

# COMPARISON OF THEORETICAL COMPLEXITIES OF TWO METHODS FOR COMPUTING ANNIHILATING IDEALS OF POLYNOMIALS

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ABSTRACT. Let  $f_1, \dots, f_p$  be polynomials in  $\mathbf{C}[x_1, \dots, x_n]$  and let  $D = D_n$  be the  $n$ -th Weyl algebra. We provide upper bounds for the complexity of computing the annihilating ideal of  $f^s = f_1^{s_1} \cdots f_p^{s_p}$  in  $D[s] = D[s_1, \dots, s_p]$ . These bounds provide an initial explanation on the differences between the running times of the two methods known to obtain the so-called Bernstein-Sato ideals.

## 1. INTRODUCTION

Fix two integers  $n \geq 1, p \geq 1$  and two sets of variables  $(x_1, \dots, x_n)$  and  $(s_1, \dots, s_p)$ . Let us consider  $f_1, \dots, f_p \in \mathbf{C}[x] = \mathbf{C}[x_1, \dots, x_n]$  and let  $D = D_n$  be the  $n$ -th Weyl algebra. A polynomial  $b(s) \in \mathbf{C}[s] = \mathbf{C}[s_1, \dots, s_p]$  is said to be a *Bernstein-Sato polynomial associated to  $f_1, \dots, f_p$*  if the following functional equation holds for a certain  $P(s) \in D[s]$ :

$$b(s)f^s = P(s)f^{s+1},$$

where  $f^s = f_1^{s_1} \cdots f_p^{s_p}$  and  $\mathbf{1} = (1, \dots, 1)$ . These polynomials form an ideal called the Bernstein-Sato ideal, noted  $\mathcal{B}_f$  or simply  $\mathcal{B}$  if no confusion arises. Analogous functional equations with respect to vectors different from  $\mathbf{1}$  yield other versions of Bernstein-Sato ideals (see for example [Bahloul(2001)]).

In [Lichtin(1988)] it is proved that  $\mathcal{B}$  is not zero. This fact is a generalization of the classical proof of Bernstein ([Bernstein(1972)]) in the algebraic setting for the case  $p = 1$ , in which  $\mathcal{B}$  is generated by the so-called *Bernstein-Sato polynomial* noted  $b_f(s)$ . The analytical case was covered in [Björk(1973)] for  $p = 1$  and [Sabbah(1987a)] and [Sabbah(1987b)] for  $p > 1$  (an interesting new proof using the Gröbner fan has been given in [Bahloul(2005)]). The roots of  $b_f(s)$  encode important algebro-geometrical data (see [Malgrange(1974)], [Hamm(1975)] or [Budur-Saito(2003)] to mention only a few) and a complete understanding of all roots for a general  $f$  is open. The case  $p > 1$  seems to be much more complex and there are conjectures on the primary decomposition of  $\mathcal{B}$ , on the conditions over  $f$  for  $\mathcal{B}$  to be principal, etc. (see for example [Maynadier(1996)]).

Until [Oaku(1997)] there were no algorithms to find the Bernstein-Sato polynomial. Since then, alternative methods have been proposed to obtain  $\mathcal{B}$  in the general case (see [Oaku and Takayama(1999)], [Bahloul(2001)] and [Briançon and Maisonobe(2002)]). These methods have a feature in common: their first step is the computation of the *annihilating ideal* of  $f^s$  in  $D[s]$ ,  $\text{Ann}_{D[s]} f^s$ . In [Castro-Ucha(2004)] some experimental evidences were given in favor of the method of Briançon-Maisonobe (BM)

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*Key words and phrases.* Complexity, Poincaré-Birkhoff-Witt algebras, Bernstein-Sato ideals. All authors partially supported by MTM2004-01165 and FQM-333.

to compute  $\text{Ann}_{D[s]}f^s$  with respect to the method of Oaku-Takayama (OT), but no clues about which facts support this advantage were provided.

Our work is a first step in order to compare theoretically both methods. We give upper bounds for the complexity of computing  $\text{Ann}_{D[s]}f^s$ , the previous requirement for both algorithms. To obtain these bounds we use the techniques and results of [Grigoriev(1990)] on the complexity of solving systems of linear equations over rings of differential operators. These extend the classical polynomial case treated in [Seidenberg(1974)]. In particular, we show that Grigoriev's construction can not be directly generalized to the algebra proposed by Briançon-Maisonobe. We prove that the complexity of computing  $\text{Ann}_{D[s]}f^s$  using the method BM is that of the calculation of a Gröbner basis in the  $n$ -th Weyl algebra with some extra  $p$  commutative variables, so  $2n + p$  variables at most. On the other hand, in the case of the method OT the calculation of such a basis is made in a  $(n + p)$ -th Weyl algebra with some extra  $2p$  variables, so  $2n + 4p$  variables altogether.

