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AN ALGORITHM FOR SYMBOLIC COMPUTATION OF CENTER MANIFOLDS.

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Abstract

A useful technique for the study of local bifurcations is the center manifold theory because a dimensional reduction is achieved. The computation of Taylor series approximations of center manifolds gives rise to several difficulties regarding the operational complexity and the computational effort. Previous works proceed in such a way that the computational effort is not optimized. In this paper an algorithm for center manifolds well suited to symbolic computation is presented. The algorithm is organized according to an iterative scheme making good use of the previous steps, thereby minimizing the number of operations. The results of two examples obtained through a REDUCE 3.2 implementation of the algorithm are included.

1. Introduction

The qualitative analysis of dynamical systems –in particular, the characterization of local bifurcations– requires auxiliary tools to facilitate its fulfilment. The center manifold theory is a useful technique because a dimensional reduction of the problem is achieved. This paper deals with the practical computation of Taylor series approximations of center manifolds.

Consider the system

$$\begin{aligned} \dot{x} &= Ax + f(x,y) \\ \dot{y} &= By + g(x,y) \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and A, B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. The functions f and g are C^r with $f(0,0) = 0$, $Df(0,0) = 0$, $g(0,0) = 0$, $Dg(0,0) = 0$. The origin is obviously a nonhyperbolic equilibrium. In this situation there exists a local invariant manifold: $y = h(x)$ with $h(0) = 0$, $Dh(0) = 0$ and h is C^r; it is the so-named center manifold. The flow on this manifold is governed by

$$\dot{x} = Ax + f(x, h(x)) \tag{1.2}$$

which constitutes the so-named reduced (n-dimensional) system. It contains all the necessary information to determine the asymptotic behavior for the flow near the origin of the $(n+m)$-dimensional system (1.1).

As the center manifold is invariant for the flow, the following equation must be held

$$M(h(x)) = Dh(x)\{Ax + f(x, h(x))\} - Bh(x) - g(x, h(x)) = 0 \tag{1.3}$$

The center manifold can be approximated as a Taylor expansion series at $x = 0$, in the following sense (Carr [2]): if a function $\phi(x)$ with $\phi(0) = 0, D\phi(0) = 0$, verifies $M(\phi(x)) = O(|x|^l)$ where $l > 1$, then $h(x) = \phi(x) + O(|x|^l)$ as $x \to 0$.

In practice we consider a polynomial approximation ϕ and its computation proceeds as follows. Let $V(k, n, m)$ denote the linear space of all m-vector functions $v(x)$ of the n-vector x which are homogeneous polynomials in x of degree k. Thus ϕ can be expressed as

$$\phi(x) = \sum_{k=2}^{k_{max}} v_k(x) \text{ where } v_k(x) \in V(k, n, m) \tag{1.4}$$

and k_{max} is the degree of accuracy. To compute $v_k(x)$ we assume that $v_i(x), \quad 2 \leq i \leq k$, have been obtained and we set $\phi_k(x) = \sum_{i=2}^{k} v_i(x)$.

If we define

$$\begin{aligned} L(h(x)) &= Dh(x)Ax - Bh(x) \\ N(h(x)) &= g(x, h(x)) - Dh(x)f(x, h(x)) \end{aligned} \tag{1.5}$$

then (1.3) can be rewritten as $L(h(x)) = N(h(x))$.

Note that L is a linear operator and $L(V(k, n, m)) \subset V(k, n, m)$ for all k. So it is required that

$$L(\phi_k(x)) = N(\phi_k(x)) + O(|x|^{k+1}) \tag{1.6}$$

and as $L(\phi_k(x)) = \sum_{i=2}^{k} L(v_i(x))$, then

$$L_k(v_k(x)) = n_k(x) \tag{1.7}$$

where L_k is L restricted to $V(k, n, m)$ and $n_k(x)$ represents the k-degree terms of Taylor expansion of $N(\phi_k(x))$ -and so $n_k(x)$ is an element of $V(k, n, m)$. The equation (1.7) constitutes a linear system to be solved in $V(k, n, m)$ whose dimension is $m \cdot \binom{k+n-1}{k}$.

In the applications (1.1) can be a large system (the value of $m + n$ is high); further, one can consider linear degeneracies of codimension greater than one (high value of n). In other cases, as in presence of symmetries, we deal with high-codimension nonlinear degeneracies, forcing a growth in the order of necessary accuracy (high value of k_{max}). In sum, the linear system (1.7) might be a very large system and so its computer algebra resolution should be effectively impossible unless a careful insight is provided.

