A new Controller for the Inverted Pendulum on a Cart

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SUMMARY

This paper presents a complete solution to the problem of swinging-up and stabilization of the inverted pendulum on a cart, with a single control law. The resulting law has two parts: first, an energy-shaping law is able to swing and maintain the pendulum up. Then, the second part introduces additional control to stop the cart and it is based on forwarding control with bounded input. The resulting control law is the sum of both parts and does not commute between different laws although there exist switches inside the controller. Copyright © 2008 John Wiley & Sons, Ltd.

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1. Introduction

The inverted pendulum is an interesting device that has attracted the attention of the nonlinear control community for many years (see [2, 8, 9, 16, 22, 23, 26, 27, 28], to mention only a few references). It is a simple underactuated system that serves as a benchmark for nonlinear control techniques. The control problems associated with it can be found in many applications, such as attitude control of a space booster on takeoff and the stability of walking robots. Two are the main problems that are usually considered: 1) swing the pendulum up from the downward position (or, even, any arbitrary initial position and velocity); and 2) stabilization around the upper vertical position once the pendulum is in a neighborhood of it. The second problem can be easily solved by means of linear methods since only small perturbations are considered. For the swing-up problem several control strategies have been presented [7, 10, 16]. Nevertheless, these solutions are not able to simultaneously solve the second problem: they maintain the saddle nature of the desired equilibrium and what they achieve is asymptotic stabilization of its stable manifold. Hence, in order to solve the full problem, an hybrid control strategy is implemented: the swing-up law is committed to carrying the pendulum to the neighborhood of the desired position (global law) and, then, the control law is switched into a stabilizing one (local law). This hybrid control works quite well in experimental frameworks. However, it has been a main theoretical challenge in nonlinear control to merge both control laws into a single one. In [19] an interesting approach is presented, which merges smoothly both local and global solutions to the problem, guarantying the stability of the resultant system. In [22] a single controller is also proposed but it requires a strategy for commutation of the reference value. The problem addressed in [18] is similar but as the pendulum swing-up controller does not guarantee stability, the techniques presented in that paper are not directly

applicable.

In this paper, we present a new control law that it is not the combination of different laws designed for solving the two subproblems separately. Instead, the new law solves simultaneously both problems. It is based on energy-shaping methods [17] but instead of just introducing damping, a combination of damping and energy injection (pumping) is needed giving rise to a pumping-damping [6] strategy. The resultant control law drives the pendulum to the desired upright position from any initial position and/or at any velocity (except a set of zero measure), and it is able to stabilize it. Previous results have been reported in [11, 3, 4]. A similar idea is used in [6].

Furthermore, in a second part of the paper, the carrying element of the pendulum is also taken into account and the full pendulum-on-a-cart system is considered. A new objective is added: the cart is desired to stop once the pendulum reaches the upright position. This problem is solved by means of the addition to the previous control law, a new term based on forwarding-with-bounded-input techniques [12, 14, 13, 24, 25]. For this, it is very useful to start with the (local) Lyapunov function provided in the first stage of the design. The resulting control law is the sum of both parts and avoids any commutation between different sub-controllers.

The rest of the paper is organized as follows. In Section 2 the problem of the pendulum on a cart is solved in two steps. This approach is based on dealing first with the simple pendulum, disregarding the cart for the moment. A controller is proposed that swings up the pendulum and stabilizes it at the upright position. In Section 3 this controller is enlarged to use in the full system in such a way that the cart is also stopped. The paper closes with a Section of conclusions.

2. Control of the pendulum subsystem

The model of the pendulum on a cart after partial linearization and normalization [21] is

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \sin x_1 - \cos x_1 u \\
\dot{x}_3 &= u,
\end{aligned}
\tag{1}
$$

where x_1 is the angular position of the pendulum with the origin at the upright position, and x_2 and x_3 are the velocities of the pendulum and the cart respectively and u is the force applied to the cart. Therefore, the system is defined on a cylindrical state space: $(x_1, x_2, x_3) \in \mathbb{S} \times \mathbb{R}^2$.

If only the pendulum is considered, the equations (1) reduce to

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \sin x_1 - \cos x_1 u.
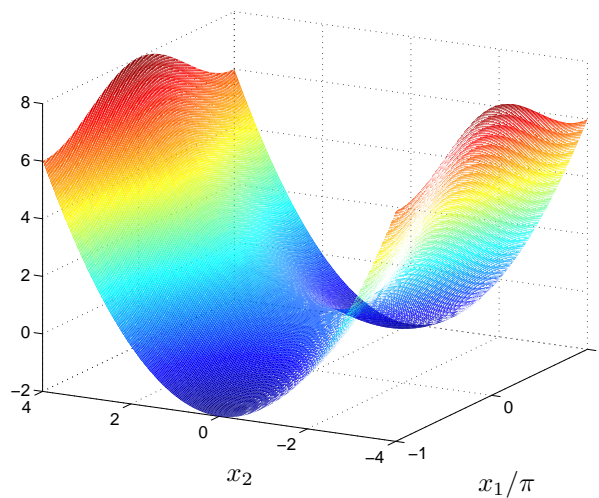\end{aligned}
\tag{2}
$$

Figure 1. Shape of the energy for the simple pendulum.

