Asymptotic behaviour of $2D-$Navier-Stokes Equations with Delays

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Some results on the asymptotic behaviour of solutions to Navier-Stokes equations when the external force contains some hereditary characteristics are proved. We show two different approaches to prove the convergence of solutions to the stationary one, when this is unique. The first is a direct method while the second is based in the Razumikhin type one.

Keywords: Stability, long-time behaviour, $2D$ Navier-Stokes equations, variable delays.

1. Introduction

The Navier-Stokes equations govern the motion of usual fluids like water, air, oil, etc. These equations have been the subject of numerous works since the first paper of Leray was published in 1933 (see Constantin & Foias 1988; Lions 1969; Temam 1979, and the references therein). In our recent work Caraballo & Real (2001) we consider a Navier-Stokes model in which the external force contains some hereditary features and prove the existence of weak solutions. These situations containing delays may appear when we want to control the system (in certain sense) by applying a force which takes into account not only the present state of the system but the history of the solution.

Another interesting problem concerns the asymptotic behaviour of the systems, since this analysis can provide useful information on the future evolution of the system. This will be the main aim of this paper.

To this end, let $\Omega \subset \mathbb{R}^2$ be an open bounded set with regular boundary Γ, and consider the following functional $2D-$Navier-Stokes problem (for further details and notations see Lions 1969 and Temam 1979):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + \sum_{i=1}^{2} u_i \dfrac{\partial u}{\partial x_i} = f - \nabla p + g(t, u_t) \quad \text{in} \quad (0, +\infty) \times \Omega, \\ \text{div } u = 0 \quad \text{in} \quad (0, +\infty) \times \Omega, \\ u = 0 \;\; \text{on} \; (0, +\infty) \times \Gamma, \\ u(0, x) = u_0(x), \quad x \in \Omega, \\ u(t, x) = \phi(t, x), \; t \in (-h, 0) \quad x \in \Omega, \end{cases}$$

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where we assume that $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u_0 the initial velocity field, f a nondelayed external force field, g another external force with some hereditary characteristics and ϕ the initial datum in the interval of time $(-h, 0)$, where h is a fixed positive number.

In Section 2, we will recall some preliminary results on the existence and uniqueness of weak solutions of our model that were proved in Caraballo & Real (2001). Then, we will complete these by proving a regularity result concerning the existence of strong solutions; such solutions will be needed in our stability analysis. In Section 3, we analyze the exponential convergence towards a unique stationary solution; such a solution exists if, for instance, the viscosity is large enough. Our main aim is to exhibit two different approaches ensuring this asymptotic behaviour: a direct method, and a Razumikhin type one. To apply the former, we need only the existence of weak solutions but must impose a restriction on the delay term, while the latter needs strong solutions but the extra assumption on the delay forcing term can be removed. A surprising fact, in principle, is that we can establish a sufficient condition on the parameters and functions appearing in the model, such that both results hold true in some particular cases.

2. Preliminaries and statement of the problem

To start, we consider the following usual abstract spaces:

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u \ = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with norm $|\cdot|$, and inner product $(\cdot, \cdot)$ where for $u, v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x) \mathrm{d}x,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with norm $\|\cdot\|$, and associated scalar product $((\cdot, \cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \mathrm{d}x.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact.

Finally, we will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V'.

Now we define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x \quad \forall\, u, v, w \in V.$$

The trilinear form satisfies

$$b(u, v, w) = -b(u, w, v), \qquad \forall\, u, v, w \in V,$$

and consequently

$$b(u, v, v) = 0, \qquad \forall\, u, v \in V,$$

$$b(u,u,v-u)-b(v,v,v-u)=-b(v-u,u,v-u), \qquad \forall\, u,v\in V.$$

We recall that $(H_0^1(\Omega))^2$ (and, as a consequence, V too) is embedded in $(L^4(\Omega))^2$, this injection being compact. Also, there exists $C_1(\Omega)>0$ such that

$$|b(u,v,w)|\le C_1(\Omega)|u|_{(L^4(\Omega))^2}\|v\|\|w\|, \qquad \forall\, u,v,w\in V.$$

Now, consider a fixed $T>0$. Let X be a Banach space. Given a function $u:(-h,T)\to X$, for each $t\in(0,T)$ we denote by u_t the function defined on $(-h,0)$ by the relation $u_t(s)=u(t+s), s\in(-h,0)$.

In order to state the problem in the correct framework, let us first establish suitable assumptions on the term in which the delay is present.

