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SEMIGROUPS OF COMPOSITION OPERATORS AND INTEGRAL OPERATORS IN SPACES OF ANALYTIC FUNCTIONS

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Abstract. We study the maximal spaces of strong continuity on $BMOA$ and the Bloch space $\mathcal{B}$ for semigroups of composition operators. Characterizations are given for the cases when these maximal spaces are $VMOA$ or the little Bloch $\mathcal{B}_0$. These characterizations are in terms of the weak compactness of the resolvent function or in terms of a specially chosen symbol g of an integral operator T_g. For the second characterization we prove and use an independent result, namely that the operators T_g are weakly compact on the above mentioned spaces if and only if they are compact.

1. Introduction

Let $\mathcal{H}(\mathbf{D})$ be the Fréchet space of all analytic functions in the unit disk endowed with the topology of uniform convergence on compact subsets of $\mathbf{D}$. We say that a Banach space X is a Banach space of analytic functions if consists of functions of $\mathcal{H}(\mathbf{D})$ such that the inclusion $i(f) = f\colon X \to \mathcal{H}(\mathbf{D})$ is continuous.

A (one-parameter) semigroup of analytic functions is any continuous homomorphism $\Phi\colon t \mapsto \Phi(t) = \varphi_t$ from the additive semigroup of nonnegative real numbers into the composition semigroup of all analytic functions which map $\mathbf{D}$ into $\mathbf{D}$. In

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other words, $\Phi = (\varphi_t)$ consists of analytic functions on $\mathbf{D}$ with $\varphi_t(\mathbf{D}) \subset \mathbf{D}$ and for which the following three conditions hold:

(1) φ_0 is the identity in $\mathbf{D}$,
(2) $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
(3) $\varphi_t \to \varphi_0$, as $t \to 0$, uniformly on compact subsets of $\mathbf{D}$.

It is well known that condition (3) above can be replaced by

(3′) For each $z \in \mathbf{D}$, $\varphi_t(z) \to z$, as $t \to 0$.

Each such semigroup gives rise to a semigroup (C_t) consisting of composition operators on $\mathcal{H}(\mathbf{D})$,

$$C_t(f) := f \circ \varphi_t, \qquad f \in \mathcal{H}(\mathbf{D}).$$

We are going to be interested in the restriction of (C_t) to certain linear subspaces $\mathcal{H}(\mathbf{D})$. Given a Banach space of analytic functions and a semigroup (φ_t), we say that (φ_t) generates a semigroup of operators on X if (C_t) is a well-defined strongly continuous semigroup of bounded operators in X. This exactly means that for every $f \in X$, we have $C_t(f) \in X$ for all $t \geq 0$ and

$$\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0.$$

Thus the crucial step to show that (φ_t) generates a semigroup of operators in X is to pass from the pointwise convergence $\lim_{t\to 0^+} f \circ \varphi_t(z) = f(z)$ on $\mathbf{D}$ to the convergence in the norm of X.

This connection between composition operators and operator semigroups opens the possibility of studying spectral properties, operator ideal properties or dynamical properties of the semigroup of operators (C_t) in terms of the theory of functions. The paper [3] can be considered as the starting point in this direction.

Classical choices of X treated in the literature are the Hardy spaces H^p, the disk algebra $A(\mathbf{D})$, the Bergman spaces A^p, the Dirichlet space $\mathcal{D}$ and the chain of spaces Q_p and $Q_{p,0}$ which have been introduced recently and which include the spaces $BMOA$, Bloch as well as their "little oh" analogues. See [26] and [28] for definitions and basic facts of the spaces and [22], [23] and [25] for composition semigroups on these spaces.

Very briefly, the state of the art is the following: (i) Every semigroup of analytic functions generates a semigroup of operators on the Hardy spaces H^p $(1 \leq p < \infty)$, the Bergman spaces A^p $(1 \leq p < \infty)$, the Dirichlet space, and on the spaces VMOA and little Bloch. (ii) No non-trivial semigroup generates a semigroup of operators in the space H^∞ of bounded analytic functions. (iii) There are plenty of semigroups (but not all) which generate semigroups of operators in the disk algebra. Indeed, they can be well characterized in different analytical terms [6].

