^2\, ds \geq 0\,.
$$

In (4.31), we divide by θ, we take limit when $\theta \to 0$ and we use the hemicontinuity (a.4) to obtain:

$$
(4.32) \qquad E\int_0^T \langle v - A(x), \theta z_2\rangle\, ds \geq 0\,, \quad \forall z_2 \in L^p(\Omega\times(0,T);V)\cap L^2(\Omega\times(0,T);H)\,,
$$

and so, $v = A(x)$ ∎

Remark: We observe that theorem 4.2 also holds when V is a separable and reflexive Banach space with $V \hookrightarrow H$.

5. Some extensions of the results

First, we shall extend theorem 4.2 to the case in which ρ, τ take negative values, since this situation usually appears in the applications. Clearly, we have to fix not only the initial value (for $t = 0$) for the solution of (PC), but also for negative t. In fact, we have the following result:

Theorem 5.1

Assume the hypotheses in theorem 4.2, but changing $(\rho.\tau)$ *by the following:*

$$
\exists\, h > 0 \ \textit{such that} \quad -h \leq \tau(t), \rho(t) \leq t\,, \ \forall t \in [0,T]\,,
$$

and let ψ be a process such that $\psi \in I^p(-h,0;V) \cap L^2(\Omega;C(-h,0;H))$ (where these spaces are defined in the obvious manner, setting $\mathcal{F}_t = \mathcal{F}_0$, $\forall t \in [-h,0]$). Then, there exists a unique process $x \in I^p(-h,T;V) \cap L^2(\Omega;C(-h,T;H))$ such that,

$$(PC)' \qquad \begin{cases} x(t) + \int_0^t \left[A(s,x(s)) + B(s,x(\tau(s))) + f(s)\right] ds \\ \qquad = \psi(0) + \int_0^t \left[C(s,x(\rho(s))) + g(s)\right] dw_s\,, \quad P-\text{a.s.}, \ \forall t \in [0,T], \\ x(t) = \psi(t)\,, \quad t \in (-h,0] \end{cases}$$

Proof. We can rewrite the equation $(PC)'$ in an equivalent form and then we apply theorem 4.2. Indeed, observe that, for example, we can split the term $\int_0^t B(s,x(\tau(s)))\,ds$ into two other terms, as follows:

$$\begin{aligned} \int_0^t B(s,x(\tau(s)))\,ds &= \int_0^t B(s,x(\tau(s)))\mathbf{1}_{\{s:\ \tau(s)\geq 0\}}\,ds \\ &\quad + \int_0^t B(s,x(\tau(s)))\mathbf{1}_{\{s:\ \tau(s)<0\}}\,ds \\ &= \int_0^t B_1(s,x(\tau(s)))\,ds \\ &\quad + \int_0^t B(s,\psi(\tau(s)))\mathbf{1}_{\{s:\ \tau(s)<0\}}\,ds\,, \end{aligned}$$

where B_1 is given (for $x \in H$, $s \in [0,T]$) by

$$B_1(s,x) = \begin{cases} B(s,x) & \text{if } \ \tau(s) \geq 0 \\ 0 & \text{si } \ \tau(s) < 0\,. \end{cases}$$

This operator B_1 obviously satisfies the same hypotheses as B, and the function f_1 given by $f_1(s) = B(s,\psi(\tau(s)))\mathbf{1}_{\{s:\ \tau(s)<0\}}$, belongs to $I^2(0,T;H)$. Also, we can do the same with the other term, and we can apply theorem 4.2 to the equation

$$\begin{aligned} x(t) &+ \int_0^t \left[A(s,x(s)) + B_1(s,x(\tau(s))) + f(s) + f_1(s)\right] ds \\ &= \psi(0) + \int_0^t \left[C_1(s,x(\rho(s))) + g(s) + g_1(s)\right] dw_s\,, \quad P-\text{a.s.}, \ \forall t \in [0,T], \quad \blacksquare \end{aligned}$$

Finally, we can extend theorem 4.2 (and so, theorem 5.1) to the case in which there are, instead the family $A(t,.)$, a finite sum of families of operators of this type. In particular, we consider q real, reflexive and separable Banach spaces V_i, $i = 1,2,...,q$, such that $V_i \hookrightarrow H \hookrightarrow V_i'$ for all $1 \leq i \leq q$. Let $V = \cap_{i=1}^q V_i$ with the norm $\|x\| = \sum_{i=1}^q \|x\|_{V_i}$. Then, if we suppose that V is separable and dense in H, we will have $V \hookrightarrow H \hookrightarrow V'$. Let $p_1, p_2, ...p_q$ be real numbers with $p_i > 1$, for all $1 \leq i \leq q$. Now, we consider the families $A_i(t,.) : V_i \to V_i'$, for all $1 \leq i \leq q$, verifying (a.1)–(a.5), for some $\alpha_i > 0$, β_i, $\lambda_i \in \mathbf{R}$. Let $A(t,.) : V \to V'$ be defined by $A(t,x) = \sum_{i=1}^q A_i(t,x)$. Then, we get the following result:

