COMPARISON OF THE LONG-TIME BEHAVIOUR OF LINEAR ITO AND STRATONOVICH PARTIAL DIFFERENTIAL EQUATIONS

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To the memory of the first author´s father

Abstract

In this paper, we point out the different long-time behaviour of stochastic partial differential equations when one considers the stochastic term in the Ito or Stratonovich sense. In particular, we prove that the Stratonovich interpretation may not produce modification in the exponential stability of the deterministic model for a wide range of stochastic perturbations, while Ito's one can give different results. In fact, some stabilization or destabilization effect can be obtained.

Keywords: Almost sure exponential stability, stabilization, destabilization, linear Ito PDE, linear Stratonovich PDE.

1 Introduction and statement of the problem

As it is well known, stochastic partial differential equations arise in the modelling of numerous problems from Physics, Biology, Chemistry,... It is assumed that the real phenomenon is better described if one considers a random or stochastic term in the equation of the model. This fact implies that we need to give a sense to the new nondeterministic equation. Two interpretation are the most commonly used in the literature: Ito's stochastic equation and Stratonovich's one. Each interpretation gives a different solution of the stochastic equation, so they provide different answers to the same problem. There exist several reasons which make reasonable both possibilities and there exists a rule which permit us to pass from one kind of equation to the other (see Arnold [1], Oksendal [10], Kunita [9],...). However, when one is analyzing the long-time behaviour of the solutions, special care should be paid to the choice of the model since the solutions of both stochastic equations can have totally different behaviour. This will be the main aim of this paper. Indeed, we are going to show that a linear deterministic model is exponentially stable if and only if when a certain class of Stratonovich noise is added to the problem, this remains exponentially stable. However, when the noise is considered in Ito's sense, several different situations are possible, that is, it may happen that the deterministic and stochastic model are both exponentially stable, or that the deterministic is unstable and the stochastic is stable (stabilization), or even the former is stable while the latter unstable (destabilization). We would like to point out that we are going to restrict ourselves to the consideration of the linear case since, in this important situation, we will be able to prove necessary and sufficient conditions for the stability of the Stratonovich stochastic equation, and criteria for the Ito one, in terms of the stability of the corresponding deterministic one, and making use of simple arguments from the theory of linear semigroups, showing once again the power of this theory in dealing with stochastic problems. No doubt at all, our treatment admits extensions to semilinear and more general nonlinear situations, however, in these cases, only sufficient conditions can be obtained. Due to these reasons, we have preferred to consider the linear case described below.

Now, we are going to give the statement of the problem we shall study in this work.

Let us consider the following deterministic evolution equation

$$\frac{du(t)}{dt} = Au(t),$$

where A is an unbounded linear operator in the real separable Hilbert space H (with norm $|\cdot|$ and inner product $(\cdot,\cdot)$), with domain $D(A)$ dense in H, i.e. $A : D(A) \subset H \longrightarrow H,$ and assume that A is the generator of a c_0-semigroup, $S(t)$. Let $u(t; 0, u_0)$ denote the solution of the following problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) \\ u(0) = u_0 \in H \end{cases} \tag{1}$$

It is well known that $u(t; 0, u_0) = S(t)u_0$, for $u_0 \in D(A)$. Moreover, if $S(t)$ is analytic, then $u(t; 0, u_0) = S(t)u_0$, for all $u_0 \in H$.

Definition 1 *Equation (1) (or the semigroup $S(t)$) is said to be exponentially stable, or simply stable, if there exist $M \geq 1, \gamma > 0$ such that*

$$|S(t)| \leq Me^{-\gamma t}, \quad \forall t \geq 0. \tag{2}$$

The following result is due to Datko [5].