In order to solve (1.7) we need a matrix representation of L_k and n_k and this task involves computational complexities. Notice that a direct substitution of $\phi_k(x)$ in the Taylor expansion of N to obtain n_k produces not only k-degree terms but lower and higher ones which are not required and consequently the computational effort would not be optimized.

Previous known works ([4], [5], [10]) essentially proceed in this way. Therefore it seems interesting to design new approaches which overcome the limitations above mentioned. In this paper an algorithm for center manifolds well suited to symbolic computation is presented. The algorithm is organized according to an iterative scheme making good use of the previous steps, thereby minimizing the number of operations and the memory requirements.

2. Description of the algorithm

In the study of the behaviour near a degenerate equilibrium of a dynamical system is of great interest to use certain coordinate changes by means of which it is possible to "simplify" its differential equation, so obtaining the so-called normal forms. These forms are simpler than initial system to the effect that nonlinear terms which are not essential have been removed.

The coordinate transformations yielding normal forms can be used for center manifolds calculations (see Chow & Hale [3]). Let us make the following near-identity transformation in (1.1):

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{h}(\bar{x}) \end{pmatrix} \tag{2.1}$$

where $\bar{x} \in \mathbb{R}^n, \bar{y} \in \mathbb{R}^m$ and $\bar{h}(0) = 0, D\bar{h}(0) = 0$. The new differential equations are:

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + \bar{f}(\bar{x}, \bar{y}) \\ \dot{\bar{y}} &= B\bar{y} + \bar{g}(\bar{x}, \bar{y}) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \bar{f}(\bar{x}, \bar{y}) =& f(\bar{x}, \bar{y} + \bar{h}(\bar{x})) \\ \bar{g}(\bar{x}, \bar{y}) =& -\{D\bar{h}(\bar{x})A\bar{x} - B\bar{h}(\bar{x})\} + \\ &+ \{g(\bar{x}, \bar{y} + \bar{h}(\bar{x})) - D\bar{h}(\bar{x})f(\bar{x}, \bar{y} + \bar{h}(\bar{x}))\} \end{aligned} \tag{2.3}$$

We choose $\bar{h}(\bar{x})$ such that $\bar{y} = 0$ were an invariant hyperplane for (2.2). This condition is equivalent to $\bar{g}(\bar{x}, 0) = 0$ and therefore, we deduce that $\bar{h}(\bar{x})$ must verify the equation (1.3) corresponding to center manifolds; from now, we identify $\bar{h}$ and h. Furthermore the system

$$\dot{\bar{x}} = A\bar{x} + \bar{f}(\bar{x}, 0)$$

becomes the reduced system. So center manifold computation for (1.1) is equivalent to calculate the transformation (2.1) leading to (2.2) with the above condition. From a geometrical point of view the playing role of coordinate transformation is to flat the center manifold.

In Meyer & Schmidt [8] and Chow & Hale [3], an approach to the transformation theory leading to normal forms –using Lie transforms– is presented. They arrive to a recursive algorithm to obtain the transformed equations from original ones. It follows a review of ideas behind their algorithm and how to use them in our problem.

Suppose the following formal expansions:

$$\begin{aligned} f(x, y) &= \sum_{k \geq 2} f_k(x, y), \quad f_k \in V(k, n+m, n) \\ g(x, y) &= \sum_{k \geq 2} g_k(x, y), \quad g_k \in V(k, n+m, m) \\ h(x) &= \sum_{k \geq 2} h_k(x), \quad h_k \in V(k, n, m) \end{aligned} \tag{2.4}$$

It must be noticed that h_k can be identified to v_k (see 1.4). And we will also suppose

$$\begin{aligned} \bar{f}(\bar{x},\bar{y}) &= \sum_{k\geq 2} \bar{f}_k(\bar{x},\bar{y}), \quad \bar{f}_k \in V(k,n+m,n) \\ \bar{g}(\bar{x},\bar{y}) &= \sum_{k\geq 2} \bar{g}_k(\bar{x},\bar{y}), \quad \bar{g}_k \in V(k,n+m,m) \end{aligned} \tag{2.5}$$

Comparing (2.3) and (1.5) it must be concluded that

$$\bar{g}_k(\bar{x},0) = -L_k(h_k(\bar{x})) + n_k(\bar{x}), \quad k \geq 2 \tag{2.6}$$

