ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL/DIFFERENCE EQUATIONS WITHOUT FAVARD'S SEPARATION CONDITION. II

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ABSTRACT. In this paper we continue the research started in a previous paper, where we proved that the linear differential equation

$$x' = A(t)x + f(t) \tag{1}$$

with Levitan almost periodic coefficients has a unique Levitan almost periodic solution, if it has at least one bounded solution and the bounded solutions of the homogeneous equation

$$x' = A(t)x \tag{2}$$

are homoclinic to zero (i.e. $\lim\limits_{|t|\to+\infty} |\varphi(t)| = 0$ for all bounded solution φ of (2)). If the coefficients of (1) are Bohr almost periodic and all bounded solutions of equation (2) are homoclinic to zero, then the equation (1) admits a unique almost automorphic solution.

In this second part we first generalise this result for linear functional differential equations (FDEs) of the form

$$x' = A(t)x_t + f(t), \tag{3}$$

as well as for neutral FDEs.

Analogous results for functional difference equations with finite delay and some classes of partial differential equations are also given.

We study the problem of existence of Bohr/Levitan almost periodic solutions of differential equations of type (3) in the context of general semi-group non-autonomous dynamical systems (cocycles), in contrast with the group non-autonomous dynamical systems framework considered in the first part.

Dedicated to Russell Johnson on his 60th birthday

1. Introduction

The aim of this paper is to analyse the existence of Bohr/Levitan almost periodic and almost automorphic solutions of linear evolution differential/difference equations (functional differential equations with finite delay, neutral functional differential equations, functional difference equations with finite delay and some classes

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of partial differential equations) with Bohr/Levitan almost periodic and almost automorphic coefficients without Favard's separation condition.

In our paper [5], this problem was studied for ordinary differential (and difference) equations (the reader can also find in our paper [5] a large bibliography about different generalizations of Favard's theory and some information about applications of Ellis semigroup in this theory). In particular, we established the following result:

The linear differential equation

$$x' = A(t)x + f(t) \tag{4}$$

with Levitan almost periodic coefficients possesses a unique Levitan almost periodic solution, if it has at least one bounded solution and the bounded solutions of the homogeneous equation

$$x' = A(t)x \tag{5}$$

are *homoclinic to zero* (i.e. $\lim\limits_{|t|\to+\infty} |\varphi(t)| = 0$ for all bounded solution φ of (5)).

If the coefficients of (4) are Bohr almost periodic, and all bounded solutions of equation (5) (respectively, of all limiting equations

$$y' = B(t)y, \tag{6}$$

where $B \in H(A) := \overline{\{A_t : t \in \mathbb{R}\}}, A_t(s) := A(t+s)$, and by bar we denote the closure in the compact–open topology) *homoclinic to zero*, then, equation (1) admits a unique almost automorphic (respectively, Bohr almost periodic) solution.

This paper is organized as follows.

In Section 2, we collect some well known facts from the theory of dynamical systems (both autonomous and non-autonomous). Namely, the notions of almost periodic (both in the Bohr and Levitan senses), almost automorphic and recurrent motions, shift dynamical systems, almost periodic and almost automorphic functions, cocycle, skew-product dynamical systems, and general non-autonomous dynamical systems. We also recall some results from our first part [5] which will be also necessary for our analysis in this paper.

Section 3 is devoted to the study of some general properties of one-sided (semigroup) non-autonomous dynamical systems which permit to extend Theorems 2.6 and 2.11 in our first part [5] to this class of systems. To this respect, it is worth pointing out that, in contrast with the situations in [5], the examples considered in the present paper only generate semi-group dynamical systems and, therefore, the theory developed in [5] cannot be applied directly to them. Moreover, in order to prove the abstract results in Section 3, new ideas and techniques are necessary. The situation is as follows. Every functional differential (respectively, functional difference or evolution partial differential equation) generates a semi-group non-autonomous dynamical system under some appropriate assumptions. Then, we prove that there exists a two-sided subsystem of the original one-sided one, but being single-valued in the positive direction and set-valued (generally speaking) in the negative direction. So, it is not possible to apply directly the theory developed in our first part [5]. Then, we prove that this system can be embedded into a Bebutov dynamical system (shift dynamical systems on the space of entire trajectories), which is a two-sided single-valued dynamical system.

This construction can be considered as a "bridge" between two-sided single-valued dynamical systems (for which we can apply Theorems 2.6 and 2.11) and one-sided dynamical systems, allowing to transport our results in part I to the context of semi-group non-autonomous dynamical systems.

In Section 4, we will apply our general results from Section 3 to the analysis of the existence of Bohr (respectively, Levitan) almost periodic and almost automorphic solutions of different classes of evolution equations (functional differential/difference equations with finite delay, neutral functional differential equations, linear partial differential equations) with Bohr almost periodic (respectively, Levitan almost periodic) and almost automorphic coefficients.

2. Preliminaries on the Theory of Dynamical Systems

We recall in this section some concepts about dynamical systems and some results from [5] which will be useful for our investigation in this paper.

2.1. **Bohr/Levitan Almost Periodic and Almost Automorphic Motions.** Let X be a complete metric space, $\mathbb{R}$ ($\mathbb{Z}$) be the group of real (integer) numbers, $\mathbb{T}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$.

Let $\mathbb{S} \subseteq \mathbb{T}$ be a sub-semigroup of the group $\mathbb{T}$ and $0 \in \mathbb{S}$. Denote by $(X, \mathbb{S}, \pi)$ *a dynamical system*, i.e. $\pi : \mathbb{S} \times X \mapsto X$ is a continuous mapping with $\pi(0, x) = x$ and $\pi(t_1 + t_2, x) = \pi(t_2, \pi(t_1, x))$ for all $x \in X$ and $t_1, t_2 \in \mathbb{S}$.

Given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon-$*shift* (respectively, an $\varepsilon-$*almost period*) of the point $x \in X$, if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$ there exists a positive number l such that in any segment of length l there is an $\varepsilon-$shift (respectively, $\varepsilon-$almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{\pi(t, x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*, where by bar we denote the closure in X.

Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} \ : \text{ such that } \{\pi(t_n, x)\} \to x \text{ and } \{t_n\} \to \infty\}$.

A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called *Levitan almost periodic* [13], if there exists a dynamical system $(Y, \mathbb{T}, \lambda)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

A point $x \in X$ is called *stable in the sense of Lagrange* (*st*.L for short), if its trajectory $\{\pi(t, x) \ : \ t \in \mathbb{T}\}$ is relatively compact.

A point $x \in X$ is called *almost automorphic* [13, 20] for the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(i) x is st.L;

(ii) there exists a dynamical system $(Y, \mathbb{T}, \lambda)$, a homomorphism h from $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \lambda)$, and an almost periodic (in the sense of Bohr) point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

Remark 2.1. *1. Every almost automorphic point $x \in X$ is also Levitan almost periodic.*

2. A Levitan almost periodic point x with relatively compact trajectory $\{\pi(t,x) : \ t \in T\}$ is also almost automorphic (see [1]–[4],[8], [10],[14] *and* [20]*). In other words, a Levitan almost periodic point x is almost automorphic, if and only if its trajectory $\{\pi(t,x) : \ t \in T\}$ is relatively compact.*

3. Let (X, T, π) and (Y, T, λ) be two dynamical systems, $x \in X$ and the following conditions be fulfilled:

(i) *a point $y \in Y$ is Levitan almost periodic;*
(ii) $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Then, the point x is also Levitan almost periodic.

4. Let $x \in X$ be a st.L point, $y \in Y$ be an almost automorphic point and $\mathfrak{N}_y \subseteq \mathfrak{N}_x$. Then, the point x is almost automorphic too.

2.2. **Shift Dynamical Systems, Almost Periodic and Almost Automorphic Functions.** Let us recall now a general method to construct dynamical systems on the space of continuous functions. This fact will be important in order to obtain a two-sided dynamical system from a one-sided one, as we explained in the Introduction.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on X, Y a complete pseudo metric space, and P a family of pseudo metrics on Y. We denote by $C(X, Y)$ the family of all continuous functions $f : X \to Y$ equipped with the compact-open topology. This topology is given by the following family of pseudo metrics $\{d^p_K\}$ $(p \in P, \ K \in \mathcal{K}(X))$, where

$$d^p_K(f, g) := \max_{x \in K} p(f(x), g(x))$$

and $\mathcal{K}(X)$ denotes the family of all compact subsets of X. For all $\tau \in \mathbb{T}$ we define a mapping $\sigma_\tau : C(X, Y) \to C(X, Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau, x))$ $(x \in X)$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = id_{C(X,Y)}$;
b. $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1 + \tau_2}, \ \forall \tau_1, \tau_2 \in \mathbb{T}$,
c. σ_τ is continuous $\forall \tau \in \mathbb{T}$.

Lemma 2.2. [7] *The mapping $\sigma : \mathbb{T} \times C(X, Y) \to C(X, Y)$, defined by the equality $\sigma(\tau, f) := \sigma_\tau f \ \ (f \in C(X, Y), \ \tau \in \mathbb{T})$, is continuous and, consequently, the triple $(C(X, Y), \mathbb{T}, \sigma)$ is a dynamical system on $C(X, Y)$.*

Consider now a useful example of dynamical system of the form $(C(X, Y), \mathbb{T}, \sigma)$.

Example 2.3. *Let $X = \mathbb{T}$, and denote by $(X, \mathbb{T}, \pi)$ a dynamical system on $\mathbb{T}$, where $\pi(t, x) := x + t$. The dynamical system $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$ is called* Bebutov's dynamical system [17] *(a dynamical system of translations, or shifts dynamical system). For example, the equality*

$$d(f,g) := \sup_{L>0} \max\{d_L(f,g), L^{-1}\},$$

where $d_L(f,g) := \max_{|t|\le L} \rho(f(t),g(t))$, *defines a complete metric (Bebutov's metric) on the space* $C(\mathbb{T},Y)$ *which is compatible with the compact open topology on* $C(\mathbb{T},Y)$.

Remark 2.4. *It is known* [17, 19] *that* $d(f,g) < \varepsilon$ *(respectively,* $d(f,g) > \varepsilon$ *or* $d(f,g) = \varepsilon$*) is equivalent to the inequality* $d_{\frac{1}{\varepsilon}}(f,g) < \varepsilon$ *(respectively,* $d_{\frac{1}{\varepsilon}}(f,g) > \varepsilon$ *or* $d_{\frac{1}{\varepsilon}}(f,g) = \varepsilon$*).*

It is said that the function $\varphi \in C(\mathbb{T},Y)$ *possesses a property* (A) if the motion $\sigma(\cdot,\varphi) : \mathbb{T} \to C(\mathbb{T},Y)$, generated by the function φ, possesses this property in the Bebutov dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$. As property (A) we can take periodicity, almost periodicity, almost automorphy, recurrence, etc.

2.3. **Cocycles, Skew-Product Dynamical Systems, and Non-Autonomous Dynamical Systems.** Let $\mathbb{T}_i$ (i=1,2) be a sub-semigroup of $\mathbb{T}$ and $0 \in \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{T}$. A triplet $\langle (X,\mathbb{T}_1,\pi), (Y,\mathbb{T}_2,\lambda), h\rangle$, where h is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\lambda)$, is called *a non-autonomous dynamical system.*

Let $(Y,\mathbb{T}_2,\lambda)$ be a dynamical system on Y, W a complete metric space, and φ a continuous mapping from $\mathbb{T}_1 \times W \times Y$ in W, possessing the following properties:

a. $\varphi(0,u,y) = u$ $(u \in W, y \in Y)$;
b. $\varphi(t+\tau,u,y) = \varphi(\tau,\varphi(t,u,y),\lambda(t,y))$ $(t,\tau \in \mathbb{T}_1,\ u \in W, y \in Y)$.

Then the triplet $\langle W,\varphi,(Y,\mathbb{T}_2,\lambda)\rangle$ (or shortly φ) is called [16] *a cocycle* on $(Y,\mathbb{T}_2,\lambda)$ with the fiber W.