It is an open problem to know whether the bound proposed in this work is reached *à la* Mayr-Meyer ([Mayr-Meyer(1982)]), that is to say, whether an example with this worst complexity can be explicitly obtained. Such an example would mean a complete answer to the question of which is the complexity of computing  $\text{Ann}_{D[s]}f^s$ , proposed by Professor N. Takayama.

## 2. PRELIMINARIES

In this section we just remind briefly some details of the methods of Briançon-Maisonobe and Oaku-Takayama.

**2.1. Method of Briançon-Maisonobe.** In this case the computations are made in the non-commutative algebra<sup>1</sup>

$$R = D[s, t] = D[s_1, \dots, s_p, t_1, \dots, t_p],$$

an extension of the  $n$ -th Weyl algebra  $D$  in which the new variables  $s, t$  satisfy the relations  $[s_i, t_j] = \delta_{ij}t_i$ . It is a *Poincaré-Birkhoff-Witt (PBW) algebra*:

**Definition 1.** A PBW algebra  $R$  over a ring  $k$  is an associative algebra generated by finitely many elements  $x_1, \dots, x_n$  verifying the relations

$$Q = \{x_j x_i = q_{ji} x_i x_j + p_{ji}, 1 \leq i < j \leq n\},$$

where each  $p_{ji}$  is a finite  $k$ -linear combination of *standard* terms  $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  and each  $q_{ji} \in k$  verifying the two following conditions:

- (1) There is an *admissible*<sup>2</sup> ordering  $\prec$  on  $\mathbf{N}^n$  such that  $\exp(p_{ji}) \prec \exp(x_j x_i)$  for every  $1 \leq i < j \leq n$ .
- (2) The standard terms  $\mathbf{x}^\alpha$ , with  $\alpha \in \mathbf{N}^n$ , form a  $k$ -basis of  $R$  as a vector space.

It is possible to compute Gröbner bases in PBW algebras. The book [Bueso et al.(2003)] is a good introduction to the subject of effective calculus in this fairly general family.

The following algorithm computes  $\mathcal{B}$ , starting from

$$I := \text{Ann}_R(f^s) = \langle s_j + f_j t_j, \partial_i + \sum_j \frac{\partial f_j}{\partial x_i} t_j, 1 \leq i \leq n, 1 \leq j \leq p \rangle.$$

<sup>1</sup>It is, in fact, the ring introduced in classical works by Malgrange and Kashiwara for  $p = 1$ .

<sup>2</sup>Here admissible means a total ordering among the elements of  $\mathbf{N}^n$  with  $\mathbf{0}$  as smallest element.

- Algorithm 2.1.* (1) Obtain  $J = \text{Ann}_{D_n[s]}f^s = \langle G_1 \cap D_n[s] \rangle$  where  $G_1$  is a Gröbner basis of  $I$  with respect to any term ordering where variables  $t_j$  are greater than the others (that is, an *elimination ordering for the  $t_j$* .)  
 (2)  $\mathcal{B} = (\langle G_2 \rangle + \langle f_1, \dots, f_p \rangle) \cap \mathbf{C}[s]$ , where  $G_2$  is a Gröbner basis of  $J$  with respect to any term ordering with  $x_i, \partial_j$  greater than  $s_l$ , for all  $i, j, l$ .

**2.2. Method of Oaku-Takayama.** All the computations are made in Weyl algebras. More precisely, we start from

$$I' = \langle t_j - f_j, \sum_{j=1}^p \frac{\partial f_j}{\partial x_i} \partial_{t_j} + \partial_i, i = 1, \dots, n, j = 1, \dots, p \rangle$$

- Algorithm 2.2.* (1) Obtain  $J' = I' \cap \mathbf{C}[t_1 \partial_{t_1}, \dots, t_n \partial_{t_n}] \langle x, \partial_x \rangle$ .  
 (2)  $J = \text{Ann}_{D_n[s]}(f^s) = J''$ , where  $J''$  denotes the ideal generated by the generators of  $J'$  after replacing each  $t_i \partial_{t_i}$  by  $-s_i - 1$ .  
 (3)  $\mathcal{B} = (\langle G_2 \rangle + \langle f_1, \dots, f_p \rangle) \cap \mathbf{C}[s]$ , where  $G_2$  is a Gröbner basis of  $J$  with respect to any term ordering with  $x_i, \partial_j$  greater than  $s_l$ , for all  $i, j, l$ .