In a general way, let X and Y be two separable Banach spaces, and $g:[0,T]\times C^0([-h,0];X)\to Y$ such that

(I) for all $\xi\in C^0([-h,0];X)$, the mapping $t\in[0,T]\to g(t,\xi)\in Y$ is measurable,

(II) for each $t\in[0,T]$, $g(t,0)=0$,

(III) there exists $L_g>0$ such that $\forall\, t\in[0,T], \quad \forall\,\xi,\eta\in C^0([-h,0];X)$

$$\|g(t,\xi)-g(t,\eta)\|_Y\le L_g\,\|\xi-\eta\|_{C^0([-h,0];X)}\,,$$

(IV) there exists $C_g>0$ such that $\forall\, t\in[0,T]$, $\forall\, u,v\in C^0([-h,T];X)$

$$\int_0^t\|g(s,u_s)-g(s,v_s)\|_Y^2\,\mathrm{d}s\le C_g\int_{-h}^t\|u(s)-v(s)\|_X^2\,\mathrm{d}s.$$

Observe that (I)-(III) imply that given $u\in C^0([-h,T];X)$, the function $g_u:t\in[0,T]\to Y$ defined by $g_u(t)=g(t,u_t)\ \forall\, t\in[0,T]$, is measurable (see Bensoussan et al. 1992) and, in fact, belongs to $L^\infty(0,T;Y)$. Then, thanks to (IV), the mapping

$$\mathcal{G}:u\in C^0([-h,T];X)\to g_u\in L^2(0,T;Y)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$ which is uniformly continuous from $L^2(-h,T;X)$ into $L^2(0,T;Y)$. From now on, we will denote $g(t,u_t)=\widetilde{\mathcal{G}}(u)(t)$ for each $u\in L^2(-h,T;X)$, and thus, $\forall\, t\in[0,T]$, $\forall\, u,v\in L^2(-h,T;X)$, we will have

$$\int_0^t\|g(s,u_s)-g(s,v_s)\|_Y^2\,\mathrm{d}s\le C_g\int_{-h}^t\|u(s)-v(s)\|_X^2\,\mathrm{d}s.$$

With the convention above, assume that $u_0\in H$, $\phi\in L^2(-h,0;V)$, $f\in L^2(0,T;V')$, $g:[0,T]\times C^0([-h,0];V)\to(L^2(\Omega))^2$ satisfies hypotheses (I)-(IV) with $X=V$, $Y=(L^2(\Omega))^2$, $L_g=L_1$ and $C_{g_1}=C_1$. For example, when the function g is defined by $g(t,\phi)=G(\phi(-\rho(t))$ for a suitable differentiable delay function ρ and a Lipschitz continuous G, the assumptions above hold (see Caraballo and Real (2001) for more details and further examples).

Thus, we are interested in the following problem:

$$\left\{\begin{array}{l}\text{To find } u\in L^2(-h,T;V)\cap L^\infty(0,T;H) \text{ such that, for all } v\in V,\\ \dfrac{\mathrm{d}}{\mathrm{d}t}(u(t),v)+\nu((u(t),v))+b(u(t),u(t),v)=\langle f(t),v\rangle+(g(t,u_t),v)\\ u(0)=u_0,\ \ u(t)=\phi(t),\ \ t\in(-h,0),\end{array}\right. \tag{2.1}$$

where the equation in (2.1) must be understood in the sense of $\mathcal{D}'(0,T)$.

Remark 2.1. *Observe that the terms in (2.1) are well defined. In particular, by hypotheses (I)-(IV), if* $u \in L^2(-h,T;V)$ *the term* $g(t,u_t)$ *defines a function in* $L^2(0,T;(L^2(\Omega)^2)$*. Moreover, if there exists a solution to this problem, then it belongs to the space* $C^0([0,T];H)$*.*

Remark 2.2. *Define the operator* $A : V \mapsto V'$ *by*

$$\langle Au, v\rangle = ((u,v)), \qquad \forall\, u, v \in V,$$

consider the operator $B : V \times V \mapsto V'$ *defined by*

$$\langle B(u,v), w\rangle = b(u,v,w), \qquad \forall\, u, v, w \in V,$$

and set $B(u) = B(u,u)$*. Then, (2.1) can be rewritten as*

$$\left\{ \begin{array}{l} \text{To find } u \in L^2(-h,T;V) \cap L^\infty(0,T;H) \text{ such that, ,} \\ \dfrac{d}{dt}u(t) + \nu Au(t) + B(u(t)) = f(t) + g(t,u_t) \text{ in } V' \\ u(0) = u_0, \ \ u(t) = \phi(t), \ \ t \in (-h,0), \end{array} \right. \tag{2.2}$$

Observe that if we denote $D(A) = (H^2(\Omega))^2 \cap V$*, then*

$$Au = -P\Delta u, \qquad \forall\, u \in D(A),$$

where P *is the orthogonal projector from* $(L^2(\Omega))^2$ *onto* H*, and there exists a constant* $C_2(\Omega) > 0$ *such that*

$$|u|_2 \le C_2(\Omega)|Au|, \qquad \forall\, u \in D(A), \tag{2.3}$$

where $|u|_2$ *denotes the norm of* u *in* $(H^2(\Omega))^2$ *(see Constantin & Foias 1988).*

In the following theorem we recall an existence and uniqueness result concerning our 2D-Navier-Stokes model with delays, completed with a statement about the existence of strong solutions.