In the absence of forcing u the only stable equilibrium point is the hanging position. The upright position is a saddle point. This fact can be illustrated using energy considerations. The

energy of the pendulum subsystem with $u = 0$, represented in Fig. 1, is the sum of potential and kinetic energies:

$$H(x_1, x_2) = \cos x_1 - 1 + \frac{x_2^2}{2}.$$

Our goal is to design a controller that is able to swing up the pendulum from (almost) all initial conditions and to maintain the pendulum at the upright position. We will base the derivation on the potential energy shaping method, choosing as desired Hamiltonian functions of the form

$$H_d(x_1, x_2) = V_d(x_1) + \frac{x_2^2}{2}, \tag{3}$$

where the desired potential energy V_d should have a single minimum at the desired upright position. A generalized Hamiltonian target system with H_d as a Hamiltonian function is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -k_a \end{bmatrix} \begin{bmatrix} D_{x_1} H_d \\ D_{x_2} H_d \end{bmatrix}, \tag{4}$$

where k_a is a damping coefficient (or even it can be a function of x_1 and x_2). With H_d as given by (3), (4) yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -V_d'(x_1) - k_a x_2. \end{aligned} \tag{5}$$

It is well known that the problem with choosing an appropriate $V_d(x_1)$ function is related to the term $\cos x_1$, affecting to the control signal, u, in the second equation of (2). For instance, the most elementary choice is $V_d = -\cos x_1$, which has an appropriate shape (a single minimum at the desired upright position), but it leads to the control law $u = 2\tan x_1$ (for the case $k_a = 0$) which cannot be implemented in the full range $|x_1| \leq \pi$ because for $x_1 = \pm\pi/2$ the feedback law is unbounded.

To solve the matching problem of the open (2) and closed (5) loop behaviors, and in order to avoid the division by $\cos x_1$, a good choice of V_d' is

$$V_d' = -\sin x_1 + \cos x_1 \beta(x_1), \qquad (6)$$

and then, for $k_a = 0$, $u = \beta(x_1)$ (the case $k_a \neq 0$ will be discussed latter). Some additional conditions should be imposed on $\beta(\cdot)$. First, $\beta(0) = 0$ to guarantee that the origin $(0,0)$ is an equilibrium of the closed-loop system. Just as the pendulum behaves in a cylindrical state space, the closed-loop system should display some periodicity. Then, it is reasonable to make $\beta(x_1) = \sin x_1 \bar{\beta}(\cos x_1)$. This choice facilitates the integration of (6) to get V_d. We must also impose that $V_d(x_1) = V_d(-x_1)$ and, therefore, $V_d'(x_1) = -V_d'(-x_1)$.

A family of functions, V_d, that fulfill these conditions is given by

$$V_d = a_0 + \cos x_1 - a_2 \cos^2 x_1 - a_3 \cos^3 x_1 - \cdots, \qquad (7)$$

which yields

$$
\begin{aligned}
V_d' \;=\; D_{x_1} V_d \;=\;\; & -\sin x_1 + 2a_2 \sin x_1 \cos x_1 + 3a_3 \sin x_1 \cos^2 x_1 + ... \\
\;=\;\; & -\sin x_1 + \sin x_1 \cos x_1 (2a_2 + 3a_3 \cos x_1 + ...),
\end{aligned}
\qquad (8)
$$

which clearly allows us to determine $\beta(x_1)$ to match this last expression with (6). Therefore, we have a family of functions, V_d, which solves the matching problem for the pendulum.
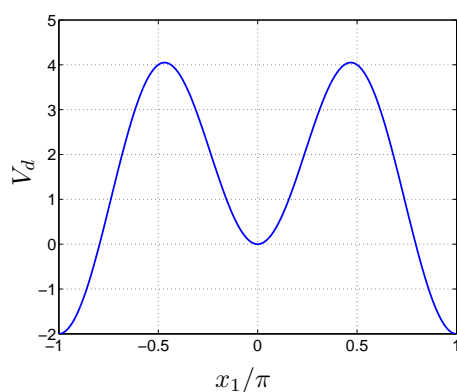
The simplest case of this family is obtained by taking $a_0 = a - 1$, $a_2 = a$ and $a_k = 0, \forall k > 2$, which leads to

$$V_d(x_1) = \cos x_1 - a \cos^2 x_1 + a - 1. \qquad (9)$$

Other choices can be found in [5]. Figure 2 shows that V_d has a minimum at the origin. It can be easily shown that this happens for $a > 0.5$.

With this V_d we obtain

$$V_d' = -\sin x_1 + 2a \cos x_1 \sin x_1,$$

Figure 2. Shape of V_d for $a = 5$.

and then with the feedback law

$$u = 2a \sin x_1, \tag{10}$$

which is defined everywhere, the matching of the open and closed loop systems is solved.