Theorem 2 *The following statements are equivalent:*

i) $S(t)$ *is stable;*

ii) $\int_0^\infty |S(t)x|dt < \infty,$ *for each,* $x \in H$*;*

iii) There exists a self-adjoint nonnegative operator $P \in \mathcal{L}(H)$ *such that*

$$2(Ax, Px) = -(x, x), \text{ for each } x \in D(A).$$

Let us now consider that a linear noise is added to the problem (1). Thus, we can consider, respectively, the Stratonovich and Ito versions of the stochastic equation:

$$\begin{cases} du(t) = Au(t)dt + Bu(t) \circ dw_t \\ \qquad u(0) = u_0 \in H, \end{cases} \tag{3}$$

$$\begin{cases} du(t) = Au(t)dt + Bu(t)dw_t \\ \qquad u(0) = u_0 \in H, \end{cases} \tag{4}$$

where, for the sake of simplicity, we assume that w_t is a real standard Wiener process, defined on the filtered and complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and B is a linear operator (bounded or not) in H. We want to mention that, although our analysis can be extended to the more general situation of considering a sum of the form $\sum_{i=1}^{d} B_i u dw_t^i$ instead of $Budw_t$, where w_t^i are mutually independent standard Wiener processes, we prefer to develop this case for the sake of clarity.

As we are interested in the stability analysis of the solutions to equations (3) and (4), we assume that, for each $u_0 \in H$, there exists a unique strong (and therefore, mild) solution of both problems (see Da Prato-Zabczyk [4] and Kunita [9] for the definitions, properties and conditions under which this holds).

2 The stability of Stratonovich equations

Let $u(t, \omega; 0, u_0)$ denote the unique solution to the problem

$$\begin{cases} du(t) = Au(t)dt + Bu(t) \circ dw_t \\ u(0) = u_0 \in H, \end{cases} \tag{5}$$

where we assume that the operator $B : D(B) \subset H \to H$ is the generator of a c_0-group, denoted $S_B(t)$, satisfying $D(A) \subset D(B)$. We now recall the following definition.

Definition 3 *The zero solution of problem (5) is said to be exponentially asymptotically stable with probability one (w.p.1) if there exist $N \subset \Omega$, $\mathbb{P}(N) = 0$, and $\alpha, \beta > 0$, such that for every $\omega \notin N$, there exists $T(\omega) > 0$ such that*

$$|u(t, \omega; 0, u_0)| \leq \beta |u_0| \exp(-\alpha t), \forall t \geq T(\omega), u_0 \in D(A)$$

The following result holds

Theorem 4 *Assume that A and $S_B(t)$ commute. Then, $u \equiv 0$ is exponentially stable as solution of Eq. (1) if and only if $u \equiv 0$ is exponentially asymptotically stable w.p.1 as solution of (5).*

Proof. Let us make the following change:

$$z(t) := z(t, \omega; 0, u_0) = S_B^{-1}(w_t(\omega))u(t, \omega; 0, u_0). \tag{6}$$

Now, it is not difficult to check that

$$\begin{aligned} dz(t) &= S_B^{-1}(w_t(\omega))du(t) - S_B^{-1}(w_t(\omega))Bu(t) \circ dw_t \\ &= S_B(-w_t(\omega))Au(t)dt \\ &= Az(t)dt, \end{aligned}$$

and $z(0) = u_0$. Consequently, the process $z(t)$ is solution to Eq. (1) $\mathbb{P}-a.s.$, in fact $z(t) = S(t)u_0$, and the solutions to (1) and (5) are related by means of (6). Now, we can prove the theorem.