In the above notation our objective is to obtain h_k, $\bar{f}_k$.
If $x = \epsilon X$, $y = \epsilon Y$, $\epsilon \in \mathbb{R}$ in (1.1), then

$$\begin{aligned} \dot{X} &= AX + \sum_{k\geq 1} F_k(X,Y)\epsilon^k/k! \\ \dot{Y} &= BY + \sum_{k\geq 1} G_k(X,Y)\epsilon^k/k! \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} F_k(X,Y) &= k!\, f_{k+1}(X,Y) \\ G_k(X,Y) &= k!\, g_{k+1}(X,Y), \quad k \geq 1 \end{aligned} \tag{2.8}$$

and they are homogeneous polynomials in (X,Y) of degree $k+1$. Also define $F_0(X,Y) = AX$ and $G_0(X,Y) = BY$.

Now consider a transformation of variables:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ H(\tilde{X}) \end{pmatrix} = \begin{pmatrix} \tilde{X} \\ \tilde{Y} + \sum_{k\geq 1} H_k(\tilde{X}) \end{pmatrix} \tag{2.9}$$

where the H_k are homogeneous in $\tilde{X}$ of degre $k+1$. Then the differential equations for $(\tilde{X},\tilde{Y})$ are

$$\begin{aligned} \dot{\tilde{X}} &= A\tilde{X} + \sum_{k\geq 1} \tilde{F}_k(\tilde{X},\tilde{Y})\epsilon^k/k! \\ \dot{\tilde{Y}} &= B\tilde{Y} + \sum_{k\geq 1} \tilde{G}_k(\tilde{X},\tilde{Y})\epsilon^k/k! \end{aligned} \tag{2.10}$$

where the $\tilde{F}_k, \tilde{G}_k$ are homogeneous polynomials in $(\tilde{X},\tilde{Y})$ of degree $k+1$. Consequently, the changes of variables $x = \epsilon X$, $y = \epsilon Y$; $\bar{x} = \epsilon\tilde{X}$, $\bar{y} = \epsilon\tilde{Y}$ and (2.9) yield the system (2.2) provided that

$$H_k(\tilde{X}) = k!\, h_{k+1}(\tilde{X}), \quad k \geq 1 \tag{2.11}$$

and so we obtain

$$\begin{aligned} \tilde{F}_k(\tilde{X},\tilde{Y}) &= k!\,\bar{f}_{k+1}(\tilde{X},\tilde{Y}) \\ \tilde{G}_k(\tilde{X},\tilde{Y}) &= k!\,\bar{g}_{k+1}(\tilde{X},\tilde{Y}) \end{aligned} \tag{2.12}$$

In fact, transforming (2.7) by the changes defined by (2.9) is equivalent to transforming (1.1) by the changes of the form (2.1). The reason justifying the above set of transformations is that the $\tilde{F}_k$, $\tilde{G}_k$ can be recursively computed from F_i, G_i, H_i, $i \leq k$; and so, the relations (2.8), (2.11) and (2.12) enable us to calculate recursively $\bar{f}_k$, $\bar{g}_k$.

We introduce now the following notation:

$$\begin{pmatrix} R(X,Y) \\ S(X,Y) \end{pmatrix} \times \begin{pmatrix} 0 \\ T(X) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial R(X,Y)}{\partial Y} T(X) \\ \dfrac{\partial S(X,Y)}{\partial Y} T(X) - \dfrac{\partial T(X)}{\partial X} R(X,Y) \end{pmatrix} \tag{2.13}$$

Notice that this convention is related to the Lie bracket operator when applied to the two particular functions above.

If we define the sequence

$$\begin{pmatrix} F_l^i \\ G_l^i \end{pmatrix}, \quad l,i = 0,1,2,\ldots$$

by the recursive relations:

$$\begin{aligned} & F_l^0 = F_l, \quad G_l^0 = G_l, \quad l = 0,1,2,\ldots \\ & \begin{pmatrix} F_l^i \\ G_l^i \end{pmatrix} = \begin{pmatrix} F_{l+1}^{i-1} \\ G_{l+1}^{i-1} \end{pmatrix} + \sum_{j=0}^{l} \binom{l}{j} \begin{pmatrix} F_{l-j}^{i-1} \\ G_{l-j}^{i-1} \end{pmatrix} \times \begin{pmatrix} 0 \\ H_{j+1} \end{pmatrix} \quad \begin{matrix} l = 0,1,2,\ldots \\ i = 1,2,3,\ldots \end{matrix} \end{aligned} \tag{2.14}$$

then it can be proved ([3], [8]):

$$\begin{pmatrix} \tilde{F}_k \\ \tilde{G}_k \end{pmatrix} = \begin{pmatrix} F_0^k \\ G_0^k \end{pmatrix}, \quad k = 1,2,\ldots \tag{2.15}$$

