ASYMPTOTIC STABILITY OF NONLINEAR STOCHASTIC EVOLUTION EQUATIONS

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Abstract

Some results on the pathwise asymptotic stability of solutions to stochastic partial differential equations are proved. Special attention is paid in proving sufficient conditions ensuring almost sure asymptotic stability with a non-exponential decay rate. The situation containing some hereditary characteristics is also treated. The results are illustred with several examples.

1 Introduction

In the analysis of the asymptotic behaviour of solutions to differential systems, one can find that a solution can be asymptotically stable but may not be exponentially. Moreover, in the nonlinear and/or nonautonomous situations it may happen that the stability can be even super-exponential (see Caraballo [1]). This fact motivated our interest in determining, if possible, the decay rate of the solutions to other solution or stationary one. Our main aim in this paper is to carry out a similar study in the context of stochastic evolution equations including the possibility that some delay terms appear in the models.

There exists a wide literature concerning pathwise exponential stability of stochastic evolution equation. We mention here, amongst others, Caraballo [2], Caraballo and Liu [3], Haussmann [7], Ichikawa [8] and Liu [11]. However, only a few works have been done concerning the non-exponential stability of these stochastic systems. It is worth mentioning the paper by Liu [9] on the polynomial stability for semilinear stochastic evolution equation with time delays (which contains, in particular, the nondelay situation). But, as far as we know, the nonlinear case has not been previously treated and it is our main interest in this paper to prove some results to cover this gap.

Another interesting question is concerned with the stabilizing effect which may be produced by the noise. Although this is not our major objective in this paper, we will show that in some occasions in which the theory previously developed by Caraballo et al. [4] does not provide exponential stabilization, we can determine stability with a non-exponential decay rate, e.g. polynomial or super-exponential.

The content of the paper is as follows. In Section 2 we first establish the framework in which our analysis is carried out in the nondelay context. We introduce the basic notations and assumptions. We also prove some sufficient conditions ensuring almost sure stability of solutions to stochastic evolution equations, and exhibit an example to illustrate these results which can also be interpreted as a non-exponential stabilization result produced by the noise. Section 3 is devoted to the establishment of a similar result for a class of nonlinear stochastic partial differential equations with delays and we also include some illustrative examples. Finally, some conclusions and remarks are written in the last Section.

2 Stability of stochastic evolution equation

Let V be a reflexive Banach space and H a real separable Hilbert space such that

$$V \subset H \equiv H^* \subset V^*$$

where the injections are continuous and dense. In addition, we also assume both V and V^* are uniformly convex.

We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V, H and V^* respectively; by $(\cdot,\cdot)$ the inner product in H, and by $\langle\cdot,\cdot\rangle$ the duality product between V and V^*.

Assume $\{\Omega,\mathcal{F},P\}$ is a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, i.e., $\mathcal{F}_0$ contains the null sets in $\mathcal{F}$ and $\mathcal{F}_t=\cap_{s>t}\mathcal{F}_s$, for all $t\geq 0$, and let us consider a real valued $\{\mathcal{F}_t\}-$Wiener process $\{W(t)\}_{t\geq 0}$.

We denote by $I^p(0,T;V)$ (for $p\geq 2$) the closed subspace of $L^p(\Omega\times(0,T),\mathcal{F}\otimes\mathcal{B}([0,T]),P\otimes \mathrm{d}t;V)$ of all stochastic processes which are $\mathcal{F}_t$-adapted for almost every t in $(0,T)$ (in what follows, a.e. t), where $\mathcal{B}([0,T])$ denotes the Borel σ-algebra of subsets in $[0,T]$. We write $L^2(\Omega;C(0,T;H))$ instead of $L^2(\Omega,\mathcal{F},P;C(0,T;H))$, where $C(0,T;H)$ denotes the space of all continuous functions from $[0,T]$ into H.

In this Section we shall consider the following infinite-dimensional stochastic differential equation in V^* and for $T>0$:

$$\begin{cases} \mathrm{d}X(t)=f(t,X(t))\mathrm{d}t+g(t,X(t))\mathrm{d}W(t), & t\in[0,T], \\ X(0)=X_0, \end{cases} \tag{1}$$

where $f(t,\cdot):V\to V^*$ is a suitable family of (nonlinear) operators (see conditions below), $g(t,\cdot):V\to H$ is another family of operators satisfying

(g1) The map $t\longmapsto g(t,x)$ is Lebesgue measurable from $(0,T)$ into $H,\forall x\in V$,

(g2) There exists $L>0$ such that

$$|g(t,x)-g(t,y)|\leq L\,\|x-y\| \quad \forall x,y\in V, \quad \text{a.e.}t.,$$

and $X_0\in L^p(\Omega,\mathcal{F}_0,P;V)$ is an arbitrarily fixed initial datum.

As we are mainly interested in the stability analysis, we shall then assume that for each $T>0$ and every $X_0\in L^p(\Omega,\mathcal{F}_0,P;V)$ there exists a process

$$X(t)\in I^p(0,T;V)\cap L^2(\Omega;C(0,T;H))$$

which is solution to (1). In other words, $X(t)$ satisfies the following equation in V^*:

$$X(t) = X_0 + \int_0^t f(s, X(s))\mathrm{d}s + \int_0^t g(s, X(s))\mathrm{d}W(s), \forall t \in [0, T],\ P - \text{a.s.} \tag{2}$$

To this end, if we assume assumptions below, we then can ensure that there exists a unique solution to this problem (1) (see Pardoux [12]):

1. Measurability: $\forall x \in V$, the map $t \in (0, T) \longmapsto f(t, x) \in V^*$ is Lebesgue measurable, a.e.t.

2. Hemicontinuity: The map $\theta \in \mathbb{R} \longmapsto \langle f(t, x + \theta y), z\rangle \in \mathbb{R}$ is continuous $\forall x, y, z \in V$, a.e.t.

3. Boundedness: There exists $c > 0$ such that

$$\|f(t,x)\|_* \leq c\,\|x\|^{p-1}\ \ \forall x \in V,\ \text{a.e.}t.$$

4. Coercivity: $\exists \alpha > 0, \lambda, \gamma \in \mathbb{R}$ such that

$$2\,\langle f(t,x), x\rangle + \|g(t,x)\|^2 \leq -\alpha\,\|x\|^p + \lambda\,|x|^2 + \gamma\ \forall x \in V,\ \text{a.e.}t.$$

5. Monotonicity:

$$-2\,\langle f(t,x) - f(t,y), x - y\rangle + \lambda\,|x - y|^2 \geq \|g(t,x) - g(t,y)\|^2\ \ \forall x, y \in V,\ \text{a.e.}t.$$

In what follows, we will assume that it holds at least the assumptions ensuring that the integrals in (2) make sense.

On the other hand, let us define some operators which will be used later on jointly with the Itô's formula.

Unless otherwise is stated, we will assume that $U(t,x)$ is a $C^{1,2}$-positive functional such that for any $x \in V$, $t \in \mathbb{R}^+$, $U'_x(t,x) \in V$, and satisfies some additional assumptions which enable us to apply Itô's formula for the process $X(t)$ solution to

(2) (see, e.g. Pardoux [12, p. 63]). We can then define operators L and Q as follows: for $x \in V$, $t \in \mathbb{R}^+$

$$LU(t,x) = U_t'(t,x) + \left\langle U_x'(t,x), f(t,x) \right\rangle + \frac{1}{2}(U_{xx}''(t,x)g(t,x), g(t,x))$$

and

$$QU(t,x) = (U_x'(t,x), g(t,x))^2$$

In the sequel, we will refer to this functional U as *an appropriate Lyapunov functional*. The following result is known as the exponential martingale inequality and will play an important role in some of the results in this paper. For the sake of simplicity, we only include the particular form which will be used in our arguments.