Another interesting case [11] is obtained with

$$V_d(x_1) = -\frac{\cos 3x_1}{3} + \frac{1}{3}, \tag{11}$$

which yields

$$V_d' = -\sin x_1 + 4\cos^2 x_1 \sin x_1,$$

and the feedback law

$$u = 4\cos x_1 \sin x_1 = 2\sin 2x_1. \tag{12}$$

Many other V_d belonging to the same family can be conceived. However, and for the sake of simplicity, in the sequel we will be only concerned with the desired potential V_d given by (9). Nevertheless, most of the results developed here can be extended to other V_d belonging to family (7), such as (11).

As it is shown in Fig. 2, function V_d given by (9) has other minima, which are undesirable. The same happens with every function of class (7). To overcome this problem, we begin adopting a damping term for the desired closed loop of the form $-kx_2 \cos x_1$ (that is, in Eq. (5) $k_a = k \cos x_1$). With this term, for values of x_1 such that $\pi/2 < |x_1| < \pi$, and due to the sign of $\cos x_1$, energy injection is produced instead of damping. With the injection of energy the pendulum tends towards the region above the horizontal. In that region, there is only a single minimum for V_d, the desired upright position. Therefore, the term $-kx_2 \cos x_1$ causes the equilibrium at the bottom of the additional minima to change from stable to unstable. It should be noted that k_a does not have a definite sign and, consequently, the closed-loop system loses the generalized Hamiltonian structure and the stability of the system has to be analyzed by other methods.

In summary, for the moment we propose a target system with the structure:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -k \cos x_1 \end{bmatrix} \begin{bmatrix} D_{x_1} H_d \\ D_{x_2} H_d \end{bmatrix}, \tag{13}$$

that is,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -V_d'(x_1) - kx_2 \cos x_1, \end{aligned} \tag{14}$$

where, if H_d is given by

$$H_d(x_1, x_2) = \cos x_1 - a \cos^2 x_1 + \frac{x_2^2}{2}, \tag{15}$$

the target system is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - 2a \cos x_1 \sin x_1 - kx_2 \cos x_1, \end{aligned} \tag{16}$$

which can clearly be matched to system (2) with the control law

$$u = 2a \sin x_1 + kx_2. \tag{17}$$

The local stability achieved with this control law can be proved using (15) as Lyapunov function.

Law (17) has been also proposed in [9], but without justification; as reported there, its actual application requires careful analysis as parameters a and k vary.

We can distinguish between two versions of the swing up problem. The first is the *strict* one, where only the transition from the hanging position (that is, for initial conditions $x_1 \approx \pi$ and $x_2 = 0$) is considered. This problem is solved with controller (17) as can be shown by means of simulations. There is a more complex version of the swing up problem, the *global* one, where almost any initial condition is taken into account. For this second version, controller (17) has a big drawback. It injects energy in any case when the pendulum is below the horizontal. This means that even in the case when the pendulum has enough energy to rotate, and then to go above the horizontal, we are still injecting more energy. In such a case, the energy injection can lead the system to a rotating limit cycle [1]. This is a very interesting phenomenon that deserves thorough analysis, but that is outside of the scope of this paper.

When a trajectory is in the basin of an undesirable minimum, we need to inject energy to force the trajectory to leave it. This means that we have to inject energy only when the energy of the pendulum is not great enough to leave this region. Recalling that V_d has the shape of Fig. 2, this undesirable well can be visualized as enclosed by the curve represented in Fig. 3. This curve is the H_d−level curve corresponding to $H_d = H^* \triangleq \frac{1}{4a} + a - 1$. But when the system is outside this region we must damp it. To that end, we define the following function:

$$\varphi(x_1, x_2) \triangleq \begin{cases} -k & \text{if } H_d(x_1, x_2) \leq H^* \text{ and } \cos x_1 < \frac{1}{2a} \\ k & \text{elsewhere,} \end{cases} \tag{18}$$

where $k > 0$ is a tuning parameter. That is, $\varphi(x_1, x_2)$ is negative only inside the regions where
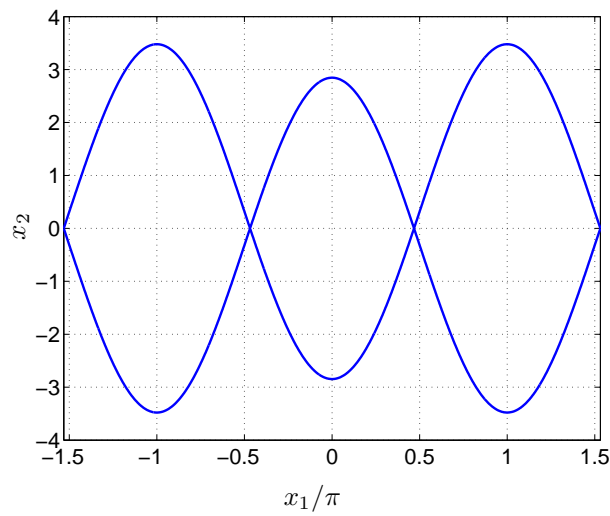
Figure 3. Curve $H_d = H^*$.