Let $\langle W,\varphi,(Y,\mathbb{T}_2,\lambda)\rangle$ be a given cocycle, let $X := W \times Y$, and define a mapping $\pi : \mathbb{T}_1 \times X \to X$ as follows: $\pi(t,(u,y)) := (\varphi(t,u,y),\lambda(t,y))$ (i.e. $\pi = (\varphi,\lambda)$). Then it is easy to see that $(X,\mathbb{T}_1,\pi)$ is a dynamical system on X, which is called *a skew-product dynamical system* [16], and $h = pr_2 : X \to Y$ is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\lambda)$ and, hence, $\langle (X,\mathbb{T}_1,\pi), (Y,\mathbb{T}_2,\lambda), h\rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W,\varphi,(Y,\mathbb{T}_2,\lambda)\rangle$ on the dynamical system $(Y,\mathbb{T}_2,\lambda)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi), (Y,\mathbb{T}_2,\lambda), h\rangle$ $(X := W \times Y)$, called *a non-autonomous dynamical system generated by the cocycle* $\langle W,\varphi,(Y,\mathbb{T}_2,\lambda)\rangle$ on $(Y,\mathbb{T}_2,\lambda)$.

Notice that non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential/difference equations since, under appropriate assumptions, every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system).

2.4. **Comparability and Uniform Comparability of Motions by the Character of Recurrence in the Sense of Shcherbakov.** Let $(\Omega,\mathbb{T},\lambda)$ be a dynamical system. A point $\omega \in \Omega$ is called (see, for example, [19] and [21]) *positively* *(negatively) stable in the sense of Poisson*, if there exists a sequence $t_n \to +\infty$ (respectively, $t_n \to -\infty$) such that $\lambda(t_n, \omega) \to \omega$. If the point ω is Poisson stable in both directions, it is called *Poisson stable*.

Let (X, h, Ω) be a fiber space, i.e. X and Ω be two metric spaces and $h : X \to \Omega$ is a homomorphism from X into Ω. The subset $M \subseteq X$ is said to be *conditionally relatively compact* [6, 7], if the pre-image $h^{-1}(\Omega') \bigcap M$ of every relatively compact subset $\Omega' \subseteq \Omega$ is a relatively compact subset of X, in particular $M_\omega := h^{-1}(\omega) \bigcap M$ is relatively compact for every ω. The set M is called *conditionally compact* if it is closed and conditionally relatively compact.

Now we recall some results proved in [5] for a group non-autonomous dynamical systems, and which will be useful for the analysis in this paper.

Lemma 2.5. [5] *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \lambda)\rangle$ be a cocycle, and $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \lambda), h\rangle$ a non-autonomous dynamical system associated to the cocycle φ. Suppose that $x_0 := (u_0, \omega_0) \in X := W \times \Omega$ and the set $Q_{(u_0,\omega_0)} := \overline{\{\varphi(t, u_0, \omega_0) \mid t \in \mathbb{T}\}}$ (respectively, $Q^+_{(u_0,\omega_0)} := \overline{\{\varphi(t, u_0, \omega_0) \mid t \in \mathbb{T}_+\}}$, where $\mathbb{T}_+ := \{t \in \mathbb{T} \mid t \geq 0\}$) is compact.*

Then the set $H(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{T}\}}$ (respectively, $\overline{\{\pi(t, x_0) \mid t \in \mathbb{T}_+\}} := H^+(x_0)$) is conditionally compact.

Let $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \lambda), h\rangle$ be a two-sided (a group) non-autonomous dynamical system, and $\omega \in \Omega$ be a Poisson stable point. Denote by

$$\mathcal{E}_\omega := \{\xi | \quad \exists \{t_n\} \in \mathfrak{N}_\omega \quad \text{such that} \quad \pi(t_n, \cdot)|_{X_\omega} \to \xi\},$$

where $X_\omega := \{x \in X | \quad h(x) = \omega\}$ and $\to$ means the pointwise convergence.

Theorem 2.6. [5] *Let X be a conditionally compact metric space and $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \lambda), h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *there exists a Poisson stable point $\omega \in \Omega$;*
(ii) $\lim\limits_{|t|\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ *for all* $x_1, x_2 \in X_\omega := h^{-1}(\omega) = \{x \in X : h(x) = \omega\}$.

Then, there exists a unique point $x_\omega \in X_\omega$ such that $\xi(x_\omega) = x_\omega$ for all $\xi \in \mathcal{E}_\omega$.

A point $x \in X$ is called [17]–[19] *comparable with $\omega \in \Omega$ by the character of recurrence* if $\mathfrak{N}_\omega \subseteq \mathfrak{N}_x$.

Remark 2.7. *If a point $x \in X$ is comparable with $\omega \in \Omega$ by the character of recurrence, and if ω is stationary (respectively, τ-periodic, recurrent, Poisson stable), then so is the point x (see* [19]*).*

Corollary 2.8. [5] *Let X be a conditionally compact metric space and $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \lambda), h\rangle$ a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *there exists a Poisson stable point $\omega \in \Omega$;*
(ii) $\lim\limits_{|t|\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ *for all* $x_1, x_2 \in X_\omega := h^{-1}(\omega) = \{x \in X : h(x) = \omega\}$.

Then, there exists a unique point $x_\omega \in X_\omega$, which is comparable with $\omega \in \Omega$ by the character of recurrence, and such that

$$\lim_{|t|\to+\infty} \rho(\pi(t,x),\pi(t,x_\omega)) = 0 \tag{7}$$

for all $x \in X_\omega$.

Corollary 2.9. [5] *Let $\omega \in \Omega$ be a stationary (respectively, τ-periodic, almost automorphic, recurrent, Levitan almost periodic, Poisson stable) point. Then under the conditions of Corollary 2.8 there exists a unique stationary (respectively, τ-periodic, almost automorphic, recurrent, Levitan almost periodic, Poisson stable) point $x_\omega \in X_\omega$ such that the equality (7) holds for all $x \in X_\omega$.*

Denote by $\mathfrak{M}_\omega := \{\{t_n\} \subset \mathbb{T} \mid \text{the sequence } \{\lambda(t_n,\omega)\} \text{ is convergent }\}$.

A point $x \in X$ is called *uniformly comparable with $\omega \in \Omega$ by the character of recurrence* (see [17]–[19]) if $\mathfrak{M}_\omega \subseteq \mathfrak{M}_x$.

Remark 2.10. *If a point $x \in X$ is uniformly comparable with $\omega \in \Omega$ by the character of recurrence, and ω is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, Poisson stable), then, so is the point x (see* [17]–[19]*).*

Theorem 2.11. [5] *Let X be a compact metric space, and $\langle (X,\mathbb{T},\pi),\ (\Omega,\mathbb{T},\lambda), h\rangle$ a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *there exists a Poisson stable point $\omega \in \Omega$;*
(ii) $\lim\limits_{|t|\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0$ *for all $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$.*

Then, there exists a unique point $x_\omega \in X_\omega$, which is uniformly comparable with $\omega \in \Omega$ by the character of recurrence, and such that (7) takes place for all $x \in X_\omega$.

Corollary 2.12. [5] *Let $\omega \in \Omega$ be a stationary (respectively, τ-periodic, Bohr almost periodic, recurrent, Poisson stable) point. Then, under the conditions of Theorem 2.11, there exists a unique stationary (respectively, τ-periodic, Bohr almost periodic, recurrent, Poisson stable) point $x_\omega \in X_\omega$ such that (7) is fulfilled for all $x \in X_\omega$.*

3. Semi-group Dynamical Systems

Now we prove similar results to the ones in [5] (and which have been stated in Section 2), but for one-sided (semi-group) non-autonomous dynamical systems. The reason is that the applications to be analysed in Section 4 only generate semigroups rather than groups.

Let $\langle (X,\mathbb{T}_+,\pi),(\Omega,\mathbb{T},\lambda),h\rangle$ (respectively, $\langle W,\varphi,(\Omega,\mathbb{T},\lambda)\rangle$, where $\varphi : \mathbb{T}_+ \times W \times \Omega \mapsto W$) be a semi-group non-autonomous dynamical system (respectively, a semi-group cocycle), where $\mathbb{T}_+ := \{t \in \mathbb{T} \mid t \geq 0\}$.

A continuous mapping $\gamma : \mathbb{T} \mapsto X$ (respectively, $\nu : \mathbb{T} \mapsto W$) is called *an entire trajectory of the semi-group dynamical system $(X,\mathbb{T}_+,\pi)$ (respectively, semi-group* *cocycle* $\langle W, \varphi, (\Omega, \mathbb{T}, \lambda)\rangle$ or shortly φ) passing through the point x (respectively, (u, ω)), if $\gamma(0) = x$ (respectively, $\nu(0) = u$) and $\pi(t, \gamma(s)) = \gamma(t + s)$ (respectively, $\varphi(t, \nu(s), \lambda(s, \omega)) = \nu(t + s)$) for all $t \in \mathbb{T}_+$ and $s \in \mathbb{T}$.

Remark 3.1. *Let* $(X, \mathbb{T}_+, \pi)$ *be a skew-product dynamical system generated by semi-group cocycle* $\langle W, \varphi, (\Omega, \mathbb{T}, \lambda)\rangle$ *(*$X := W \times \Omega$ *and* $\pi := (\varphi, \lambda)$*, i.e.* $\pi(t, (u, \omega)) := (\varphi(t, u, \omega), \lambda(t, \omega))$*), then* γ *is an entire trajectory of the semi-group dynamical system* $(X, \mathbb{T}_+, \pi)$ *passing through the point* $x = (u, \omega)$ *if and only if* $\gamma = (\nu, Id_\Omega)$*, where* ν *is an entire trajectory of the semi-group cocycle* $\langle W, \varphi, (\Omega, \mathbb{T}, \lambda)\rangle$ *passing through the point* (u, ω) *and* Id_Ω *is the identity mapping acting onto* Ω*.*

Lemma 3.2. *Consider a semi-group cocycle* $\langle W, \varphi, (\Omega, \mathbb{T}, \lambda)\rangle$ *and suppose that the following conditions hold:*

(i) *the positive semi-trajectory* $\varphi(\mathbb{T}_+, u_0, \omega_0)$ *of the cocycle* φ *is relatively compact, i.e. the set* $Q := \overline{\varphi(\mathbb{T}_+, u_0, \omega_0)}$ *is compact;*
(ii) *the point* $\omega_0 \in \Omega$ *is Poisson stable in the positive direction.*

Then, there exist a point $u \in Q$ *and an entire trajectory* ν *of the semi-group cocycle* φ *passing through the point* $x := (u, \omega_0)$ *such that* $\nu(\mathbb{T}) \subseteq Q$*.*

Proof. Since the point $\omega_0 \in \Omega$ is Poisson stable in the positive direction, then there is a sequence $\{\tau_n\} \to +\infty$ such that $\lambda(\tau_n, \omega_0) \to \omega_0$. Denote by ν_n the function from $C(\mathbb{T}, W)$ defined by the equality $\nu_n(t) = \varphi(t + \tau_n, u_0, \omega_0)$ for all $t \geq -\tau_n$ and $\nu_n(t) = u_0$ for all $t \leq -\tau_n$. We will show that the sequence $\{\nu_n\}$ is relatively compact in $C(\mathbb{T}, W)$. To this end it is sufficient to show that the sequence $\{\nu_n\}$ is equi-continuous on every interval $[-l, l]$ $(l > 0)$, because $\nu_n(\mathbb{T}) \subseteq Q$ by definition of ν_n, and Q is a compact subset of W. If we suppose that it is not true, then there exist $l_0, \varepsilon_0 > 0$, $\delta_n \to 0$ and $t_n^1, t_n^2 \in [-l_0, l_0]$ such that

$$|t_n^1 - t_n^2| < \delta_n \text{ and } \rho(\nu_n(t_n^1), \nu_n(t_n^2)) \geq \varepsilon_0 \tag{8}$$

for all $n \in \mathbb{N}$. Without loss of generality we can assume that $\tau_n \geq l_0$ and, consequently, $\nu_n(t) = \varphi(t + \tau_n, u_0, \omega_0)$ for all $t \in [-l_0, l_0]$. From the last equality and (8) we have

$$\begin{aligned} \varepsilon_0 \leq \rho(\varphi(t_n^1 + \tau_n, u_0, \omega_0), \varphi(t_n^2 + \tau_n, u_0, \omega_0)) = \\ \rho(\varphi(t_n^1 + l_0, \varphi(\tau_n - l_0, u_0, \omega_0), \lambda(\tau_n - l_0, \omega_0)), \\ \varphi(t_n^2 + l_0, \varphi(\tau_n - l_0, u_0, \omega_0), \lambda(\tau_n - l_0, \omega_0)). \end{aligned} \tag{9}$$

Due to our assumptions, $\{\sigma(\tau_n - l_0, \omega_0)\} \to \sigma(-l_0, \omega_0)$ and the sequence $\{\varphi(\tau_n - l_0, u_0, \omega_0)\}$ can be considered convergent. Let $\bar{u} := \lim_{n\to\infty} \varphi(\tau_n - l_0, u_0, \omega_0)$ and $\bar{t} := \lim_{n\to\infty} t_n^1 = \lim_{n\to\infty} t_n^2$. Then, taking limits in (9) as $n \to \infty$, we obtain

$$\varepsilon_0 \leq \rho(\varphi(l_0 + \bar{t}, \bar{u}, \lambda(-l_0, \omega_0)), \varphi(l_0 + \bar{t}, \bar{u}, \lambda(-l_0, \omega_0))) = 0.$$

This contradiction proves our statement.