Assume that problem (1) is exponentially stable. This means that there exist $M \geq 1, \gamma > 0$ such that $|S(t)| \leq Me^{-\gamma t}$, for all $t \geq 0$. On the other hand, as the operator B is the generator of a c_0-group, it is well known (see Pazy [11]) that there exist $\sigma \in \mathbb{R}, b \geq 1$, such that $|S_B(t)| \leq be^{\sigma|t|}$, for all $t \in \mathbb{R}$. Then, since $\mathbb{P} - a.s.$

$$\lim_{t \to +\infty} \frac{|w_t(\omega)|}{t} = 0, \tag{7}$$

there exists $N \subset \Omega, \mathbb{P}(N) = 0$ such that if $\omega \notin N$, then

$$\lim_{t \to +\infty} (\gamma - \sigma \frac{|w_t(\omega)|}{t}) = \gamma,$$

and, there exists $T(\omega)$ such that for all $t \geq T(\omega)$

$$\gamma - \sigma \frac{|w_t(\omega)|}{t} \geq \frac{\gamma}{2}.$$

Thus, given $u_0 \in D(A)$, $\omega \notin N$, and taking into account that $z(t) = S(t)u_0$,

$$\begin{aligned} |u(t, \omega; 0, u_0)| &= |S_B(w_t(\omega))z(t)| \\ &\leq Mbe^{\sigma|w_t(\omega)|}e^{-\gamma t}|u_0| \\ &\leq Mb|u_0|e^{-(\gamma - \frac{\sigma|w_t(\omega)|}{t})t} \\ &\leq Mb|u_0|e^{-\gamma_0 t}, \quad \forall t \geq T(\omega), \end{aligned}$$

where $\gamma_0 = \gamma/2$. Therefore, $u \equiv 0$ is exponentially asymptotically stable w.p.1 as solution of Eq. (5).

Conversely, if B is an operator such that $u \equiv 0$ is exponentially asymptotically stable w.p.1 as solution of Eq. (5), there exist $N_0 \subset \Omega, \mathbb{P}(N_0) = 0$, and $\alpha, \beta > 0$, such that if $\omega \notin N_0$, there exists $T_0(\omega) > 0$ such that

$$|u(t,\omega;0,u_0)| \leq \beta |u_0| e^{-\alpha t}, \quad \forall t \geq T_0(\omega)$$

Now, for a fixed $\omega \notin N_0$, (6) implies

$$\begin{aligned} |S(t)u_0| &\leq |S_B(-w_t(\omega))u(t,\omega;0,u_0)|, \\ &\leq b\beta |u_0| e^{-(\alpha - \frac{\sigma|w_t(\omega)|}{t})t}, \forall t \geq T_0(\omega). \end{aligned}$$

On the other hand, from (7) we can assure the existence of $N_1 \subset \Omega$, $P(N_1) = 0$, such that for $\omega \notin N_1$, there exists $T_1(\omega) > 0$ such that for all $t \geq T_1(\omega)$, it holds

$$\alpha - \frac{\sigma|w_t(\omega)|}{t} \geq \frac{\alpha}{2}.$$

Denoting $N = N_0 \bigcup N_1$ and taking a fixed $\omega \notin N$, it easily follows that

$$|S(t)u_0| \leq M e^{-\gamma t}|u_0|, \quad \forall t \geq \widetilde{T},$$

where $\widetilde{T} = \widetilde{T(\omega)} = \max\{T_0(\omega), T_1(\omega)\}$. The proof is now complete. ■

Remark 5 *Notice that what we have proved is:*

i) *If the deterministic system (1) is exponentially stable, then for all linear stochastic perturbations (in the Stratonovich sense) of the kind considered in Theorem 4, the perturbed system remains exponentially stable.*

ii) *Conversely, if there exists a perturbation of the mentioned kind that makes the stochastic system become exponentially asymptotically stable, then the deterministic system must be exponentially stable too.*

In conclusion, we can afirm that stochastic perturbations of this type do not modify the long-time behaviour of the solutions of the deterministic problem.

