An exponential growth condition in H^2 for the pullback attractor of a non-autonomous reaction-diffusion equation

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Abstract

Some exponential growth results for the pullback attractor of a reaction-diffusion when time goes to $-\infty$ are proved in this paper. First, a general result about $L^p \cap H_0^1$ exponential growth is established. Then, under additional assumptions, an exponential growth condition in H^2 for the pullback attractor of the non-autonomous reaction-diffusion equation is also deduced.

Key words: reaction-diffusion equations, non-autonomous (pullback) attractors, invariant sets, H^2-exponential growth.
Mathematics Subject Classifications (2000): 35B41, 35Q35

1 Introduction and setting of the problem

Let us consider the following problem for a non-autonomous reaction-diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \triangle u = f(u) + h(t) \ \text{ in } \ \Omega \times (\tau, +\infty)\,, \\ u = 0 \ \text{ on } \ \partial\Omega \times (\tau, +\infty)\,, \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $f \in C^1(\mathbb{R})$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. We assume that there exist positive constants α_1, α_2, k, l, and $p > 2$ such that

$$-k - \alpha_1 |s|^p \leq f(s)s \leq k - \alpha_2 |s|^p, \quad \forall s \in \mathbb{R}, \tag{2}$$

$$f'(s) \leq l, \quad \forall s \in \mathbb{R}. \tag{3}$$

Let us denote

$$\mathcal{F}(s) := \int_0^s f(r)dr.$$

Then, there exist positive constants $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ and $\widetilde{k}$ such that

$$-\widetilde{k} - \widetilde{\alpha}_1 |s|^p \leq \mathcal{F}(s) \leq \widetilde{k} - \widetilde{\alpha}_2 |s|^p, \qquad \forall s \in \mathbb{R}. \tag{4}$$

It is well-known (see, e.g. [8] or [11]) that under the conditions above, for any initial condition $u_\tau \in L^2(\Omega)$, there exists a unique solution $u(\cdot) = u(\cdot; \tau, u_\tau)$ of (1), i.e., a unique function $u \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C^0([\tau, T]; L^2(\Omega))$ for all $T > \tau$, such that

$$u(t) - \int_\tau^t \Delta u(s)\, ds = u_\tau + \int_\tau^t (f(u(s)) + h(s))\, ds \quad \forall\, t \geq \tau,$$

where the equality must be understood in the sense of the dual of $H_0^1(\Omega) \cap L^p(\Omega)$.

Therefore, we can define a process $U = \{U(t, \tau),\ \tau \leq t\}$ in $L^2(\Omega)$ as

$$U(t, \tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in L^2(\Omega), \quad \forall \tau \leq t. \tag{5}$$

A pullback attractor for the process U defined by (5) (cf. [3], [4], [5]) is a family $\mathcal{A} = \{\mathcal{A}(t) :\ t \in \mathbb{R}\}$ of compact subsets of $L^2(\Omega)$ such that

a) $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $\tau \leq t$,(invariance property),
b) $\lim_{\tau \to -\infty} \sup_{u_\tau \in B} \inf_{v \in \mathcal{A}(t)} |U(t, \tau)u_\tau - v| = 0$, for all $t \in \mathbb{R}$, for any bounded subset $B \subset L^2(\Omega)$, (pullback attraction),

where $|\cdot|$ denotes the norm in $L^2(\Omega)$.

It can be proved (see, for instance, [2] and [7]) that, under the above conditions, if in addition h satisfies

$$\int_{-\infty}^t e^{\lambda_1 s} |h(s)|^2\, ds < +\infty \quad \forall\, t \in \mathbb{R}, \tag{6}$$

where λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition in Ω, then there exists a pullback attractor for the process U defined by (5), and satisfying

$$\lim_{\tau\to-\infty}\left(e^{\lambda_1\tau}\sup_{v\in\mathcal{A}(\tau)}|v|^2\right)=0. \tag{7}$$

Several studies on this model have already been published (see, for example, [1], [6], [9], [10], [12]).