Thus, the sequence $\{\nu_n\}$ is relatively compact in $C(\mathbb{T}, W)$ and, consequently, without loss of generality we can suppose that it is convergent. Denote by ν its limit. Then, it is easy to see that $\nu(0) = u := \lim_{n\to\infty} \varphi(\tau_n, u_0, \omega_0)$ and $\nu(t+s) = \lim_{n\to\infty} \varphi(t+$

$s+\tau_n,u_0,\omega_0) = \lim\limits_{n\to\infty}\varphi(t,\varphi(s+\tau_n,u_0,\omega_0),\lambda(s+\tau_n,\omega_0)) = \varphi(t,\nu(s),\lambda(s,\omega_0))$ for all $t\in\mathbb{T}_+$ and $s\in\mathbb{T}$. □

Remark 3.3. *1. Consider a semi-group cocycle $\langle W,\varphi,(\Omega,\mathbb{T},\lambda)\rangle$ and suppose that the positive semi-trajectory $\varphi(\mathbb{T}_+,u_0,\omega_0)$ of the cocycle φ is relatively compact. Let $x_0:=(u_0,\omega_0)$, then the subset $H^+(x_0):=\overline{\{\pi(t,x_0)\mid t\in\mathbb{T}_+\}}$ is conditionally compact in the bundle fiber space (X,h,Ω) ($X:=W\times\Omega$ and $h:=pr_2:X\mapsto\Omega$), because $H^+(x_0)\subseteq Q\times\Omega$ ($Q:=\overline{\varphi(\mathbb{T}_+,u_0,\omega_0)}$).*

2. Under the conditions of Lemma 3.2 the non-autonomous dynamical system $\langle(X,\mathbb{T}_+,\pi),(\Omega,\mathbb{T},\lambda),h\rangle$, generated by the cocycle φ, has at least one entire trajectory γ with $h(\gamma(0))=\omega_0$ such that $\gamma(\mathbb{T})\subseteq H^+(x_0)$.

Lemma 3.4. *Consider a semi-group dynamical system $\langle(X,\mathbb{T}_+,\pi),(\Omega,\mathbb{T},\lambda),h\rangle$ and suppose that the following conditions hold:*

- (i) *the subset $H^+(x_0)\subseteq X$ is conditionally compact;*
- (ii) *the point $\omega_0:=h(x_0)\in\Omega$ is Poisson stable in the positive direction.*

Then, there exist a point $x\in H^+(x_0)$ and an entire trajectory γ of the semi-group dynamical system $(X,\mathbb{T}_+,\pi))$ passing through the point x such that $\gamma(\mathbb{T})\subseteq H^+(x_0)$.

Proof. Since the point $\omega_0\in\Omega$ is Poisson stable in the positive direction, then there is a sequence $\{\tau_n\}\to+\infty$ such that $\lambda(\tau_n,\omega_0)\to\omega_0$. Denote by γ_n the function from $C(\mathbb{T},X)$ defined by equality $\gamma_n(t)=\pi(t+\tau_n,x_0)$ for all $t\ge-\tau_n$ and $\gamma_n(t)=x_0$ for all $t\le-\tau_n$. We will show that the sequence $\{\gamma_n\}$ is relatively compact in $C(\mathbb{T},X)$. Let $l>0$, and $\Omega^{'}\subseteq\Omega$ be a compact subset containing the sequence $\{\omega_n\}$ ($\omega_n:=\lambda(\tau_n,\omega_0)$) and its limit ω_0). Since the set $\lambda([-l,l],\Omega^{'})$ is compact, then the set $M\cap h^{-1}(\lambda([-l,l],\Omega^{'}))$ is compact as well. In particular, the set $\cup_{n=1}^{\infty}\gamma_n([-l,l])\subseteq M\cap h^{-1}(\lambda([-l,l],\Omega^{'}))$ is relatively compact. Now we will verify that the sequence $\{\gamma_n\}$ is equi-continuous on the interval $[-l,l]$. If we suppose that it is not true, then there exist $\varepsilon_0>0$, $\delta_n\to$ and $t_n^1,t_n^2\in[-l,l]$ such that

$$|t_n^1-t_n^2|<\delta_n \ \text{ and } \ \rho(\gamma_n(t_n^1),\gamma_n(t_n^2))\ge\varepsilon_0 \tag{10}$$

for all $n\in\mathbb{N}$. Without loss of generality, we may consider that the sequence $\{\gamma_n(-l)\}$ is convergent, and denote its limit by $\bar{x}$. From inequality (10) we have

$$\varepsilon_0\le\rho(\gamma_n(t_n^1),\gamma_n(t_n^1))=\rho(\pi(l+t_n^1,\gamma_n(-l)),\pi(l+t_n^2,\gamma_n(-l))). \tag{11}$$

Passing to the limit in equation (11) as $n\to\infty$, and taking into consideration (13), we obtain $\varepsilon_0\le\rho(\pi(l+\bar{t},\bar{x}),\pi(l+\bar{t},\bar{x}))=0$, where $\bar{t}:=\lim\limits_{n\to\infty}t_n^1=\lim\limits_{n\to\infty}t_n^2$. This contradiction proves our statement. Thus, the sequence $\{\gamma_n\}$ is equi-continuous on $[-l,l]$, and the set $\cup_{n=1}^{\infty}\gamma_n([-l,l])$ is relatively compact. Taking into account that the number $l>0$ is arbitrary we conclude that the sequence $\{\gamma_n\}$ is relatively compact in $C(\mathbb{T},X)$. We may suppose that the sequence $\{\gamma_n\}$ is convergent. Denote by $\gamma:=\lim\limits_{n\to\infty}\gamma_n$, then $g(0)=x:=\lim\limits_{n\to\infty}\pi(\tau_n,x_0)$ and γ is an entire trajectory of $(X,\mathbb{T}_+,\pi)$ such that $\gamma(\mathbb{T})\subseteq H^+(x_0)$, because by construction $\gamma_n(\mathbb{T})\subseteq H^+(x_0)$ for all $n\in\mathbb{N}$. □

This result will allow us to construct below a group non-autonomous dynamical system on the space of entire trajectories which is induced by a semigroup non-autonomous dynamical system.

The entire trajectory γ of the semigroup dynamical system $(X, \mathbb{T}_+, \pi)$ is said to be *comparable with $\omega \in \Omega$ by the character of recurrence* ($(\Omega, \mathbb{T}, \lambda)$ is a two-sided dynamical system) if $\mathfrak{N}_\omega \subseteq \mathfrak{N}_\gamma$, where $\mathfrak{N}_\gamma := \{\{t_n\} \subset \mathbb{R} \mid$ such that the sequence $\{\gamma(t+t_n)\}$ converges uniformly with respect to t on every compact from $\mathbb{T}$, i.e. it converges in the space $C(\mathbb{T}, X)$.

Remark 3.5. *Let $\langle (X, \mathbb{T}_+, \pi), (\Omega, \mathbb{T}, \lambda), h\rangle$ be a semi-group non-autonomous dynamical system, and M a subset of X. Denote by $\tilde{M} := \{x \in M \mid$ there exists at least one entire trajectory γ of $(X, \mathbb{T}_+, \pi)$ passing through the point x with condition $\gamma(\mathbb{T}) \subseteq M\}$. It is easy to see that the set $\tilde{M}$ is invariant, i.e. $\pi(t, \tilde{M}) = \tilde{M}$ for all $t \in \mathbb{T}_+$. Moreover, $\tilde{M}$ is the maximal invariant set which is contained in M.*

Denote by $\Phi(M)$ the family of all entire trajectories γ of a semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ with condition $\gamma(\mathbb{T}) \subseteq M$.

Lemma 3.6. *Assume that $\langle (X, \mathbb{T}_+, \pi), (\Omega, \mathbb{T}, \lambda), h\rangle$ is a semi-group non-autonomous dynamical system, M a conditionally compact subset of X, and $\tilde{M} \neq \emptyset$. Then, the following statements hold:*

(i) *the set $\tilde{M}$ is closed;*
(ii) *$\tilde{\Omega} := h(\tilde{M})$ is a closed and invariant subset of $(\Omega, \mathbb{T}, \lambda)$;*
(iii) *$\Phi(M)$ is a closed and shift invariant subset of $C(\mathbb{T}, M)$ and, consequently, a shift dynamical system $(\Phi(M), \mathbb{T}, \sigma)$ is induced on $\Phi(M)$ by the Bebutov dynamical system $(C(\mathbb{T}, M), \mathbb{T}, \sigma)$;*
(iv) *the mapping $H : \Phi(M) \mapsto \tilde{\Omega}$ defined by the equality $H(\gamma) := h(\gamma(0))$ is a homomorphism of the dynamical system $(\Phi(M), \mathbb{T}, \sigma)$ onto $(\tilde{\Omega}, \mathbb{T}, \lambda)$, i.e. the map H is continuous and*

$$H(\sigma(t, \gamma)) = \lambda(t, H(\gamma)) \tag{12}$$

for all $\gamma \in \Phi(M)$ and $t \in \mathbb{T}$;
(v) *the set $\Phi(M)$ is conditionally compact with respect to $(\Phi(M), \tilde{\Omega}, H)$;*
(vi) *if $\gamma_1, \gamma_2 \in \Phi(M)$ and $h(\gamma_1(0)) = h(\gamma_2(0))$, then the following conditions are equivalent:*
   a. $\lim\limits_{|t| \to +\infty} \rho(\gamma_1(t), \gamma_2(t)) = 0$, *where ρ is the distance on X;*
   b. $\lim\limits_{|t| \to +\infty} d(\sigma(t, \gamma_1), \sigma(t, \gamma_2)) = 0$, *where d is the Bebutov distance on $\Phi(M)$.*

Proof. (i) Let $x := \lim\limits_{n \to \infty} x_n$ with $x_n \in \tilde{M}$. Then, for each $n \in \mathbb{N}$, there exists $\gamma_n \in \Phi(M)$ such that $\gamma_n(0) = x_n$. We will show that the sequence $\{\gamma_n\}$ is relatively compact in $C(\mathbb{T}, M)$, i.e. with respect to the compact-open topology. Let $l > 0$, and $\Omega' \subseteq \Omega$ be a compact subset containing the sequence $\{\omega_n\}$ ($\omega_n := h(x_n)$) and its limit $\omega := h(x)$. Since the set $\lambda([-l, l], \Omega')$ is compact, then the set $M \cap h^{-1}(\lambda([-l, l], \Omega'))$ is also compact. In particular, the set $\cup_{n=1}^{\infty} \gamma_n([-l, l]) \subseteq M \cap h^{-1}(\lambda([-l, l], \Omega'))$ is relatively compact. Now we will check that the sequence $\{\gamma_n\}$

is equi-continuous on the interval $[-l,l]$. If we suppose that it is not true, then there exist $\varepsilon_0>0$, $\delta_n\to$ and $t_n^1,t_n^2\in[-l,l]$ such that

$$|t_n^1-t_n^2|<\delta_n \quad\text{and}\quad \rho(\gamma_n(t_n^1),\gamma_n(t_n^2))\ge\varepsilon_0 \tag{13}$$

for all $n\in\mathbb{N}$. Without loss of generality, we may consider that the sequence $\{\gamma_n(-l)\}$ is convergent and denote its limit by $\bar{x}$. From inequality (13) we have

$$\varepsilon_0\le\rho(\gamma_n(t_n^1),\gamma_n(t_n^1))=\rho(\pi(l+t_n^1,\gamma_n(-l)),\pi(l+t_n^2,\gamma_n(-l))). \tag{14}$$

Passing to the limit in (14) as $n\to\infty$, and taking into account (13), we obtain $\varepsilon_0\le\rho(\pi(l+\bar{t},\bar{x}),\pi(l+\bar{t},\bar{x}))=0$, where $\bar{t}:=\lim\limits_{n\to\infty}t_n^1=\lim\limits_{n\to\infty}t_n^2$. This is a contradiction which proves our statement. Thus the sequence $\{\gamma_n\}$ is equi-continuous on $[-l,l]$ and the set $\cup_{n=1}^{\infty}\gamma_n([-l,l])$ is relatively compact. Noticing that the number $l>0$ is arbitrary, we conclude that the sequence $\{\gamma_n\}$ is relatively compact in $C(\mathbb{T},M)$. We may suppose that the sequence $\{\gamma_n\}$ is convergent. Denote by $\gamma:=\lim\limits_{n\to\infty}\gamma_n$, then $g(0)=x$ and γ is an entire trajectory of $(X,\mathbb{T}_+,\pi)$ such that $\gamma(\mathbb{T})\subseteq M$. This means that the point x belongs to $\tilde{M}$, i.e. $\tilde{M}$ is closed.