Remark 6 *a) Observe that if we consider the particular case* $Bu = \sigma u$*, for some* $\sigma \in \mathbb{R}$*, and all* $u \in H$*, the* c_0-*group* $S_B(t)$ *is given by* $S_B(t) = e^{\sigma t} I$*, and the hypotheses in Theorem 4 are fulfilled. Therefore, the null solution of (1) is exponentially stable iff the null solution to Eq. (5) is exponentially asymptotically stable w.p.1 for all* $\sigma \in \mathbb{R}$*.*

b) It is worth mentioning that, in the finite dimensional case, Arnold [2] proves that the deterministic system $dx(t) = Ax(t)dt$ *can be stabilized by a suitable Stratonovich linear noise if and only if trace* $A < 0$*. However, as far as we know, a similar result in infinite dimension remains as an open question and it seems difficult to develop a similar analysis to the one in Arnold [2]. Nevertheless, if we consider stochastic perturbations in Ito's sense much more can be obtained as we are going to show in the next Section.*

3 The stability of Ito equations

In this Section, we will prove that the long-time behaviour of the deterministic problem can be modified if we consider the stochastic perturbation in the Ito sense.

Let us now consider the Ito equation

$$\begin{cases} du(t) = Au(t)dt + Bu(t)dw_t \\ u(0) = u_0 \in H. \end{cases} \tag{8}$$

It is well known that this is equivalent to the following Stratonovich formulation (see Kunita [9])

$$\begin{cases} du(t) = (Au(t) - \frac{1}{2}B^2u(t))dt + Bu(t) \circ dw_t \\ u(0) = u_0 \in H, \end{cases} \tag{9}$$

where we assume that the linear operator $B : D(B) \subset H \to H$ satisfies $D(A) \subset D(B^2)$ and generates a c_0-group. Let us denote $C = A - \frac{1}{2}B^2$ and assume that C is the generator of a c_0- semigroup $S_C(t)$, which automatically holds if, for instance, $B \in \mathcal{L}(H)$ (see, Pazy [11] or Curtain and Pritchard [3]). By the virtue of the analysis in the preceding Section, we can assure that, under the additional hypothesis of commuting A and $S_B(t)$, Eq. (8) is exponentially stable (or, equivalently, (9) is exponentially stable) if and only if $S_C(t)$ is stable. Consequently, the following three cases are possible:

1. $S(t)$ is stable and the operator B is such that $S_C(t)$ is also stable (the stability of the deterministic system remains under the stochastic perturbation)

2. $S(t)$ is not stable but the operator B makes $S_C(t)$ be stable (a stabilization effect has been produced by the random perturbation)

3. $S(t)$ is stable but $S_C(t)$ is not stable (the deterministic system has been destabilized by the noise)

Let us now prove some result concerning these possibilities in the following subsections.

3.1 Case 1: stability results

We can first prove the following result concerning stability.

Theorem 7 *a) Assume $S(t)$ is stable (i.e. it holds (2)) and $B \in \mathcal{L}(H)$ generates a c_0-group. Then, the semigroup $S_C(t)$ is stable if*

$$|B|^2_{\mathcal{L}(H)} < \frac{2\gamma}{M}.$$

b) In particular, if in addition to the hypotheses in a), the operator B is defined as $B(u) = bu$, for all $u \in H$, where $b \in \mathbb{R}$, then, the semigroup $S_C(t)$ is stable whatever b be.

Proof.

a) It is known (see Curtain-Pritchard [3], Theorem 10.9 page 210) that $S_C(t)$ satisfies

$$|S_C(t)|_{\mathcal{L}(H)} \leq Me^{(-\gamma + \frac{M}{2}|B|^2_{\mathcal{L}(H)})t}.$$

Thus, if $-\gamma + \frac{M}{2}|B|^2_{\mathcal{L}(H)} < 0$, $S_C(t)$ is stable.

b) In the particular case $B(u) = bu$, we have that $S_C(t) = e^{-\frac{b^2}{2}t}S(t)$. So, we easily get that $|S_C(t)| \leq Me^{-(\gamma+\frac{b^2}{2})t}$, and for all $b \in \mathbb{R}$ it follows that $\gamma + \frac{b^2}{2} > 0$. ■

We immediately can prove the following consequence.