Recently, in [4], the study of semigroups of composition operators in the framework of the space $BMOA$ was initiated. The present paper can be considered as a sequel of [4]. In section 3 we state some general facts about the maximal subspace of strong continuity $[\varphi_t, X]$ for the composition semigroup induced by (φ_t) on an abstract Banach space X of analytic functions satisfying certain conditions. In section 4 we consider the maximal subspaces $[\varphi_t, BMOA]$ and $[\varphi_t, \mathcal{B}]$. In particular, for the case of the Bloch space we show that no non-trivial composition semigroup is strongly continuous on the whole space $\mathcal{B}$ thus $[\varphi_t, \mathcal{B}]$ is strictly contained in $\mathcal{B}$. We also show that that the equality $[\varphi_t, BMOA] = VMOA$ or $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ is equivalent to the weak compactness of the resolvent operator $\mathcal{R}(\lambda, \Gamma)$ for the composition semigroup $\{C_t\}$ acting on $VMOA$ (respectively on $\mathcal{B}_0$). Here Γ denotes the infinitesimal generator of $\{C_t\}$ and for each λ in the resolvent set of Γ, $\mathcal{R}(\lambda, \Gamma)$ denotes the (bounded) resolvent operator

$$\mathcal{R}(\lambda, \Gamma) = (\lambda - \Gamma)^{-1}.$$

In section 5 we study an integral operator T_g on $BMOA$ and $\mathcal{B}$ and we show in particular that its compactness and weak compactness are equivalent (Theorem 6). This result for the case of $BMOA$ was also obtained independently by different methods in [13]. In section 6 we apply these results for T_g for a special choice of the symbol $g = \gamma$ (defined later in Definition 4) to obtain a characterization of the cases of equality $[\varphi_t, BMOA] = VMOA$ and $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ in terms of γ (Corollary 2). In the final section 7 we make some additional observations concerning the Koenigs function of the semigroup in relation to space of strong continuity, and state some related open questions.

2. Background

If (φ_t) is a semigroup, then each map φ_t is univalent. The infinitesimal generator of (φ_t) is the function

$$G(z) = \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t}, \quad z \in \mathbf{D}.$$

This convergence holds uniformly on compact subsets of $\mathbf{D}$ so $G \in \mathcal{H}(\mathbf{D})$. Moreover G satisfies

$$G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z)\frac{\partial \varphi_t(z)}{\partial z}, \quad z \in \mathbf{D},\ t \geq 0. \tag{1}$$

Further G has a representation

$$G(z) = (\bar{b}z - 1)(z - b)P(z), \quad z \in \mathbf{D}, \tag{2}$$

where $b \in \overline{\mathbf{D}}$ and $P \in \mathcal{H}(\mathbf{D})$ with $\operatorname{Re} P(z) \geq 0$ for all $z \in \mathbf{D}$. If G is not identically null, the couple (b, P) is uniquely determined from (φ_t) and the point b is called the Denjoy–Wolff point of the semigroup. We want to mention that this point plays a crucial role in the dynamical behavior of the semigroup (see [23, 7]).

Recall also the notion of Koenigs function associated with a semigroup. For every non-trivial semigroup (φ_t) with generator G, there exists a unique univalent function $h \colon \mathbf{D} \to \mathbf{C}$, called the *Koenigs function* of (φ_t), such that

1. If the Denjoy–Wolff point b of (φ_t) is in $\mathbf{D}$ then $h(b) = 0$, $h'(b) = 1$ and

$$h(\varphi_t(z)) = e^{G'(b)t}h(z) \ \text{ for all } z \in \mathbf{D} \text{ and } t \geq 0. \tag{3}$$

Moreover,

$$h'(z)G(z) = G'(b)h(z), \quad z \in \mathbf{D}. \tag{4}$$

2. If the Denjoy–Wolff point b of (φ_t) is on $\partial\mathbf{D}$ then $h(0) = 0$ and

$$h(\varphi_t(z)) = h(z) + t \ \text{ for all } z \in \mathbf{D} \text{ and } t \geq 0. \tag{5}$$

Moreover,

$$h'(z)G(z) = 1, \quad z \in \mathbf{D}. \tag{6}$$

For the sake of completeness and to fix notations, we present a quick review of basic properties of $BMOA$, $VMOA$, the Bloch space $\mathcal{B}$, and the little Bloch space $\mathcal{B}_0$.