(ii) Let us now prove that the set $\tilde{\Omega}=h(\tilde{M})$ is closed. Indeed, let $\omega:=\lim\limits_{n\to\infty}\omega_n$, where $\omega_n\in\tilde{\Omega}$ and $\Omega'\subseteq\Omega$ is a compact subset such that $\omega,\omega_n\in\Omega'$ for all $n\in\mathbb{N}$. Since $\omega_n\in\tilde{\Omega}=h(\tilde{M})$, then there exists $x_n\in\tilde{M}$ such that $\omega_n=h(x_n)$. On the other hand, $\{x_n\}\subset M\cap h^{-1}(\Omega')$ and, consequently, it is relatively compact. We may suppose that the sequence $\{x_n\}$ is convergent, and denote by x its limit. It is clear that $x\in\tilde{M}$, because $\tilde{M}$ is closed. It is now sufficient to note that $h(x)=\omega$ and, consequently, $\omega\in\tilde{\Omega}$. From the equality

$$\lambda(t,\tilde{\Omega})=\lambda(t,h(\tilde{M}))=h(\pi(t,\tilde{M}))=h(\tilde{M})=\tilde{\Omega},\quad\text{for all}\quad t\in\mathbb{T}_+,$$

the invariance of the set $\tilde{\Omega}$ follows.

(iii) Let $\{\gamma_n\}\subset\Phi(M)$ and $\gamma:=\lim\limits_{n\to\infty}\gamma_n$. Then $\pi(t,\gamma(s))=\lim\limits_{n\to\infty}\pi(t,\gamma_n(s))=\lim\limits_{n\to\infty}\gamma_n(t+s)=\gamma(t+s)$ for all $t\in\mathbb{T}_+$ and $s\in\mathbb{T}$, i.e. $\gamma\in\Phi(M)$ which means that $\Phi(M)$ is closed in the space $C(\mathbb{T},M)$. To establish the shift invariance of $\Phi(M)$ it is sufficient to note that $\pi(t,\sigma(\tau,\gamma)(s))=\pi(t,\gamma(\tau+s))=\gamma(t+\tau+s)=\sigma(\tau,\gamma)(t+s)$ for all $t\in\mathbb{T}_+$ and $s\in\mathbb{T}$ and, consequently, $\sigma(\tau,\gamma)\in\Phi(M)$ for all $\gamma\in\Phi(M)$ and $\tau\in\mathbb{T}$.

(iv) The continuity of the mapping H follows from its definition, because $H=Ev\circ h$ is the composition of two continuous mappings: $h:X\mapsto\Omega$ and $Ev:C(\mathbb{T},X)\mapsto X$, defined by equality $Ev(\gamma):=\gamma(0)$. To finish the proof of this fourth statement we note that $\lambda(t,H(\gamma))=\lambda(t,h(\gamma(0)))=h(\pi(t,\gamma(0)))=h(\gamma(t))=h(\sigma(t,\gamma)(0))=H(\sigma(t,\gamma))$ for all $t\in\mathbb{T}$ and $\gamma\in\Phi(M)$.

(v) We will prove now that the set $\Phi(M)$ is conditionally compact with respect to $(\Phi(M),\ \tilde{\Omega},\ H)$. Let $\Omega'\subseteq\tilde{\Omega}$ be a compact subset, and $\{\gamma_n\}$ a sequence in $\Phi(M)\cap h^{-1}(\Omega')$. Reasoning as in the proof of the first statement, we can prove that the sequence $\{\gamma_n\}$ is relatively compact.

(vi) Finally, we will establish the last statement. The implication b.$\Rightarrow$ a. is evident. Now, we prove that a. $\Rightarrow$ b. If we suppose that it is not true, then there exist $\varepsilon_0 > 0$ and $|t_n| \to +\infty$ such that

$$d(\sigma(t_n, \gamma_1), \sigma(t_n, \gamma_2)) \ge \varepsilon_0, \quad \forall n \ge 1. \tag{15}$$

Thanks to Remark 2.4, the inequality (15) is equivalent to

$$\max_{|t| \le \frac{1}{\varepsilon_0}} \rho(\gamma_1(t + t_n), \gamma_2(t + t_n)) \ge \varepsilon_0.$$

Since the mappings γ_1 and γ_2 are continuous on $\mathbb{T}$, then there exists a sequence $\{\tau_n\} \subset [-\frac{1}{\varepsilon_0}, \frac{1}{\varepsilon_0}]$ such that

$$\varepsilon_0 \le \max_{|t| \le \frac{1}{\varepsilon_0}} \rho(\gamma_1(t + t_n), \gamma_2(t + t_n)) = \rho(\gamma_1(\tau_n + t_n), \gamma_2(\tau_n + t_n)). \tag{16}$$

It is clear that $|\tau_n + t_n| \to +\infty$ as $n \to \infty$, and passing to the limit in (16) as $n \to \infty$, and taking into account condition a., we obtain $\varepsilon_0 \le 0$. This contradiction proves our statement, and the lemma is completely proved. □

Corollary 3.7. *Under the conditions of Lemma 3.6, if the set M is compact, then $\Phi(M)$ is also compact.*

Theorem 3.8. *Let $\langle (X, \mathbb{T}_+, \pi), (\Omega, \mathbb{T}, \lambda), h \rangle$ be a semi-group non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

a. *there exists a point $x_0 \in X$ such that $H^+(x_0)$ is conditionally compact;*
b. *the point $\omega_0 := h(x_0) \in \Omega$ is Poisson stable;*
c.
$$\lim_{|t| \to +\infty} \rho(\gamma_1(t), \gamma_2(t)) = 0$$
for all entire trajectories γ_1 and γ_2 of the semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ with the conditions: $\gamma_i(\mathbb{T}) \subseteq H^+(x_0)$ and $h(g_1(0)) = h(g_2(0)) = \omega_0$.

Then, there exists a unique entire trajectory γ of $(X, \mathbb{T}_+, \pi)$ possessing the following properties:

(i) $\gamma(\mathbb{T}) \subseteq H^+(x_0)$;
(ii) $h(\gamma(0)) = \omega_0$;
(iii) *γ is comparable with $\omega_0 \in \Omega$ by the character of recurrence.*

Proof. Let $M := H^+(x_0)$. Then, by Lemma 3.4, we have $\Phi(M) \ne \emptyset$. Consider the group non-autonomous dynamical system $\langle (\Phi(M), \mathbb{T}, \sigma), (\tilde{\Omega}, \mathbb{T}, \lambda), H \rangle$ constructed in the proof of Lemma 3.6. By Lemma 3.4 the point ω_0 belongs to $\tilde{\Omega}$. According to Lemma 3.6 all conditions of Corollary 2.8 are fulfilled and, consequently, we obtain the existence of at least one entire trajectory γ of the dynamical system $(X, \mathbb{T}_+, \pi)$ possessing the properties listed in the theorem. To finish the proof of the theorem it is sufficient to show that there exists at most one entire trajectory of $(X, \mathbb{T}_+, \pi)$ with the properties listed in the theorem. Let γ_1 and γ_2 be two entire trajectories with necessary properties. In particular, $\gamma_i(\mathbb{T}) \subseteq H^+(x_0)$ and $\mathfrak{N}_{\omega_0} \subseteq \mathfrak{N}_{\gamma_i}$ $(i = 1, 2)$. Under the conditions of the theorem we have

$$\lim_{|t| \to +\infty} d(\sigma(t, \gamma_1), \sigma(t, \gamma_2)) = 0.$$

On the other hand, there exists a sequence $\{t_n\} \in \mathfrak{N}_{\omega_0} \subseteq \mathfrak{N}_{\gamma_i}$ $(i = 1, 2)$ such that $|t_n| \to +\infty$ and, consequently, we have

$$d(\gamma_1, \gamma_2) = \lim_{n\to+\infty} d(\sigma(t_n, \gamma_1), \sigma(t_n, \gamma_2)) = 0,$$

i.e. $\gamma_1 = \gamma_2$, and the theorem is therefore proved. □

Corollary 3.9. *Assume the Hypotheses of Theorem 3.8, as well as*

$$\lim_{|t|\to+\infty} \rho(\gamma_1(t), \gamma_2(t)) = 0$$

for all entire trajectories γ_1 and γ_2 of the semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ with the conditions: $\gamma_i(\mathbb{T})$ (i=1,2) is conditionally compact and $h(g_1(0)) = h(g_2(0)) = \omega_0$.

Then, there exists a unique entire trajectory γ of $(X, \mathbb{T}_+, \pi)$, which is comparable with $\omega_0 \in \Omega$ by the character of recurrence, and which satisfies the following properties:

(i) *$\gamma(\mathbb{T})$ is conditionally compact;*
(ii) *$h(\gamma(0)) = \omega_0$.*

Proof. This statement can be proved by a slight modification of the arguments in the proof of Theorem 3.8. □

Corollary 3.10. *Let $\omega_0 \in \Omega$ be a stationary (τ-periodic, almost automorphic, recurrent, Levitan almost periodic, Poisson stable) point. Then, under the conditions of Theorem 3.8 there exists a unique stationary (τ-periodic, almost automorphic, recurrent, Levitan almost periodic, Poisson stable) entire trajectory γ of the dynamical system $(X, \mathbb{T}_+, \pi)$ such that $\gamma(\mathbb{T}) \subseteq H^+(x_0)$.*

Proof. This statement follows directly from Theorem 3.8 and Remark 2.7. □

The entire trajectory γ of the semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ is said to be *uniformly comparable* with the point ω of the group dynamical system $(\Omega, \mathbb{T}, \lambda)$ by the character of recurrence, if $\mathfrak{M}_\omega \subseteq \mathfrak{M}_\gamma$, where $\mathfrak{M}_\gamma := \{\{t_n\} \subset \mathbb{T} \mid$ the sequence $\sigma(t_n, \gamma)$ converges in the space $C(\mathbb{T}, X)\}$.

Theorem 3.11. *Let $\langle (X, \mathbb{T}_+, \pi), (\Omega, \mathbb{T}, \lambda), h\rangle$ be a semi-group non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

a) *there exists $x_0 \in X$ such that $H^+(x_0)$ is compact;*
b) *the point $\omega_0 := h(x_0) \in \Omega$ is recurrent;*
c)
$$\lim_{|t|\to+\infty} \rho(\gamma_1(t), \gamma_2(t)) = 0$$
for all entire trajectories γ_1 and γ_2 of the semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ with the conditions: $\gamma_i(\mathbb{T}) \subseteq H^+(x_0)$ and $h(g_1(0)) = h(g_2(0))$.

Then, there exists a unique entire trajectory γ of $(X, \mathbb{T}_+, \pi)$ possessing the following properties:

(i) *$\gamma(\mathbb{T}) \subseteq H^+(x_0)$;*

(ii) $h(\gamma(0)) = \omega_0$;
(iii) γ *is uniformly comparable by the character of recurrence with* $\omega_0 \in \Omega$.