Lemma 1 *Assume $X(t)$ is a solution to (1). Suppose $g(t,x)$ satisfies conditions (g1) and (g2), $U(t,x)$ is an appropriate Lyapunov functional and T, α, β are any positive constants. Then*

$$P\left\{ \sup_{0 \le t \le T} \left[\int_0^t \left(U_x'(s, X(s)), g(s, X(s))\right) dW(s) - \int_0^t \frac{\alpha}{2} QU(s, X(s))\, ds \right] > \beta \right\} \le e^{-\alpha\beta}.$$

Proof. See, for instance, Liu and Truman [10, Lemma 3.8.1]. ■

We will introduce a precise definition of almost sure stability with general decay function $\lambda(t)$ based on the concept of generalized Lyapunov exponent (see Caraballo [1] for a related concept in the deterministic framework).

Definition 2 *Let $\lambda(t)$ be a positive function defined for sufficiently large $t > 0$, say $t \ge T > 0$, and satisfying that $\lambda(t) \uparrow +\infty$ as $t \to +\infty$. The solution $X(t)$ to (1) (defined in the future, i.e. for t large enough) is said to decay to zero almost surely with decay function $\lambda(t)$ and order at least $\gamma > 0$, if its generalized Lyapunov exponent is less than or equal to $-\gamma$ with probability one, i.e.*

$$\limsup_{t \to \infty} \frac{\log |X(t)|}{\log \lambda(t)} \le -\gamma, \ \ P - a.s.$$

If in addition 0 *is solution to (1), the zero solution is said to be almost surely asymptotically stable with decay function* $\lambda(t)$ *and order at least* γ*, if every solution to (1) decays to zero almost surely with decay function* $\lambda(t)$ *and order at least* γ*.*

Remark 3 *Clearly, replacing in the above definition the decay function* $\lambda(t)$ *by a certain suitable* $O(e^t)$ *leads to the usual exponential stability definition.*

Now, we can prove a first sufficient condition ensuring almost sure stability of the solution of (1) with certain decay rate.

Theorem 4 *Let* $U(t,x)$ *be an appropriate Lyapunov functional. Assume that* $\log\lambda(t)$ *is uniformly continuous over* $t \in [T, +\infty)$ *and there exists a constant* $\tau \geq 0$ *such that*

$$\limsup_{t\to\infty} \frac{\log\log t}{\log\lambda(t)} \leq \tau.$$

Assume that there exist two continuous non-negative functions $\varphi_1(t)$*,* $\varphi_2(t)$*, constants* $q > 0,\ m \geq 0, \mu, \nu, \theta \in \mathbb{R}$ *and a non-increasing function* $\xi(t) > 0$ *such that*

(a) $|x|^q \lambda(t)^m \leq U(t,x),\ (t,x) \in \mathbb{R}^+ \times V$.

(b) $LU(t,x) + \xi(t)QU(t,x) \leq \varphi_1(t) + \varphi_2(t)U(t,x),\ (t,x) \in \mathbb{R}^+ \times V$.

(c)

$$\limsup_{t\to\infty} \frac{\log \int_0^t \varphi_1(s)\mathrm{d}s}{\log\lambda(t)} \leq \nu, \qquad \limsup_{t\to\infty} \frac{\int_0^t \varphi_2(s)\mathrm{d}s}{\log\lambda(t)} \leq \theta$$

$$\liminf_{t\to\infty} \frac{\log\xi(t)}{\log\lambda(t)} \geq -\mu.$$

Then, every solution $X(t)$ *to (1) defined in the future satisfies*

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log\lambda(t)} \leq -\frac{m - [\theta + (\nu \vee (\mu+\tau))]}{q}, \quad P - a.s.$$

In particular, if $m > \theta + (\nu \vee (\mu+\tau))$*, the solution* $X(t)$ *decays to zero almost surely with decay function* $\lambda(t)$ *and order at least* $\gamma = (m - [\theta + (\nu \vee (\mu+\tau))])/q$*.*

Proof. We will apply Itô's formula to the function $U(t, x)$ and the process $X(t)$. Noticing the definitions of L and Q, we can derive that

$$U(t, X(t)) = U(0, X_0) + \int_0^t LU(s, X(s))\,\mathrm{d}s + \int_0^t (U_x'(s, X(s)), g(s, X(s)))\mathrm{d}W(s). \tag{3}$$

Now, from the uniform continuity of $\log \lambda(t)$, we can ensure that for each $\varepsilon > 0$ there exist two positive integers $N = N(\varepsilon)$ and $k_1(\varepsilon)$ such that if $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}$, $k \ge k_1(\varepsilon)$, it follows

$$\left|\log \lambda\left(\frac{k}{2^N}\right) - \log \lambda(t)\right| \le \varepsilon.$$

On the other hand, due to the exponential martingale inequality

$$P\left\{\omega : \sup_{0\le t\le w}\left[M(t) - \int_0^t \frac{u}{2} QU(s, X(s))\mathrm{d}s\right] > v\right\} \le e^{-uv},$$

for any positive constants u, v and w, where

$$M(t) = \int_0^t (U_x'(s, X(s))\,, g(s, X(s)))\mathrm{d}W(s).$$

In particular, for the preceding $\varepsilon > 0$, if we set

$$u = 2\xi\left(\frac{k}{2^N}\right), \quad v = \xi\left(\frac{k}{2^N}\right)^{-1} \log\frac{k-1}{2^N}, \quad w = \frac{k}{2^N}, \quad k = 2, 3, ...$$

we can then apply the Borel-Cantelli lemma to obtain that for almost all $\omega \in \Omega$, there exists an integer $k_0(\varepsilon, \omega) > 0$ such that

$$\begin{aligned}\int_0^t (U_x'(s, X(s))\,, g(s, X(s)))\mathrm{d}W(s) &\le \xi\left(\frac{k}{2^N}\right)^{-1} \log\frac{k-1}{2^N} \\ &\quad + \xi\left(\frac{k}{2^N}\right)\int_0^t QU(s, X(s))\mathrm{d}s \\ &\le \xi\left(\frac{k}{2^N}\right)^{-1} \log\frac{k-1}{2^N} \\ &\quad + \int_0^t \xi(s) QU(s, X(s))\mathrm{d}s\end{aligned}$$

for $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon,\omega)$. Substituting this into (3) and using condition (b), we deduce

$$U(t,X(t)) = U(0,X_0) + \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} + \int_0^t \varphi_1(s)\mathrm{d}s + \int_0^t \varphi_2(s)U(s,X(s))\,\mathrm{d}s,\ P-\text{a.s.} \tag{4}$$

for $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon,\omega)$. Consequently, by the virtue of Gronwall's lemma, it follows

$$U(t,X(t)) = \left(U(0,X_0) + \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} + \int_0^t \varphi_1(s)\mathrm{d}s\right) \times \exp\left(\int_0^t \varphi_2(s)\,\mathrm{d}s\right),\ P-\text{a.s.}$$

for $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon,\omega)$.