Theorem 5.2

In the precedent situation for the family $A(t,.)$ *, if we assume the hypotheses in theorem 5.1 for* B , C , f , g , ρ , τ *, and we consider a process*

$$\psi \in \bigcap_{i=1}^{q} I^{p_i}(-h,0;V_i) \cap L^2(\Omega; C(-h,0;H)) ,$$

then, there exists a unique process solution of $(PC)'$ *in* $\cap_{i=1}^{q} I^{p_i}(-h,T;V_i) \cap L^2(\Omega; C(-h,T;H))$.

∎

6. Examples

Let $\mathcal{O}$ be an open subset of $\mathbf{R}^N$ with regular boundary (for example, of class C^2). We consider the Sobolev spaces $W^{1,p}(\mathcal{O})$, $W_0^{1,p}(\mathcal{O})$, and when $p=2$, it is usual to denote these spaces by $H^1(\mathcal{O})$, $H_0^1(\mathcal{O})$ respectively. In this Section, $\mathcal{O}$ will be bounded and we denote $H = L^2(\mathcal{O})$. Observe that, when $\mathcal{O}$ is bounded, then $L^p(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ and $W^{1,p}(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$, $\forall p \geq 2$.

Let us see some motivating examples in order to justify our work.

Example 6.1.– An equation arising in population biology.

Let $\mathcal{O}$ be a given geographic domain (also called habitat) and suppose that we want to study the evolution of the frequency of a certain genetic character from the population inside $\mathcal{O}$. If $u(t,x)$ denotes the frequency at the instant t and the point $x \in \mathcal{O}$, Fleming [7], from discrete models in time and space, proposes the stochastic PDE

$$\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) + F(u(t,x)) = \xi(t,x) \, , \quad t \geq 0 \, , \quad x \in \mathcal{O} \tag{6.1}$$

for modelling the evolution of that phenomenon.

The term $\Delta u = \sum_{1 \leq i \leq N} \frac{\partial^2 u}{\partial x_i^2}$, in (6.1), depends on the migratory movements of the population in $\mathcal{O}$ following a normal law of dispersion (this is also the usual hypothesis in deterministic models). We assume that F takes the form

$$F(u(t,x)) = \beta(t,x)|u(t,x)|u(t,x) + \gamma(t,x)u(t,x) \, ,$$

where β , $\gamma \in L^\infty((0,T) \times \mathcal{O})$, for a fixed $T > 0$. This term can be interpreted as the selective advantage (or disadvantage) of the gene that we are studying. Finally, $\xi(t,x)$ (also called the stochastic genetic derivative) shows the random fluctuactions in the transfer of genetic characters from the present to the rising generation. In fact, ξ usually has the form $\xi(t,x) = C(u(t,x))\dot{w}_t$, where w_t is a Wiener process and C is a Hilbert valued operator.

In conclusion, from (6.1) we can state that the evolution of the frequency is determined by:

a) The diffusion Δu of the gene because of the migratory movements.

b) The production of the gene $-F(u)$ due to the actual population and the dispersion velocity.

c) The random alterations in the hereditary process: $\xi(t,x)$.

However, in many situations the random alterations in c) depends on the frequency evaluated in a previous instant $\rho(t)\,(\leq t)$, since the past history of the process determines the future behaviour. Thus, in this case the equation (6.1) takes the following form:

$$\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) + F(u(t,x)) = C(u(\rho(t),x))\,\dot{w}_t\,, \quad t \geq 0\,, \quad x \in \mathcal{O}\,. \tag{6.2}$$

Now, we are going to put this problem in a suitable form for applying our theory.