Proof. Let $M := H^+(x_0)$. Then, by Lemma 3.4 we have that $\Phi(M) \neq \emptyset$. Consider the group non-autonomous dynamical system $\langle (\Phi(M), \mathbb{T}, \sigma), (\tilde{\Omega}, \mathbb{T}, \lambda), H \rangle$ (see Lemma 3.6 and its proof). By Lemma 3.4 the point ω_0 belongs to $\tilde{\Omega}$. According to Lemma 3.6, all conditions of Theorem 2.11 are fulfilled and, consequently, we obtain the existence of at least one entire trajectory γ of the dynamical system $(X, \mathbb{T}_+, \pi)$ which is uniformly comparable with $\omega_0 \in \Omega$ by the character of recurrence, and $\gamma(\mathbb{T}) \subseteq H^+(x_0)$. To finish the proof it is sufficient to show that there exists at most one entire trajectory of $(X, \mathbb{T}_+, \pi)$ with the properties (i)-(iii). Let γ_1 and γ_2 be two entire trajectories satisfying (i)-(iii). In particular, $\gamma_i(\mathbb{T}) \subseteq H^+(x_0)$ and $\mathfrak{M}_{\omega_0} \subseteq \mathfrak{M}_{\gamma_i}$ $(i = 1, 2)$. Then we also have $\mathfrak{N}_{\omega_0} \subseteq \mathfrak{N}_{\gamma_i}$ $(i = 1, 2)$. From assumption c) we obtain

$$\lim_{|t| \to +\infty} d(\sigma(t, \gamma_1), \sigma(t, \gamma_2)) = 0.$$

On the other hand, there exists a sequence $\{t_n\} \in \mathfrak{N}_{\omega_0} \subseteq \mathfrak{N}_{\gamma_i}$ $(i = 1, 2)$ such that $|t_n| \to +\infty$ and, consequently,

$$d(\gamma_1, \gamma_2) = \lim_{n \to +\infty} d(\sigma(t_n, \gamma_1), \sigma(t_n, \gamma_2)) = 0,$$

i.e. $\gamma_1 = \gamma_2$, and proof is complete. □

Corollary 3.12. *In addition to assumptions in Theorem 3.11, suppose that*

$$\lim_{|t| \to +\infty} \rho(\gamma_1(t), \gamma_2(t)) = 0$$

for all entire trajectories γ_1 *and* γ_2 *of the semi-group dynamical system* $(X, \mathbb{T}_+, \pi)$ *with the conditions:* $\gamma_i(\mathbb{T})$ *(i=1,2) is conditionally compact and* $h(g_1(0)) = h(g_2(0))$.

Then, there exists a unique entire trajectory γ *of* $(X, \mathbb{T}_+, \pi)$, *which is uniformly comparable with* $\omega_0 \in \Omega$ *by the character of recurrence, and such that* $\gamma(\mathbb{T})$ *is relatively compact.*

Proof. This statement follows by a slight modification of the proof of Theorem 3.11. □

Corollary 3.13. *Let* $\omega_0 \in \Omega$ *be a stationary (respectively,* τ*-periodic, Bohr almost periodic, almost automorphic, recurrent) point. Then, under the conditions of Theorem 3.11, there exists a unique stationary (respectively,* τ*-periodic, Bohr almost periodic, almost automorphic, recurrent) entire trajectory* γ *of the dynamical system* $(X, \mathbb{T}_+, \pi)$ *such that* $\gamma(\mathbb{T}) \subseteq H^+(x_0)$.

Proof. This statement follows easily from Theorem 3.11, and Remarks 2.7 and 2.10. □

4. Compatible and Uniformly Compatible Solutions of Linear Functional Differential/Difference Equations

In this section we will establish interesting dynamical properties for some classes of linear differential/difference systems with delays, as well as for some linear partial differential equations.

4.1. **Linear Functional-Differential Equations with Finite Delay.** Let $r > 0$, and denote by $C([a,b],\mathbb{R}^n)$ the Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}^n$ with the sup-norm. For $[a,b] := [-r,0]$ we put $\mathcal{C} := C([-r,0],\mathbb{R}^n)$. Let $c \in \mathbb{R}, a \geq 0$, and $u \in C([c-r,c+a],\mathbb{R}^n)$. We define $u_t \in \mathcal{C}$ for any $t \in [c,c+a]$ by the relation $u_t(\theta) := u(t+\theta), -r \geq \theta \geq 0$. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{C},\mathbb{R}^n)$ be the Banach space of all linear operators that act from $\mathcal{C}$ into $\mathbb{R}^n$ and equipped with the operator norm, let $C(\mathbb{R},\mathfrak{A})$ be the space of all operator-valued functions $A : \mathbb{R} \to \mathfrak{A}$ with the compact-open topology, and let $(C(\mathbb{R},\mathfrak{A}),\mathbb{R},\sigma)$ be the group dynamical system of shifts on $C(\mathbb{R},\mathfrak{A})$. Let $H(A) := \overline{\{A_\tau \mid \tau \in \mathbb{R}\}}$, where A_τ is the shift of the operator-valued function A by τ and the bar denotes the closure in $C(\mathbb{R},\mathfrak{A})$.

Example 4.1. *Consider the non-homogeneous linear functional-differential equation with finite delay*

$$u' = A(t)u_t + f(t), \tag{17}$$

and its corresponding homogeneous linear equation

$$u' = A(t)u_t, \tag{18}$$

where $A \in C(\mathbb{R},\mathfrak{A})$ *and* $f \in C(\mathbb{R},\mathbb{R}^n)$.

Denote by $\varphi(t,u,(A,f))$ the solution of equation (17) defined on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) with the initial condition $\varphi(0,u,(A,f)) = u \in \mathcal{C}$, i.e. $\varphi(s,u,(A,f)) = u(s)$ for all $s \in [-r,0]$. By $\tilde{\varphi}(t,u,(A,f))$ we will denote below the trajectory of equation (17), corresponding to the solution $\varphi(t,u,(A,f))$, i.e. the mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into $\mathcal{C}$, defined by $\tilde{\varphi}(t,u,(A,f))(s) := \varphi(t+s,u,(A,f))$ for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and $s \in [-r,0]$.

Let $\varphi(t,u_i,(A,f))$ $(i=1,2)$ be two solutions of equation (17), then

$$\lim_{|t|\to\infty} |\varphi(t,u_1,(A,f)) - \varphi(t,u_2,(A,f))| = \lim_{|t|\to\infty} |\tilde{\varphi}(t,u_1,(A,f)) - \tilde{\varphi}(t,u_2,(A,f))|_{\mathcal{C}},$$

where $|\cdot|_{\mathcal{C}}$ (respectively, $|\cdot|$) is the norm on the space $\mathcal{C}$ (respectively, $\mathbb{R}^n$).

Along with equation (17) (respectively, (18)) we consider the family of equations

$$v' = B(t)v_t + g(t) \tag{19}$$

(respectively,

$$v' = B(t)v_t \quad), \tag{20}$$

where $(B,g) \in H(A,f) := \overline{\{(B_\tau,f_\tau) \mid \tau \in \mathbb{R}\}}$. (respectively, $B \in \overline{\{B_\tau \mid \tau \in \mathbb{R}\}} := H(A)$). Let $\tilde{\varphi}(t,v,(B,g))$ (respectively, $\tilde{\varphi}(t,v,B)$ be the trajectory of equation (19) (respectively, (20)) satisfying the condition $\tilde{\varphi}(0,v,(B,g)) = v$ (respectively, $\tilde{\varphi}(0,v,B) = v$), and defined for all $t \geq 0$. Let $Y := H(A,f)$ and denote the group dynamical system of shifts on $H(A,f)$ by $(Y,\mathbb{R},\sigma)$. Let $X := \mathcal{C} \times Y$, and let

$\pi := (\tilde{\varphi}, \sigma)$ be the dynamical system on X defined by the equality $\pi(\tau, (v, (B, g))) := (\tilde{\varphi}(\tau, v, (B, g)), B_\tau)$. The semi-group non-autonomous system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ $(h := pr_2 : X \to Y)$ is generated by equation (17).

Then we can prove the following result.

Lemma 4.2. *Let $\varphi(t, u, (A, f))$ be a solution of equation (17), which is bounded on $\mathbb{R}_+$, and let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R},), h\rangle$ be a non-autonomous dynamical system generated by equation (17). Then, the set $\overline{\{(\tilde{\varphi}(t, u, (A, f)), (A_t, f_t)) \mid t \ge 0\}} := H(u, (A, f))$ is conditionally compact with respect to (X, h, Y).*

Proof. Let K be an arbitrary compact subset of $Y := H(A, f)$ and $\{x_k\} := \{(u_k, (A_k, f_k))\}$ be a subsequence from $h^{-1}(K) \cap H(u, (A, f))$. Since $\{(A_k, f_k)\} \subseteq K \subseteq H(A, f)$ and the subset K is compact, then we can suppose that the sequence $\{(A_k, f_k)\}$ is convergent in $C(\mathbb{R}, \mathfrak{A}) \times C(\mathbb{R}, \mathbb{R}^n)$. On the other hand, for every $k \in \mathbb{N}$, there exists $t_k \in \mathbb{R}_+$ such that

$$|u_k - \tilde{\varphi}(t_k, u, (A, f))|_{\mathcal{C}} \le 1/k \tag{21}$$

and

$$\rho((A_{t_k}, f_{t_k}), (A_k, f_k)) \le 1/k. \tag{22}$$

Denote by

$$\psi_k(t) := \varphi(t + t_k, u, (A, f)) = \varphi(t, \tilde{\varphi}(t_k, u, (A, f)), (A_{t_k}, f_{t_k})) \tag{23}$$

for all $t \in \mathbb{R}_+$. By the inequality (22) we may suppose that the sequence $\{(A_{t_k}, f_{t_k})\}$ is convergent. Since the solution $\varphi(t, u, (A, f))$ is bounded on $\mathbb{R}_+$, from the equality (23) it follows that the sequence $\{\psi_k\}$ is bounded and equi-continuous on every compact $[-r, l]$ $(l > 0)$ subset of $[-r, +\infty)$. Thus, the sequence $\{\tilde{\psi}_k(t)\}:=\{\tilde{\varphi}(t, \tilde{\varphi}(t_k, u, (A, f)), (A_{t_k}, f_{t_k}))\}$ is relatively compact in $C(\mathbb{R}_+, \mathcal{C})$ and, consequently, the sequence $\{\tilde{\varphi}(t_k, u, (A, f))\} = \{\tilde{\psi}_k(0)\} \subset \mathcal{C}$ is relatively compact too. From the inequality (22) and the relatively compactness of the sequence $\{\tilde{\varphi}(t_k, u, (A, f))\}$, it follows that the sequence $\{u_k\}$ is also relatively compact. $\square$

A solution $\varphi \in C(\mathbb{R}, \mathbb{R}^n)$ of equation (17) is called *compatible by the character of recurrence*, if $\mathfrak{N}_{(A,f)} \subseteq \mathfrak{N}_\varphi$, where $\mathfrak{N}_{(A,f)} := \{\{t_n\} \subset \mathbb{R} \mid (A_{t_n}, f_{t_n}) \to (A, f)\}$ (respectively, $\mathfrak{N}_\varphi := \{\{t_n\} \subset \mathbb{R} \mid \varphi_{t_n} \to \varphi\}$).

Remark 4.3. *1. Note that the sequence $\{\varphi_{t_n}\}$ converges in $C(\mathbb{R}, \mathbb{R}^n)$, if and only if the sequence $\{\tilde{\varphi}_{t_n}\}$ converges in $C(\mathbb{R}, \mathcal{C})$, where $\tilde{\varphi} : \mathbb{R} \mapsto \mathcal{C}$ is defined by the equality $\tilde{\varphi}(t)(s) := \varphi(t + s)$ for all $s \in [-r, 0]$ and $t \in \mathbb{R}$.*

2. As a consequence of the previous statement it holds that $\mathfrak{N}_\varphi = \mathfrak{N}_{\tilde{\varphi}}$.

Theorem 4.4. *Let $(A, f) \in C(\mathbb{R}, \mathfrak{A}) \times C(\mathbb{R}, \mathbb{R}^n)$ be Poisson stable. Suppose that the following conditions hold:*

(i) *equation (17) admits a solution $\varphi(t, u_0, (A, f))$ which is bounded on $\mathbb{R}_+$;*
(ii) *all the solutions of equation (18), which are bounded on $\mathbb{R}$, tend to zero as the time tends to ∞, i.e. $\lim\limits_{|t|\to+\infty} |\varphi(t, u, A)| = 0$ if $\varphi(t, u, A)$ is bounded on $\mathbb{R}$.*

Then, equation (17) possesses a unique compatible solution $\varphi(t,\bar{u},(A,f))$ which is bounded on $\mathbb{R}$.