energy must be injected, and positive elsewhere. As it will be seen below, function $\varphi(x_1, x_2)$ will determine the sign of damping. Therefore, a pumping-damping energy law will be obtained. With definition (18), the following target system is proposed:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \sin x_1 - 2a \cos x_1 \sin x_1 - \varphi(x_1, x_2) x_2 \cos^2 x_1.
\end{aligned}
\tag{19}
$$

The control law that matches system (2) with target system (19) is

$$
u = 2a \sin x_1 + \varphi(x_1, x_2) x_2 \cos x_1.
\tag{20}
$$

This control law is valid in the whole cylindrical state space. Energy is injected inside the undesirable wells whiles the system is damped elsewhere.

Let us study the behavior of system (19). First, the equilibrium points are easily determined: $(0, 0), (\pm \arccos(1/2a), 0)$ and $(\pm\pi, 0)$. If $a > 0.5$ the origin is stable and the equilibrium points at $x_1 = \pm\pi$ are unstable. The stability of the other equilibrium points is more difficult to study since the switching curve passes through them. It will be analyzed below.

It is easy to see that

$$\dot{H}_d = -\varphi(x_1, x_2)x_2^2 \cos^2 x_1,$$

which means that as the sign of φ changes, injection of energy or damping is produced. A Lyapunov function candidate –strictly speaking, this is not a true Lyapunov function because it is not positive definite; nevertheless, the following analysis is valid since (21) is bounded from below– is

$$V = \varphi(x_1, x_2)(H_d - H^*) = \varphi(x_1, x_2)\left(\cos x_1 - a\cos^2 x_1 + \frac{x_2^2}{2} - H^*\right). \qquad (21)$$

It must be taken into account that at the switching curve –that is $\{(x_1, x_2)|H_d(x_1, x_2) = H^*, \cos x_1 < 1/(2a)\}$– function V is not differentiable. Outside this switching curve we have the convenient relation

$$\dot{V} = -\varphi^2 x_2^2 \cos^2 x_1 \leq 0.$$

The behavior around the switching curve can be analyzed using energy considerations: as this curve is (part of) a level curve and the energy increases when the energy is lower that the one corresponding to this level, and it decreases when the energy is higher, it can be deduced that the curve is attractive. Since commutations occur at this curve, a sliding motion is produced along it. The direction of motion is determined by the equation $\dot{x}_1 = x_2$. Therefore, we can see that the sliding motion makes the equilibrium points at $x_1 = \pm \arccos(1/2a)$ to be attractive since the sliding manifold directs the motion towards them. This fact is undesirable and we can conclude that controller (20) does not work properly. In the following, two modifications of this control law are proposed in order to improve the behavior of the closed-loop system: first, the stability of the undesirable equilibrium points is removed and then, the sliding motion is avoided.

2.1. Achieving almost-global stability

In order to avoid the attractiveness of the undesirable equilibrium points $x_1 = \pm \arccos(1/2a)$, the switching curve is changed from $H_d = H^*$ to $H_d = H^* + \varepsilon$ with $0 < \varepsilon \ll 1$. This change is performed substituting the switching function φ by

$$\varphi_\varepsilon(x_1, x_2) \triangleq \begin{cases} -k & \text{if } H_d(x_1, x_2) \leq H^* + \varepsilon \text{ and } \cos x_1 < \frac{1}{2a} \\ k & \text{elsewhere.} \end{cases}$$

The control law is now:

$$u = \underbrace{2a \sin x_1}_{u_c} + \underbrace{\varphi_\varepsilon(x_1, x_2) x_2 \cos x_1}_{u_d}. \tag{22}$$

Now, the same analysis can be performed with the Lyapunov function candidate

$$V = \varphi_\varepsilon(x_1, x_2) (H_d - H^* - \varepsilon) \tag{23}$$

The behavior of the system can now be explained with the help of Fig. 4. In this figure the level curves corresponding to $H_d = H^*$ and $H_d = H^* + \varepsilon$ are plotted. The sliding motion is now along the latter of these curves. It can be shown that the undesirable equilibrium points $(\pm \arccos(1/2a), 0)$ are outside the sliding manifold. The stability of these points can be analyzed studying the system linearization concluding that they are non-hyperbolic saddles. Therefore, they do not preclude the almost-global stability property. Nevertheless, this property is not proved yet. Notice that with the choice (23), the Lyapunov function candidate is continuous along the new switching curve (the dashed curve of Fig. 4). However, function (23) is not continuous along the segments $\overline{AB}$ and $\overline{CD}$ in Fig. 4. Therefore, the stability analysis must be performed carefully.