We remark that the computations (2.14) can be accomplished by considering the so-called Lie triangle:

$$\begin{array}{cccccc} Z_0^0 & & & & & \\ Z_1^0 & Z_0^1 & & & & \\ Z_2^0 & Z_1^1 & Z_0^2 & & & \\ Z_3^0 & Z_2^1 & Z_1^2 & Z_0^3 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \qquad \text{where } Z_l^i = \begin{pmatrix} F_l^i \\ G_l^i \end{pmatrix},$$

and each element can be calculated by using the elements in the column one step to the left and up. From (2.15) the searched elements are Z_0^k, which are on the diagonal of Lie triangle. Note that in each row the terms involved have always the same degree.

Remember that our objective is to obtain h_k, $\bar{f}_k$, $k \geq 2$, and now, since (2.11), (2.12), it is the same to compute H_k, $\tilde{F}_k$, $k \geq 1$. From (2.12), the condition $\bar{g}(\bar{x}, 0) = 0$ becomes $\tilde{G}_k(\tilde{X}, 0) = 0$, $k \geq 1$, and then we can write (see 2.6 and 2.11):

$$\tilde{G}_k(\tilde{X}, 0) = G_0^k(\tilde{X}, 0) = k! \{-L_{k+1}(\frac{H_k(\tilde{X})}{k!}) + n_{k+1}(\tilde{X})\} = 0, \quad k \geq 1 \tag{2.16}$$

We recognize in (2.16) the equation satisfying the k-approximation of the center manifold, which is obtained in a recursive way as the second component of Z_0^k element on the diagonal of Lie triangle. Furthermore, the first component of Z_0^k is precisely $\tilde{F}_k$ (see (2.15)) which leads us to the reduced system.

We can rewrite (2.16) as

$$L_{k+1}(H_k(\tilde{X})) = N_{k+1}(\tilde{X}), \quad k \geq 1 \tag{2.17}$$

where $N_{k+1}(\cdot) = k!\, n_{k+1}(\cdot)$. A key observation is that we can split the algorithm in two branches, i.e. it is more convenient to compute L_{k+1} on one hand and N_{k+1} on the other hand. Thus, our aim will be now to obtain separately the two sides of (2.17).

We now turn to perform some adaptations which permits us to achieve N_{k+1}. For that we set

$$Z_{k-i}^i = W_{k-i}^i + \begin{pmatrix} 0 \\ -L_{k+1}(H_k(\tilde{X})) \end{pmatrix}, \quad k \geq 1, \quad 1 \leq i \leq k \tag{2.18}$$

and then it can be strictly proved that a recursive relation analogous to (2.14) holds for the W's. In fact the last term in the summatory leading to Z_{k-1}^1 (i.e. with $j = k-1$) becomes

$$Z_0^0 \times \begin{pmatrix} 0 \\ H_k(\tilde{X}) \end{pmatrix} = \begin{pmatrix} 0 \\ -L_{k+1}(H_k(\tilde{X})) \end{pmatrix} \tag{2.19}$$

and then,

$$W_{k-1}^1 = \begin{pmatrix} F_k \\ G_k \end{pmatrix} + \sum_{j=0}^{k-2} \binom{k-1}{j} \begin{pmatrix} F_{k-j-1} \\ G_{k-j-1} \end{pmatrix} \times \begin{pmatrix} 0 \\ H_{j+1} \end{pmatrix} \tag{2.20}$$

Furthermore, taking into account that

$$\begin{pmatrix} 0 \\ S(X) \end{pmatrix} \times \begin{pmatrix} 0 \\ T(X) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we obtain

$$Z_l^j \times \begin{pmatrix} 0 \\ H_{j+1} \end{pmatrix} = W_l^j \times \begin{pmatrix} 0 \\ H_{j+1} \end{pmatrix} \tag{2.21}$$

and therefore

$$W^i_{k-i} = W^{i-1}_{k-i+1} + \sum_{j=0}^{k-i} \binom{k-i}{j} W^{i-1}_{k-i-j} \times \begin{pmatrix} 0 \\ H_{j+1} \end{pmatrix}, \quad 2 \le i \le k \tag{2.22}$$

for all $k \ge 1$. With this notation, we construct a similar triangle without the first column:

$$\begin{array}{ccccc} W^1_0 & & & & \\ W^1_1 & W^2_0 & & & \\ W^1_2 & W^2_1 & W^3_0 & & \\ W^1_3 & W^2_2 & W^3_1 & W^4_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Note that (2.16) together with (2.18) implies that the first n components of W^k_0 and Z^k_0 are the same, giving us $\bar{F}_k$, and the last m components now provide us N_{k+1}. This strategy along with the determination of L_{k+1} for each k (see (3.1)) allows us the setting of the linear system (2.17).