*Remark 1.* The second step above is, as in Algorithm 2.1, the elimination of all the variables but  $(s_1, \dots, s_p)$ . Often the bottleneck to obtain the Bernstein-Sato ideal is this step. As far as we know, the example for  $p = 2$  with  $f_1 = x^2 + y^3, f_2 = x^3 + y^2$  is intractable for the available computer algebra systems.

The computation of

$$I' \cap \mathbf{C}[t_1 \partial_{t_1}, \dots, t_n \partial_{t_n}] \langle x, \partial_x \rangle$$

uses  $2n + 4p$  variables, as new variables  $u_j, v_j$  for  $1 \leq j \leq p$  are introduced. More precisely, the main calculation is an elimination of these new variables for the ideal

$$\langle t_j - u_j f_j, \sum_{j=1}^p \frac{\partial f_j}{\partial x_i} u_j \partial_{t_j} + \partial_i, 1 - u_j v_j, 1 \leq i \leq n, 1 \leq j \leq p, \rangle.$$

### 3. COMPLEXITY

In [Grigoriev(1990)] a bound for the degree of the solutions of a general system of linear equations over the Weyl algebra is given, with a procedure somewhat similar to the one of [Seidenberg(1974)]. In this section we shall see how much of the work of Grigoriev is applicable to our PBW algebra  $R$  of 2.1.

The construction has two different steps. In the first, the given system is reduced to another system in a diagonal form. In the second, it is shown how to normalize the new system in order to eliminate, successively, the variables.

We need a technical lemma to reduce the system to a diagonal form. This lemma comes from Grigoriev's paper (see [Grigoriev(1990), Lemma 1]), but we will write it in a more general way. Here  $\text{deg}$  means the *total degree* of a term, that is, the sum of the exponents of all of its variables.

**Lemma 1.** *Let  $A$  be a  $(m-1) \times m$  matrix with entries in a Poincaré-Birkhoff-Witt algebra  $S$  with a basis of  $p$  elements. If  $\text{deg}(a_{ij}) \leq d$ , there exists a nonzero-vector  $f = (f_1, \dots, f_m) \in S^m$  such that  $Af = 0$  and  $\text{deg}(f) \leq 2p(m-1)d = N$ .*

*Proof.* Consider the linear space  $T \subset S^m$  of vectors  $c = (c_1, \dots, c_m) \in S^m$  such that  $\text{deg}(c) \leq N$ . We have  $\dim(T) = \binom{N+p}{p} m$ . For any vector  $c \in T$  it

is clear that  $\deg(Ac) \leq N + d$ . If we consider now the vector space  $\gamma$  of vectors  $e = (e_1, \dots, e_{m-1}) \in S^{m-1}$  such that  $\deg(e) \leq N + d$ , we have  $\dim(\gamma) = \binom{N+d+p}{p} (m-1)$ . We prove that  $\dim(\gamma) < \dim(T)$ .

$$\begin{aligned} & \binom{N+d+p}{p} / \binom{N+p}{p} = \\ & \frac{N+d+p}{N+p} \frac{N+d+p-1}{N+p-1} \dots \frac{N+d+1}{N+1} \leq \left( \frac{N+d+1}{N+1} \right)^p. \end{aligned}$$

It is enough to see  $\left( \frac{N+d+1}{N+1} \right)^p < 1 + \frac{1}{m-1}$ . This inequality follows from:

$$\begin{aligned} \left( 1 + \frac{1}{m-1} \right)^{\frac{1}{p}} &> 1 + \frac{1}{p(m-1)} + \frac{1}{2} \frac{1}{p} \left( \frac{1}{p} - 1 \right) \left( \frac{1}{m-1} \right)^2 > \\ &> 1 + \frac{1}{2p(m-1)} > 1 + \frac{d}{N+1} \end{aligned}$$

□

If we work in a noetherian domain (not necessarily commutative), we can always define the *rank* of a finite module as in [Stafford(1978)]. Given a square matrix in a Poincaré-Birkhoff-Witt algebra we say that it is *non-singular* if it has maximal rank. In this case we can obtain a left quasi-inverse with the previous lemma:

**Lemma 2.** *Given a  $m \times m$  non-singular matrix  $B$  over a PBW algebra  $S$  as in Lemma 1, it has a left quasi-inverse matrix  $G$  over  $S$ , such that  $\deg(G) \leq N$ .*