In the following, it will be proved that, for small enough ε, when the system reaches the points of discontinuity for (23), the trajectory evolves towards the origin. Due to the symmetry

of the problem only the case of $x_2 > 0$ will be considered. The reasoning will start at the upper

half of segment $\overline{AB}$ that, as it will be seen below, is a worse case than starting at the upper half

segment $\overline{CD}$. As $x_2 > 0$ the system will evolve towards the right. In this region $\dot{H}_d > 0$ and the

system will evolve towards the sliding curve (the dashed curve of the figure). After reaching

$x_1 = \pi$ it will continue through the sliding curve from $x_1 = -\pi$ towards $x_1 = -\arccos(1/2a)$.

Therefore, the upper half of segment $\overline{CD}$ (with $x_2 > 0$) will eventually be reached. The value

of the energy function at this point will be close to $H^* + \varepsilon$. When the system starts on this

segment as $x_2 > 0$ it will evolve towards the right but, in this region, $\dot{H}_d < 0$. If ε is small

enough the level curve $H_d = H^*$ (the solid curve) will be reached before reaching segment $\overline{AB}$

(notice that the damping does not depend on ε. Once this level curve is reached and crossed

the system can not go out of the compact set $\{x : H_d(x) < H^*\}$ where $\dot{V} \leq 0$ and asymptotic

stability is guaranteed.

Since no other points of discontinuity exist, the following proposition can be stated:

Proposition 1. *The origin of system (2) with control law (22) is almost GAS.*

2.2. Avoiding the sliding mode

In many control applications sliding modes are not admissible due to the chattering

phenomena. The control law proposed in the previous section presents sliding motions due

to the discontinuity along the level curve $H_d = H^* + \varepsilon$. One way of solving this problem is

proposed in this section. The idea is to modify the switching function in such a way that it

is equal to zero at both sides of the switching curve. Thus, the switching function will be

continuous at the switching curve and no sliding motions can occur. This can be accomplished
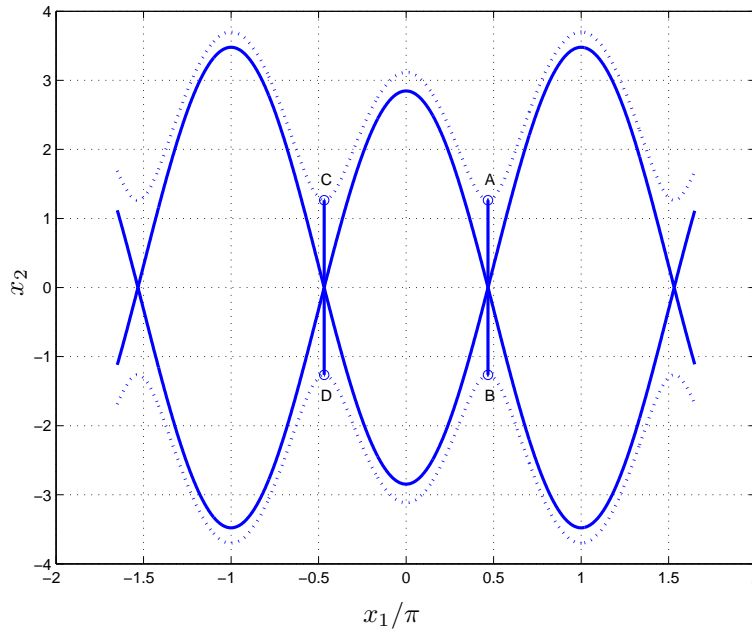
Figure 4. Level curves $H_d = H^*$ (solid) and $H_d = H^* + \varepsilon$ (dotted).

multiplying function φ_ε by $|H_d(x_1, x_2) - H^* - \varepsilon|$ when $\cos x_1 < 1/(2a)$. In this way u_d will be equal to zero at the level curve $H_d = H^* + \varepsilon$ avoiding the sliding motion. In this way, the system will tend asymptotically to this level curve. Since at $x \in [-\pi/2, \pi/2]$ it is desired that the trajectories cross the level curve, function φ_ε should not change at this interval. All this results in the new switching function:

$$\tilde{\varphi}_\varepsilon(x_1, x_2) \triangleq \begin{cases} k\big(H_d(x_1, x_2) - H^* - \varepsilon\big) & \text{if } \cos x_1 < \frac{1}{2a} \\ k & \text{elsewhere} \end{cases}$$

The resulting control law is

$$u = \underbrace{2a \sin x_1}_{u^c} + \underbrace{\tilde{\varphi}_\varepsilon(x_1, x_2)x_2 \cos x_1}_{u^d}, \tag{24}$$

where u_c can be interpreted as a conservative control law and u_d a dissipation/injection term.