In practice, the final objective is usually the reduced system and it should be noticed that computing the first n components of the next row in the above triangle up to W^0_{k+1}, the $(k+1)$-approximation to the reduced equation is obtained.

3. Programming aspects

The above approach permits us to set up a computer algebra algorithm which proceeds iteratively up to a settled order. It is possible to implement the algorithm by selecting the appropriate primitives of a computer algebra system merely reproducing the mentioned steps. However, as noted in the introduction, it is more efficient and less expensive to use a vectorial representation of the functions involved.

We constrain ourselves to work in $V(k+1,n,m)$ choosing an ordered basis. In particular, we will use a lexicographic ordered basis.

3.1. Representation of L_{k+1}

Let us denote d the number of different $(k+1)$-degree monomials in n-vector x and let $\mathcal{P}_{k+1} = \{p^1, p^2, \ldots, p^d\}$ be the ordered set of n-indices with module $k+1$. If $\mathbf{e}^m_l$ stands for the l-th element of a canonical m-dimensional basis, then

$$\mathcal{B}_{k+1} = \{x^{p^i} \cdot \mathbf{e}^m_l : 1 \le i \le d, \quad p^i \in \mathcal{P}_{k+1}, \quad 1 \le l \le m\}$$

is a basis of $V(k+1,n,m)$ where $x^{p^i} = x_1^{p^i_1} x_2^{p^i_2} \cdots x_n^{p^i_n}$.

To determine the matrix representation of L_{k+1} over $\mathcal{B}_{k+1}$ observe that

$$L_{k+1}(x^{p^i}\cdot \mathbf{e}_l^m) = \frac{\partial(x^{p^i}\cdot \mathbf{e}_l^m)}{\partial x}Ax - Bx^{p^i}\cdot \mathbf{e}_l^m =$$

$$= (-b_{1l}x^{p^i}, -b_{2l}x^{p^i}, \ldots, \sum_{s=1}^{n}\sum_{r=1}^{n}(p_s^i a_{sr}x_r\frac{x^{p^i}}{x_s}) - b_{ll}x^{p^i}, \ldots, -b_{ml}x^{p^i})^T \qquad (3.1.1)$$

where

$$\frac{x^{p^i}}{x_s} = \begin{cases} x_1^{p_1^i}x_2^{p_2^i}\cdots x_s^{p_s^i-1}\cdots x_n^{p_n^i}, & \text{if } p_s^i \geq 1, \\ x^0, & \text{otherwise.} \end{cases}$$

Thus (3.1.1) provides the image of $((l-1)d+1)$-th basic vector in $\mathcal{B}_{k+1}$. To obtain the $\mathcal{B}_{k+1}$-representation of above m-vector we can organize the matrix representation of L_{k+1} as a matrix of $m \times m$ blocks, each block being a $d \times d$ matrix. Then we can identify

$$L_{k+1} \equiv \begin{pmatrix} \mathcal{A}-b_{11}I & -b_{12}I & \cdots & -b_{1m}I \\ -b_{21}I & \mathcal{A}-b_{22}I & \cdots & -b_{2m}I \\ \vdots & \vdots & \ddots & \vdots \\ -b_{m1}I & -b_{m2}I & \cdots & \mathcal{A}-b_{mm}I \end{pmatrix} \qquad (3.1.2)$$

where I is the identity of order d and $\mathcal{A}$ is a square matrix independent on l which arises from the double sumnatory in (3.1.1).

Regarding (3.1.2), it is obvious the importance of the structure of matrix B. In fact, if B is triangular we can solve the corresponding system (2.17) by means of a backward-substitution process. Without loss of generality, we can suppose that the matrix B is in its Jordan form, and then we might adopt specific methods to solve (2.17).

The matrix A conditiones strongly the structure of $\mathcal{A}$ that should make possible in several typical cases we might adopt specific resolution methods. In any case $\mathcal{A}$ is certainly sparse and a deeper study of its structure can be of interest.