More precisely, we proved in [1] that, under the above conditions, if Ω is regular enough, then for any $\tau\in\mathbb{R}$ the set $\mathcal{A}(\tau)$ is a bounded subset of $L^p(\Omega)\cap H_0^1(\Omega)$, and if moreover $h\in W^{1,2}_{loc}(\mathbb{R};L^2(\Omega))$, then $\mathcal{A}(\tau)$ is also a bounded subset of $H^2(\Omega)$. Therefore, the aim of this paper is to continue with the analysis of this model in the sense of proving that the family $\mathcal{A}(\tau)$ satisfies also an exponential growth condition on the space $L^p(\Omega)\cap H_0^1(\Omega)$, and finally in $H^2(\Omega)$ provided some additional assumptions are fulfilled.

This will be carried out in the next section where we first prove an exponential growth condition for the attractor $\mathcal{A}(\tau)$ in $L^p(\Omega)\cap H_0^1(\Omega)$ when $\tau\to-\infty$. We also prove, under appropriate additional assumptions, an exponential growth condition in $H^2(\Omega)$ for $\mathcal{A}(\tau)$.

2 An exponential growth condition for the pullback attractor.

First, we recall a lemma (see [8]) which is necessary for the proof of our results.

Lemma 2.1 *Let X,Y be Banach spaces such that X is reflexive, and the inclusion $X\subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0,T;X)$ such that $u_n\rightharpoonup u$ weakly in $L^q(t_0,T;X)$ for some $q\in[1,+\infty)$ and $u\in C^0([t_0,T];Y)$.*

Then, $u(t)\in X$ for all $t\in[t_0,T]$ and

$$\|u(t)\|_X\le\sup_{n\ge1}\|u_n\|_{L^\infty(t_0,T;X)}\qquad\forall t\in[t_0,T].$$

We will denote by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$, by $\|\cdot\|=|\nabla\cdot|$ the norm in $H_0^1(\Omega)$, by $\|\cdot\|_{H^2(\Omega)}$ the norm in $H^2(\Omega)$, and by $\|\cdot\|_{L^p(\Omega)}$ the norm in $L^p(\Omega)$. We will use $\langle\cdot,\cdot\rangle$ to denote either the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ or between $L^{p'}(\Omega)$ and $L^p(\Omega)$.

For each integer $n \geq 1$, we denote by $u_n(t) = u_n(t; \tau, u_\tau)$ the Galerkin approximation of the solution $u(t; \tau, u_\tau)$ of (1), which is given by

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j, \tag{8}$$

and is the solution of

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) = \langle \Delta u_n(t), w_j \rangle + (f(u_n(t)), w_j) + (h(t), w_j), \\ (u_n(\tau), w_j) = (u_\tau, w_j) \qquad j = 1, .., n, \end{cases} \tag{9}$$

where $\{w_j : j \geq 1\}$ is the Hilbert basis of $L^2(\Omega)$ formed by the eigenfunctions associated to $-\Delta$ in $H_0^1(\Omega)$.

We prove the following result.

Theorem 1 *Assume that $f \in C^1(\mathbb{R})$ satisfies (2) and (3). Suppose moreover that $\Omega \subset \mathbb{R}^N$ is a bounded C^κ domain, with $\kappa \geq \max(2, N(p-2)/2p)$, $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, and condition (6) holds. Then $\mathcal{A}(\tau)$ satisfies*

$$\lim_{\tau \to -\infty} \left\{ e^{\lambda_1 \tau} \left(\sup_{v \in \mathcal{A}(\tau)} \|v\|^2 + \sup_{v \in \mathcal{A}(\tau)} \|v\|^p_{L^p(\Omega)} \right) \right\} = 0. \tag{10}$$

PROOF. From the inequality (9) of [1], for any $t \geq \tau$ we have

$$\begin{aligned} |u_n(r)|^2 &+ \int_\tau^r \|u_n(s)\|^2 \, ds + \int_\tau^r \|u_n(s)\|^p_{L^p(\Omega)} \, ds \\ &\leq C_1 \left(|u_\tau|^2 + \int_\tau^t |h(s)|^2 \, ds + (t - \tau) \right), \end{aligned} \tag{11}$$

for all $r \in [\tau, t]$, and all $n \geq 1$, where $C_1 := \dfrac{\max\left\{1, \lambda_1^{-1}, 2k\,|\Omega|\right\}}{\min\{1, 2\alpha_2\}}$.