Theorem 2.3. *Let us consider* $u_0 \in H$, $\phi \in L^2(-h,0;V)$, $f \in L^2(0,T;V')$, *and assume that* $g : [0,T] \times C^0([-h,0];V) \to (L^2(\Omega))^2$ *satisfies hypotheses (I)-(IV) with* $X = V$, $Y = (L^2(\Omega))^2$, $L_g = L_1$ *and* $C_g = C_1$*. Also, we assume that the following condition (V) holds:*

(V) If v^m *converges weakly to* v *in* $L^2(-h,T;V)$ *and strongly in* $L^2(-h,T;H)$*, then* $g(\cdot, v^m_\cdot)$ *converges weakly to* $g(\cdot, v_\cdot)$ *in* $L^2(0,T;V')$*.*

Then,

a) There exists a unique weak solution to (2.1) which, in addition, belongs to the space $C^0([0,T];H)$*.*

b) If $f \in L^2(0,T;(L^2(\Omega))^2)$ *and* $u_0 \in V$*, then, the solution* u *to (2.1) is a strong solution, that is,*

$$u \in L^2(0,T;D(A)) \cap C^0([0,T];V) \text{ and } u' \in L^2(0,T;H). \tag{2.4}$$

In particular, if $\phi \in C^0([-h,0];V)$ *and* $u_0 = \phi(0)$*, then* $u \in C^0([-h,T];V)$*.*

Proof. For the proof of a), see Caraballo & Real (2001).

To prove b), observe that if $f \in L^2(0,T;(L^2(\Omega))^2)$ and we denote $\tilde{f}(t) = P(f(t)+g(t,u_t))$, then, for all $v \in V$, u satisfies

$$\begin{cases} u \in L^2(0,T;V) \cap L^\infty(0,T;H), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(u(t),v) + \nu((u(t),v)) + b(u(t),u(t),v) = (\tilde{f}(t),v) \text{ in } \mathcal{D}'(0,T), \\ u(0) = u_0, \end{cases} \tag{2.5}$$

with $\tilde{f} \in L^2(0,T;H)$, and consequently, if $u_0 \in V$, u satisfies (2.4) (see Temam 1979, 1995). □

3. Stability of stationary solutions

First of all, we will prove a result ensuring the existence of stationary solutions to our Navier-Stokes model when the delay term has a special form, provided the viscosity is large enough. Then, we will prove that when this stationary solution is unique all the solutions to our problem converge to it exponentially fast. This result requires a strong assumption on the delay function, which we will relax later on by using a different approach, namely, a Razumikhin type argument (see Razumikhin (1956), (1960) and Hale and Lunel (1995) for a modern and nice presentation of the method in the finite-dimensional case), but for this it will be necessary to deal with strong solutions. In the sequel, λ_1 will denote the first eigenvalue of A.

(*a*) *Existence and uniqueness of stationary solutions*

Let us consider the following equation

$$\frac{du}{dt} + \nu Au + B(u) = f + g(t,u_t), \tag{3.1}$$

with $f \in V'$ independent of t. A stationary solution to (3.1) is a $u^\star$ such that

$$\nu Au^\star + B(u^\star) = f + g(t,u^\star)$$

for all $t \geq 0$. Now we will prove a result ensuring existence and uniqueness of stationary solutions when the delay term has a particular form, namely, we will assume that the term g is given by

$$g(t,u_t) = G(u(t-\rho(t))),$$

with $G : \mathbb{R}^2 \to \mathbb{R}^2$ a function satisfying $G(0) = 0$ and such that there exists $L_1 > 0$ for which

$$|G(u) - G(v)|_{\mathbb{R}^2} \leq L_1|u-v|_{\mathbb{R}^2}, \forall u,v \in \mathbb{R}^2,$$

and $\rho \in C^1([0,+\infty))$, $\rho(t) \geq 0$ for all $t \geq 0$, $h = \sup_{t\geq 0}\rho(t) \in (0,+\infty)$ and $\rho_* = \sup_{t\geq 0}\rho'(t) < 1$. Observe that this situation is within our framework and satisfies our assumptions (Conditions (I)-(V)) ensuring the existence and uniqueness of solutions (see Caraballo & Real 2001).

Now we are interested in proving that there exist stationary solutions to our equation (3.1), i.e., there exists $u^\star \in V$ such that

$$\nu Au^\star + B(u^\star) = f + G(u^\star).$$

We can prove the following result.

Theorem 3.1. *Suppose that G satisfies the conditions above and $\nu > \lambda_1^{-1} L_1$. Then,*
(*a*) *for all $f \in V'$ there exists a stationary solution to (3.1);*
(*b*) *if $f \in (L^2(\Omega))^2$, the stationary solutions belong to $D(A)$;*
(*c*) *there exists a constant $C_3(\Omega) > 0$ such that if $(\nu - \lambda_1^{-1} L_1)^2 > C_3(\Omega)\|f\|_{V'}$, then the stationary solution to (3.1) is unique.*

Proof. (a) Let $f \in V'$ be fixed. By the Lax-Milgram Theorem, it is easy to see that for each $z \in V$ given, there exists a unique $u \in V$ such that

$$\nu((u,v)) + b(z,u,v) = \langle f, v\rangle + (G(z), v), \quad \forall\, v \in V. \tag{3.2}$$