Let $V_1 = H_0^1(\mathcal{O})$, $V_2 = L^3(\mathcal{O})$, $K = \mathbf{R}$, w_t a standard Wiener process and the operator families $A_1(t,.) : V_1 \to V_1'$, $A_2(t,.) : V_2 \to V_2' = L^{3/2}(\mathcal{O})$ defined by

$$\begin{aligned}\langle A_1(t,u),v\rangle = \sum_{i,j=1}^{N} \int_{\mathcal{O}} \alpha_{ij}(t,x)\frac{\partial u(t,x)}{\partial x_i}\frac{\partial v(t,x)}{\partial x_j}\,dx \\ + \int_{\mathcal{O}} \gamma(t,x)u(x)v(x)\,dx\,, \quad u,v \in V_1\end{aligned}$$

and, for $u \in V_2$, $A_2(t,u)(x) = \beta(t,x)|u(x)|u(x)$ a.e. in $\mathcal{O}$, where α_{ij}, β, $\gamma \in L^\infty((0,T)\times\mathcal{O})$ and there exist positive numbers α and β such that

$$\sum_{i,j=1}^{N} \alpha_{ij}(t,x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{N} \xi_i^2 \;\text{ a.e. in } (0,T)\times\mathcal{O}\,,\; \forall(\xi_1,...,\xi_N) \in \mathbf{R}^N\,, \tag{6.3}$$

$$\beta(t,x) \geq \beta > 0 \;\text{ a.e. in } (0,T)\times\mathcal{O}\,. \tag{6.4}$$

Let $C : H \to H$ be given by $C(u)(x) = \varphi(u(x))$ for u in H and x in $\mathcal{O}$, where $\varphi : \mathbf{R} \to \mathbf{R}$ verifies $\varphi(0) = 0$ and there exists a positive constant c such that

$$|\varphi(x) - \varphi(y)| \leq c|x-y| \quad \forall x,y \in \mathbf{R}\,. \tag{6.5}$$

¿From (6.3), it is easy to check that conditions (a.1)–(a.5) hold for A_1. ¿From (6.4), it follows that A_2 also verifies (a.1)–(a.5). Finally, (6.5) yield (c.1)–(c.3).

Consequently, given ρ as in theorem 5.1 (for instance, $\rho(t) = t-h$) and ψ in $I^2(-h,0;V_1)\cap I^3(-h,0;V_2)\cap L^2(\Omega;C(-h,0;H))$, there exists a unique process u in $I^2(-h,T;V_1)\cap I^3(-h,T;V_2)\cap L^2(\Omega;C(-h,T;H))$ such that

$$\begin{cases} du(t,x) - \displaystyle\sum_{ij=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(t,x)\frac{\partial u(t,x)}{\partial x_j}\right) dt + \gamma(t,x)u(t,x)\,dt \\ +\beta(t,x)|u(t,x)|u(t,x)\,dt = \varphi(u(\rho(t),x))\,dw_t\,, \;\text{ in } (0,T)\times\mathcal{O}\,, \\ u(t,x) = \psi(t,x)\,, \;\text{ in } (-h,0)\times\mathcal{O}\,, \\ u(t,x) = 0\,, \;\text{ in } (-h,T)\times\partial\mathcal{O}\,. \end{cases} \tag{6.6}$$

Remark: The equation in (6.6) is a distributed parameter stochastic version of a classic equation arising in population biology:

$$dN(t) = aN(t)(1 - bN(t))\,dt + N(\rho(t))\,dw_t\,,\quad a, b \in \mathbf{R}\,.$$

Observe that our motivating problem is included here taking $\alpha_{ij} = 1$. The condition $u(t,x) = 0$ in $(-h,T) \times \partial\mathcal{O}$ can be interpreted as the character rarely appears in $\partial\mathcal{O}$ since outside $\mathcal{O}$ there exists a population without that character.

Remark: We note that (6.6) is also used for modelling the diffusion of a product concentration across a membrane (see Viot [16]). For that, we have to set

$$F(u) = \sigma \frac{u}{1 + |u|}\,,\ \sigma > 0\,,\ C(u) = u\,,$$

and so, given $f \in I^2(-h,0;H)$, ρ as in theorem 5.1, $\psi \in I^2(-h,0;H_0^1(\mathcal{O})) \cap L^2(\Omega; C(-h,0;H))$ and w_t a $L^2(\mathcal{O})$–Wiener process with incremental covariance W with kernel $q \in L^\infty(\mathcal{O}\times\mathcal{O})$ (see Viot [16] for details), there exists a unique process u in $I^2(-h,T;H_0^1(\mathcal{O})) \cap L^2(\Omega; C(-h,T;H))$ such that

$$\begin{cases} du(t,x) - \Delta u(t,x)\,dt + \sigma \dfrac{u(t,x)}{1 + |u(t,x)|}\,dt + f(t)\,dt = u(\rho(t),x)\,dw_t\,,\ \text{ in } (0,T)\times\mathcal{O}\,, \\ u(t,x) = \psi(t,x)\,,\ \text{ in } (-h,0)\times\mathcal{O}\,, \\ u(t,x) = 0\,,\ \text{ in } (-h,T)\times\partial\mathcal{O}\,. \end{cases}$$

Example 6.2.– An example for applying theorem 5.2.