$BMOA$ is the Banach space of all analytic functions in the Hardy space H^2 whose boundary values have bounded mean oscillation. There are many characterizations of this space but we will use the one in terms of Carleson measures (see [28, 12]). Namely, a function $f \in H^2$ belongs to $BMOA$ if and only if there exists a constant $C > 0$ such that

$$\int_{R(I)} |f'(z)|^2(1-|z|^2)\, dA(z) \leq C|I|,$$

for any arc $I \subset \partial\mathbf{D}$, where $R(I)$ is the Carleson rectangle determined by I, that is,

$$R(I) := \left\{ re^{i\theta} \in \mathbf{D} \colon 1 - \frac{|I|}{2\pi} < r < 1 \text{ and } e^{i\theta} \in I \right\}.$$

As usual, $|I|$ denotes the length of I and $dA(z)$ the normalized Lebesgue measure on $\mathbf{D}$. The corresponding $BMOA$ norm is

$$\|f\|_{BMOA} := |f(0)| + \sup_{I\subset\partial\mathbf{D}} \left(\frac{1}{|I|} \int_{R(I)} |f'(z)|^2(1-|z|^2)\, dA(z) \right)^{1/2}.$$

Trivially, each polynomial belongs to $BMOA$. The closure of all polynomials in $BMOA$ is denoted by $VMOA$. Alternatively, $VMOA$ is the subspace of $BMOA$ formed by those $f \in BMOA$ such that

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_{R(I)} |f'(z)|^2(1-|z|^2)\, dA(z) = 0.$$

Particular and quite interesting examples of members of $VMOA$ are provided by functions in the Dirichlet space $\mathcal{D}$, which is the space of those functions $f \in \mathcal{H}(\mathbf{D})$ such that $f' \in L^2(\mathbf{D}, dA)$ with norm $\|f\|_{\mathcal{D}} = |f(0)|^2 + \|f'\|^2_{L^2(\mathbf{D},\, dA)}$. In fact, for every $f \in \mathcal{D}$,

$$\frac{1}{|I|} \int_{R(I)} |f'(z)|^2(1-|z|^2)\, dA(z) \leq 2 \int_{R(I)} |f'(z)|^2\, dA(z) \to 0,$$

as $|I| \to 0$, so $\mathcal{D}$ is contained in $VMOA$. This last inequality also implies that there is an absolute constant C such that

$$\|f\|_{BMOA} \leq C\|f\|_{\mathcal{D}} \tag{7}$$

for each $f \in \mathcal{D}$.

A holomorphic function $f \in \mathcal{H}(\mathbf{D})$ is said to belong to the *Bloch space* $\mathcal{B}$ whenever $\sup_{z\in\mathbf{D}} \left(1-|z|^2\right) |f'(z)| < \infty$. It is well-known that $\mathcal{B}$ is a Banach space when it is endowed with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbf{D}} \left(1-|z|^2\right) |f'(z)|.$$

The closure of the polynomials in $\mathcal{B}$ is called the *little Bloch space* and it is denoted by $\mathcal{B}_0$. It is also well-known that $f \in \mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1} \left(1-|z|^2\right) |f'(z)| = 0.$$

For more information on these Banach spaces, we refer the reader to the excellent monographs [12] or [28].