The same previous analysis could be performed to the system with control law (22)

substituting the sliding motion by an asymptotic motion to the switching line. In order to compute $\dot{V}$ now, it must be taken into account that $\tilde{\varphi}_\varepsilon$ depends on x_1 and x_2. It is easily seen that $\dot{V} = -2\tilde{\varphi}_\varepsilon^2 x_2^2 \cos^2 x_1$ when $\cos x_1 < 1/2a$. Therefore, using the previous ideas, the following proposition can be proved:

Proposition 2. *The origin of system (2) with control law (24) is almost GAS.*

The behavioral difference with respect to the previous law is that the control law does not present sliding motions. The cost of this nice nature is that the energy injection/damping near the switching curve is very small and, thus, the system will be slower. Another drawback of this approach is that, now, commutation between different control laws instead of between different values of a tuning parameter takes place. Nevertheless, the commuting control laws have been derived using an only idea trying to solve both the swing-up and the stabilization problems.

Figures 5 and 6 show the behavior of the system for different values of parameter k. Figure 7 illustrates how the trajectory starting at $x_1 = \pi$ and $x_2 = 0$ leaves the undesirable well and falls into the desired one.

3. Stopping the cart

As we saw in the previous section, control law (22) solves the swing up problem for the pendulum subsystem without considering the cart. To cope with the cart, we add a new term, v, to the control law (24)

$$u = 2a \sin x_1 + \widetilde{\varphi}_\varepsilon x_2 \cos x_1 + v. \tag{25}$$
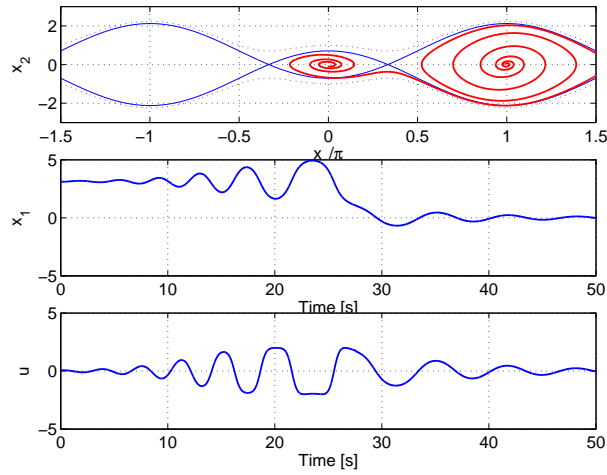
Figure 5. Results of a simulation with $a = 1$, $\varepsilon = .25$ and $k = 0.2$. Initial conditions: hanging position
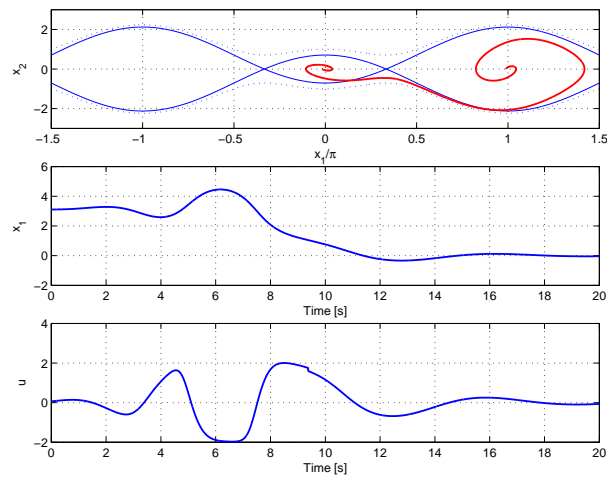
at rest.

Figure 6. Results of a simulation with $a = 1$, $\varepsilon = .25$ and $k = 0.6$. Initial conditions: hanging position
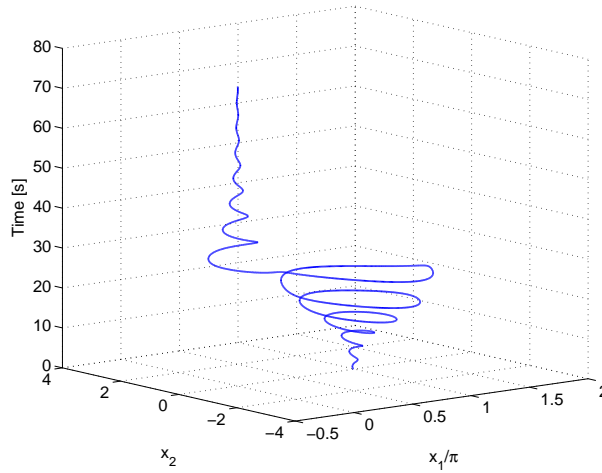
at rest.

Figure 7. Trajectory leaving the point $x_1 = \pi$ and $x_2 = 0$ and reaching the up-right position, for

$$k = 0.2.$$

When this control law is applied to system (1), we obtain

$$\dot{x}_1 \;=\; x_2 \tag{26}$$

$$\dot{x}_2 \;=\; \sin x_1 - 2a \sin x_1 \cos x_1 - \widetilde{\varphi}_\varepsilon x_2 \cos^2 x_1 - \cos x_1 v \tag{27}$$

$$\dot{x}_3 \;=\; 2a \sin x_1 + \widetilde{\varphi}_\varepsilon x_2 \cos x_1 + v. \tag{28}$$

The need for term v is clear because, without it, variable x_3 is not fed back and it would evolve

without control, driven by the motions of x_1 and x_2 produced while the pendulum is reaching

the upright position. Once the pendulum has reached the inverted position, then $x_1 = x_2 = 0$,

but $x_3 \neq 0$. In effect,

$$x_3 = \int (2a \sin x_1 + \widetilde{\varphi}_\varepsilon x_2 \cos x_1) dt \neq 0$$

and, therefore, as x_3 is not equal to zero, the cart suffers a drift which leaves its motion

unbounded.