*Proof.* There is no vector  $b \neq 0$ , in  $\mathcal{R}^m$  such that  $bB = 0$ . If we consider the matrix  $B^{(i)}$  obtained from  $B$  deleting its  $i$ -th column, using Lemma 1 we obtain a vector  $g_i \neq 0$  such that  $g_i B^{(i)} = 0$  and  $\deg(g_i) \leq N$ , so the matrix  $G$  which has  $g_i$  as its  $i$ -th row, for  $i = 1, \dots, m$ , is a left quasi-inverse of  $B$ . □

**Lemma 3.** *Given a system of linear equations over a PBW algebra defined by a  $m \times s$  matrix  $A$  of rank  $r$  with its elements  $\deg(a_{ij}) \leq d$ , we can always construct a matrix  $C$  that defines an equivalent system, and such that*

$$(1) \quad CA = \begin{pmatrix} C_1 & 0 \\ C_2 & E \end{pmatrix} A = \left( \begin{array}{ccc|c} a_1 & & 0 & \\ & \ddots & & * \\ 0 & & a_r & \\ \hline & & 0 & 0 \end{array} \right)$$

where  $E$  is the identity matrix.

*Proof.*  $C_1$  is the left quasi-inverse of the submatrix of  $A$  of maximal rank  $r$  (after reordering the rows or columns of  $A$  if it were necessary).  $C_2$  is constructed with the requirement on the left lower corner to be zero. The right lower corner is zero by the definition of rank. □

Thanks to this lemma, we can assume that our system is equivalent to a system in diagonal form:

$$a_k V_k + \sum_{r+1 \leq l \leq s} a_{k,l} V_l = b_k, \quad 1 \leq k \leq r, \quad \deg(a_k), \deg(a_{k,l}), \deg(b_k) \leq 2pmd.$$

Once the system is in diagonal form, we need to normalize it. To do this, we construct some syzygies, applying Lemma 1 to the submatrix of the first  $r$  columns and the column  $l > r$ . There always exist  $h^{(l)}, h_1^{(l)}, \dots, h_r^{(l)}$  such that:

$$a_k h_k^{(l)} + a_{k,l} h^{(l)} = 0, \quad 1 \leq k \leq r \quad \deg(h^{(l)}), \deg(h_i^{(l)}) \leq 4p^2 m^2 d$$

The result that gives the normalization in the Weyl algebra is the following one:

**Lemma 4** ([Grigoriev(1990)], Lemma 4). *Given  $g_1, \dots, g_t \in D$  a family of elements, there is a nonsingular linear transformation of  $2n$ -dimensional space with basis  $x_1, \dots, x_n, \partial_1, \dots, \partial_n$  under which:*

$$\begin{aligned} x_i &\rightarrow \Gamma_{x_i} = \sum_{j=1}^n \gamma_{i,j}^{(1,1)} x_j + \sum_{j=1}^n \gamma_{i,j}^{(1,2)} \partial_j; \\ \partial_i &\rightarrow \Gamma_{\partial_i} = \sum_{j=1}^n \gamma_{i,j}^{(2,1)} x_j + \sum_{j=1}^n \gamma_{i,j}^{(2,2)} \partial_j \end{aligned}$$

such that the following relations hold:

$$\Gamma_{x_i} \Gamma_{\partial_i} = \Gamma_{\partial_i} \Gamma_{x_i} - 1; \quad \Gamma_{x_i} \Gamma_{x_j} = \Gamma_{x_j} \Gamma_{x_i};$$

$$\Gamma_{\partial_i} \Gamma_{\partial_j} = \Gamma_{\partial_j} \Gamma_{\partial_i}; \quad \Gamma_{\partial_i} \Gamma_{x_j} = \Gamma_{x_j} \Gamma_{\partial_i}, \quad i \neq j,$$

and if we denote by  $\Gamma_{g_i}$  the transformed of  $g_i$  with the indicated linear transformation, we have  $\Gamma_{g_i} = \partial_n^{\deg(g_i)} + \widetilde{\Gamma}_{g_i}$ .

*Remark 2.* The main fact in the proof of Lemma 4 is that the matrices of the linear transformations defined by the relations in the Weyl algebra are a transitive group. Let us see why we can not assure the existence of such a normalization lemma to every PBW algebra.