Proof. First of all, we prove that equation (17) admits at most one compatible solution. Indeed, if we suppose that it is not true, then there are at least two compatible solutions $\varphi(t,u_i,(A,f))$ (i=1,2 and $u_1 \neq u_2$) defined and bounded on $\mathbb{R}$. Since (A,f) is Poisson stable, then $\psi(t) := \varphi(t,u_1,(A,f)) - \varphi(t,u_2,(A,f))$ ($t \in \mathbb{R}$) is also Poisson stable. On the other hand, $\psi(t) = \varphi(t,u_1-u_2,A)$ is a solution of equation (18), bounded on $\mathbb{R}$, and, consequently, $\lim\limits_{|t|\to+\infty} |\psi(t)| = 0$. This fact and the Poisson stability of ψ imply that $\psi(t) = 0$ for all $t \in \mathbb{R}$. In particular, $u_1 - u_2 = \psi(0) = 0$. This contradiction proves our statement.

Now we will prove that equation (17) admits at least one compatible solution. Denote by $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ the non-autonomous dynamical system generated by equation (17) (see Example 4.1). Then, by Lemma 4.2, the positively invariant set $H^+(x_0) \subset X$ (where $x_0 := (u_0,(A,f)) \in X := \mathcal{C}\times H(A,f)$ and $H^+(x_0) := \overline{\{\pi(t,x_0) \mid t \in \mathbb{R}_+\}}$) is conditionally compact. Consider now $x_1,x_2 \in H^+(x_0)\cap X_{(A,f)}$, where $X_{(A,f)} := \mathcal{C}\times\{(A,f)\}$ (i.e. $x_i = (u_i,(A,f))$ and $u_i \in \mathcal{C}$ (i=1,2)). Then

$$\lim_{|t|\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = \lim_{|t|\to+\infty} |\varphi(t,u_1,(A,f)) - \varphi(y,u_2,(A,f))|_{\mathcal{C}} = 0.$$

Now, to finish the proof, it is sufficient to refer to Theorem 3.8 and Corollary 3.9. □

Corollary 4.5. *Under the conditions of Theorem 4.4, if $(A,f) \in C(\mathbb{R},\mathfrak{A})\times C(\mathbb{R},\mathbb{R}^n)$ is τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then equation (17) admits a unique τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable) solution.*

Proof. This statement follows from Theorem 4.4 and Corollary 3.10. □

Corollary 4.6. *Under the conditions of Theorem 4.4, if $(A,f) \in C(\mathbb{R},\mathfrak{A})\times C(\mathbb{R},\mathbb{R}^n)$ is almost automorphic, then equation (17) admits a unique almost automorphic solution.*

Proof. Since the function $\varphi(t,\bar{u},(A,f))$ is bounded on $\mathbb{R}$, and the functions $A \in C(\mathbb{R},\mathfrak{A})$ and $f \in C(\mathbb{R},\mathbb{R}^n)$ are bounded on $\mathbb{R}$, then $\varphi(t,\bar{u},(A,f))$ is uniformly continuous on $\mathbb{R}$. Thus $\bar{\varphi} := \varphi(\cdot,\bar{u},(A,f)) \in C(\mathbb{R},\mathbb{R}^n)$ is a Lagrange stable point of the dynamical system $(C(\mathbb{R},\mathbb{R}^n),\mathbb{R},\sigma)$. On the other hand, by Corollary 4.5 the function $\bar{\varphi}$ is Levitan almost periodic and, consequently, it is almost automorphic. □

Corollary 4.7. *Under the conditions of Theorem 4.4, if $(A,f) \in C(\mathbb{R},\mathfrak{A})\times C(\mathbb{R},\mathbb{R}^n)$ is Bohr almost periodic, then equation (17) admits a unique almost automorphic solution.*

Proof. This statement follows from Corollary 4.6 because every Bohr almost periodic function is almost automorphic. □

A solution $\varphi \in C(\mathbb{R}, \mathbb{R}^n)$ of equation (17) is called (see [17],[19]) *uniformly compatible by the character of recurrence* if $\mathfrak{M}_{(A,f)} \subseteq \mathfrak{M}_{\varphi}$, where $\mathfrak{M}_{(A,f)} := \{\{t_n\} \subset \mathbb{R} \mid$ such that the sequence $\{(A_{t_n}, f_{t_n})\}$ is convergent$\}$ (respectively, $\mathfrak{M}_{\varphi} := \{\{t_n\} \subset \mathbb{R} \mid$ such that the sequence $\{\varphi_{t_n}\}$ is convergent$\}$).

Remark 4.8. *Observe that the first item of Remark 4.3 implies $\mathfrak{M}_{\varphi} = \mathfrak{M}_{\tilde{\varphi}}$.*

Theorem 4.9. *Let $(A, f) \in C(\mathbb{R}, \mathfrak{A}) \times C(\mathbb{R}, \mathbb{R}^n)$ be recurrent. Suppose that the following conditions hold:*

(i) *equation (17) admits a solution $\varphi(t, u_0, (A, f))$ which is bounded on $\mathbb{R}_+$;*
(ii) *for all $B \in H(A)$ the solutions of equation (20), which are bounded on $\mathbb{R}$, tend to zero as the time tends to ∞, i.e. $\lim\limits_{|t| \to +\infty} |\varphi(t, u, B)| = 0$ if $\varphi(t, u, B)$ is bounded on $\mathbb{R}$.*

Then, equation (17) has a unique uniformly compatible solution $\varphi(t, \bar{u}, (A, f))$.

Proof. Note that under the conditions of the theorem, equation (17) admits at most one uniformly compatible solution. Indeed, every uniformly compatible solution is compatible. On the other hand, by Theorem 4.4, equation (17) admits a unique compatible solution.

Now we prove that equation (17) admits, at least, a uniformly compatible solution. Indeed, since the function $\varphi(t, u_0, (A, f))$ is bounded on $\mathbb{R}_+$, and the functions $A \in C(\mathbb{R}, \mathfrak{A})$ and $f \in C(\mathbb{R}, \mathbb{R}^n)$ are bounded on $\mathbb{R}$, then $\varphi(t, u_0, (A, f))$ is uniformly continuous on $\mathbb{R}_+$. Thus, the trajectory $\tilde{\varphi}(t, u_0, (A, f))$ of equation (17) is relatively compact on $\mathbb{R}_+$. Denote by $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ the semi-group non-autonomous dynamical system generated by equation (17). Under the conditions of the theorem, the positively invariant set $H^+(x_0) \subset X$ (where $x_0 := (u_0, (A, f)) \in X$ and $H^+(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{R}_+\}}$) is compact. Let now $x_1, x_2 \in H^+(x_0) \cap X_{(B,g)}$, where $(B, g) \in H(A, f)$ and $X_{(B,g)} := \mathcal{C} \times \{(B, g)\}$ (i.e. $x_i = (u_i, (B, g))$ and $u_i \in \mathcal{C}$ (i=1,2)), then

$$\lim_{|t| \to +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = \lim_{|t| \to +\infty} |\varphi(t, u_1, (B, g)) - \varphi(y, u_2, (B, g))|_{\mathcal{C}} = 0.$$

To finish the proof it is sufficient to refer to Theorem 3.11 and Corollary 3.12. □

Corollary 4.10. *Under the conditions of Theorem 4.9, if $(A, f) \in C(\mathbb{R}, \mathfrak{A}) \times C(\mathbb{R}, \mathbb{R}^n)$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (17) admits a unique τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.*

Proof. This statement follows from Theorem 4.9 and Corollary 3.13. □

4.2. **Neutral Linear Functional-Differential Equations.**

Example 4.11. *Now consider the neutral functional-differential equation*

$$\frac{d}{dt} D u_t = A(t) u_t + f(t) \tag{24}$$

and its corresponding homogeneous linear equation

$$\frac{d}{dt}Du_t = A(t)u_t, \tag{25}$$

where $A \in C(\mathbb{R},\mathfrak{A})$, and the operator $D \in \mathfrak{A}$ is atomic at zero (see [11, p.67]*).*

Along with equation (24) (respectively, (25)), we consider the family of equations

$$\frac{d}{dt}Dv_t = B(t)v_t + g(t) \tag{26}$$

(respectively,

$$\frac{d}{dt}Dv_t = B(t)v_t \quad), \tag{27}$$

where $(B,g) \in H(A,f) := \overline{\{(B_\tau, f_\tau) \mid \tau \in \mathbb{R}\}}$ (respectively, $B \in \overline{\{B_\tau \mid \tau \in \mathbb{R}\}} := H(A)$). Let $\varphi(t,v,(B,g))$ (respectively, $\varphi(t,v,B)$) be the trajectory of equation (26) (respectively, (27)) satisfying the condition $\varphi(0,v,(B,g)) = v$ (respectively, $\varphi(0,v,B) = v$) and defined for all $t \geq 0$. Let $Y := H(A,f)$, and denote by $(Y,\mathbb{R},\sigma)$ the group dynamical system of shifts on $H(A,f)$. Let $X := \mathcal{C} \times Y$, and let $\pi := (\varphi,\sigma)$ be the dynamical system on X defined by the equality $\pi(\tau,(v,(B,g))) := (\varphi(\tau,v,(B,g)), B_\tau, g_\tau)$. The non-autonomous system $\langle (X,\mathbb{R}_+,\pi), (Y,\mathbb{R},\sigma), h\rangle$ ($h := pr_2 : X \to Y$) is generated by equation (24).

Lemma 4.12. *Let $\varphi(t,u,(A,f))$ be a (respectively, defined and bounded on $\mathbb{R}$) solution of equation (24) bounded on $\mathbb{R}_+$. Assume that the operator D is stable* [11, p.287] *and $H(A,f)$ is a compact subset of $C(\mathbb{R},\mathfrak{A}) \times C(\mathbb{R},\mathbb{R}^n)$. Then $\tilde{\varphi}(\mathbb{R}_+,u,(A,f))$ (respectively, $\tilde{\varphi}(\mathbb{R},u,(A,f))$) is a relatively compact subset of $\mathcal{C}$.*

Proof. This statement can be proved by modifying slightly the proof of Theorem 6.1 from [11, p.293]. □

Theorem 4.13. *Let $(A,f) \in C(\mathbb{R},\mathfrak{A}) \times C(\mathbb{R},\mathbb{R}^n)$ be recurrent. Suppose that the following conditions hold:*

(i) *equation (24) admits a solution $\varphi(t,u_0,(A,f))$ bounded on $\mathbb{R}_+$;*
(ii) *for all $B \in H(A)$, the bounded on $\mathbb{R}$ solutions of equation (25) tend to zero as the time tends to ∞, i.e. $\lim_{|t|\to+\infty} |\varphi(t,u,B)| = 0$ if $\varphi(t,u,B)$ is bounded on $\mathbb{R}$.*

Then, equation (24) possesses a unique uniformly compatible solution $\varphi(t,\bar{u},(A,f))$.

Proof. Let us first prove that equation (24) admits, at least, a uniformly compatible solution. Indeed, since the function $\varphi(t,u_0,(A,f))$ is bounded on $\mathbb{R}_+$, and the functions $A \in C(\mathbb{R},\mathfrak{A})$ and $f \in C(\mathbb{R},\mathbb{R}^n)$ are bounded on $\mathbb{R}$, then $\varphi(t,u_0,(A,f))$ is uniformly continuous on $\mathbb{R}_+$. By Lemma 4.12, $\tilde{\varphi}(\mathbb{R}_+,u_0,(A,f))$ is relatively compact in $\mathcal{C}$. Denote by $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ the semi-group non-autonomous dynamical system generated by equation (24). Under the conditions of the theorem, the positively invariant set $H^+(x_0) \subset X$ (where $x_0 := (u_0,(A,f)) \in X$ and $H^+(x_0) := \overline{\{\pi(t,x_0) \mid t \in \mathbb{R}_+\}}$) is compact. Let now $x_1, x_2 \in H(x_0) \cap X_{(B,g)}$, where $(B,g) \in H(A,f)$ and $X_{(B,g)} := \mathcal{C} \times \{(B,g)\}$ (i.e. $x_i = (u_i,(B,g))$ and $u_i \in \mathcal{C}$ (i=1,2)), then

$$\lim_{|t|\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = \lim_{|t|\to+\infty} |\tilde{\varphi}(t,u_1,(B,g)) - \tilde{\varphi}(y,u_2,(B,g))|_{\mathcal{C}} = 0.$$

Now to finish the proof it is sufficient to refer to Theorem 3.11 and Corollary 3.12. □

Corollary 4.14. *Under the conditions of Theorem 4.13 if $(A,f) \in C(\mathbb{R},\mathfrak{A}) \times C(\mathbb{R},\mathbb{R}^n)$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then the equation (24) admits a unique τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.*

Proof. This statement follows from Theorem 4.13 and Corollary 3.13. □

4.3. **Linear Functional-Difference Equations with Finite Delay.** Let $r \in \mathbb{Z}_+$. We would like to point out that, although we will use the same notation used in Subsection 4.1, we must be careful since here the intervals $[a,b]$ are considered as subsets of $\mathbb{Z}$. Accordingly, we will denote by $C([a,b],E)$ the Banach space of all functions $\varphi : [a,b]\,(\subset \mathbb{Z}) \to E$ with the sup-norm. For $[a,b] := [-r,0]$ we put $\mathcal{C} := C([-r,0],E)$. Let $c,a \in \mathbb{Z},\ a \geq 0$, and $u \in C([c-r,c+a],E)$. We define $u_t \in C$ for any $t \in [c,c+a]$ by the relation $u_t(s) := u(t+s), -r \leq s \leq 0$. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{C},E)$ be the Banach space of all linear operators that act from $\mathcal{C}$ into E, equipped with the operator norm. Let $C(\mathbb{Z},\mathfrak{A})$ be the space of all operator-valued functions $A : \mathbb{Z} \to \mathfrak{A}$ with the compact-open topology, and let $(C(\mathbb{Z},\mathfrak{A}),\mathbb{Z},\sigma)$ be the group dynamical system of shifts on $C(\mathbb{Z},\mathfrak{A})$. Let $H(A) := \overline{\{A_\tau \mid \tau \in \mathbb{Z}\}}$, where A_τ is the shift of the operator-valued function A by τ and the bar denotes closure in $C(\mathbb{Z},\mathfrak{A})$.