On the other hand, thanks to condition (c) and the uniform continuity of $\log\lambda(t)$,

$$\textstyle\int_0^t \varphi_1(s)\mathrm{d}s \le \lambda(t)^{\nu+\varepsilon},\quad \int_0^t \varphi_2(s)\mathrm{d}s \le (\theta+\varepsilon)\log\lambda(t),\quad \xi\left(\frac{k}{2^N}\right)^{-1} \le \mathrm{e}^{\varepsilon(\mu+\varepsilon)}\lambda(t)^{\mu+\varepsilon}$$

for $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}$, $k \ge k_1(\varepsilon)$. Owing to the assumption on $\lambda(t)$,

$$\log\frac{k-1}{2^N} \le \log t \le \lambda(t)^{\tau+\varepsilon} \quad \text{for} \quad \frac{k-1}{2^N} \le t \le \frac{k}{2^N}.$$

Hence, for almost all $\omega \in \Omega$

$$\log U(t,X(t)) \le \log(U(0,X_0) + \lambda(t)^{\mu+\tau+2\varepsilon}\mathrm{e}^{\varepsilon(\mu+\varepsilon)} + \lambda(t)^{\nu+\varepsilon}) + (\theta+\varepsilon)\log\lambda(t)$$

for $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon,\omega) \vee k_1(\varepsilon)$, which immediately implies

$$\limsup_{t\to\infty}\frac{\log U(t,X(t))}{\log\lambda(t)} \le [(\mu+\tau+2\varepsilon)\vee(\nu+\varepsilon)] + \theta + \varepsilon,\ P-\text{a.s.} \tag{5}$$

As $\varepsilon > 0$ is arbitrary, (5) yields to

$$\limsup_{t\to\infty}\frac{\log U(t,X(t))}{\log\lambda(t)} \le [\nu\vee(\mu+\tau)] + \theta,\ P-\text{a.s.}$$

Finally,

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)}\leq-\frac{m-[\theta+(\nu\vee(\mu+\tau))]}{q},\ \ P-\text{a.s.}$$

which finishes the proof. ■

Remark 5 *Note that, in the preceding theorem we have assumed that* $\log\lambda(t)$ *is uniformly continuous over* $t\in[T,+\infty)$ *and there exists a constant* $\tau\geq 0$ *such that*

$$\limsup_{t\to\infty}\frac{\log\log t}{\log\lambda(t)}\leq\tau.$$

However, when the functional $QU(t,x)$ *is also bounded below (see next Theorem 6), it is not necessary to impose the uniform continuity of* $\log\lambda(t)$ *provided that we assume a stronger hypothesis on the growing rate of* $\lambda(t)$.

Theorem 6 *Let* $U(t,x)$ *be an appropriate Lyapunov functional. Assume that* $X(t)$ *is a solution to (1) satisfying that* $|X(t)|\neq 0$ *for all* $t\geq 0$ *and* $P-$*a.s. provided* $|X_0|\neq 0$ $P-$*a.s. Assume there exist two continuous functions* $\varphi_1(t)\in\mathbb{R}$, $\varphi_2(t)\geq 0$, *and constants* $q>0, m\geq 0, \nu\geq 0,\ \mu\geq 0, \theta\in\mathbb{R}$ *such that*

(a) $|x|^q\lambda(t)^m\leq U(t,x)$, $(t,x)\in\mathbb{R}^+\times V$.

(b) $LU(t,x)\leq\varphi_1(t)U(t,x)$, $(t,x)\in\mathbb{R}^+\times V$.

(c) $QU(t,x)\geq\varphi_2(t)U(t,x)^2$, $(t,x)\in\mathbb{R}^+\times V$.

(d)

$$\limsup_{t\to\infty}\frac{\int_0^t\varphi_1(s)\mathrm{d}s}{\log\lambda(t)}\leq\theta,\quad\liminf_{t\to\infty}\frac{\int_0^t\varphi_2(s)\mathrm{d}s}{\log\lambda(t)}\geq 2\nu$$

$$\limsup_{t\to\infty}\frac{\log t}{\log\lambda(t)}\leq\frac{\mu}{2}.$$

Then, for every $\alpha\in(0,1)$, it holds

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)}\leq-\frac{m-(\alpha^{-1}\mu+\theta-\nu(1-\alpha))}{q},\ P-a.s.$$

In particular, if $m > \alpha^{-1}\mu + \theta - \nu(1-\alpha)$, *the solution* $X(t)$ *decays to zero almost surely with decay function* $\lambda(t)$ *and order at least*

$$\gamma_\alpha = \left(m - \left(\alpha^{-1}\mu + \theta - \nu(1-\alpha)\right)\right)/q.$$

Proof. Fix $|X_0| \neq 0$ $P-$a.s. Then, thanks to Itô's formula

$$\log\ U(t, X(t)) = \log U(0, X_0) + M(t) + \int_0^t \frac{LU(s, X(s))}{U(s, X(s))} \mathrm{d}s - \frac{1}{2}\int_0^t \frac{QU(s, X(s))}{U(s, X(s))^2}\,\mathrm{d}s \tag{6}$$

where

$$M(t) = \int_0^t \frac{1}{U(s, X(s))}(U_x'(s, X(s)), g(s, X(s)))\mathrm{d}W(s).$$

Due to the exponential martingale inequality,

$$P\left\{\omega : \sup_{0\le t\le w}\left[M(t) - \int_0^t \frac{u}{2}\frac{QU(s, X(s))}{U(s, X(s))^2}\mathrm{d}s\right] > v\right\} \le e^{-uv}$$

for any positive constants u, v and w. In particular, taking $0 < \alpha < 1$ and setting

$$u = \alpha, \quad v = 2\alpha^{-1}\log(k-1), \quad w = k, \quad k = 2, 3, ...$$

we can apply Borel-Cantelli's lemma to obtain that, for almost all $\omega \in \Omega$, there exists an integer $k_0(\varepsilon, \omega) > 0$ such that

$$M(t) \le 2\alpha^{-1}\log(k-1) + \frac{\alpha}{2}\int_0^t \frac{QU(s, X(s))}{U(s, X(s))^2}\mathrm{d}s$$

for $0 \le t \le k$, $k \ge k_0(\varepsilon, \omega)$. Substituting this into (6) and using condition (c), we get that

$$\log U(t, X(t)) \le \log U(0, X_0) + 2\alpha^{-1}\log(k-1) + \int_0^t \varphi_1(s)\mathrm{d}s - \frac{1}{2}(1-\alpha)\int_0^t \varphi_2(s)\,\mathrm{d}s$$

for $0 \le t \le k$, $k \ge k_0(\varepsilon, \omega)$. Now, using condition (d), it follows

$$\log U(t, X(t)) \le \log U(0, X_0) + \frac{(\mu+\varepsilon)}{\alpha}\log\lambda(t) + (\theta+\varepsilon)\log\lambda(t) - \frac{1}{2}(1-\alpha)(2\nu-\varepsilon)\log\lambda(t)$$

for $k-1 \leq t \leq k$, $k \geq k_0(\varepsilon,\omega)$, which implies that

$$\limsup_{t\to\infty} \frac{\log U(t,X(t))}{\log \lambda(t)} \leq \alpha^{-1}(\mu+\varepsilon)+\theta+\varepsilon-\frac{1}{2}(1-\alpha)(2\nu-\varepsilon),\ P-\text{a.s.}$$

Taking into account that $\varepsilon > 0$ is arbitrary and using (a) we can deduce

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log \lambda(t)} \leq -\frac{m-(\alpha^{-1}\mu+\theta-\nu(1-\alpha))}{q},\ P-\text{a.s.},$$

as required. ■

Remark 7 *Observe that, as the decay order of the solution in the preceding theorem depends on the parameter α, an interesting question is concerned with the possibility of determining the biggest value for γ_α. To this respect, one can check that this optimal value depends on the relation between μ and ν. Let us explain this in more detail. Indeed, in order to find the optimal value $\gamma^* = \sup_{0<\alpha<1} \gamma_\alpha$, we need to find out the minimum value f^* for the function $f(\alpha) = \alpha^{-1}\mu+\theta-\nu(1-\alpha)$ when the parameter $\alpha \in (0,1)$, and consequently it will hold that $\gamma^* = (m-f^*)/q$. Now, by straightforward computations, it is not difficult to check that*

$$f^* = \begin{cases} 2(\mu\nu)^{1/2}+\theta-\nu, & \text{if } 0 \leq \mu < \nu, \\ \mu+\theta, & \text{if } \nu \leq \mu, \end{cases}$$

which implies that

$$\gamma^* = \begin{cases} \frac{m-\left[2(\mu\nu)^{1/2}+\theta-\nu\right]}{q}, & \text{if } 0 \leq \mu < \nu, \\ \frac{m-[\mu+\theta]}{q}, & \text{if } \nu \leq \mu, \end{cases}$$