3.2. Computing N_{k+1}

Now we will denote d' the number of different $(k+1)$-degree monomials in $(n+m)$-vector (x,y) and let $\mathcal{Q}_{k+1} = \{q^1, q^2, \ldots, q^{d'}\}$ be the ordered set of $(n+m)$-indices with module $k+1$ while $\mathbf{e}_l^{n+m}$ stands for the l-th element of a canonical basis in $\mathbb{R}^{n+m}$. To represent adequately the W's expressions and (F_k, G_k), we can construct a basis of $V(k+1, n+m, n+m)$ by

$$\mathcal{B}'_{k+1} = \{(x,y)^{q^i}\cdot \mathbf{e}_l^{n+m} : 1 \leq i \leq d', \quad q^i \in \mathcal{Q}_{k+1}, \quad 1 \leq l \leq n+m\}$$

In this context to perform (2.20) and (2.22), we will split the corresponding expresions in terms of basic elements and as the $\times$-operation is clearly linear it is interesting to verify its behaviour over those elements. Thus if $1 \leq l_2 \leq m$, $q \in \mathcal{Q}_{k_1}$, and $p \in \mathcal{P}_{k_2}$ one can obtain

$$(x,y)^q \cdot \mathbf{e}_{l_1}^{n+m} \times \begin{pmatrix} 0 \\ x^p \cdot e_{l_2}^m \end{pmatrix} =$$

$$= \begin{cases} q_{n+l_2} \dfrac{(x,y)^q}{y_{l_2}} x^p \cdot e_{l_1}^{n+m} - p_{l_1}(x,y)^q \dfrac{x^p}{x_{l_1}} \cdot e_{n+l_2}^{n+m}, & \text{for } 1 \le l_1 \le n, \\ q_{n+l_2} \dfrac{(x,y)^q}{y_{l_2}} x^p \cdot e_{n+l_1}^{n+m}, & \text{for } n+1 \le l_1 \le n+m. \end{cases} \tag{3.2.1}$$

where

$$\frac{(x,y)^q}{y_{l_2}} = \begin{cases} x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n} y_1^{q_{n+1}} \cdots y_{l_2}^{q_{n+l_2}-1} \cdots y_m^{q_{n+m}}, & \text{if } q_{n+l_2} \ge 1, \\ (x,y)^0, & \text{otherwise.} \end{cases}$$

The above expressions enable us to work only with vectorial coefficients instead of the corresponding polynomial by means of their representation in $\mathcal{B}_{k_1}$, $\mathcal{B}'_{k_2}$. For that it is useful to have defined some auxiliar procedures to handle the basic elements. One can argument that with this aproach we waste the possibilities of symbolic computation, but the experience suggests that in most cases computer algebra cannot be effectively used by merely transcribing formulas. Note that symbolic computation is still needed because the mentioned coefficients can involve additional parameters. If we adopt no such scheme (i.e. working directly memory polynomials) we can exhaust rapidly the memory space (thereby increasing the number of "garbage collection" or what should be worse, causing a machine "hang-up"). Furthermore the frequent built-in function calls (v.g. derivatives) would be very time-consuming.

3.3. Program structure

Thus we can summarize the algorithm as follows:

(a) Read data problem A, B, f, g (see (1.1)).
(b) Set $k_{\max}$, the wanted accuracy degree.
(c) Build the basis $\mathcal{B}'_k$ for $2 \le k \le k_{\max}$. Note that $\mathcal{B}_k$ is easily localized into $\mathcal{B}'_k$.
(d) Set up the vectorial coefficients of data functions f, g.
(e) Loop: for $k = 1, \ldots, k_{\max} - 1$
  (e.1) Determine $\mathcal{A}$ corresponding to $k+1$
  (e.2) Compute W_{k-1}^1 according to (2.20) and (3.2.1)
  (e.3) Loop: for $l = 2, \ldots, k$ compute W_{k-l}^l following (2.22) and (3.2.1)
  (e.4) Segregate N_{k+1} from W_0^k
  (e.5) Solve (2.17) using (3.1.2) to obtain H_k.
(f) Write results.

4. Computational results

We have obtained a first implementation of the above algorithm on REDUCE 3.2 [7]. Now we present the results achieved for two examples. We want to mention that in both

cases the use of the algorithm has been crucial. Previous calculations by hand (we do not recommend it) or by reproducing (1.3) directly on a computer algebra system required tedious work, even though a deeper analysis to remove unnecessary terms had been made.