Moreover, taking $v = u$ in (3.2), it follows that

$$\nu\|u\| \le \|f\|_{V'} + \lambda_1^{-1} L_1 \|z\|. \tag{3.3}$$

Take $k > 0$ such that $k(\nu - \lambda_1^{-1} L_1) \ge \|f\|_{V'}$, and denote

$$\mathcal{C} = \{z \in V;\ \|z\| \le k\}.$$

Then, $\mathcal{C}$ is a convex and compact subset of $(L^4(\Omega))^2$, and moreover, by (3.3), the mapping $z \mapsto u$, defined by means of (3.2), maps $\mathcal{C}$ into $\mathcal{C}$. If we see that this mapping is continuous in $\mathcal{C}$ with the topology induced by $(L^4(\Omega))^2$, then, thanks to Schauder's Theorem, we will obtain the existence of a fixed point in $\mathcal{C}$ for the mapping, and obviously this fixed point is a stationary solution to (3.1). The continuity of the mapping $z \mapsto u$ can be seen as follows. Let $z_i \in \mathcal{C}$ and $u_i \in \mathcal{C}$ be such that

$$\nu((u_i, v)) + b(z_i, u_i, v) = \langle f, v\rangle + (G(z_i), v), \quad \forall\, v \in V, \quad i = 1, 2.$$

Then, it is easy to obtain that

$$\begin{aligned} \nu\|u_1 - u_2\|^2 &= b(z_2 - z_1, u_1, u_1 - u_2) + (G(z_1) - G(z_2), u_1 - u_2) \\ &\le kC_1(\Omega)|z_1 - z_2|_{L^4(\Omega))^2}\|u_1 - u_2\| \\ &\quad + L_1\lambda_1^{-1/2}|z_1 - z_2|\|u_1 - u_2\|. \end{aligned} \tag{3.4}$$

As $V \subset (L^4(\Omega))^2$ and $(L^4(\Omega))^2 \subset (L^2(\Omega))^2$ with continuous injections, the continuity of the mapping $z \mapsto u$ in $\mathcal{C}$, with respect to the topology induced by $(L^4(\Omega))^2$, is a direct consequence of (3.4).

(*b*) Notice that if $f \in (L^2(\Omega))^2$, then every stationary solution u_* to (3.1) is also a solution to (2.1), but with initial data $u_0 = \phi(t) = u_*$ for $t \in [-h, 0)$, and forcing term $\tilde{f} = P(f + G(u_*)) \in H \subset L^2(0, T; H)$. Thus, we can apply the standard regularity results from the theory of the Navier-Stokes equations without delays.

(*c*) Let $f \in V$ be given, and let u_1 and u_2 be two stationary solutions to (3.1). Then, arguing as we did for the inequality (3.4), we obtain

$$\nu\|u_1 - u_2\|^2 \le C_1(\Omega)|u_1 - u_2|_{L^4(\Omega))^2}\|u_1\|\|u_1 - u_2\| + L_1\lambda_1^{-1/2}|u_1 - u_2|\|u_1 - u_2\|. \tag{3.5}$$

But

$$\begin{aligned} \nu\|u_1\|^2 &= \langle f, u_1\rangle + (G(u_1), u_1) \\ &\le \|f\|_{V'}\|u_1\| + \lambda_1^{-1} L_1 \|u_1\|^2, \end{aligned}$$

Thus,

$$(\nu - \lambda_1^{-1}L_1)\|u_1\| \leq \|f\|_{V'}.$$

Using this last inequality, and the continuous injection of V into $(L^4(\Omega))^2$, we obtain easily from (3.5) that there exists $C_3(\Omega) > 0$ such that

$$(\nu - \lambda_1^{-1}L_1)^2\|u_1 - u_2\|^2 \leq C_3(\Omega)\|f\|_{V'}\|u_1 - u_2\|^2.$$

This completes the proof of the theorem. □

Remark 3.2. *a) Although we have proved the existence of a unique stationary solution in this particular situation, this may well happen in more general cases.*
b) Also, it is worth pointing out that the regularity result in part (b) of the previous theorem does not depend on the particular form of the term g we have chosen; this is clear from the argument used in the proof.

(*b*) *Exponential convergence of solutions: a direct approach*

Now we will prove that, under appropriate assumptions, our model has a unique stationary solution, u_∞, and every weak solution approaches u_∞ exponentially fast as t goes to $+\infty$.

Theorem 3.3. *Assume that the forcing term $g(t, u_t)$ is given by $g(t, u_t) = G(u(t-\rho(t))$ with $\rho \in C^1(\mathbb{R}^+; [0, h])$ such that $\rho'(t) \leq \rho_* < 1$ for all $t \geq 0$. Then, there exist two constants $k_i > 0$, $i = 1, 2$, depending only on Ω, such that if $f \in (L^2(\Omega))^2$ and $\nu > \lambda_1^{-1}L_1$ satisfy in addition*

$$2\nu\lambda_1 > \frac{(2-\rho_*)L_1}{1-\rho_*} + \frac{k_1|f|}{\nu - \lambda_1^{-1}L_1} + \frac{k_2|f|^3}{\nu^2(\nu - \lambda_1^{-1}L_1)^3}, \tag{3.6}$$

then there is a unique stationary solution u_∞ of (3.1) and every solution of (2.1) converges to u_∞ exponentially fast as $t \to +\infty$. More exactly, there exist two positive constants C and λ, such that for all $u_0 \in H$ and $\phi \in L^2(-h, 0; V)$, the solution u of (2.1) with $f(t) \equiv f$ satisfies

$$|u(t) - u_\infty|^2 \leq Ce^{-\lambda t}\left(|u_0 - u_\infty|^2 + \|\phi - u_\infty\|^2_{L^2(-h,0;V)}\right), \tag{3.7}$$

for all $t \geq 0$.