System (26)–(28) is a cascade with feedforward structure. In effect, making $[x_1 \; x_2]^T = \xi$

and $x_3 = z$, system (26)–(28) can be written in the standard feedforward form

$$
\begin{aligned}
\dot{z} &= h(\xi) + g_1(\xi)v \\
\dot{\xi} &= f(\xi) + g_2(\xi)v,
\end{aligned}
\tag{29}
$$

where the equations have been reordered as it is usual. The upper equation in (29) is the one of the cart, and the lower one corresponds to the pendulum. Moreover, it should be noted that $\dot{\xi} = f(\xi)$ is almost GAS, with a Lyapunov function obtained in the previous section.

The problem now is to determine a feedback $v(x_1, x_2, x_3)$ such that the speed of the cart is controlled at the same time that the upright position of the pendulum maintains its stability. The feedforward form of system (26)–(28) suggests applying conventional forwarding to get the control law $v(x_1, x_2, x_3)$. Unfortunately, the application of this method in this case leads to a partial differential equation that it is hard to solve.

One way to overcome this difficulty is to use saturation functions in the forwarding design method [24, 25]. Here we use the approach proposed by Astolfi and Kaliora [12, 14, 13]. Furthermore, it has the advantage of having a meaningful physical interpretation. Following Astolfi and Kaliora's suggestion, we consider the control law

$$
v = -\varepsilon_2 \sigma \left(\frac{\lambda x_3}{\varepsilon_2} \right),
\tag{30}
$$

where σ is the saturating function $\sigma(y) = \mathrm{sgn}(y) \min\{|y|, 1\}$, λ is a design parameter, and ε_2 is a small positive constant. Here we use ε_2 to distinguish it from the ε introduced in the previous section when the switching function $\widetilde{\varphi}_\varepsilon$ was defined. Therefore,

$$
u = 2a \sin x_1 + \widetilde{\varphi}_\varepsilon x_2 \cos x_1 - \varepsilon_2 \sigma \left(\frac{\lambda x_3}{\varepsilon_2} \right).
\tag{31}
$$

The purpose of the new term v is explained in the following. Notice that all the nonlinearities of the closed-loop system, except the saturation that appears in v, do not depend on x_3. First of

all, consider the case when the pendulum is near the upright position with small velocity –i.e. x_1 and x_2 are close to zero– with any value of the cart speed x_3. In this situation, and due to the former fact, the system can be linearized around $(x_1, x_2) = (0, 0)$ resulting a linear system combined with a saturation. This kind of system is well-known (it belongs to the class known as Lure's systems) and, thus, parameter λ can be tuned in order to stabilize this system. Then, if the pendulum reaches a neighborhood of $(x_1, x_2) = (0, 0)$ and it remains there, the cart will eventually stop. The problem now is to see if there is any guarantee that the pendulum will eventually arrive at this neighborhood and that it will stay there. Notice that with $v = 0$ this is true and notice also that $|v| \leq \varepsilon_2$ so it can be very small by design. Law (31) will work if the behavior of the pendulum for small values of v is similar to the one corresponding to $v = 0$. Fortunately, this question has been rigourously formulated fifteen years ago (see [20]). This desired characteristic of the pendulum subsystem is fulfilled if it is *input to state stable* (ISS). There exist nice and powerful Lyapunov-like results that help us to guarantee that system (26)–(27) is ISS (with restrictions) as stated the following proposition.

Proposition 3. *System (26)–(27) is locally ISS for* $||x|| < 0.095\pi$.

Sketch of the proof: Choose as Lyapunov function candidate the function

$$V(x_1, x_2) = \cos x_1 - a \cos^2 x_1 + \frac{x_2^2}{2} + a - 1 + \varepsilon_v x_1 x_2.$$

It is straightforward to see that it fulfills the conditions of Theorem 5.2 in [15] for $||x|| < 0.095\pi$.

∎

As we are interested in a global law that is even able to swing up the pendulum, we have to include another discontinuity: we choose $v = 0$ in order to reach the region for which system (26)–(27) is locally ISS. Once this region is reached, we use control law (31) in order to also

stabilize (28) according to the procedure proposed in [13].