Example 4.15. *Consider the non-homogeneous linear functional-difference equation with finite delay (see, for example,* [15, 22]*)*

$$u(t+1) = A(t)u_t + f(t) \tag{28}$$

and corresponding homogeneous linear equation

$$u(t+1) = A(t+1)u_t, \tag{29}$$

where $A \in C(\mathbb{Z},\mathfrak{A})$ and $f \in C(\mathbb{Z},E)$.

Remark 4.16. *1. Denote by $\varphi(t,u,(A,f))$ the solution of equation (28) defined on $\mathbb{Z}_+$ (respectively, on $\mathbb{Z}$) with initial condition $\varphi(0,u,(A,f)) = u \in \mathcal{C}$. By $\tilde{\varphi}(t,u,(A,f))$ we will denote below the trajectory of equation (28), corresponding to the solution $\varphi(t,u,(A,f))$, i.e. the mapping from $\mathbb{Z}_+$ (respectively, $\mathbb{Z}$) into $\mathcal{C}$, defined by equality $\tilde{\varphi}(t,u,(A,f))(s) := \varphi(t+s,u,(A,f))$ for all $t \in \mathbb{Z}_+$ (respectively, $t \in \mathbb{Z}$) and $s \in [-r,0]$.*

2. Let $\varphi(t,u_i,(A,f))$ ($i = 1,2$) be two solutions of equation (28), then

$$\lim_{|t|\to\infty} |\varphi(t,u_1,(A,f)) - \varphi(t,u_2,(A,f))| = \lim_{|t|\to\infty} |\tilde{\varphi}(t,u_1,(A,f)) - \tilde{\varphi}(t,u_2,(A,f))|_{\mathcal{C}}.$$

Along with equation (28) (respectively, (29)) we consider the family of equations

$$v(t+1) = B(t)v_t + g(t) \tag{30}$$

(respectively,

$$v(t+1) = B(t)v_t \quad), \tag{31}$$

where $(B,g) \in H(A,f) := \overline{\{(B_\tau, f_\tau) \mid \tau \in \mathbb{Z}\}}$. *(respectively,* $B \in \overline{\{B_\tau \mid \tau \in \mathbb{Z}\}}$ $:= H(A)$*). Let* $\tilde{\varphi}(t,v,(B,g))$ *(respectively,* $\tilde{\varphi}(t,v,B)$ *be the solution of equation (30) (respectively, (31)) satisfying the condition* $\tilde{\varphi}(0,v,(B,g)) = v$ *(respectively,* $\tilde{\varphi}(0,v,B) = v$*) and defined for all* $t \geq 0$. *Let* $Y := H(A,f)$ *and denote the group dynamical system of shifts on* $H(A,f)$ *by* $(Y,\mathbb{Z},\sigma)$. *Let* $X := \mathcal{C} \times Y$ *and let* $\pi := (\varphi,\sigma)$ *be the dynamical system on* X *defined by the equality* $\pi(\tau,(v,(B,g))) := (\tilde{\varphi}(\tau,v,(B,g)), B_\tau, g_\tau)$. *The semi-group non-autonomous system* $\langle (X,\mathbb{Z}_+,\pi), (Y,\mathbb{Z},\sigma), h\rangle$ $(h := pr_2 : X \to Y)$ *is generated by equation (28).*

Lemma 4.17. *Let* $\varphi(n,u,(A,f))$ *be a solution of equation (28) which is relatively compact on* $\mathbb{R}_+$*, and let* $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z},\sigma),h\rangle$ *be a non-autonomous dynamical system generated by equation (28). Then, the set*

$$H(u,(A,f)) := \overline{\{(\tilde{\varphi}(\tau,u,(A,f)),(A_\tau,f_\tau)) \mid \tau \geq 0\}}$$

is conditionally compact with respect to (X,h,Y).

Proof. This statement is obvious. □

A solution $\varphi \in C(\mathbb{Z},E)$ of equation (28) is said to be *compatible by the character of recurrence* if $\mathfrak{N}_{(A,f)} \subseteq \mathfrak{N}_\varphi$, where $\mathfrak{N}_{(A,f)} := \{\{t_k\} \subset \mathbb{Z} \mid (A_{t_k}, f_{t_k}) \to (A,f)\}$ (respectively, $\mathfrak{N}_\varphi := \{\{t_k\} \subset \mathbb{Z} \mid \varphi_{t_k} \to \varphi\}$).

Theorem 4.18. *Let* $(A,f) \in C(\mathbb{Z},\mathfrak{A}) \times C(\mathbb{Z},E)$ *be Poisson stable. Suppose that the following conditions hold:*

- (i) *equation (28) admits a relatively compact on* $\mathbb{Z}_+$ *solution* $\varphi(t,u_0,(A,f))$*;*
- (ii) *all the relatively compact on* $\mathbb{Z}$ *solutions of equation (29) tends to zero as the time tends to* ∞*, i.e.* $\lim\limits_{|t|\to+\infty} |\varphi(t,u,A)| = 0$ *if* $\varphi(n,u,A)$ *is relatively compact on* $\mathbb{Z}$.

Then, equation (28) has a unique compatible solution $\varphi(t,\bar{u},(A,f))$.

Proof. First of all, we will prove that under the conditions of the theorem, equation (28) admits, at most, a compatible solution. If we suppose that it is not true, then there are at least two compatible solutions $\varphi(t,u_i,(A,f))$ (i=1,2 and $u_1 \neq u_2$) defined and bounded on $\mathbb{Z}$. Since (A,f) is Poisson stable, then $\psi(t) := \varphi(t,u_1,(A,f)) - \varphi(t,u_2,(A,f))$ $(t \in \mathbb{Z})$ is also Poisson stable. On the other hand, $\psi(t) = \varphi(t,u_1-u_2,A)$ is a relatively compact on $\mathbb{Z}$ solution of equation (29) and, consequently, $\lim\limits_{|t|\to+\infty} |\psi(t)| = 0$. From the last equality and the Poisson stability of ψ we obtain $\psi(t) = 0$ for all $t \in \mathbb{Z}$. In particular, $u_1 - u_2 = \psi(0) = 0$. This contradiction proves our statement.

Now we will prove that equation (28) admits at least one compatible solution. Denote by $\langle (X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z},\sigma),h\rangle$ the non-autonomous dynamical system generated by equation (28) (see Example 4.15). By Lemma 4.17, under the conditions of the theorem, the positively invariant set $H^+(x_0) \subset X$ (where $x_0 := (u_0,(A,f)) \in X :=$

$C \times H(A,f)$ and $H^+(x_0) := \overline{\{\pi(t,x_0) \mid t \in \mathbb{Z}_+\}}$) is conditionally compact. Let now $x_1, x_2 \in H^+(x_0) \cap X_{(A,f)}$, where $X_{(A,f)} := \mathcal{C} \times \{(A,f)\}$ (i.e. $x_i = (u_i,(A,f))$ and $u_i \in \mathcal{C}$ (i=1,2)), then

$$\lim_{|t|\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = \lim_{|t|\to+\infty} |\varphi(t,u_1,(A,f)) - \varphi(t,u_2,(A,f))|_{\mathcal{C}} = 0.$$

Now to finish the proof it is sufficient to refer to Theorem 3.8 and Corollary 3.9. □

Corollary 4.19. *Under the conditions of Theorem 4.18, if $(A,f) \in C(\mathbb{Z},\mathfrak{A}) \times C(\mathbb{Z},E)$ is τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then equation (28) admits a unique τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable) solution.*

Proof. This statement follows from Theorem 4.18 and Corollary 3.10. □

Corollary 4.20. *Under the conditions of Theorem 4.18, if $(A,f) \in C(\mathbb{Z},\mathfrak{A}) \times C(\mathbb{Z},E)$ is almost automorphic, then equation (28) admits a unique almost automorphic solution.*

Proof. Since the function $\varphi(t,\bar{u},(A,f))$ is relatively compact on $\mathbb{Z}$, then $\bar{\varphi} := \varphi(\cdot,\bar{u},(A,f)) \in C(\mathbb{Z},E)$ is a Lagrange stable point of dynamical system $(C(\mathbb{Z},),\mathbb{Z},\sigma)$. On the other hand, by Corollary 4.5 the function $\bar{\varphi}$ is Levitan almost periodic and, consequently, it is almost automorphic. □

Corollary 4.21. *Under the conditions of Theorem 4.18, if $(A,f) \in C(\mathbb{Z},\mathfrak{A}) \times C(\mathbb{Z},E)$ is Bohr almost periodic, then equation (28) admits a unique almost automorphic solution.*

Proof. This statement follows from Corollary 4.20 because every Bohr almost periodic function is almost automorphic. □

A solution $\varphi \in C(\mathbb{Z},E)$ of equation (28) is called (see [17], [19]) *uniformly compatible by the character of recurrence* if $\mathfrak{M}_{(A,f)} \subseteq \mathfrak{M}_\varphi$, where $\mathfrak{M}_{(A,f)} := \{\{t_k\} \subset Z \mid$ such that the sequence $\{(A_{t_k}, f_{t_k})\}$ is convergent$\}$ (respectively, $\mathfrak{M}_\varphi := \{\{t_k\} \subset Z \mid$ such that the sequence $\{\varphi_{t_k}\}$ is convergent$\}$).

Theorem 4.22. *Let $(A,f) \in C(\mathbb{Z},\mathfrak{A}) \times C(\mathbb{Z},E)$ be recurrent. Suppose that the following conditions hold:*

- (i) *equation (28) admits a solution $\varphi(t,u_0,(A,f))$ relatively compact on $\mathbb{Z}_+$;*
- (ii) *for all $B \in H(A)$, the relatively compact on $\mathbb{Z}$ solutions of equation (31) tend to zero as the time goes to ∞, i.e. $\lim_{|t|\to+\infty} |\varphi(t,u,B)| = 0$ provided that $\varphi(t,u,B)$ is relatively compact on $\mathbb{Z}$.*

Then, equation (17) has a unique uniformly compatible solution $\varphi(t,\bar{u},(A,f))$.

Proof. Note that, under the conditions of Theorem, equation (28) admits, at most, a uniformly compatible solution. Indeed, observe that every uniformly compatible solution is compatible and, on the other hand, by Theorem 4.18, equation (28) admits, at most, a compatible solution.