Proof. Let $f \in (L^2(\Omega))^2$ be fixed. Consider u, the solution of (2.2) for $f(t) \equiv f$, and let $u_\infty \in D(A)$ be a stationary solution to (3.1). We set $w(t) = u(t) - u_\infty$, and observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) + \nu Aw(t) + B(u(t)) - B(u_\infty) = G(u(t-\rho(t)) - G(u_\infty).$$

Now fix a positive λ to be determined later on. By standard computations we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\lambda t}|w(t)|^2) &= \lambda\mathrm{e}^{\lambda t}|w(t)|^2 + \mathrm{e}^{\lambda t}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 \\
&\leq \mathrm{e}^{\lambda t}(\lambda|w(t)|^2 - 2\nu\|w(t)\|^2 + 2b(w(t), w(t), u_\infty) \\
&\quad +2L_1|w(t-\rho(t))||w(t)|) \\
&\leq \lambda_1^{-1}\mathrm{e}^{\lambda t}(\lambda + L_1 - 2\nu\lambda_1)\|w(t)\|^2 \\
&\quad +2\mathrm{e}^{\lambda t}|b(w(t), w(t), u_\infty)| + L_1\mathrm{e}^{\lambda t}|w(t-\rho(t))|^2. \qquad (3.8)
\end{aligned}$$

Obviously
$$|b(w(t),w(t),u_\infty)| \le c|w(t)|\|w(t)\||u_\infty|_\infty, \tag{3.9}$$
where we denote by $|u_\infty|_\infty$ the norm of u_∞ in $(L^\infty(\Omega))^2$. Observe that $H^2(\Omega) \subset L^\infty(\Omega)$ with continuous injection, thus, using (2.3), we obtain the existence of a constant $c_1 > 0$ depending only on Ω such that
$$|b(w(t),w(t),u_\infty)| \le c_1\lambda_1^{-1/2}\|w(t)\|^2|Au_\infty|. \tag{3.10}$$
On the other hand,
$$\begin{aligned}\nu|Au_\infty| &\le |f| + |G(u_\infty)| + |B(u_\infty)| \\ &\le |f| + L_1|u_\infty| + c'\|u_\infty\||u_\infty|_\infty,\end{aligned}$$
and consequently, from the continuous injection of $H^2(\Omega)$ into $L^\infty(\Omega)$, the inequality (2.3), and the Gagliardo-Nirenberg interpolation inequality, we obtain
$$\nu|Au_\infty| \le |f| + L_1|u_\infty| + c''\|u_\infty\||u_\infty|^{1/2}|Au_\infty|^{1/2}. \tag{3.11}$$
But
$$c''\|u_\infty\||u_\infty|^{1/2}|Au_\infty|^{1/2} \le \frac{(c'')^2\lambda_1^{-1/2}}{2\nu}\|u_\infty\|^3 + \frac{\nu}{2}|Au_\infty|,$$
and thus, from (3.11) we deduce
$$|Au_\infty| \le \frac{2}{\nu}|f| + \frac{2L_1\lambda_1^{-1/2}}{\nu}\|u_\infty\| + \frac{(c'')^2\lambda_1^{-1/2}}{\nu^2}\|u_\infty\|^3. \tag{3.12}$$
Now, as
$$\begin{aligned}\nu\|u_\infty\|^2 &= (f,u_\infty) + (G(u_\infty),u_\infty) \\ &\le |f|\lambda_1^{-1/2}\|u_\infty\| + L_1\lambda_1^{-1}\|u_\infty\|^2,\end{aligned}$$
we obtain from (3.12)
$$\begin{aligned}|Au_\infty| &\le \frac{2}{\nu}|f| + \frac{2L_1\lambda_1^{-1}}{\nu(\nu-L_1\lambda_1^{-1})}|f| + \frac{(c'')^2\lambda_1^{-2}}{\nu^2(\nu-L_1\lambda_1^{-1})^3}|f|^3 \\ &= \frac{2}{(\nu-L_1\lambda_1^{-1})}|f| + \frac{(c'')^2\lambda_1^{-2}}{\nu^2(\nu-L_1\lambda_1^{-1})^3}|f|^3. \end{aligned} \tag{3.13}$$
From (3.8), (3.10), (3.13), and denoting
$$k_1 = 4c_1\lambda_1^{1/2}, \qquad k_2 = 2c_1\lambda_1^{-3/2}(c'')^2,$$
it follows that
$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\lambda t}|w(t)|^2) \\ &\le \lambda_1^{-1}\mathrm{e}^{\lambda t}\left(\lambda + L_1 - 2\nu\lambda_1 + \frac{k_1|f|}{(\nu-L_1\lambda_1^{-1})} + \frac{k_2|f|^3}{\nu^2(\nu-L_1\lambda_1^{-1})^3}\right)\|w(t)\|^2 \\ &\quad + L_1\mathrm{e}^{\lambda t}|w(t-\rho(t))|^2. \end{aligned} \tag{3.14}$$