In summary, the resultant control law is given by:

$$u = u_e + v$$

where u_e is given in (25) and v is given by

$$v = \begin{cases} 0 & \text{for } ||x|| \geq 0.095\pi \\ \varepsilon_2 \sigma \left(\frac{\lambda x_3}{\varepsilon_2} \right) & \text{for } ||x|| < 0.095\pi \end{cases} \tag{32}$$

Since u_e introduces a discontinuity, the final control signal will have two points of discontinuity: the first one when $\cos x_1 = 1/(2a)$ (due to the discontinuity of $\widetilde{\varphi}_\varepsilon$) and the second one when $||x|| = 0.095\pi$ (due to the discontinuity introduced in v). Notice that the gap of both discontinuities can be made arbitrarily small reducing ε and ε_2. However, making ε_2 small deteriorates the performance of the system, in the sense that it may take a long time to stop the cart. In any case, this discontinuity can be avoided modulating (30) by a function that goes to zero when $||x||$ is larger than a certain value.

It should be realized that the ideas of [14] can be recursively applied and, thus, the position of the cart could also be controlled in a further step. Nevertheless, for simplicity, this step is omitted here.

The performance of the proposed strategy can be observed in the simulation that appears in Fig. 8 which corresponds to the values $x(0) = (0.99\pi, 0, 0)$, $a = 1$, $\varepsilon = 0.5$, $k = 0.5$, $\varepsilon_2 = 0.1$ and $\lambda = 0.1$. The top left graph shows the projection of the trajectory into the (x_1, x_2) plane. It can be seen that the trajectory tends to the origin of this plane. The time evolutions of x_1 and x_3 are plotted in the top-right and the bottom-left graphs respectively. It can be seen these two variables (and, consequently, also x_2) eventually tend to zero. The bottom right graph

shows the evolution of u. The vertical dotted lines represent the instant when the switch in $\tilde{\varphi}_\varepsilon$ occurs (left line) and when the ISS region is entered (right line).
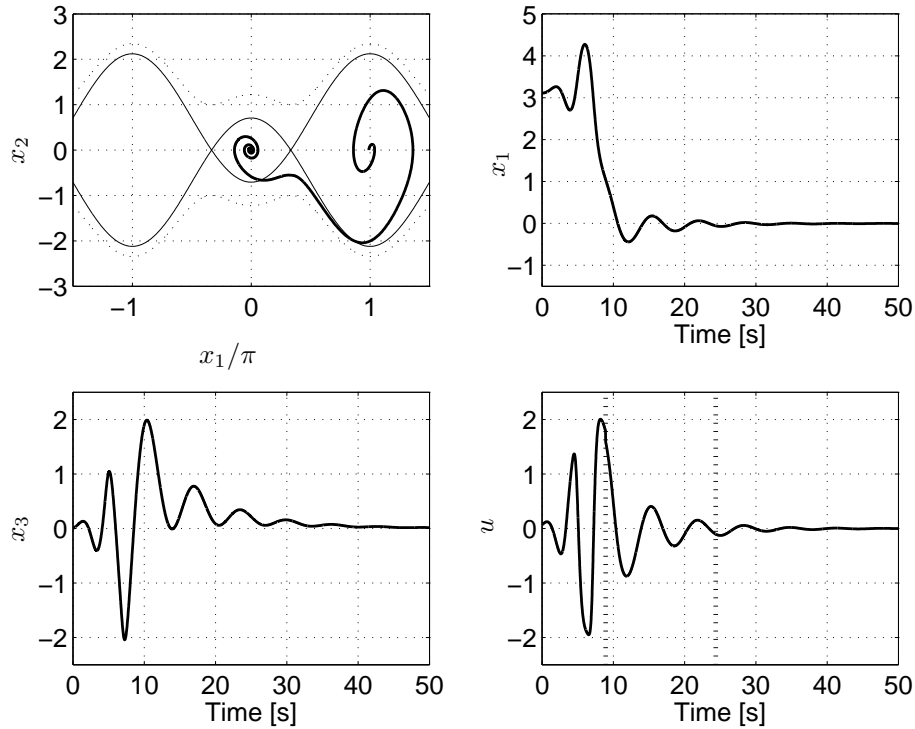
Figure 8. Simulation results for the full controller.

As it was expected, the pendulum evolves towards a neighborhood of the desired position and, then, the cart slowly decelerates (so the pendulum does not fall) until it eventually stops. This two-time-scale behavior can be regarded as a natural solution of the problem considered.

4. Conclusions

In this paper, we have presented a single controller able to swing up the pendulum on a cart from the hanging position to the upright position in an inverted pendulum. A design procedure

with two main steps has been introduced. In the first one, a control law has been obtained that drives the pendulum to the desired upright position, disregarding the cart for the moment. This control law belongs to the family of the energy shaping methods. The proposed control law includes a pumping-damping term in such a way that instead of adding positive damping in all the state space, negative damping (that is, energy injection) will be added inside some undesirable regions. In this way, the system tends to leave these regions and carried on to the region where the desired equilibrium point stands. In the second step, this control law is extended to take into account the cart. To accomplish this, forwarding design via saturation functions has been used. Thus, a full controller has been obtained that is able to control both the pendulum and the cart.

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