Corollary 8 *Assume that the semigroup $S(t)$ is stable and the operator A commutes with the c_0- group generated by $B \in \mathcal{L}(H)$. Then, Eq.(8) is stable provided $|B|^2_{\mathcal{L}(H)} < \frac{2\gamma}{M}$.*

Proof. It follows from a) in the preceding theorem and the equivalence between the stability of semigroup $S_C(t)$, Eq. (9) and (8). ■

Remark 9 *Notice that Theorem 7 gives a different and simpler proof of some results in Haussmann [6] and Ichikawa [7] in the particular case of commuting operators A and $S_B(t)$ (the group generated by B). Moreover, in the case considered in part b) our results improves theirs since we get stability for all $b \in \mathbb{R}$, while they only can assure exponential stability for small values of b.*

3.2 Case 2: stabilization results

Firstly, we are going to prove that when the semigroup $S(t)$ is not stable, we can always choose a suitable linear operator B such that $S_C(t)$ becomes stable.

Theorem 10 *Assume A generates a c_0- semigroup $S(t)$ and consider the linear operator B defined as $Bu = bu$, for some $b \in \mathbb{R}$. Then, the semigroup $S_C(t)$ generated by $C = A - \frac{1}{2}B^2$ is stable provided b is large enough.*

Proof. Indeed, it is known that there exists $M \geq 1$ and $\rho \in \mathbb{R}$ such that $|S(t)| \leq Me^{\rho t}, \forall t \geq 0$. Thus, by similar computations as in part b) in Theorem 7, we can easily get that $|S_C(t)| \leq Me^{(\rho-\frac{b^2}{2})t}, \forall t \geq 0$, and taking b large enough, we obtain that $\rho - \frac{b^2}{2} < 0$, and S_C is therefore stable. ■

However, it is not necessary to set $B(u) = bu$ in the theorem in order to get stabilization, as can be deduced from the next more general result.

It is known (and not difficult to prove) that, if $S(t)$ is a c_0-semigroup with infinitesimal generator A on H, there exists a nonnegative number $\alpha \geq 0$ such that

$$(Ax, x) \leq \alpha|x|^2, \forall x \in D(A),$$

if and only if A generates a c_0-semigroup $S(t)$ such that $|S(t)| \leq e^{\alpha t}, t \geq 0$.

We can now prove the following.

Theorem 11 *Assume that $B : D(B) \subset H \to H$ is a linear (bounded or unbounded) operator with $D(A) \subset D(B)$. Suppose that the two following hypotheses also hold:*

i) There exists $\beta \in \mathbb{R}$ such that

$$(Ax, x) + \frac{1}{2}|Bx|^2 \leq \beta|x|^2, \forall x \in D(A), \tag{10}$$

(which immediately holds when $B \in \mathcal{L}(H)$, by setting $\beta = \alpha + \frac{1}{2}|B|^2$).

ii) There exists $b, \widetilde{b} \in \mathbb{R}, 0 \leq b \leq \widetilde{b}$, such that

$$b|x|^2 \leq (x, Bx) \leq \widetilde{b}|x|^2, \forall x \in D(B). \tag{11}$$

Then, for every $u_0 \in D(A), u_0 \neq 0$, such that the solution $u(t) = u(t, \omega; 0, u_0)$ to Eq. (8) satisfies $|u(t)| \neq 0$, for all $t \geq 0, \mathbb{P}-a.s.$, it holds

$$\limsup_{t\to+\infty} \frac{1}{t} \log|u(t; u_0)|^2 \leq -(b^2 - \beta), \mathbb{P}-a.s.$$