Also, integrating inequality (10) of [1] with respect to s from τ to r, we obtain

$$\begin{aligned} (r - \tau) &\left(\|u_n(r)\|^2 + \|u_n(r)\|^p_{L^p(\Omega)} \right) \\ &\leq C_2 \left(\int_\tau^r \|u_n(s)\|^2 \, ds + \int_\tau^r \|u_n(s)\|^p_{L^p(\Omega)} \, ds \right) \\ &\quad + \frac{(t - \tau)}{\min\{1, 2\widetilde{\alpha}_2\}} \int_\tau^t |h(s)|^2 \, ds \\ &\quad + \frac{4\widetilde{k}}{\min\{1, 2\widetilde{\alpha}_2\}} |\Omega| \, (t - \tau), \end{aligned} \tag{12}$$

for any $t \geq \tau$, all $r \in [\tau, t]$, and all $n \geq 1$, where $C_2 := \dfrac{\max\{1, 2\widetilde{\alpha}_1\}}{\min\{1, 2\widetilde{\alpha}_2\}}$.

From (11) and (12) we now obtain that

$$
\begin{aligned}
(r-\tau)\left(\|u_n(r)\|^2+\|u_n(r)\|_{L^p(\Omega)}^p\right) \leq\ & C_1C_2\left(|u_\tau|^2+\int_\tau^t |h(s)|^2\,ds+(t-\tau)\right) \\
&+\frac{(t-\tau)}{\min\{1,2\widetilde{\alpha}_2\}}\int_\tau^t |h(s)|^2\,ds \\
&+\frac{4\widetilde{k}}{\min\{1,2\widetilde{\alpha}_2\}}\,|\Omega|\,(t-\tau), \qquad (13)
\end{aligned}
$$

for any $t \geq \tau$, all $r \in [\tau, t]$, and all $n \geq 1$.
In particular, from (13) we deduce

$$
\|u_n(r)\|^2+\|u_n(r)\|_{L^p(\Omega)}^p \leq C_3\left(|u_\tau|^2+\int_\tau^{\tau+2}|h(s)|^2\,ds+1\right), \qquad (14)
$$

for all $r \in [\tau+1, \tau+2]$, and any $n \geq 1$, where

$$
C_3 := \max\left\{C_1C_2+\frac{2}{\min\{1,2\widetilde{\alpha}_2\}},\, 2C_1C_2+\frac{8\widetilde{k}}{\min\{1,2\widetilde{\alpha}_2\}}\,|\Omega|\right\}.
$$

It is well known (see [8] or [11]) that $u_n(\cdot)=u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot)=u(\cdot;\tau,u_\tau)$ in $L^2(\tau,t;H_0^1(\Omega))\cap L^p(\tau,t;L^p(\Omega))$, for all $t>\tau$. Thus, from (14) and Lemma 2.1, we in particular obtain

$$
\|u(\tau+1)\|^2+\|u(\tau+1)\|_{L^p(\Omega)}^p \leq C_3\left(|u_\tau|^2+\int_\tau^{\tau+2}|h(s)|^2\,ds+1\right).
$$

Multiplying this inequality by $e^{\lambda_1(\tau+1)}$ and using (5), we have

$$
\begin{aligned}
& e^{\lambda_1(\tau+1)}\left(\|U(\tau+1,\tau)u_\tau\|^2+\|U(\tau+1,\tau)u_\tau\|_{L^p(\Omega)}^p\right) \qquad (15) \\
& \leq C_3e^{\lambda_1}\left(e^{\lambda_1\tau}|u_\tau|^2+\int_\tau^{\tau+2}e^{\lambda_1 s}|h(s)|^2\,ds+e^{\lambda_1\tau}\right),
\end{aligned}
$$

for all $\tau \in \mathbb{R}$, and all $u_\tau \in L^2(\Omega)$.