If we consider the PBW algebra defined by Briancon and Maisonobe for  $p = 1$ , that is

$$R = \mathbf{C}[s, t, x_1, \dots, x_n, \partial_1, \dots, \partial_n],$$

a general linear transformation as the one appearing in Lemma 4 has the form:

$$\begin{aligned} s &\rightarrow \Gamma_s = \alpha_1 s + \beta_1 t + \sum_{j=1}^n \gamma_j^{(s,1)} x_j + \sum_{j=1}^n \gamma_j^{(s,2)} \partial_j \\ t &\rightarrow \Gamma_t = \alpha_2 s + \beta_2 t + \sum_{j=1}^n \gamma_j^{(t,1)} x_j + \sum_{j=1}^n \gamma_j^{(t,2)} \partial_j \\ x_i &\rightarrow \Gamma_{x_i} = \alpha_i^{(1)} s + \beta_i^{(1)} t + \sum_{j=1}^n \gamma_{i,j}^{(1,1)} x_j + \sum_{j=1}^n \gamma_{i,j}^{(1,2)} \partial_j \\ \partial_i &\rightarrow \Gamma_{\partial_i} = \alpha_i^{(2)} s + \beta_i^{(2)} t + \sum_{j=1}^n \gamma_{i,j}^{(2,1)} x_j + \sum_{j=1}^n \gamma_{i,j}^{(2,2)} \partial_j \end{aligned}$$

and it has to verify the following relations:

$$\begin{aligned} (1) \Gamma_s \Gamma_t &= \Gamma_t \Gamma_s + \Gamma_t; & (2) \Gamma_s \Gamma_{x_i} &= \Gamma_{x_i} \Gamma_s; & (3) \Gamma_s \Gamma_{\partial_i} &= \Gamma_{\partial_i} \Gamma_s; \\ (4) \Gamma_t \Gamma_{x_i} &= \Gamma_{x_i} \Gamma_t; & (5) \Gamma_t \Gamma_{\partial_i} &= \Gamma_{\partial_i} \Gamma_t; & (6) \Gamma_{x_i} \Gamma_{\partial_i} &= \Gamma_{\partial_i} \Gamma_{x_i} - 1; \\ (7) \Gamma_{x_i} \Gamma_{x_j} &= \Gamma_{x_j} \Gamma_{x_i}; & (8) \Gamma_{\partial_i} \Gamma_{\partial_j} &= \Gamma_{\partial_j} \Gamma_{\partial_i}; & (9) \Gamma_{x_i} \Gamma_{\partial_j} &= \Gamma_{\partial_j} \Gamma_{x_i} \end{aligned}$$

From relation (1), we obtain  $\alpha_2 = \gamma_j^{(t,1)} = \gamma_j^{(t,2)} = 0$  for all  $j$ , so  $\Gamma_t = \beta_2 t$ . The transformation must be nonsingular, so we must have  $\beta_2 \neq 0$ , and again using (1) we deduce that  $\alpha_1 = 1$ . Using (4), we obtain that  $\alpha_i^{(1)} = 0$  for all  $i$ . This, together with (5) implies that  $\alpha_i^{(2)} = 0$  for all  $i$ .

From relation (2) ( $\Gamma_s$  commutes with  $\Gamma_{x_i}$ ) we have  $\beta_i^{(1)} = 0$ , and relation (3) gives  $\beta_i^{(2)} = 0$ . Due to relations (6) to (9) (between  $\Gamma_{x_i}$  and  $\Gamma_{\partial_j}$ ) we have that the submatrix

$$\begin{pmatrix} \gamma_{i,j}^{(1,1)} & \gamma_{i,j}^{(1,2)} \\ \gamma_{i,j}^{(2,1)} & \gamma_{i,j}^{(2,2)} \end{pmatrix}$$

verifies the relations of Lemma 4, and in addition, from the relations with  $\Gamma_s$  it verifies

$$\sum \gamma_i^{(s,1)} \gamma_{i,i}^{(1,2)} = \sum \gamma_i^{(s,2)} \gamma_{i,i}^{(1,1)} \quad \sum \gamma_i^{(s,1)} \gamma_{i,i}^{(2,2)} = \sum \gamma_i^{(s,2)} \gamma_{i,i}^{(2,1)}.$$

So it is clear that we can not normalize with respect to the variables in  $R$ . Thus we can not repeat the second step of the process to a general PBW algebra in the way that it appears in [Grigoriev(1990)].

It is an open problem to obtain a general bound for the solutions of a general linear system over any PBW algebra or, at least, to give such a bound for  $R$ . We give up on this general problem at this point: with the aim of obtaining a bound for the complexity of the annihilating ideal of  $f^s$ , we will treat only the particular case of one equation of the type produced by the definition of the ideal  $I$  in section 2.1 or  $I'$  in section 2.2. In both cases we want to measure the complexity of computing Gröbner bases (in different rings) and we will do it by considering the equivalent problem of computing the *syzygies* of the generators of our respective ideals.

*Remark 3.* In OT algorithm the calculations are computed in a Weyl algebra of  $2n + 4p$  variables, or more precisely in a commutative polynomial ring with  $n + 3p$ ,  $(x, u, v, t)$  commutative variables extended with  $n + p$ ,  $(\partial_x, \partial_t)$  “differential” variables. Let us denote by  $A$  this algebra. The complexity of computing the annihilating ideal of  $f^s$  is bounded by the complexity of computing a Gröbner basis in  $A$ .