Let $V_1 = H_0^1(\mathcal{O})$, $V_2 = L^4(\mathcal{O})$, $p_1 = 2$, $p_2 = 4$. We consider the operators $A_1 : V_1 \to V_1'$ and $A_2 : V_2 \to V_2' = L^{4/3}(\mathcal{O})$ defined as

$$\begin{aligned} \langle A_1 u, v\rangle &= \sum_{i=1}^N \int_{\mathcal{O}} \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x)\,dx\,,\ \forall u, v \in V_1 \\ A_2(u) &= u^3\,,\ \forall u \in V_2\,. \end{aligned}$$

Let $K = \mathbf{R}$, w_t a standard Wiener process and let B, C be given by $B(u)(x) = \varphi_1(u(x))$, $C(u)(x) = \varphi_2(u(x))$, $\forall u \in H$, where $\varphi_1, \varphi_2 : \mathbf{R} \to \mathbf{R}$ verifies $\varphi_1(0) = \varphi_2(0) = 0$, and there exists positive constants c_1, c_2 such that

$$|\varphi_i(x) - \varphi_i(y)| \le c_i|x - y|\,,\ \forall x, y \in \mathbf{R}\,, i = 1, 2\,.$$

It is easy to prove that (a.1)–(c.3) hold with $p = 2$, $\lambda = 0$, $\alpha = 2$. Consequently, for $f = g = 0$, $\psi \in I^2(-h,0;V_1) \cap I^4(-h,0;V_2) \cap L^2(\Omega; C(-h,0;H))$, and ρ, τ as in theorem 5.1, we get that there exists a unique process u in

$$I^2(-h,T;V_1) \cap I^4(-h,T;V_2) \cap L^2(\Omega; C(-h,T;H)),$$

such that

$$
\begin{aligned}
du(t,x) - \Delta u(t,x)\,dt + u^3(t,x)\,dt + \varphi_1(u(\tau(t),x))\,dt& \\
= \varphi_2(u(\rho(t),x))\,dw_t\,, \ \text{ in } \ (0,T)\times\mathcal{O}\,,& \\
u(t,x) = \psi(t,x) \ \text{ in } \ (-h,0)\times\mathcal{O}\,,& \\
u(t,x) = 0 \ \text{ in } \ (-h,T)\times\partial\mathcal{O}\,.&
\end{aligned}
$$

Example 6.3.– A stochastic non–linear monotone parabolic equation.

Let $p>2$. Now, we consider $V=W_0^{1,p}(\mathcal{O})$, and we define $A:V\to V'$ by

$$
\langle A(u),v\rangle = \sum_{i=1}^N \int_{\mathcal{O}} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx + \int_{\mathcal{O}} |u|^{p-2}uv\,dx\,, \quad \forall u,v\in V\,,
$$

and let B, C, ρ, τ, w_t as in example 6.2. It is easy to check that (a.1)–(a.5) hold, with $\alpha = 1/2, \lambda = 0$ and $\beta = N+1$. Consequently, given $\psi \in I^p(-h,0;V)\cap L^2(\Omega;C(-h,0;H))$ and f, $g\in I^2(0,T;H)$, there exists a unique process u in $I^p(-h,T;V)\cap L^2(\Omega;C(-h,T;H))$ solution of $(PC)'$. In other words,

$$
\begin{aligned}
du(t,x) = {}& \left(\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u(t,x)}{\partial x_i}\right|^{p-2}\frac{\partial u(t,x)}{\partial x_i}\right)dt + |u(t,x)|^{p-2}u(t,x)\right)dt \\
& + (\varphi_1(u(\tau(t),x)) + f(t,x))\,dt + (\varphi_2(u(\rho(t),x)) + g(t,x))\,dw_t\,, \ \text{ in } \ (0,T)\times\mathcal{O}\,, \\
u(t,x) = {}& \psi(t,x)\,, \ \text{ in } \ (-h,0)\times\mathcal{O}\,, \\
u(t,x) = {}& 0\,, \ \text{ in } \ (-h,T)\times\partial\mathcal{O}\,.
\end{aligned}
$$

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