Proof. Take $u_0 \in D(A), u_0 \neq 0$, such that $|u(t)| \neq 0, \mathbb{P} - a.s.$ From Ito's formula we get

$$\begin{aligned} |u(t)|^2 &= |u_0|^2 + 2\int_0^t (u(s), Au(s))ds + 2\int_0^t (u(s), Bu(s))dw_s \\ &\quad + \int_0^t |Bu(s)|^2 ds, \end{aligned}$$

and, once again Ito's formula yields

$$\begin{aligned} \log|u(t)|^2 &= \log|u_0|^2 + 2\int_0^t \frac{(u(s), Au(s)) + \frac{1}{2}|Bu(s)|^2}{|u(s)|^2} ds + 2\int_0^t \frac{(u(s), Bu(s))}{|u(s)|^2} dw_s \\ &\quad -\frac{1}{2}\int_0^t \frac{4(u(s), Bu(s))^2}{|u(s)|^4} ds \\ &\leq \log|u_0|^2 + 2\int_0^t (\beta - b^2)ds + 2\int_0^t \frac{(u(s), Bu(s))}{|u(s)|^2} dw_s. \end{aligned}$$

Observing now that the last term is a continuous local martingale vanishing at $t = 0$, and taking into account that $(x, Bx) \leq \widetilde{b}|x|^2, \forall x \in D(B)$, it easily follows from the law of the iterated logarithm that

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \frac{(u(s), Bu(s))}{|u(s)|^2} dw_s = 0, \mathbb{P} - a.s.$$

and, consequently,

$$\limsup_{t\to+\infty} \frac{1}{t} \log |u(t)|^2 \leq -(b^2 - \beta), \mathbb{P} - a.s. \blacksquare$$

Remark 12 *Notice that in the preceding proof we have not used the the linearity of operators A and B. Consequently, the theorem remains true for more general nonlinear operators satisfying (10) and (11).*

Corollary 13 *In addition to the hypotheses in Theorem 11, assume that A and the group $S_B(t)$ commute and that $b^2 - \beta > 0$. Then, the semigroup $S_C(t)$ is stable.*

Proof. It follows from the equivalence between the stability of the semigroup $S_C(t)$, Eq. (9) and (8). $\blacksquare$

Example.

As an application of these results, we shall exhibit the situation considered by Kwiecinska [8], and we are going to obtain the same stabilization result. Consequently, the result in [8] is only a particular case of our more general setting.

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d (d \leq 3)$ with $C^\infty-$boundary, and consider the following stochastic heat equation:

$$\begin{cases} du(t,x) = (\Delta u(t,x) + \alpha u(t,x))dt + \gamma u(t,x)dw_t, t > 0, x \in \mathcal{O}, \\ u(t,x) = 0, t > 0, x \in \partial\mathcal{O}, \\ u(0,x) = u_0(x), x \in \mathcal{O}, \end{cases} \tag{12}$$

where Δ denotes the Laplacian operator. Let us consider $H = L^2(\mathcal{O})$ and denote $A = \Delta + \alpha I$, $B = \gamma I$. Then, $D(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$. Let $\lambda_0 > 0$ denote the first eigenvalue of the Laplacian operator. It is then well known that

$$|v| \leq \lambda_0^{-1/2} ||v||, \forall v \in H_0^1(\mathcal{O}),$$

where $||\cdot||$ denotes the usual norm in $H_0^1(\mathcal{O})$. Taking these facts into account, it is not difficult to obtain that,

$$(Av, v) + \frac{1}{2}|Bv|^2 \leq (\alpha - \lambda_0 + \frac{1}{2}\gamma^2)|v|^2, \ \forall v \in D(A),$$

and

$$(v, Bv) = \gamma|v|^2, \forall v \in H.$$

So, we can apply Theorem 11, taking $\beta = \alpha - \lambda_0 + \frac{1}{2}\gamma^2$ and $b = \gamma$. Therefore, we obtain exponential stability w.p.1 if $b^2 - \beta > 0$, or equivalently if

$$2(\alpha - \lambda_0) - \gamma^2 < 0.$$

If $\alpha < \lambda_0$, which means that the null solution of the deterministic equation (i.e. Eq. (12) with $\gamma = 0$) is exponentially stable, then for all $\gamma \in \mathbb{R}$, the null solution of the stochastic equation (12) remains exponentially stable w.p.1.