Recall that the complexity in the Weyl algebra is given by the following theorem:

**Theorem 1** (Th. 6,[Grigoriev(1990)]). *Given a solvable system in the Weyl algebra  $D_n$ :*

$$\sum_{1 \leq l \leq s} u_{k,l} V_l = w_k, \quad 1 \leq k \leq m$$

*with  $\deg(u_{k,l}), \deg(w_k) \leq d$ . There exists a solution with  $\deg(V_l) < (md)^{2^{O(n)}}$*

As we said before in the Briançon-Maisonobe ring  $R$  we can not construct a similar algorithm to bound the degree of a solution for a system in general. But in our very special case, our problem is equivalent to computing the solutions of the equation:

$$(s_1 + f_1 t_1) V_1 + \dots + (s_p + f_p t_p) V_p + (\partial_1 + \sum_j \frac{\partial f_j}{\partial x_1} t_j) V_{p+1} + \dots + (\partial_n + \sum_j \frac{\partial f_j}{\partial x_n} t_j) V_{p+n} = 0$$

To simplify notation we write the precedent equation as  $\sum_l Q_l V_l = 0$ .

**Theorem 2.** *Given  $f = (f_1, \dots, f_p)$ , the computation of the annihilating ideal of  $f^s$  in the Briançon-Maisonobe algebra  $R = D[s_1, \dots, s_p, t_1, \dots, t_p]$  can be reduced to the computation of the syzygies of the generators  $\partial_i + \sum_j \frac{\partial f_j}{\partial x_i} t_j$  in the Weyl algebra  $D[t_1, \dots, t_p]$ .*

*Proof.* Trying to repeat Grigoriev's ideas, the first step is the reduction of the system to one in diagonal form. Due to the fact that we have only one equation this step is done. Then, we need to compute  $h_1^{(l)}, h^{(l)}$  for  $2 \leq l \leq n+p$  such that:

$$\begin{aligned} (s_1 + f_1 t_1) h_1^{(2)} + (s_2 + f_2 t_2) h^{(2)} &= 0 \\ \vdots & \\ (s_1 + f_1 t_1) h_1^{(p)} + (s_p + f_p t_p) h^{(p)} &= 0 \\ (s_1 + f_1 t_1) h_1^{(p+1)} + (\partial_1 + \sum_j \frac{\partial f_j}{\partial x_1} t_j) h^{(p+1)} &= 0 \\ \vdots & \\ (s_1 + f_1 t_1) h_1^{(p+n)} + (\partial_n + \sum_j \frac{\partial f_j}{\partial x_n} t_j) h^{(p+n)} &= 0 \end{aligned}$$

It is easy to see that

$$\begin{aligned} [s_i + f_i t_i, s_j + f_j t_j] &= 0 \\ [s_i + f_i t_i, \partial_j + \sum_l \frac{\partial f_l}{\partial x_j} t_l] &= s_i (\sum_l \frac{\partial f_l}{\partial x_j} t_l) + f_i t_i \partial_j - \partial_j f_i t_i - (\sum_l \frac{\partial f_l}{\partial x_j} t_l) s_i = \\ &= t_i s_i \frac{\partial f_i}{\partial x_j} + t_i \frac{\partial f_i}{\partial x_j} + \sum_{l \neq i} t_l s_i \frac{\partial f_l}{\partial x_j} + t_i f_i \partial_j - t_i f_i \partial_j - t_i \frac{\partial f_i}{\partial x_j} - \sum_l \frac{\partial f_l}{\partial x_j} t_l s_i = 0 \end{aligned}$$

and we obtain  $h^{(l)} = s_1 + f_1 t_1$  for all  $l \geq 2$ .

These are the elements we need to normalize, and they are almost in normal form with respect to the variable  $s_1$ . This form is required to make the division of the solutions  $V_l$ ,  $l \geq 2$  by  $h^{(l)}$  with respect to a lexicographical ordering with leading term  $s_1$ . We obtain a remainder  $\bar{V}_l$  such that  $\deg_{s_1}(\bar{V}_l) < \deg_{s_1}(h^{(l)}) = 1$ , so  $s_1$  does not appear in  $\bar{V}_l$ . So  $V_l = h^{(l)} \bar{V}_l + \bar{V}_l$ , and adding the relation  $Q_1 h_1^{(l)} + Q_l h^{(l)} = 0$  multiplied by  $-\bar{V}_l$  to our initial equation, we obtain:

$$Q_1 \bar{V}_1 + Q_2 \bar{V}_2 + \dots + Q_{n+p} \bar{V}_{n+p} = 0$$

with  $Q_i, \bar{V}_i$  without  $s_1$  for  $i \geq 2$ , so  $\bar{V}_1 = 0$ , where  $\bar{V}_1 = V_1 - h_1^{(2)} \bar{V}_2 - \dots - h_1^{(n+p)} \bar{V}_{n+p}$ . We have then the new equation:

$$Q_2 \bar{V}_2 + \dots + Q_{n+p} \bar{V}_{n+p} = 0$$

in a Briançon-Maisonobe algebra  $\mathbf{C}[s_2, \dots, s_p, t_1, \dots, t_p, x, \partial]$ .

Repeating the process for  $Q_2, \dots, Q_p$ , we reduce our problem to solving:

$$(\partial_1 + \sum_j \frac{\partial f_j}{\partial x_1} t_j) V_{p+1} + \dots + (\partial_n + \sum_j \frac{\partial f_j}{\partial x_n} t_j) V_{p+n} = 0$$

in the Weyl algebra  $D[t_1, \dots, t_p]$ .  $\square$

*Remark 4.* As a consequence of Theorem 2, the bound for the complexity of computing the annihilating ideal of  $f^s$  in  $R$  is bounded by the complexity of computing a Gröbner basis in a Weyl algebra with  $3p$  variables less than the one required by OT method. Although the complexity of computing these objects in any case is known to be double exponential (with respect to the number of variables and the total degree of the generators of the ideal) by Theorem 1, it is clear that the reduction of  $3p$  variables in BM method is a significant advantage, both theoretically and in practice as it is shown in the examples (see [Castro-Ucha(2004)]).

TABLE 1. CPU times for the computation of  $Annf^s$ 

$f$	Briançon-Maisonobe's method	Oaku-Takayama's method
$x^3 + xy^2 + z^2$	< 0.01s	0.39s
$x^4 + y^3 + z^2$	< 0.01s	0.39s
$yx^3 + y^3 + z^2$	0.06s	3.97s
$x^3 + y^2 + z^2$	< 0.01s	0.02s
$x^5 + y^2 + z^2$	< 0.01s	4.66s
$x^7 + y^2 + z^2$	< 0.01s	298.56s
$x^4 + y^5 + xy^4$	0.56s	E (> 12h)

TABLE 2. CPU times for the computation of  $Annf_1^{s_1} f_2^{s_2}$ 

$f_1$	$f_2$	Briançon-Maisonobe's method	Oaku-Takayama's method
$x^3 + y^2$	$x^2 + y^3$	0.72s	6363.97s
$x^5 + y^3$	$x^3 + y^5$	3.53s	E (> 6h)
$x^7 + y^5$	$x^5 + y^7$	11.84s	E (> 6h)
$x^3 + y^2$	$xz + y$	< 0.01s	9.73s
$x^5 + y^2$	$xz + y$	< 0.01s	1568.59 s
$x^{11} + y^5$	$xz + y$	3s	E (> 6h)

TABLE 3. CPU times for the computation of  $Annf_1^{s_1} \cdots f_p^{s_p}$ 

$f_1$	$f_2$	$f_3$	Briançon-Maisonobe's method	Oaku-Takayama's method
$x + y$	$x - y$	$x^2 + y$	< 0.01s	29.46s
$x + y$	$x^2 + y$	$x + y^2$	2.64s	E
$x + y$	$x^2 + y$	$x^2 + y^3$	116.24s	E
$x + y$	$x^2 + y$	$x^3 + y^2$	1728.41s	E

#### 4. APPENDIX: EXPERIMENTAL DATA

In the tables we give some examples for which it is clear the superiority of Briançon-Maisonobe's method. They have been tested<sup>3</sup> using SINGULAR::PLURAL 2.1 (see [Greuel et al.(2003)]) in a PC Pentium IV, 1Gb RAM and 3.06GHz running under Windows XP.

SINGULAR::PLURAL 2.1 is a system for non-commutative general purpose, so the calculations in our algebras are not supposed to be optimal. We present the following data only for the sake of comparing both methods in the same system. In the case of [Briançon and Maisonobe(2002)] method we have used a pure lexicographical ordering, while for [Oaku and Takayama(1999)] we have used typical elimination ordering. These are the orderings with best results for each case.

#### REFERENCES

[Bahloul(2001)] Bahloul, R. Algorithm for computing Bernstein-Sato ideals associated with a polynomial mapping. *Journal of Symbolic Computation* **32**, 643–662, 2001.