Remark 8 *Note that Theorem 9 below permits us to avoid the restriction on the growing rate on the decay function when $QU(t,x)$ is also bounded above by a suitable bound.*

Theorem 9 *Let $U(t,x) \in C^{1,2}(\mathbb{R}^+ \times H; \mathbb{R}^+)$ be an appropriate Lyapunov functional. Assume that there exist three continuous functions $\varphi_1(t) \in \mathbb{R}$, $\varphi_2(t) \geq 0$, $\varphi_3(t) \geq 0$, and constants $q > 0, m \geq 0, \nu > 0$, $\mu \geq 0$ and $\theta \in \mathbb{R}$ such that*

(a) $|x|^q \lambda(t)^m \le U(t,x)$, $(t,x) \in \mathbb{R}^+ \times V$.

(b) $LU(t,x) \le \varphi_1(t) U(t,x)$, $(t,x) \in \mathbb{R}^+ \times V$.

(c) $\varphi_2(t) U(t,x)^2 \le QU(t,x) \le \varphi_3(t) U(t,x)^2$, $(t,x) \in \mathbb{R}^+ \times V$.

(d)

$$\limsup_{t\to\infty} \frac{\int_0^t \varphi_1(s)\mathrm{d}s}{\log \lambda(t)} \le \theta, \qquad \liminf_{t\to\infty} \frac{\int_0^t \varphi_2(s)\mathrm{d}s}{\log\, \lambda(t)} \ge 2\nu$$

$$\limsup_{t\to\infty} \frac{\int_0^t \varphi_3(s)\mathrm{d}s}{\log \lambda(t)} \le \mu.$$

Then, if $X(t)$ is a solution to (1) *defined in the future and satisfying that $|X(t)| \ne 0$ for all $t \ge 0$ and $P-$a.s. provided $|X_0| \ne 0$ $P-$a.s., it holds*

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log \lambda(t)} \le -\frac{m - (\theta - \nu)}{q}, \quad P - a.s.$$

In particular, if $m > \theta - \nu$, the solution $X(t)$ decays to zero almost surely with decay function $\lambda(t)$ and order at least $\gamma = (m - (\theta - \nu))/q$.

Proof. Fix $|X_0| \ne 0$ $P-$a.s. Then, Itô's formula implies again (6). Using conditions (b) and (c) we obtain

$$\log U(t, X(t)) \le \log U(0, X_0) + M(t) + \int_0^t \varphi_1(s)\mathrm{d}s - \frac{1}{2}\int_0^t \varphi_2(s)\,\mathrm{d}s. \qquad (7)$$

Now, condition (d) and (7) imply

$$\log U(t, X(t)) \le \log U(0, X_0) + M(t) + (\theta + \varepsilon)\log \lambda(t) - \frac{1}{2}(2\nu - \varepsilon)\log \lambda(t),$$

and

$$\limsup_{t\to\infty} \frac{\log U(t, X(t))}{\log \lambda(t)} \le \limsup_{t\to\infty} \frac{M(t)}{\log \lambda(t)} + \theta + \varepsilon - \frac{1}{2}(2\nu - \varepsilon),\ P - \text{a.s.}$$

Let us denote by $\langle M(t) \rangle$ the quadratic variation process associated to $M(t)$. From our assumptions we can deduce that $M(t)$ is a local martingale vanishing at $t = 0$. Moreover, condition (c) implies

$$\int_0^t \varphi_2(s)\,\mathrm{d}s \le \langle M(t) \rangle = \int_0^t \frac{QU(s, X(s))}{U(s, X(s))^2} ds \le \int_0^t \varphi_3(s)\,\mathrm{d}s.$$

Now, as $\nu > 0$, it follows that $\lim_{t\to\infty} \langle M(t)\rangle = +\infty$ and by means of the strong law of large numbers we obtain

$$\lim_{t\to\infty} \frac{M(t)}{\langle M(t)\rangle} = 0, \ \ P - \text{a.s.}$$

Taking into account that, for t large enough,

$$\begin{aligned} \frac{|M(t)|}{\log \lambda(t)} &= \frac{|M(t)|}{\langle M(t)\rangle} \frac{\langle M(t)\rangle}{\log \lambda(t)} \\ &\leq \frac{|M(t)|}{\langle M(t)\rangle} \frac{\int_0^t \varphi_3(s)\, \mathrm{d}s}{\log \lambda(t)}, \end{aligned}$$

we easily deduce that, from assumption (d), it follows

$$\limsup_{t\to\infty} \frac{M(t)}{\log \lambda(t)} = 0, \ \ P - \text{a.s.}$$

and, consequently,

$$\limsup_{t\to\infty} \frac{\log U(t, X(t))}{\log \lambda(t)} \leq \theta + \varepsilon - \frac{1}{2}(2\nu - \varepsilon), \ \ P - \text{a.s.}$$

Since the constant $\varepsilon > 0$ is arbitrary, we can affirm that

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log \lambda(t)} \leq -\frac{m - (\theta - \nu)}{q}, \ \ P - \text{a.s.}$$

and the proof is therefore complete. ■

Remark 10 *It is worth pointing out that this result allows us to establish some kind of stabilization effect produced by the noise on deterministic systems. To this respect, Caraballo et al. [4] proved some results on the exponential stabilization of deterministic (and stochastic) systems when a suitable noise appears in the equations. However, it may happen that the noise does not cause exponential stability (or at least we are not able to know if this has happened) and, consequently, it would be very interesting to investigate if the noise has produced a different kind of stability (e.g. polynomial, logarithmic, or even super-exponential). As this will be the aim of a subsequent paper, we only include here an example to illustrate our ideas.*

Example 1. Let us consider the following problem

$$\begin{cases} \mathrm{d}X(t) = A(t)X(t)\mathrm{d}t + g(t, X(t))\mathrm{d}W(t), & t > 0, \\ X(0) = X_0, \end{cases}$$

where operators $A(t)$ and g are defined as follows. We consider an open and bounded set $\mathcal{O} \subset \mathbb{R}^N$ with regular boundary and let $2 \leq p < +\infty$. Consider also the Sobolev spaces $V = W_0^{1,p}(\mathcal{O})$, $H = L^2(\mathcal{O})$ with their usual norms, inner product and duality. The monotone family of operators $A(t) \colon V \to V^*$ is then defined by

$$\langle v, A(t)u\rangle = -\sum_{i=1}^{N} \int_{\mathcal{O}} \left| \frac{\partial u(x)}{\partial x_i} \right|^{p-2} \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx + \int_{\mathcal{O}} \frac{a}{1+t} u(x)v(x)dx \ \ \forall u, v \in V\,,$$

where $a \in \mathbb{R}$, and $g(t,u) = b\,(1+t)^{-1/2}\,u$, $b \in \mathbb{R}$, $u \in H$ for all $t \in \mathbb{R}^+$.

Now, consider the function $U(t,u) = |u|^2$, $u \in H$ and let us compute $LU(t,u)$ and $QU(t,u)$.