4.1. Example 1 ($n = 3, m = 1, k_{\max} = 4$)

In [1], a system dynamics model representing the evolution of three urban zones with a diffusion mechanism among zones is presented. After a certain change of variables, the model can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -3u & 0 & 0 \\ 0 & -3u & 0 \\ 0 & 0 & a_0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + a_1 \begin{pmatrix} \frac{-x^2}{\sqrt{6}} + \frac{y^2}{\sqrt{6}} + \frac{2xz}{\sqrt{3}} \\ \frac{2xy}{\sqrt{6}} + \frac{2yz}{\sqrt{3}} \\ \frac{x^2}{\sqrt{3}} + \frac{y^2}{\sqrt{3}} + \frac{z^2}{\sqrt{3}} \end{pmatrix}$$

$$+ a_2 \begin{pmatrix} \frac{x^3}{2} + \frac{xy^2}{2} + xz^2 - \frac{x^2z}{\sqrt{2}} + \frac{y^2z}{\sqrt{2}} \\ \frac{y^3}{2} + yz^2 - \frac{x^2y}{2} + \frac{y^2z}{\sqrt{2}} + \sqrt{2}xyz \\ \frac{-x^3}{3\sqrt{2}} + \frac{xy^2}{\sqrt{2}} + \frac{z^3}{3} + x^2z + y^2z \end{pmatrix} \tag{4.1.1}$$

$$+ a_3 \begin{pmatrix} \frac{1}{\sqrt{6}}(-\frac{5x^4}{6} + x^2y^2 + \frac{y^4}{2} + \frac{8xz^3}{3\sqrt{2}} + 2z^2(y^2 - x^2) + \frac{4x\bar{z}(x^2 + y^2)}{\sqrt{2}}) \\ \frac{1}{\sqrt{2}}(\frac{2y(x^3 + 3xy^2)}{3\sqrt{3}} + \frac{8yz^3}{3\sqrt{6}} + \frac{4xyz^2(x^2 + y^2)}{\sqrt{3}} + \frac{4yz(x^2 + y^2)}{\sqrt{6}}) \\ \frac{1}{\sqrt{3}}(\frac{x^4}{2} + x^2y^2 + \frac{y^4}{2} + \frac{z^4}{3} + 2z^2(x^2 + y^2) - \frac{4z(x^3 - 3xy^2)}{3\sqrt{2}}) \end{pmatrix}$$

where x, y, z are related with the urban development of each zone, u stands for the diffusion coefficient and a_0, a_1, a_2, a_3 are parameters defining the nonlinearity involved. We adopt u as the bifurcation parameter, and then we must add to (4.1.1) a new equation: $\dot{u} = 0$.

It can be easily seen that the critical value of the bifurcation parameter is $u_c = 0$ and that (4.1.1) is in correspondence with (1.1) if we set $x_1 = x, \quad x_2 = y, \quad x_3 = u, \quad y_1 = z$, where A is now a tridimensional zero-matrix and $B = (a_0)$.

The model is equivariant under a simmetry group (the dihedral group D_3) and this fact made possible –by using a specific basis obtained with the use of complex variables– to compute by hand the corresponding center manifold. We present here the results obtained up to fourth order with our algorithm that are in full concordance with the previous calculations [9]:

$$
\begin{aligned}
h(x,y,u) = &-\frac{a_1}{\sqrt{3}a_0}y^2 - \frac{a_1}{\sqrt{3}a_0}x^2 + 6\frac{a_1}{\sqrt{3}a_0^2}y^2u - \sqrt{2}\frac{a_0a_2+2a_1^2}{2a_0^2}xy^2 \\
&+ 6\frac{a_1}{\sqrt{3}a_0^2}x^2u + \sqrt{2}\frac{a_0a_2+2a_1^2}{6a_0^2}x^3 - 36\frac{a_1}{\sqrt{3}a_0^3}y^2u^2 \\
&- \sqrt{2}\frac{a_0a_3+a_1a_2}{2\sqrt{6}a_0^2}y^4 + 3\sqrt{2}\frac{3a_0a_2+10a_1^2}{2a_0^3}xy^2u - 36\frac{a_1}{\sqrt{3}a_0^3}x^2u^2 \\
&- \sqrt{2}\frac{a_0a_3+a_1a_2}{\sqrt{6}a_0^2}x^2y^2 - \sqrt{2}\frac{3a_0a_2+10a_1^2}{2a_0^3}x^3u - \sqrt{2}\frac{a_0a_3+a_1a_2}{2\sqrt{6}a_0^2}x^4 + \cdots
\end{aligned}
\tag{4.1.2}
$$