<sup>3</sup>The CPU times must be considered as approximations: as it is explained in the SINGULAR::PLURAL 2.1 Manual, the command `timer` is not absolutely reliable due to the shortcomings of the Windows operating system.



- [Bahloul(2005)] Bahloul, R. Démonstration constructive de l'existence de polynômes de Bernstein-Sato pour plusieurs fonctions analytiques. *Compositio Math.* **141**, no. 1, 175–191, 2005 (to appear).
- [Bernstein(1972)] Bernstein, I.N. The analytic continuation of generalized functions with respect to a parameter. *Funct. Anal.* **6** 273–285, 1972.
- [Björk(1973)] Björk, J.E. Dimensions over Algebras of Differential Operators. Preprint, 1973.
- [Briançon and Maisonobe(2002)] Briançon, J. and Maisonobe, Ph. *Remarques sur l'idéal de Bernstein associé à des polynômes*. PUMA, 650 (preprint). 2002.
- [Budur-Saito(2003)] Budur, N. and Saito M. Multiplier ideals,  $V$ -filtration and spectrum. arXiv:math.AG/0305118, 2003.
- [Bueso et al.(2003)] Bueso, J.L. Gómez-Torrecillas, J. and Verschoren, A. *Algorithmic methods in non-commutative algebra. Applications to quantum groups*. Mathematical modelling. Theory and applications **17**, Kluwer Academic Publishers. 2004.
- [Castro-Ucha(2004)] Castro-Jiménez, F.J. and Ucha-Enríquez, J.M. On the computation of Bernstein-Sato ideals. *J. Symbolic Comput.* **37**(5), 629–639. 2004.
- [Greuel et al.(2003)] G.-M. Greuel, V. Levandovskyy, and H. Schönemann. SINGULAR::PLURAL 2.1. A Computer Algebra System for Noncommutative Polynomial Algebras. Centre for Computer Algebra, University of Kaiserslautern (2003). <http://www.singular.uni-kl.de/plural>.
- [Grigoriev(1990)] Grigoriev, D. Complexity of solving systems of linear equations over the rings of differential operators. In *Effective methods in algebraic geometry (Castiglione, 1990)*, Progr. Math., **94**. Birkhauser Boston, MA 195–202. 1991.
- [Hamm(1975)] Hamm, H.A. Remarks on asymptotic integrals, the polynomial of I.N. Bernstein and the Picard-Lefschetz monodromy. In *Several complex variables (Proc. Sympos. Pure Math. Vol XXX, Part 1, Williams Coll., Williamstown, Mass. 1975)*, 31–35. Amer. Math. Soc., Providence, R.I. 1977.
- [Kashiwara(1976)] Kashiwara, M. B-functions and holonomic systems. Rationality of roots of B-functions. *Invent. Math.* **38** (1) 33–53. 1976.
- [Lichtin(1988)] Lichtin, B. Generalized Dirichlet series and b-functions. *Compositio Math* **65** 81–120. 1988.
- [Malgrange(1974)] Malgrange, B. Le polynôme de Bernstein d'une singularité isolée. In *Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974)*. Lecture Notes in Math. **459** Springer-Verlag, New York 98–119. 1975.
- [Maynadier(1996)] Maynadier, H. Equations fonctionnelles pour une intersection complète quasi-homogène à singularité isolée et un germe semi-quasi-homogène, Thèse, Nice-sophia Antipolis, 1996.
- [Mayr-Meyer(1982)] Mayr, E.W. and Meyer, A.R. The complexity of the word problems for commutative semigroups and polynomial ideals. *Adv. in Math.* **46**(3), 305–329. 1982.
- [Oaku(1997)] Oaku, T. An algorithm of computing b-functions. *Duke Math. J.* **87** 115–132 1997.
- [Oaku and Takayama(1999)] Oaku, T. Takayama, N. An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation. *Journal of Pure and Applied Algebra* **139** 201–233 1999.
- [Sabbah(1987a)] Sabbah, C. Proximité évanescence I. La structure polaire d'un D-Module. Appendice en collaboration avec F.J. Castro Jiménez. *Compositio Math.* **62**, 283–328. 1987.
- [Sabbah(1987b)] Sabbah, C. Proximité évanescence II. Equations fonctionnelles pour plusieurs fonctions analytiques. *Compositio Math.* **64**, 213–241. 1987.
- [Seidenberg(1974)] Seidenberg, A. Constructions in algebra. *Trans. Amer. Math. Soc.* **197** 273–313. 1974.
- [Stafford(1978)] Stafford, J. T. Module structure of Weyl algebras. *J. London Math. Soc.* (2) **18**, no. 3, 429–442. 1978.

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