On the one hand, it easily follows

$$LU(t,u) = 2\langle u, A(t)u\rangle + |g(t,u)|^2 = -2\|u\|^p + \frac{2a+b^2}{1+t}|u|^2, \ \ u \in V, \tag{8}$$

so we can set $\varphi_1(t) = \left(2a + b^2\right)/(1+t)$. On the other hand,

$$QU(t,u) = (2u, b(1+t)^{-1/2}u)^2 = 4b^2\,(1+t)^{-1}\,|u|^4\,,$$

so that $\varphi_2(t) = \varphi_3(t) = 4b^2\,(1+t)^{-1}$.

In order to apply now the exponential stabilization result in Caraballo et al. [4] (in fact, Theorem 2.2 or the Remark following this theorem), we need to determine constants δ_0 and ρ_0 satisfying

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \varphi_1(s)\mathrm{d}s \leq \delta_0, \quad \liminf_{t\to+\infty} \frac{1}{t} \int_0^t \varphi_2(s)\mathrm{d}s \geq \rho_0,$$

and the result in [4] ensures that

$$\limsup_{t\to\infty} \frac{1}{t} \log|X(t)|^2 \leq -(2\rho_0 - \delta_0), \quad P - \text{a.s.}$$

It is easy to compute that in this example both constants are equal to 0, so that we do not know whether the solution decays to zero exponentially or not. However, we can apply the preceding theorem and prove, at least, asymptotic decay with a lower decay rate. Indeed, taking $\lambda(t) = t, m = 0, q = 2$ we can easily check that the assumptions in the last theorem hold with $\theta = 2a + b^2, \nu = 2b^2, \mu = 4b^2$, and therefore $-(m - (\theta - \nu))/q = \left(2a - b^2\right)/2$. We have then proved asymptotic decay to zero with decay function $\lambda(t) = t$ and order at least $\frac{b^2}{2} - a$ provided b is large enough (in fact, whenever $b^2 > 2a$).

3 Stability of stochastic delay evolution equation

In this section we shall investigate the almost sure stability for a class of stochastic functional evolution equations (which, in particular, includes the case of stochastic evolution equations with variable and distributed delays). The main objective is to develop a general theory similar to the one in the case without delays in the preceding Section by also using the Lyapunov functional technique. It is worth mentioning that Caraballo et al. proved in [5] a particular result on the exponential stability of stochastic partial functional differential equations by a Razumikhin type of argument considering the usual quadratic Lyapunov function. The analysis which will be carried out in this Section completes and improves that one.

In a similar way as we did in Section 2, given $h \geq 0$, $p \geq 2$ and $T > 0$, we will denote by $I^p(-h, T; V)$ the closed subspace of $L^p(\Omega \times [-h, T], \mathcal{F} \otimes \mathcal{B}([-h, T]), \mathrm{d}P \otimes \mathrm{d}t; V)$ of all $\mathcal{F}_t$−adapted processes for a.e. t where we will set $\mathcal{F}_t = \mathcal{F}_0$ for $t < 0$. We will also write $L^2(\Omega; C(-h, T; H))$ instead of $L^2(\Omega, \mathcal{F}, P; C(-h, T; H))$ and $C(-h, T; H)$ denotes now the space of all continuous functions from $[-h,T]$ into H.

Let $C_H = C(-h, 0; H)$ with norm $|\psi|_{C_H} = \sup_{-h \leq s \leq 0} |\psi(s)|$, $\psi \in C_H$; $L^p_V = L^p([-h, 0], V)$ and $L^p_H = L^p([-h, 0], H)$.

On the other hand, given a stochastic process

$$X(t) \in I^p(-h,T;V) \cap L^2(\Omega; C(-h,T;H)),$$

we associate with another stochastic process

$$X_t : \Omega \longmapsto L^p_V \cap C_H$$

by means of the usual relation $X_t(s)(\omega) = X(t+s)(\omega)$, $0 \leq t \leq T$, $-h \leq s \leq 0$.

Let us consider the following stochastic evolution equation in V^*:

$$\begin{cases} \mathrm{d}X(t) = (A(t,X(t)) + F(t,X_t))\,\mathrm{d}t + G(t,X_t)\mathrm{d}W(t), t \in [0,T], \\ X(0) = \psi(t),\ t \in [-h,0] \end{cases} \tag{9}$$

where $T > 0$ and the initial datum $\psi \in I^p(-h,0;V) \cap L^2(\Omega; C_H)$.

As our major interest in this Section is the stability analysis of solutions to (9), we shall assume that for each $\psi \in I^p(-h,0;V) \cap L^2(\Omega; C_H)$ there exists a process

$$X(t) \in I^p(-h,T;V) \cap L^2(\Omega; C(-h,T;H))$$

which is solution to (9) for every $T > 0$, in other words, $X(t)$ satisfies the following integral equation in V^* :

$$\begin{cases} X(t) = \psi(0) + \int_0^t (A(s,X(s)) + F(s,X_s))\mathrm{d}s + \int_0^t G(s,X_s)\mathrm{d}W(s), \quad \forall t \in [0,T], \\ X(t) = \psi(t),\ \forall t \in [-h,0]. \end{cases}$$

This happens, for instance, if $A(t,\cdot) : V \to V^*$ is a family of (nonlinear) operators defined a.e.t. and fulfilling the assumptions described in the preceding Section (measurability, hemicontinuity, boundedness, monotonicity and coercivity), $F(t,\cdot) : [0,T] \times C_H \to V^*$ is a family of Lipschitz continuous operators defined a.e.t, and $G : [0,T] \times C_H \to H$ is another family of Lipschitz operators defined a.e.t (see Caraballo et al. [5] and also Caraballo et al. [6] for two detailed discussions on the existence and uniqueness of solutions for this situation).

We can now prove our stability result.

Theorem 11 *Let $U(t,x)$ be an appropriate Lyapunov functional. Assume that* $\log\lambda(t)$ *is uniformly continuous on* $t \in [T,+\infty)$ *and there exists a constant* $\tau \geq 0$ *such that*

$$\limsup_{t\to\infty} \frac{\log\log t}{\log\lambda(t)} \leq \tau.$$

Assume that there exist constants $q > 0,\ m \geq 0, \mu \geq 0,\ \nu, \theta \in \mathbb{R}$, *a non-increasing function* $\xi(t) > 0$ *and two continuous non-negative functions* $\varphi_1(t),\ \varphi_2(t)$ *such that*

(a) $|x|^q \lambda(t)^m \leq U(t,x),\ \forall (t,x) \in \mathbb{R}^+ \times V$.

(b) *For a solution* $X(t)$ *to* (9) *defined in the future it holds*

$$\begin{aligned}
&\int_0^t U'_s(s,X(s))\mathrm{d}s + \int_0^t \left\langle U'_x(s,X(s)), A(s,X(s)) + F(s,X_s)\right\rangle \mathrm{d}s \\
&+ \frac{1}{2}\int_0^t (U''_{xx}(s,X(s))G(s,X_s), G(s,X_s))\mathrm{d}s + \int_0^t \xi(s)(U'_x(s,X(s)), G(s,X_s))^2 \mathrm{d}s \\
&\leq \int_0^t \varphi_1(s)\mathrm{d}s + \int_0^t \varphi_2(s)\, U(s,X(s))\mathrm{d}s + c(\psi),
\end{aligned}$$

where $c(\psi)$ *is a constant depending on the initial datum* ψ.

(c)

$$\limsup_{t\to\infty} \frac{\log \int_0^t \varphi_1(s)\mathrm{d}s}{\log\lambda(t)} \leq \nu, \quad \limsup_{t\to\infty} \frac{\int_0^t \varphi_2(s)\mathrm{d}s}{\log\lambda(t)} \leq \theta$$

$$\liminf_{t\to\infty} \frac{\log\xi(t)}{\log\lambda(t)} \geq -\mu.$$

Then,

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log\lambda(t)} \leq -\frac{m - [\theta + ((\mu+\tau)\vee\nu)]}{q}, \quad P - \text{a.s.}$$

In particular, if $m > \theta + ((\mu+\tau)\vee\nu)$ *the solution* $X(t)$ *decays to zero almost surely with decay function* $\lambda(t)$ *and order at least* $\gamma = (m - [\theta + ((\mu+\tau)\vee\nu)])\,/q$.