We will now prove that, under the conditions of the theorem, equation (28) admits, at least, a uniformly compatible solution. Indeed, since the function $\varphi(t, u_0, (A, f))$ is relatively compact on $\mathbb{Z}_+$, then $\tilde{\varphi}(\mathbb{Z}_+, u_0, (A, f))$ is relatively compact in $\mathcal{C}$. Denote by $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}, \sigma), h\rangle$ the semi-group non-autonomous dynamical system generated by equation (28). Under the conditions of the theorem, the positively invariant set $H^+(x_0) \subset X$ (where $x_0 := (u_0, (A, f)) \in X$ and $H^+(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{Z}_+\}}$) is compact. Let now $x_1, x_2 \in H^+(x_0) \cap X_{(B,g)}$, where $(B, g) \in H(A, f)$ and $X_{(B,g)} := \mathcal{C} \times \{(B, g)\}$ (i.e. $x_i = (u_i, (B, g))$ and $u_i \in \mathcal{C}$ (i=1,2)), then

$$\lim_{|t|\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = \lim_{|t|\to+\infty} |\varphi(t, u_1, (B, g)) - \varphi(t, u_2, (B, g))|_{\mathcal{C}} = 0.$$

The proof follows from Theorem 3.11 and Corollary 3.12. □

Corollary 4.23. *Under the conditions of Theorem 4.22, if $(A, f) \in C(\mathbb{Z}, \mathfrak{A}) \times C(\mathbb{Z}, E)$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (28) admits a unique τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.*

Proof. This statement follows from Theorem 4.22 and Corollary 3.13. □

Remark 4.24. *1. We point out that analogous results to Theorems 4.18 and 4.22 were established in our previous paper* [5] *for ordinary difference equations (i.e. $r = 0$) under the following condition (C): the operator $B(t)$ is invertible for all $B \in H(A)$ and $t \in \mathbb{Z}$.*
2. As a consequence of Theorems 4.18, 4.22 and Corollaries 4.19–4.21, 4.23 it follows that the results from Subsection 4.2 in [5] *also hold without imposing condition (C).*

4.4. **Linear partial differential equations.** As our final application, let us consider the differential equation

$$x' = A(t)x + f(t) \tag{32}$$

with unbounded coefficients. Let $A \in C(\mathbb{R}, \Lambda)$, where Λ is a complete metric space of linear closed operators that act on the Banach space E (for example, $A := \{A_0 + B \mid B \in [E]\}$, where A_0 is a closed operator that acts on E, and by $[E]$ we denote, as in [5], the Banach space of all bounded linear operators acting on the Banach space E with the operator norm). Consider the H-class

$$y' = B(t)y + g(t) \tag{33}$$

of equation (32), where $(B, g) \in H(A, f)$. We assume that the following conditions are fulfilled for equation (32) and its H-class:

(i) for any $u \in E$ and $(B, g) \in H(A, f)$, equation (33) has precisely one solution $\varphi(0, u, B, g)$ which is defined on $\mathbb{R}_+$, and satisfies $\varphi(0, u, (B, g)) = u$;
(ii) the map $\varphi : (t, v, (B, g)) \to \varphi(t, v, (B, g))$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}, \Lambda) \times C(\mathbb{R}, E)$.

Below we consider a class of differential equation which satisfies conditions (i) and (ii).

Example 4.25. *Consider the differential equation*

$$u' = (A_0 + A(t))u + f(t), \tag{34}$$

where A_0 is the generator of a C_0–semigroup $\{S(t)\}_{t\ge 0}$, $A \in C(\mathbb{R}, [E])$ and $f \in C(\mathbb{R}, E)$.

The results of [9, 12, 13] *imply that equation (34) satisfies conditions (i) and (ii).*

Consider the corresponding homogeneous equation

$$u' = (A_0 + A(t))u, \tag{35}$$

where $A \in C(\mathbb{R}, [E])$. Along with equations (34) and (35), we consider also the H-class of equation (34) (respectively, (35)), that is, the family of equations

$$v' = (A_0 + B(t))v + g(t), \tag{36}$$

(respectively,

$$v' = (A_0 + B(t))v \quad) \tag{37}$$

with $(B, g) \in H(A, f) := \overline{\{(A_\tau, f_\tau) \mid \tau \in \mathbb{R}\}}$ (respectively, $B \in H(A)$), $A_\tau(t) = A(t+\tau)$, $f_\tau(t) := f(t+\tau)$ and $t \in \mathbb{R}$, where the bar denotes the closure in $C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ (respectively, $C(\mathbb{R}, [E])$). Let $\varphi(t, v, (B, g))$ (respectively, $\varphi(t, v, B)$) be the solution of equation (36) (respectively, (37)) that satisfies the condition $\varphi(0, v, (B, g)) = v$ (respectively, $\varphi(0, v, B) = v$).

We put $Y := H(A, f)$ and denote the dynamical system of shifts on $H(A, f)$ by $(Y, \mathbb{R}, \sigma)$. We put $X := E \times Y$ and define a dynamical system on X by setting $\pi(t, (v, B, g)) := (\varphi(t, v, (B, g)), B_t, g_t)$ for all $(v, (B, g)) \in E \times Y$ and $t \in \mathbb{R}_+$. Then $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ is a semi-group non-autonomous dynamical system, where $h := pr_2 : X \to Y$.

Applying the results of Sections 2–3 to this system, we obtain the following statements.

Theorem 4.26. *Let $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ be Poisson stable. Suppose that the following conditions hold:*

- (i) *$\varphi(t, u_0, (A_0, A, f))$ is a relatively compact on $\mathbb{R}_+$ solution of equation (34), i.e. the set $Q_{(u_0,(A_0,A,f))} := \overline{\varphi(\mathbb{R}_+, u_0, (A_0, A, f))}$ is compact in E;*
- (ii) *all relatively compact on $\mathbb{R}$ solutions of equation (35) tends to zero as the time tends to ∞, i.e. $\lim\limits_{|t|\to+\infty} |\varphi(t, u, A_0, A)| = 0$ if $\varphi(t, u, A_0, A)$ is relatively compact on $\mathbb{R}$ (this means that the set $\varphi(\mathbb{R}, u, A_0, A))$ is relatively compact in E).*

Then, equation (34) has a unique compatible solution $\varphi(t, \bar{u}, (A_0, A, f))$ with values in the compact $Q_{(u_0,(A_0,A,,f))}$.

Proof. Denote by $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ the semi-group non-autonomous dynamical system, generated by equation (34) (see Example 4.25). By Lemma 2.5, the positively invariant set $H^+(x_0) \subset X$ (where $x_0 := (u_0, (A_0, A, f)) \in X$ and $H^+(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{R}_+\}}$) is conditionally compact. Now, to finish the proof, it is sufficient to apply Theorem 3.8. $\square$

Corollary 4.27. *Under the conditions of Theorem 4.26, if* $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ *is* τ*-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then equation (34) admits a unique* τ*-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable) solution.*

Proof. This statement follows from Theorem 4.26 and Corollary 3.10. □

Theorem 4.28. *Let* $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ *be recurrent. Suppose that the following conditions hold:*

- (i) *the equation (34) admits a relatively compact on* $\mathbb{R}_+$ *solution* $\varphi(t, u_0, (A_0, A, f))$*;*
- (ii) *for all* $B \in H(A)$ *the relatively compact on* $\mathbb{R}$ *solutions of equation (37) tends to zero as the time tends to* ∞*, i.e.* $\lim\limits_{|t|\to+\infty} |\varphi(t, u, A_0, B)| = 0$ *if* $\varphi(t, u, A_0, B)$ *is relatively compact on* $\mathbb{R}$ *.*

Then, equation (34) has a unique uniformly compatible solution $\varphi(t, \bar{u}, (A_0, A, f))$ *with values in the compact* $Q_{(u_0,(A_0,A,f))}$*.*

Proof. Denote by $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ the semi-group non-autonomous dynamical system generated by equation (34). Under the conditions of the theorem, the positive invariant set $H^+(x_0) \subset X$ is compact. Now, it is sufficient to refer to Theorem 3.11. □

Corollary 4.29. *Under the conditions of Theorem 4.28, if* $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ *is* τ*-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (34) admits a unique* τ*-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.*

Proof. This statement follows from Theorem 4.28 and Corollary 3.13. □

Remark 4.30. *It is well known (see, for example,* [9, 12, 13]*) that some classes of partial differential equations (PDEs) can be written in the form (34) and, consequently, Theorems 4.26 and 4.28 and Corollaries 4.27 and 4.29 are applicable to these PDEs.*

Remark 4.31. *1. Note that in* [5] *(Part I of our investigation in this field) one can find two examples which illustrate some of our general results:*

- (i) *a scalar Bohr almost periodic linear homogeneous equation for which all solutions are bounded on* $\mathbb{R}$ *and converge to* 0 *as* $|t| \to +\infty$ *(see Example 4.8* [5]*);*
- (ii) *a bi-dimensional Levitan almost periodic linear homogeneous system for which all solutions are bounded on* $\mathbb{R}$ *and converge to* 0 *as* $|t| \to +\infty$ *(see Example 4.9* [5]*).*

2. It is easy to construct a bi-dimensional almost automorphic linear homogeneous system for which all solutions are bounded on $\mathbb{R}$ *and converge to* 0 *as* $|t| \to +\infty$ *with the slight modification of Example 4.9 from* [5].

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References

[1] B.R. Basit, A connection between the almost periodic functions of Levitan and almost automorphic functions, *Vestnik Moskov. Univ. Ser. I Mat. Meh.* 26 (1971), no. 4, 11–15.
[2] B. Basit, Les fonctions abstraites presques automorphiques et presque périodiques au sens de Levitan, et leurs différence, *Bull. Sci. Math. (2),* 101 (1977), no. 2, 131–148.
[3] S. Bochner, A new approach to almost periodicity, *Proc. Nat. Acad. Sci. U.S.A.* 48 (1962), 2039–2043.
[4] I.U. Bronsteyn, *Extensions of Minimal Transformation Group,* Noordhoff, 1979.
[5] T. Caraballo, D.N. Cheban, Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. I, *J. Differential Equations*, (2008), doi: 10.1016/j.jde.2008.04.001, in press.
[6] D.N. Cheban, Global pullback attractors of $\mathbb{C}$-analytic non-autonomous dynamical systems, *Stochastics and Dynamics*, 1 (2001), no. 4, 511-536.
[7] D.N. Cheban, *Global Attractors of Non-Autonomous Dissipative Dynamical Systems*, Interdisciplinary Mathematical Sciences 1, River Edge, NJ: World Scientific, 2004.
[8] D. N. Cheban, Levitan almost periodic and almost automorphic solutions of V-monotone differential equations, *Journal of Dynamics and Differential Equations,* (DOI:10.1007/s10884-008-9101-x, 2008), in press.
[9] C. Chicone, Y. Latushkin, *Evolution Semigroups in Dynamicals Systems and Differential Equations,* Amer. Math. Soc., Providence, RI, 1999.
[10] J. Egawa, A characterization of almost automorphic functions, *Proc. Japan Acad. Ser. A Math. Sci.* 61 (1985), no. 7, 203–206.
[11] J. K. Hale, *Theory of Functional-Differential Equations*, Springer-Verlag, New-York Heidelber Berlin, 1977.
[12] D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Lecture Notes in Mathematics, No.840, Springer-Verlag, New York 1981.
[13] B.M. Levitan, V.V. Zhikov, *Almost Periodic Functions and Differential Equations,* Cambridge Univ. Press, London, 1982.
[14] P. Milnes, Almost automorphic functions and totally bounded groups, *Rocky Mountain J. Math.* 7 (1977), no. 2, 231–250.
[15] Satoru Murakami, Representation of solutions of linear functional difference equations in phase space. *Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996). Nonlinear Anal.* 30 (1997), no. 2, 1153–1164.
[16] G.R. Sell, *Lectures on Topological Dynamics and Differential Equations,* Van Nostrand–Reinbold, London, 1971.
[17] B.A. Shcherbakov, *Topologic Dynamics and Poisson Stability of Solutions of Differential Equations*, Ştiinţa, Chişinău, 1972. (In Russian)
[18] B.A. Shcherbakov, The comparability of the motions of dynamical systems with regard to the nature of their recurrence, *Differential Equations* 11 (1975), no. 7, 1246–1255.
[19] B.A. Shcherbakov, *Poisson Stability of Motions of Dynamical Systems and Solutions of Differential Equations*, Ştiinţa, Chişinău, 1985. (In Russian)
[20] W. Shen, Y. Yi, Almost automorphic and almost periodic dynamics in skew-product semiflows, *Mem. Amer. Math. Soc.* 136 (1998), no. 647.

[21] C.S. Sibirsky, *Introduction to Topological Dynamics,* Noordhoff, Leyden, 1975.
[22] Y. Song, Periodic and almost periodic solutions of functional difference equations with finite delays. *Advances in Difference Equationss,* V.2007, ID 68023 (doi: 10.1155/2007/68023).

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