Now, taking into account the properties of the function ρ, we deduce that if we denote $\tau(t) = t - \rho(t)$, the function τ is strictly increasing in $[0, +\infty)$, and that there exists a $\mu > 0$ such that $\tau^{-1}(t) \leq t + \mu$ for all $t \geq -\rho(0)$. Thus, by the change of variable $\eta = s - \rho(s) = \tau(s)$, we have

$$\begin{aligned}\int_0^t \mathrm{e}^{\lambda s}|w(s-\rho(s))|^2\,\mathrm{d}s &= \int_{-\rho(0)}^{t-\rho(t)} \mathrm{e}^{\lambda\tau^{-1}(\eta)}|w(\eta)|\frac{1}{\tau'(\tau^{-1}(\eta))}\,\mathrm{d}\eta \\ &\leq \frac{\mathrm{e}^{\lambda\mu}}{1-\rho_*}\int_{-h}^{t} \mathrm{e}^{\lambda\eta}|w(\eta)|^2\,\mathrm{d}\eta. \qquad (3.15)\end{aligned}$$

If (3.6) is satisfied, then there exists $\lambda > 0$ small enough such that

$$\lambda + L_1 - 2\nu\lambda_1 + \frac{k_1|f|}{(\nu - L_1\lambda_1^{-1})} + \frac{k_2|f|^3}{\nu^2(\nu - L_1\lambda_1^{-1})^3} + \frac{L_1\mathrm{e}^{\lambda\mu}}{1-\rho_*} \geq 0,$$

integrating (3.14) over the interval $[0, t]$, and taking into account (3.15), we deduce that for this $\lambda > 0$

$$\mathrm{e}^{\lambda t}|w(t)|^2 \leq |w(0)|^2 + \frac{L_1\mathrm{e}^{\lambda\mu}}{1-\rho_*}\int_{-h}^{0} \mathrm{e}^{\lambda\eta}|w(\eta)|^2\mathrm{d}\eta,$$

and thus (3.7) is satisfied. The uniqueness of u_∞ follows from the fact that if $\hat{u}_\infty$ is another stationary solution of (3.1), then $u(t) \equiv \hat{u}_\infty$ is a solution of (2.1) with $u_0 = \hat{u}_\infty$ and $\phi = \hat{u}_\infty$, and consequently, applying (3.7) and making $t \to +\infty$, one has $|\hat{u}_\infty - u_\infty|^2 \leq 0$. □

Remark 3.4. *Notice that the result holds true for more general delay terms. For instance, if g satisfies*

$$\int_0^t \mathrm{e}^{\epsilon s}|g(s,u_s) - g(s,v_s)|^2\mathrm{d}s \leq C(h)\int_{-h}^{t} \mathrm{e}^{\epsilon s}|u(s)-v(s)|^2\mathrm{d}s, \qquad (3.16)$$

for sufficiently small $\epsilon > 0$, and the existence of a unique stationary solution is known.

(*c*) *Exponential convergence of solutions: a Razumikhin approach*

In the previous section we proved a result on the exponential convergence of weak solutions to the unique stationary solution when the delay term g has a particular form, but we needed to impose a restriction on the delay function, so that our proof worked appropriately or that condition (3.16) holds. However, it is possible to prove a result for more general forcing terms and relax this restriction by using a different method which is also much used in dealing with the stability properties of delay differential equations. This method was firstly developed by Razumikhin (see Razumikhin (1956), (1960)) in the context of ordinary differential functional equations, and has already been applied to some stochastic ODEs and PDEs (e.g. Caraballo et al. 2000). However, one interesting point to be noted is that this method requires some kind of continuity concerning the operators in the model and the solutions. This will allow us to prove a result that weaken the assumptions on the delay function or condition (3.16), but concerns only the strong solutions to (2.1).

Theorem 3.5. *Assume that g satisfies conditions (I)-(V) for any $T > 0$, with X and Y as in Theorem 2.3, and that moreover for all $\xi \in C^0([-h, 0]; V)$ the mapping $t \in [0, +\infty) \mapsto g(t, \xi) \in (L^2(\Omega))^2$ is continuous. Suppose that for a given $\nu > 0$ and $f \in (L^2(\Omega))^2$ there exists a stationary solution u_∞ of (3.1) such that for some $\lambda > 0$*

$$\begin{aligned}
&-\nu\langle A(\phi(0) - u_\infty), \phi(0) - u_\infty\rangle - \langle B(\phi(0)) - B(u_\infty), \phi(0) - u_\infty\rangle \\
&+ (g(t, \phi) - g(t, u_\infty), \phi(0) - u_\infty) \\
&\leq -\lambda|\phi(0) - u_\infty|^2, \quad t \geq 0,
\end{aligned} \tag{3.17}$$

whenever $\phi \in C^0([-h, 0]; V)$ satisfies

$$\|\phi - u_\infty\|^2_{C(-h,0;H)} \leq \mathrm{e}^{\lambda h}|\phi(0) - u_\infty|^2. \tag{3.18}$$

Then, the stationary solution u_∞ of (3.1) is unique, and for all $\psi \in C^0([-h, 0]; V)$, the strong solution $u(t; \psi)$ to (2.1) corresponding to this initial datum satisfies

$$|u(t; \psi) - u_\infty|^2 \leq \mathrm{e}^{-\lambda t}\|\psi - u_\infty\|^2_{C^0([-h,0];H)}, \ \forall t \geq 0. \tag{3.19}$$