Proof. By applying once again Itô's formula we obtain

$$
\begin{aligned}
U(t,X(t)) = U(0,\psi(0)) &+ \int_0^t U_s'(s,X(s))\mathrm{d}s \\
&+ \int_0^t \left\langle U_x'(s,X(s)), A(s,X(s)) + F(s,X_s)\right\rangle \mathrm{d}s \\
&+ \frac{1}{2}\int_0^t (U_{xx}''(s,X(s))G(s,X_s), G(s,X_s))\mathrm{d}s \\
&+ \int_0^t (U_x'(s,X(s)), G(s,X_s))\mathrm{d}W(s). \qquad (10)
\end{aligned}
$$

Due to the uniform continuity of $\log\lambda(t)$ and given $\varepsilon > 0$, there exist two positive integers $N = N(\varepsilon)$ and $k_1(\varepsilon)$ such that for $\frac{k-1}{2^N} \leq t \leq \frac{k}{2^N}$, $k \geq k_1(\varepsilon)$, it follows

$$
\left|\log\lambda\left(\frac{k}{2^N}\right) - \log\lambda(t)\right| \leq \varepsilon.
$$

On the other hand, the exponential martingale inequality implies

$$
P\left\{\sup_{0\leq t\leq w}\left[M(t) - \frac{u}{2}\int_0^t (U_x'(s,X(s)), G(s,X_s))^2\mathrm{d}s\right] > v\right\} \leq e^{-uv}
$$

for any positive constants u, v and w, where

$$
M(t) = \int_0^t (U_x'(s,X(s))\,, G(s,X_s))\mathrm{d}W(s).
$$

In particular, for the preceding $\varepsilon > 0$, taking

$$
u = 2\xi\left(\frac{k}{2^N}\right), \quad v = \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}, \quad w = \frac{k}{2^N}, \quad k = 2,3,...
$$

we can then apply the Borel-Cantelli lemma to obtain that for almost all $\omega \in \Omega$, there exists an integer $k_0(\varepsilon,\omega) > 0$ such that

$$
\begin{aligned}
\int_0^t (U_x'(s,X(s))\,, G(s,X_s))\mathrm{d}W(s) &\leq \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} \\
&\qquad + \xi\left(\frac{k}{2^N}\right)\int_0^t (U_x'(s,X(s)), G(s,X_s))^2\mathrm{d}s \\
&\leq \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} \\
&\qquad + \int_0^t \xi(s)(U_x'(s,X(s)), G(s,X_s))^2\mathrm{d}s
\end{aligned}
$$

for $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon, \omega)$. Substituting this into (10) we see that P-a.s.

$$
\begin{aligned}
U(t, X(t)) \le U(0, \psi(0)) &+ \xi \left(\frac{k}{2^N}\right)^{-1} \log \frac{k-1}{2^N} + \int_0^t U_s'(s, X(s)) \mathrm{d}s \\
&+ \int_0^t \left\langle U_x'(s, X(s)), A(s, X(s)) + F(s, X_s) \right\rangle \mathrm{d}s \\
&+ \frac{1}{2} \int_0^t (U_{xx}''(s, X(s)) G(s, X_s), G(s, X_s)) \mathrm{d}s \\
&+ \int_0^t \xi(s) (U_x'(s, X(s)), G(s, X_s))^2 \mathrm{d}s
\end{aligned}
$$

for $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon, \omega)$. Using condition (b) we derive that P-a.s.

$$
\begin{aligned}
U(t, X(t)) \le U(0, \psi(0)) &+ \xi \left(\frac{k}{2^N}\right)^{-1} \log \frac{k-1}{2^N} + \int_0^t \varphi_1(s) \mathrm{d}s \\
&+ \int_0^t \varphi_2(s)\, U(s, X(s)) \mathrm{d}s + c(\psi)
\end{aligned}
$$

for $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon, \omega)$. So by virtue of the Gronwall lemma,

$$
\begin{aligned}
|X(t)|^q \, \lambda(t)^m \le \Bigg(& U(0, \psi(0)) + \xi \left(\frac{k}{2^N}\right)^{-1} \log \frac{k-1}{2^N} \\
&+ \int_0^t \varphi_1(s) \mathrm{d}s + c(\psi) \Bigg) \exp\left(\int_0^t \varphi_2(s)\, \mathrm{d}s \right)
\end{aligned}
$$

for $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon, \omega)$. Therefore, noticing condition (c) and following a similar argument as the one in the proof of theorem 4, we have that $P-$a.s.

$$
\begin{aligned}
\log(|X(t)|^q \, \lambda(t)^m) \le \log &\left(U(0, \psi(0)) + \lambda(t)^{\mu+\tau+2\varepsilon} \mathrm{e}^{\varepsilon(\mu+\varepsilon)} + \lambda(t)^{\nu+\varepsilon} + c(\psi) \right) \\
&+ (\theta + \varepsilon) \log \lambda(t)
\end{aligned}
$$

for $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}$, $k \ge k_1(\varepsilon)$. Hence

$$
\limsup_{t \to \infty} \frac{\log(|X(t)|^q \, \lambda(t)^m)}{\log \lambda(t)} \le [(\mu + \tau + 2\varepsilon) \vee (\nu + \varepsilon)] + \theta + \varepsilon, \;\; P - \text{a.s.}
$$

As $\varepsilon > 0$ is arbitrary, we immediately obtain

$$
\limsup_{t \to \infty} \frac{\log |X(t)|}{\log \lambda(t)} \le - \frac{m - [\theta + ((\mu + \tau) \vee \nu)]}{q}, \; P - \text{a.s.}
$$

The proof is therefore complete. ■

Remark 12 *Observe that assumption (b) in the preceding theorem seems different from the ones in theorems in the previous Section. However, it is possible to establish a stronger hypothesis (but easier to check in applications) implying this. Indeed, given* $U(t,x) \in C^{1,2}(\mathbb{R}^+ \times H; \mathbb{R}^+)$ *we can define the following operators* $\widetilde{L}$ *and* $\widetilde{Q}$ *acting on* $L^p_V \cap C_H$*, that is, for* $\Phi \in L^p_V \cap C_H$ *and* $t \in \mathbb{R}^+$*, we set*

$$\begin{aligned}\widetilde{L}U(t,\Phi) &= U'_t(t,\Phi(0)) + \big\langle U'_x(t,\Phi(0)), A(t,\Phi(0)) + F(t,\Phi)\big\rangle \\ &\quad + \frac{1}{2}(U''_{xx}(t,\Phi(0))g(t,\Phi), g(t,\Phi))\end{aligned}$$

and

$$\widetilde{Q}U(t,\Phi) = (U'_x(t,\Phi(0)), g(t,\Phi))^2.$$

Then, if we assume that

$$\widetilde{L}U(t,\Phi) + \xi(t)\widetilde{Q}U(t,\Phi) \le \varphi_1(t) + \varphi_2(t)U(t,\Phi(0)),$$

it immediately implies condition (b).

Finally, we shall include a couple of examples to illustrate our results.