As $\mathcal{A}(\tau+1)=U(\tau+1,\tau)\mathcal{A}(\tau)$, it follows from (15) that

$$
\begin{aligned}
& e^{\lambda_1(\tau+1)}\left(\|v\|^2+\|v\|_{L^p(\Omega)}^p\right) \\
& \leq C_3e^{\lambda_1}\left(e^{\lambda_1\tau}\sup_{w\in\mathcal{A}(\tau)}|w|^2+\int_\tau^{\tau+2}e^{\lambda_1 s}|h(s)|^2\,ds+e^{\lambda_1\tau}\right),
\end{aligned}
$$

for all $v \in \mathcal{A}(\tau+1)$, and any $\tau \in \mathbb{R}$.

Finally, this inequality implies

$$
\begin{aligned}
& e^{\lambda_1 \tau}\left(\|v\|^2+\|v\|_{L^p(\Omega)}^p\right) \\
& \leq C_3 e^{\lambda_1}\left(e^{\lambda_1(\tau-1)} \sup _{w \in \mathcal{A}(\tau-1)}|w|^2+\int_{\tau-1}^{\tau+1} e^{\lambda_1 s}|h(s)|^2 d s+e^{\lambda_1(\tau-1)}\right),
\end{aligned} \tag{16}
$$

for all $v \in \mathcal{A}(\tau)$, and any $\tau \in \mathbb{R}$. Taking into account (6) and (7), from (16) we obtain (10).

Theorem 2 *In addition to the assumptions in Theorem 1, assume moreover that* $h \in W_{loc}^{1,2}(\mathbb{R}; L^2(\Omega))$, *and satisfies*

$$
\lim_{\tau \rightarrow-\infty} e^{\lambda_1 \tau} \int_\tau^{\tau+1}|h'(s)|^2 d s=0 \tag{17}
$$

and

$$
\lim_{\tau \rightarrow-\infty} e^{\lambda_1 \tau}|h(\tau)|^2=0. \tag{18}
$$

Then $\mathcal{A}(\tau)$ *satisfies that*

$$
\lim_{\tau \rightarrow-\infty}\left(e^{\lambda_1 \tau} \sup_{v \in \mathcal{A}(\tau)}\|v\|_{H^2(\Omega)}^2\right)=0. \tag{19}
$$

PROOF. From inequality (11) in [1], taking $t=\tau+3$ and $\varepsilon=2$, we have

$$
\begin{aligned}
\left|u_n'(r)\right|^2 \leq (4l+3) & \int_{\tau+1}^{\tau+3}\left|u_n'(s)\right|^2 d s \\
& +\int_{\tau+1}^{\tau+3}\left|h'(s)\right|^2 d s,
\end{aligned} \tag{20}
$$

for all $r \in[\tau+2, \tau+3]$, and any $n \geq 1$.

Analogously, and if we take $s=\tau+1$ and $r=t=\tau+3$ in inequality (10) of [1], we have

$$
\begin{aligned}
& \int_{\tau+1}^{\tau+3}\left|u_n'(s)\right|^2 d s+\left\|u_n(\tau+3)\right\|^2+2 \widetilde{\alpha}_2\left\|u_n(\tau+3)\right\|_{L^p(\Omega)}^p \\
& \quad \leq\left\|u_n(\tau+1)\right\|^2+\int_\tau^{\tau+3}|h(s)|^2 d s+4 \widetilde{k}|\Omega|+2 \widetilde{\alpha}_1\left\|u_n(\tau+1)\right\|_{L^p(\Omega)}^p,
\end{aligned} \tag{21}
$$

for all $n \geq 1$.

From (21) and (20), we obtain

$$\begin{aligned}|u_n'(r)|^2 &\leq (4l+3)\left(\|u_n(\tau+1)\|^2 + 2\widetilde{\alpha}_1 \|u_n(\tau+1)\|_{L^p(\Omega)}^p\right)\\ &\quad + (4l+3)\left(\int_\tau^{\tau+3} |h(s)|^2\, ds + 4\widetilde{k}\,|\Omega|\right)\\ &\quad + \int_{\tau+1}^{\tau+3} |h'(s)|^2\, ds,\end{aligned}$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$.

Owing to this inequality and (14), there exists a constant $\widetilde{C}_1 > 0$ such that

$$|u_n'(r)|^2 \leq \widetilde{C}_1 \left(|u_\tau|^2 + \int_\tau^{\tau+3} \left(|h(s)|^2 + |h'(s)|^2\right) ds + 1\right), \tag{22}$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$.