Proof. Suppose there exists an initial datum $\psi \in C^0([-h, 0]; V)$ such that (3.19) does not hold. Then, denoting

$$\sigma = \inf\{t > 0;\ |u(t; \psi) - u_\infty|^2 > \mathrm{e}^{-\lambda t}\|\psi - u_\infty\|^2\},$$

we obtain that

$$\mathrm{e}^{\lambda t}|u(t; \psi) - u_\infty|^2 \leq \mathrm{e}^{\lambda\sigma}|u(\sigma; \psi) - u_\infty|^2 = ||\psi - u_\infty||^2_{C^0([-h,0];H)}, \tag{3.20}$$

for all $0 \leq t \leq \sigma$, and there is a sequence $\{t_k\}_{k\geq1}$ in $\mathbb{R}^+$ such that $t_k \downarrow \sigma$, as $k \to \infty$, and

$$\mathrm{e}^{\lambda t_k}|u(t_k; \psi) - u_\infty|^2 > \mathrm{e}^{\lambda\sigma}|u(\sigma; \psi) - u_\infty|^2. \tag{3.21}$$

On the other hand, by virtue of (3.20) it is easy to deduce

$$|u(\sigma + \theta; \psi) - u_\infty|^2 \leq \mathrm{e}^{\lambda h}|u(\sigma; \psi) - u_\infty|^2,$$

for all $-h \leq \theta \leq 0$, which, in view of assumption (3.17), immediately implies that

$$\begin{aligned}
&-\nu\langle A(u(\sigma; \psi) - u_\infty), u(\sigma; \psi) - u_\infty\rangle - \langle B(u(\sigma; \psi)) - B(u_\infty), u(\sigma; \psi) - u_\infty\rangle \\
&+ (g(\sigma, u_\sigma(\cdot; \psi)) - g(\sigma, u_\infty), u(\sigma; \psi) - u_\infty) \\
&\leq -\lambda|u(\sigma; \psi) - u_\infty|^2.
\end{aligned} \tag{3.22}$$

As $u(\cdot; \psi) \in C^0([-h, +\infty); V)$, by the continuity of the operators in the problem,

there exists $\epsilon_* >$ such that for all $\epsilon \in (0, \epsilon_*]$,

$$\begin{aligned}
&-\nu\langle A(u(t; \psi) - u_\infty), u(t; \psi) - u_\infty\rangle - \langle B(u(t; \psi)) - B(u_\infty), u(t; \psi) - u_\infty\rangle \\
&+ (g(t, u_t(\cdot; \psi)) - g(t, u_\infty), u(t; \psi) - u_\infty) \\
&\leq -\lambda|u(t; \psi) - u_\infty|^2, \quad t \geq 0,
\end{aligned} \tag{3.23}$$

for all $t \in [\sigma, \sigma+\epsilon]$. Thus, if we denote by $w(t) = u(t;\psi) - u_\infty$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w(t)|^2 &= -\nu\langle Aw(t), w(t)\rangle - \langle B(u(t;\psi)) - B(u_\infty), w(t)\rangle \\ &\quad + (g(t, u_t(\cdot;\psi)) - g(t, u_\infty), w(t)) \end{aligned}$$

for all $t \in [\sigma, \sigma+\epsilon]$, and after integrating we obtain

$$\begin{aligned} &\mathrm{e}^{\lambda(\sigma+\epsilon)}|w(\sigma+\epsilon;\psi)|^2 - \mathrm{e}^{\lambda\sigma}|u(\sigma;\psi) - u_\infty|^2 \\ &= \int_\sigma^{\sigma+\epsilon} \lambda \mathrm{e}^{\lambda t}|w(t;\psi)|^2 \mathrm{d}t \\ &\quad + \int_\sigma^{\sigma+\epsilon} \mathrm{e}^{\lambda t}\left(-2\nu\langle Aw(t), w(t)\rangle - 2\langle B(u(t;\psi)) - B(u_\infty), w(t)\rangle\right)\mathrm{d}t \\ &\quad + \int_\sigma^{\sigma+\epsilon} \mathrm{e}^{\lambda t}\left(g(t, u_t(\cdot;\psi)) - g(t, u_\infty), w(t)\right)\mathrm{d}t \\ &\leq 0. \end{aligned}$$

However, this contradicts (3.21), so (3.19) must be true.