Example 2. Consider the following one dimensional model with constant time delay

$$\begin{cases} dX(t) &= \left[-\dfrac{q}{1+t}X(t) + \dfrac{1}{1+t}X(t-h)\right]\mathrm{d}t + (1+t)^{-q}\mathrm{d}W(t),\ \mathrm{t} \in [0,T], \\ X(t) &= \psi(t),\ t \in [-h,0], \end{cases}$$

where $q > 1$ and $T, h > 0$. This problem can be set in our formulation by taking $V = H = \mathbb{R}$, $p = 2$. We will write C instead of C_H. From the standard theory on stochastic differential equations with delays, it is straightforward that the preceding problem has a unique solution for each initial datum fixed in the space $I^2(-h,0;\mathbb{R}) \cap L^2(\Omega; C)$.

Define for $u \in \mathbb{R}$ and $\phi \in C$, $A(t,u) = -\dfrac{qu}{1+t}$, $F(t,\phi) = \dfrac{1}{1+t}\phi(-h)$ and $G(t,\phi) = (1+t)^{-q}$, $t \in [0,T]$.

Now, we consider $U(t,y) = (1+t)^{2q}|y|^2$. Then, it it easy to check that for arbitrary $\delta > 1$, $\xi(t) = \dfrac{1}{4(1+t)^{\delta}}$, we have

$$
\begin{aligned}
&\int_0^t U_s'(s,X(s))\mathrm{d}s + \int_0^t \left\langle U_x'(s,X(s)), A(s,X(s)) + F(s,X_s)\right\rangle \mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t (U_{xx}''(s,X(s))G(s,X_s), G(s,X_s))\mathrm{d}s \\
&\quad + \int_0^t \frac{1}{4(1+s)^{\delta}}(U_x'(s,X(s)), G(s,X_s))^2\mathrm{d}s \\
&\le \int_0^t \mathrm{d}s + \int_0^t \frac{1}{(1+s)^{\delta}}U(s,X(s))\mathrm{d}s + \int_0^t 2(1+s)^{2q-1}X(s)X(s-h)\mathrm{d}s \\
&\le \int_0^t \mathrm{d}s + \int_0^t \frac{1}{(1+s)^{\delta}}U(s,X(s))\mathrm{d}s + \int_0^t \frac{1}{(1+s)}U(s,X(s))\mathrm{d}s \\
&\quad + \int_0^t (1+s)^{2q-1}|X(s-h)|^2\,\mathrm{d}s.
\end{aligned}
$$

We now estimate the last integral:

$$
\begin{aligned}
\int_0^t (1+s)^{2q-1}|X(s-h)|^2\,\mathrm{d}s &= |\psi|_C^2\int_{-h}^0 (1+s+h)^{2q-1}\mathrm{d}s \\
&\quad + \int_0^{t-h}(1+r+h)^{2q-1}|X(r)|^2\,\mathrm{d}r \\
&\le c(\psi) + (1+h)^{2q-1}\int_0^t \frac{1}{1+s}U(s,X(s))\mathrm{d}s,
\end{aligned}
$$

where we have denoted $c(\psi) = \frac{1}{2q}((1+h)^{2q}-1)|\psi|_C^2$ and have used the inequality

$$
(1+r+h) \le (1+r)(1+h), r, h \ge 0.
$$

Then,

$$
\varphi_1(t) = 1, \quad \varphi_2(t) = \frac{1}{(1+t)^{\delta}} + \frac{(1+h)^{2q-1}+1}{(1+t)}.
$$

By some easy computations, we can check that

$$
\tau = 0, \quad \nu = 1, \quad \theta = (1+h)^{2q-1}+1, \quad \mu = \delta.
$$

Hence, by virtue of theorem 11 it follows

$$\limsup_{t\to\infty}\frac{\log(|X(t)|)}{\log(1+t)}\leq-\frac{2q-((1+h)^{2q-1}+1+\delta)}{2},\ \ P-\text{a.s.}$$

As the constant $\delta>1$ is arbitrary, we immediately obtain

$$\limsup_{t\to\infty}\frac{\log(|X(t)|)}{\log(1+t)}\leq-\frac{2q-((1+h)^{2q-1}+2)}{2},\ \ P-\text{a.s.}$$

Thus, the zero solution is almost sure polynomially stable with decay function $(1+t)$ and order at least $\left(2q-((1+h)^{2q-1}+2)\right)/2$ whenever $q>\left(2+(1+h)^{2q-1}\right)/2$.

It is worth pointing out that the value of q is restricting the maximal admissible value for the time lag h. The larger q, the larger h is allowed to be.

Example 3. Consider the semilinear stochastic heat equation with finite time-lags r_1, r_2 $(h>r_1,r_2\geq 0)$,

$$\begin{cases} dX(t,x) &= \left[\gamma\frac{\partial^2X(t,x)}{\partial x^2}+\int_{-r_1}^0\left(\alpha_1X(t+u,x)+\alpha_2\frac{\partial X}{\partial x}(t+u,x)\right)\beta(u)\mathrm{d}u\right]\mathrm{d}t \\ &+\ \alpha(X(t))\frac{X(t-r_2,x)}{1+|X(t,x)|}\mathrm{d}W(t), \\ X(t,x) &= \psi(t,x),\ t\in[-h,0],\ x\in[0,\pi], \\ X(t,0) &= X(t,\pi)=0,\ t\geq 0. \end{cases}$$

where $\gamma>0$, $\alpha_1,\alpha_2\geq 0$; $\alpha(\cdot):\mathbb{R}\to\mathbb{R}$, $\beta:[-r_1,0]\to\mathbb{R}$ are two bounded Lipschitz continuous functions with $|\alpha(x)|\leq K$, $|\beta(u)|\leq M$, $x\in\mathbb{R}$, $u\in[-r_1,0]$, $M,K>0$. Define $V=H_0^1[0,\pi]$, $H=L^2[0,\pi]$ and denote by $\|\cdot\|$ and $|\cdot|$ the norms in V and H respectively; by $(\cdot,\cdot)$ the inner product in H.

This problem can be put within our formulation by denoting $A(t,v)(x)=\gamma\frac{\mathrm{d}^2v(x)}{\mathrm{d}x^2}$, for $v\in V$, $x\in[0,\pi]$; $F(t,\phi)(x)=\int_{-r_1}^0\left(\alpha_1\phi(u)(x)+\alpha_2\frac{\mathrm{d}\phi(u)(x)}{\mathrm{d}x}\right)\beta(u)\mathrm{d}u$ and $G(t,\phi)(x)=\alpha(\phi(0))\frac{\phi(-r_2)(x)}{1+|\phi(0)(x)|}$, for $\phi\in C_H$, $t\geq 0$, $x\in[0,\pi]$.

We will consider $U(t,y)=\mathrm{e}^{mt}|y|^2$ (where $m>0$ is a fixed but arbitrary constant) which immediately satisfies the whole assumptions required to apply Itô's formula.

It is easy to check that, if we take $\xi(t) = \dfrac{1}{4\mathrm{e}^{mt}}$, then

$$
\begin{aligned}
&\int_0^t U_s'(s, X(s))\mathrm{d}s + \int_0^t \left\langle U_x'(s, X(s)), A(s, X(s)) + F(s, X_s)\right\rangle \mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t (U_{xx}''(s, X(s))G(s, X_s), G(s, X_s))\mathrm{d}s \\
&\quad + \int_0^t \frac{1}{4\mathrm{e}^{ms}}(U_x'(s, X(s)), G(s, X_s))^2\mathrm{d}s \\
&\leq \int_0^t mU(s, X(s))\mathrm{d}s + 2K^2\int_0^t \mathrm{e}^{ms}\, |X(s-r_2)|^2\, \mathrm{d}s \\
&\quad + \int_0^t \left(2X(s)\mathrm{e}^{ms}, \alpha_1\int_{-r_1}^0 X(s+u)\beta(u)\mathrm{d}u\right)\mathrm{d}s \\
&\quad + \int_0^t \left(2X(s)\mathrm{e}^{ms}, \alpha_2\int_{-r_1}^0 \frac{\partial X(s+u)}{\partial x}\beta(u)\mathrm{d}u\right)\mathrm{d}s - 2\gamma\int_0^t \mathrm{e}^{ms}\, \|X(s)\|^2\, \mathrm{d}s \\
&\triangleq \int_0^t mU(s, X(s))\mathrm{d}s + 2K^2 I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