From inequality (13) of [1], and thanks to (22), we have

$$\begin{aligned}|\Delta u_n(r)|^2 &\leq 8\widetilde{C}_1 \left(|u_\tau|^2 + \int_\tau^{\tau+3} \left(|h(s)|^2 + |h'(s)|^2\right) ds + 1\right) + 8\,|h(r)|^2\\ &\quad + 4l^2\,|u_n(r)|^2 + 4\,(f(0))^2\,|\Omega|,\end{aligned}$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$, and therefore, by (11) we obtain that there exists a constant $\widetilde{C}_2 > 0$ such that

$$\begin{aligned}&|\Delta u_n(r)|^2\\ &\leq \widetilde{C}_2 \left(|u_\tau|^2 + \int_\tau^{\tau+3} \left(|h(s)|^2 + |h'(s)|^2\right) ds + 1 + \sup_{r\in[\tau+2,\tau+3]} |h(r)|^2\right),\end{aligned} \tag{23}$$

for all $r \in [\tau+2, \tau+3]$, and any $n \geq 1$.

It is well known that, in particular, $u_n(\cdot) = u_n(\cdot;\tau,u_\tau)$ converges weakly to $u(\cdot) = u(\cdot;\tau,u_\tau)$ in $L^2(\tau+2,\tau+3;H_0^1(\Omega))$ and $u(\cdot;\tau,u_\tau) \in C^0([\tau+2,\tau+3];H_0^1(\Omega))$. Then, by Lemma 2.1, inequality (23) and the equivalence of the norms $|\Delta v|$ and $\|v\|_{H^2(\Omega)}$, we have that there exists a constant $\widetilde{C}_3 > 0$ such that

$$\begin{aligned}&\|u(r;\tau,u_\tau)\|_{H^2(\Omega)}^2\\ &\leq \widetilde{C}_3 \left(|u_\tau|^2 + \int_\tau^{\tau+3} \left(|h(s)|^2 + |h'(s)|^2\right) ds + 1 + \sup_{r\in[\tau+2,\tau+3]} |h(r)|^2\right),\end{aligned} \tag{24}$$

for all $r \in [\tau+2, \tau+3]$, any $\tau \in \mathbb{R}$, and $u_\tau \in L^2(\Omega)$.

Now, observe that by Cauchy inequality,

$$|h(r)| \leq |h(\tau+2)| + \left(\int_{\tau+2}^{\tau+3} |h'(s)|^2\, ds\right)^{1/2},$$

for all $r \in [\tau+2, \tau+3]$. Thus, from (24), and using (5), we deduce that there exists a constant $\widetilde{C}_4 > 0$ such that

$$\|U(\tau+2,\tau)u_\tau\|^2_{H^2(\Omega)} \leq \widetilde{C}_4 \left(|u_\tau|^2 + \int_\tau^{\tau+3} \left(|h(s)|^2 + |h'(s)|^2 \right) ds + |h(\tau+2)|^2 + 1 \right),$$

for all $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$.

From this inequality, and the fact that $\mathcal{A}(\tau) = U(\tau, \tau-2)\mathcal{A}(\tau-2)$, we obtain

$$\|v\|^2_{H^2(\Omega)} \leq \widetilde{C}_4 \left(\sup_{w \in \mathcal{A}(\tau-2)} |w|^2 + \int_{\tau-2}^{\tau+1} \left(|h(s)|^2 + |h'(s)|^2 \right) ds + |h(\tau)|^2 + 1 \right), \tag{25}$$

for all $v \in \mathcal{A}(\tau)$, and any $\tau \in \mathbb{R}$.

Now, thanks to (6), (7), (17) and (18), we obtain (19) from (25).

Remark 3 *In theorems 1 and 2, the pullback attraction property is not needed. In fact, both theorems are also valid for any family* $\{\mathcal{A}(\tau) : \tau \in \mathbb{R}\}$ *of nonempty subsets of* $L^2(\Omega)$ *satisfying* (7) *and the semi-invariance property*

$$\mathcal{A}(\tau+n) \subset U(\tau+n, \tau)\mathcal{A}(\tau),$$

for all $\tau \in \mathbb{R}$ *and any integer* $n \geq 1$.

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