The uniqueness of the stationary solution is deduced in the same way as in Theorem 3.3. □

Remark 3.6. *We wish now to provide a sufficient condition which implies (3.17) and that would be easier to check in applications.*

Corollary 3.7. *Assume that g satisfies conditions (I)-(V) for any $T > 0$, with X and Y as in Theorem 2.3, and that for all $\xi \in C^0([-h, 0]; V)$ the mapping $t \in [0, +\infty) \mapsto g(t, \xi) \in (L^2(\Omega))^2$ is continuous. Suppose $\nu > 0$ and $f \in (L^2(\Omega))^2$ are given so that there exists a stationary solution u_∞ of (3.1). There exist two constants, $k_i > 0$, $i = 1, 2$, depending only on Ω, such that if*

$$2\nu\lambda_1 > 2L_1 + \frac{k_1|f|}{\nu - \lambda_1^{-1}L_1} + \frac{k_2|f|^3}{\nu^2(\nu - \lambda_1^{-1}L_1)^3}, \tag{3.24}$$

then, the stationary solution u_∞ of (3.1) is unique, and for all $\psi \in C^0([-h, 0]; V)$, the strong solution to (2.1) corresponding to this initial datum, $u(t;\psi)$, satisfies (3.19), i.e.

$$|u(t;\psi) - u_\infty|^2 \leq \mathrm{e}^{-\lambda t}\|\psi - u_\infty\|^2_{C^0([-h,0];H)}, \ \forall t \geq 0.$$

Proof. Let $\phi \in C^0([-h, 0]; V)$ be such that

$$\|\phi - u_\infty\|^2_{C^0([-h,0];H)} \leq \mathrm{e}^{\lambda h}|\phi(0) - u_\infty|^2, \tag{3.25}$$

where $\lambda > 0$ is a constant to be chosen later on. Then,

$$\begin{aligned} &-\nu\langle A(\phi(0) - u_\infty), \phi(0) - u_\infty\rangle - \langle B(\phi(0)) - B(u_\infty), \phi(0) - u_\infty\rangle \\ &+ (g(t,\phi) - g(t, u_\infty), \phi(0) - u_\infty) \\ \leq\ & -\nu\|\phi(0) - u_\infty\|^2 - b(\phi(0) - u_\infty, u_\infty, \phi(0) - u_\infty) \\ &+ L_1\|\phi - u_\infty\|_{C(-h,0;H)}|\phi(0) - u_\infty| \\ \leq\ & -\nu\|\phi(0) - u_\infty\|^2 + L_1\lambda_1^{-1}\mathrm{e}^{\lambda h}\|\phi(0) - u_\infty\|^2 \\ &+ |b(\phi(0) - u_\infty, \phi(0) - u_\infty, u_\infty)|. \end{aligned}$$

Now, using (3.10) and (3.13), and the notation used in the proof of Theorem 3.3, it follows immediately that

$$\begin{aligned} &-\nu \langle A(\phi(0)-u_\infty), \phi(0)-u_\infty\rangle - \langle B(\phi(0))-B(u_\infty), \phi(0)-u_\infty\rangle \\ &\quad +(g(t,\phi)-g(t,u_\infty), \phi(0)-u_\infty) \\ &\le \left(-\nu + L_1\lambda_1^{-1}\mathrm{e}^{\lambda h} + \frac{k_1\lambda_1^{-1}|f|}{2(\nu-\lambda_1^{-1}L_1)} + \frac{k_2\lambda_1^{-1}|f|^3}{2\nu^2(\nu-\lambda_1^{-1}L_1)^3},\right)\|\phi(0)-u_\infty\|^2. \end{aligned} \tag{3.26}$$

Then, if (3.24) is fulfilled, there exists $\lambda > 0$ such that

$$\lambda\lambda_1^{-1} - \nu + L_1\lambda_1^{-1}\mathrm{e}^{\lambda h} + \frac{k_1\lambda_1^{-1}|f|}{2(\nu-\lambda_1^{-1}L_1)} + \frac{k_2\lambda_1^{-1}|f|^3}{2\nu^2(\nu-\lambda_1^{-1}L_1)^3} \ge 0,$$

and, for this fixed λ, we can obtain from (3.26)

$$\begin{aligned} &-\nu \langle A(\phi(0)-u_\infty), \phi(0)-u_\infty\rangle - \langle B(\phi(0))-B(u_\infty), \phi(0)-u_\infty\rangle \\ &\quad +(g(t,\phi)-g(t,u_\infty), \phi(0)-u_\infty) \\ &\le -\lambda\lambda_1^{-1}\|\phi(0)-u_\infty\|^2 \\ &\le -\lambda|\phi(0)-u_\infty|^2. \end{aligned}$$

The proof is now complete. □

Remark 3.8. *Notice that Theorem 3.3 and Corollary 3.7 ensure exponential convergence of solutions under very similar sufficient conditions. In fact, when the function g is defined as* $g(t,u_t) = G(u(t-\rho(t)))$, *assumption (3.6) coincides with (3.24) when* $\rho_* = 0$ *(i.e. when the delay function* ρ *is nonincreasing), but if* $0 < \rho_* < 1$ *then (3.6) implies (3.24).*

Conclusions

We have proved some results concerning the asymptotic behaviour of solutions to a two dimensional Navier-Stokes model containing delay forcing terms. We have shown that two different techniques can be applied to get sufficient conditions ensuring the exponential convergence towards the unique stationary solution (when it exists). However, these results can be considered as some preliminary ones in the analysis of the global behaviour of this model. In fact, the uniqueness of stationary solutions holds when the viscosity is large enough, so it is very interesting to analyse the behaviour of the system when the viscosity is small. As in the conventional model, it will be possible to prove the existence of a global attractor describing the long term dynamics of the system, although new concepts (e.g. attractors for non-autonomous dynamical systems) will be needed on this occasion. This problem is being analysed actually by the authors and will result in the forthcoming paper Caraballo et al. (2003).

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