On the one hand

$$
\begin{aligned}
I_1 &\leq |\psi|_{C_H}^2 \int_{-r_2}^0 \mathrm{e}^{m(r_2+u)}\mathrm{d}u + \int_0^{t-r_2} \mathrm{e}^{mr_2}U(s, X(s))\mathrm{d}s \\
&\leq r_2\mathrm{e}^{mr_2}\, |\psi|_{C_H}^2 + \int_0^t \mathrm{e}^{mr_2}U(s, X(s))\mathrm{d}s.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
I_2 &\leq \alpha_1\int_0^t U(s, X(s))\mathrm{d}s + \alpha_1\int_0^t \mathrm{e}^{ms}\left|\int_{-r_1}^0 X(s+u)\beta(u)\mathrm{d}u\right|^2 \mathrm{d}s \\
&\leq \alpha_1\int_0^t U(s, X(s))\mathrm{d}s + \alpha_1 r_1\int_0^t \mathrm{e}^{ms}\left(\int_{-r_1}^0 |X(s+u)|^2\,\beta^2(u)\mathrm{d}u\right)\mathrm{d}s \\
&\leq \alpha_1\int_0^t U(s, X(s))\mathrm{d}s + \alpha_1 r_1 M^2\int_0^t \mathrm{e}^{ms}\left(\int_{s-r_1}^s |X(u)|^2\,\mathrm{d}u\right)\mathrm{d}s \\
&\leq \alpha_1\int_0^t U(s, X(s))\mathrm{d}s \\
&\qquad + \alpha_1 r_1^2 M^2\left[|\psi|_{C_H}^2\left(\int_{-r_1}^0 \mathrm{e}^{m(u+r_1)}\mathrm{d}u\right) + \int_0^t \mathrm{e}^{mr_1}U(s, X(s))\mathrm{d}s\right] \\
&\leq \alpha_1\int_0^t U(s, X(s))\mathrm{d}s + \alpha_1 r_1^3 M^2\mathrm{e}^{mr_1}\, |\psi|_{C_H}^2 \\
&\qquad + \alpha_1 r_1^2 M^2\int_0^t \mathrm{e}^{mr_1}U(s, X(s))\mathrm{d}s,
\end{aligned}
$$

and, finally

$$
\begin{aligned}
I_3 &\leq \alpha_2 \int_0^t U(s,X(s))\mathrm{d}s + \alpha_2 \int_0^t \mathrm{e}^{ms} \left| \int_{-r_1}^0 \frac{\partial X(s+u)}{\partial x} \beta(u)\mathrm{d}u \right|^2 \mathrm{d}s \\
&\leq \alpha_2 \int_0^t U(s,X(s))\mathrm{d}s + \alpha_2 r_1 \int_0^t \mathrm{e}^{ms} \left(\int_{-r_1}^0 \|X(s+u)\|^2 \beta^2(u)\mathrm{d}u \right) \mathrm{d}s \\
&\leq \alpha_2 \int_0^t U(s,X(s))\mathrm{d}s + \alpha_2 r_1 M^2 \int_0^t \mathrm{e}^{ms} \left(\int_{s-r_1}^s \|X(u)\|^2 \mathrm{d}u \right) \mathrm{d}s \\
&\leq \alpha_2 \int_0^t U(s,X(s))\mathrm{d}s \\
&\quad + \alpha_2 r_1^2 M^2 \left[\|\psi\|^2_{L^2_V} \int_{-r_1}^0 \mathrm{e}^{m(u+r_1)}\mathrm{d}u + \int_0^t \mathrm{e}^{mr_1}\mathrm{e}^{ms} \|X(s)\|^2 \mathrm{d}s \right] \\
&\leq \alpha_2 \int_0^t U(s,X(s))\mathrm{d}s + \alpha_2 r_1^3 M^2 \mathrm{e}^{mr_1} \|\psi\|^2_{L^2_V} \\
&\quad + \alpha_2 r_1^2 M^2 \int_0^t \mathrm{e}^{mr_1}\mathrm{e}^{ms} \|X(s)\|^2 \mathrm{d}s.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^t U_s'(s,X(s))\mathrm{d}s + \int_0^t \left\langle U_x'(s,X(s)), A(s,X(s)) + F(s,X_s) \right\rangle \mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t (U_{xx}''(s,X(s))G(s,X_s), G(s,X_s))\mathrm{d}s \\
&\quad + \int_0^t \frac{1}{4\mathrm{e}^{ms}} (U_x'(s,X(s)), G(s,X_s))^2 \mathrm{d}s \\
&\leq (m + 2K^2\mathrm{e}^{mr_2} + (\alpha_1+\alpha_2)(1 + r_1^2 M^2 e^{mr_1}) - 2\gamma) \int_0^t U(s,X(s))\mathrm{d}s + c(\psi),
\end{aligned}
$$

if we suppose that $\alpha_2 r_1^2 M^2 \mathrm{e}^{mr_1} - 2\gamma < 0$, i.e., $\gamma > \frac{\alpha_2 r_1^2 M^2 e^{mr_1}}{2}$, and where

$$
c(\psi) = (2K^2 r_2 e^{mr_2} + \alpha_1 r_1^3 M^2 e^{mr_1}) \, |\psi|^2_{C_H} + \alpha_2 r_1^3 M^2 e^{mr_1} \|\psi\|^2_{L^2_V}.
$$

Then, we obtain

$$
\varphi_1(t) = 0, \;\; \varphi_2(t) = m + 2K^2\mathrm{e}^{mr_2} - 2\gamma + (\alpha_1+\alpha_2)(1 + r_1^2 M^2 e^{mr_1}).
$$

Therefore, constants in theorem 11 can be chosen as follows

$$
\tau = 0, \;\; \nu = 0, \;\; \theta = m + 2K^2\mathrm{e}^{mr_2} - 2\gamma + (\alpha_1+\alpha_2)(1 + r_1^2 M^2 e^{mr_1}), \;\; \mu = m,
$$

whence we deduce that $P-$a.s.

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{t}\leq-\frac{m-(m+2K^2\mathrm{e}^{mr_2}-2\gamma+(\alpha_1+\alpha_2)(1+r_1^2M^2e^{mr_1})+m)}{2},$$

i.e., $P-$a.s.

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{t}\leq-\frac{(-m+2\gamma-2K^2\mathrm{e}^{mr_2}-(\alpha_1+\alpha_2)(1+r_1^2M^2e^{mr_1}))}{2}.$$

Since $m>0$ can be chosen arbitrarily, we immediately obtain, by means of a continuity argument, that whenever $\gamma>\frac{2K^2+(\alpha_1+\alpha_2)(1+r_1^2M^2)}{2}$, the solution is almost surely exponentially stable with at least order $(2\gamma-2K^2-(\alpha_1+\alpha_2)(1+r_1^2M^2))/2$. It is remarkable that no restriction is needed on the time lag r_2.

4 Conclusions and final remarks

Some results on the pathwise asymptotic stability for stochastic partial differential equations have been proved. The main results provide some sufficient conditions to guarantee almost sure stability with a general decay rate. Also the situation containing some hereditary features has been considered, and we pointed out the possibility of proving some stabilization results with non-exponential decay, which we plan to work in a subsequent paper.

Another point is that, although we have only considered the case of a real Wiener process, the results can be extended to deal with the Hilbert valued situation. However, we have preferred to consider this framework for the sake of clarity.

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