4.2. Example 2 ($n=4, m=1, k_{\max}=3$)

Our second example arise from the study of an electronic circuit partially analyzed in [6]. The equations of the model are:

$$
\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}-(\beta+\nu)/r & \beta/r & 0\\ \beta & -\beta & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}-\frac{a_3}{r}x^3 + \frac{b_3}{r}(y-x)^3\\ -b_3(y-x)^3\\ 0\end{pmatrix} \tag{4.2.1}
$$

where x, y, z are the state variables corresponding to voltages and currents in the circuit and r, a_3, b_3 are additional parameters while β, ν are bifurcation parameters. We try to compute the center manifold corresponding to $\beta_c = -\nu_c = \sqrt{r}$. In such case, the linear part of (4.2.1) presents a double-zero eigenvalue with Riesz index 2 and a simple eigenvalue equal to $-\sqrt{r}$. To achieve a formulation according to (1.1), we must perform some preparations.

Firstly we make a translation over β, ν: $\bar{\beta} = \beta - \beta_c$, $\bar{\nu} = \nu - \nu_c$, so that bifurcation occurs in a neighbourhood of $\bar{\beta} = 0$, $\bar{\nu} = 0$. Also we must include in (4.2.1) the equations $\dot{\bar{\beta}} = 0$, $\dot{\bar{\nu}} = 0$. Furthermore, we make a change of variables

$$
\begin{pmatrix}x\\ y\\ z\end{pmatrix} = P\begin{pmatrix}x_1\\ x_2\\ y_1\end{pmatrix}, \quad x_3 = \bar{\beta}, \quad x_4 = \bar{\nu}
$$

which leads us to the Jordan form of linear part in (4.2.1) (P is the matrix of principal vectors for $\beta = \beta_c$, $\nu = \nu_c$) and now we are in correspondence with (1.1).

In a previous work [4], we followed the approach in [5] but we exhausted the computing facilities at our disposal without success. Only after a strong guidance of the symbolic computations we achieved our purpose. By using the algorithm we obtain the solution in a few cpu-minutes of microVAX-II:

$$
\begin{aligned}
h(x_1,x_2,x_3,x_4) = & \frac{-r+1}{\sqrt{r}r^3}x_2x_4 + \frac{r+1}{\sqrt{r}r^3}x_2x_3 - \frac{1}{r^3}x_1x_4 - \frac{r+1}{r^3}x_1x_3 - \frac{(r-2)^2}{r^6}x_2x_4^2 \\
& + \frac{r^3+2r^2-5r-8}{r^6}x_2x_3x_4 - \frac{2r^3+7r^2+9r+4}{r^6}x_2x_3^2 \\
& + \frac{-a_3r^3+3a_3r^2+6(b_3-a_3)r+6(b_3+a_3)}{\sqrt{r}r^5}x_2^3 + \frac{-r+3}{\sqrt{r}r^5}x_1x_4^2 \\
& + \frac{r^2+5r+6}{\sqrt{r}r^5}x_1x_3x_4 + \frac{r^3+4r^2+6r+3}{\sqrt{r}r^5}x_1x_3^2 \\
& - 3\frac{a_3r^2+2(b_3-a_3)r+2(b_3+a_3)}{r^5}x_1x_2^2 \\
& + 3\frac{(b_3-a_3)r+(b_3+a_3)}{\sqrt{r}r^4}x_1^2x_2 - \frac{b_3r+(b_3+a_3)}{r^4}x_1^3
\end{aligned}
\qquad (4.2.2)
$$

5. Concluding remarks

A new algorithm for symbolic computation of center manifolds is introduced. Using an algorithm to compute normal forms, we derive a recursive algorithm to calculate center manifolds. Rand and Keith [10] use this approach but not arriving to an iterative scheme and so making not advantages of full capability of normal form transformations.

In our opinion this algorithm is a good exponent of the way computer algebra must be guided to perform effectively a complex calculation. We have tested the algorithm with different examples already solved by other means (two of them are presented here) and it has overcame several computational difficulties in previous approaches. The program used –written in REDUCE 3.2– is available at request to authors.

Future research should be directed to some enhacements already mentioned. In particular, we must investigate what polynomial internal representation is best for our purposes and the possible ameliorations depending on the actual bifurcations involved. Further in presence of simmetry a choice of adequate bases should be fruitful, by lowering the dimension of vectorial representation managed.

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