But, if $\alpha > \lambda_0$ (i.e. the zero solution of the deterministic heat equation is not stable), we can choose γ large enough, such that $2(\alpha - \lambda_0) - \gamma^2 < 0$, and the trivial solution of the deterministic equation becomes exponentially stable w.p.1.

3.3 Case 3: destabilization

Observe that, in our preceding analysis which refers to cases 1 and 2, when the operator B is bounded (or unbounded but satisfying coercivity (10) and condition (11) with some $b > 0$), we have proved some stability and stabilization results. If we now consider the possibility of being $(x, Bx) = 0$, for all $x \in D(B)$, then even destabilization can be obtained. Let us illustrate this with the following example.

Consider the following heat equation in one dimension:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \nu \frac{\partial^2 u(t,x)}{\partial x^2} + r_0 u(t,x), t > 0, 0 < x < \pi, \\ u(t,0) = u(t,\pi) = 0, t > 0, \\ u(0,x) = u_0(x), x \in [0,\pi]. \end{cases} \tag{13}$$

This problem can be formulated in our framework by setting $H = L^2([0,\pi])$, $A = \nu \frac{\partial^2}{\partial x^2} + r_0$, and it follows that $D(A) = H_0^1([0,\pi]) \cap H^2([0,\pi])$ (see Haussmann [6] or Ichikawa [7] for similar situations). It is clear that this problem can be explicitly solved yielding to

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-(\nu n^2 - r_0)t} \sin nx,$$

where $u_0(x) = \sum_{n=1}^{\infty} a_n \sin nx$. Hence, we obtain exponential stability if and only if $r_0 < \nu n^2$ for all $n \in \mathbb{N}$, i.e. iff $r_0 < \nu$.

Consider now the stochastically perturbed problem

$$\begin{cases} du(t,x) = Au(t,x)dt + Bu(t,x)dw_t \\ u(0,x) = u_0(x), \end{cases} \tag{14}$$

where B is the operator defined as $Bu(x) = \delta \frac{\partial u(x)}{\partial x}$, for $u \in H_0^1([0,\pi]), \delta \in \mathbb{R}$. Clearly, operators A and B satisfy the hypotheses in Theorem 11 with $b = 0$ and $\beta = \frac{\delta^2}{2} - \nu + r_0 \geq 0$, if we choose δ such that $\frac{\delta^2}{2} < \nu$ and $\frac{\delta^2}{2} - \nu + r_0 \geq 0$. Indeed, it easily follows $(Bv, v) = 0$, for all $v \in H_0^1([0,\pi])$, and

$$(Av, v) + \frac{1}{2}|Bv|^2 \leq (\frac{\delta^2}{2} - \nu + r_0)|v|^2, \forall v \in H_0^1([0,\pi]) \cap H^2([0,\pi]).$$

Thus, Theorem 11 implies

$$\limsup_{t\to\infty} \frac{1}{t} \log |u(t,u_0)|^2 \leq \beta, \mathbb{P} - a.s.$$

so, in general, we can not assure exponential stability. Moreover, what happens in this occasion is that the semigroup generated by $C = A - \frac{1}{2}B^2$ is not stable. Indeed, observe that the stability of problem (14) is equivalent to the stability of

$$\begin{cases} du(t,x) = Cu(t,x)dt + Bu(t,x) \circ dw_t \\ u(0,x) = u_0(x), \end{cases}$$

or, by the virtue of Theorem 4, to the stability of the deterministic problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = (\nu - \frac{b^2}{2})\frac{\partial^2 u(t,x)}{\partial x^2} + r_0 u(t,x), t > 0, 0 < x < \pi, \\ u(t,0) = u(t,\pi) = 0, t > 0, \\ u(0,x) = u_0(x), x \in [0,\pi], \end{cases}$$

which is exponentially stable iff $r_0 < \nu - \frac{\delta^2}{2}$. So, the noise has destabilized the deterministic exponentially stable system.

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