Finally at this point we would like to mention some information about univalent functions that will be needed later. Let $h\colon \mathbf{D}\to\mathbf{C}$ be univalent. Then $h\in BMOA$ if and only if $h\in\mathcal{B}$, and $h\in VMOA$ if and only if $h\in\mathcal{B}_0$. Further $h\in\mathcal{B}$ if and only if the discs that can be inscribed in the range of h have bounded radii and $h\in\mathcal{B}_0$ if and only if the radii of these discs tend to 0 as their center moves to ∞. In addition if $h\colon\mathbf{D}\to\mathbf{C}$ is univalent and non-vanishing then $\log(h(z))\in BMOA$, while if $h(b)=0$ for some $b\in\mathbf{D}$ then $\log\frac{h(z)}{z-b}\in BMOA$. Additional information can be found in [17].

3. The space $[\varphi_t, X]$

Given a semigroup of analytic functions (φ_t) and a Banach space X of analytic functions on the unit disk, we are interested on the maximal closed subspace of X, denoted by $[\varphi_t, X]$, on which (φ_t) generates a strongly continuous semigroup (C_t) of composition operators. The existence of such a maximal subspace, as well as analytical descriptions of it, will be discussed in this section.

The next result for a Banach space X of analytic functions is contained, for the special case $X=BMOA$, in [4]. The proof follows the same lines as the one in [4], and for this reason we omit the details.

Proposition 1. *Let (φ_t) be a semigroup of analytic functions and X a Banach space of analytic functions such that $C_t\colon X\to X$ are bounded for all $t\geq 0$ and $\sup_{t\in[0,1]}\|C_t\|_X = M<\infty$. Then there exists a closed subspace Y of X such that (φ_t) generates a semigroup of operators on Y and such that any other subspace of X with this property is contained in Y.*

Definition 1. We denote by $[\varphi_t, X]$ the maximal subspace consisting of functions $f\in X$ such that $\lim_{t\to 0^+}\|f\circ\varphi_t - f\|_X = 0$.

It is easy to see that if Z is any closed subspace of $[\varphi_t, X]$ which is invariant under (C_t) (i.e. $C_t(Z)\subset Z$ for every $t\geq 0$) then (φ_t) generates a semigroup of operators on Z.

Definition 2. Given a semigroup (φ_t) with generator G and a Banach space of analytic functions X we define

$$\mathcal{D}(\varphi_t, X) := \{f\in X\colon Gf'\in X\}.$$

Clearly $\mathcal{D}(\varphi_t, X)$ is a linear subspace of X.

Theorem 1. *Let (φ_t) be a semigroup with generator G and X a Banach space of analytic functions which contains the constant functions and such that $M=\sup_{t\in[0,1]}\|C_t\|_X<\infty$. Then,*

$$[\varphi_t, X] = \overline{\mathcal{D}(\varphi_t, X)}.$$

Proof. Let us show first $[\varphi_t, X]\subseteq\overline{\mathcal{D}(\varphi_t, X)}$. We may assume that (φ_t) is not trivial. Denote by Γ the infinitesimal generator of the operator semigroup (C_t) acting on the Banach space $[\varphi_t, X]$, and by $D(\Gamma)$ its domain. We will show that if $f\in D(\Gamma)$ then $Gf'\in X$. Indeed if $f\in D(\Gamma)$ then $\Gamma(f)\in[\varphi_t, X]\subseteq X$ and

$$\lim_{t\to 0^+}\left\|\frac{1}{t}(C_t(f)-f)-\Gamma(f)\right\|_X = 0.$$

Since convergence in the norm of X implies uniform convergence on compact subsets of $\mathbf{D}$ and therefore in particular pointwise convergence, for each $z \in \mathbf{D}$ we have

$$\begin{aligned}\Gamma(f)(z) &= \lim_{t\to 0^+} \frac{f(\varphi_t(z)) - f(z)}{t} = \lim_{t\to 0^+} \frac{f(\varphi_t(z)) - f(\varphi_0(z))}{t} \\ &= \frac{\partial f \circ \varphi_t(z)}{\partial t}\bigg|_{t=0} = f'(\varphi_0(z))\, \frac{\partial \varphi_t(z)}{\partial t}\bigg|_{t=0} = f'(z)G(z),\end{aligned}$$

that is, $Gf' = \Gamma(f) \in X$, and thus $D(\Gamma) \subset \{f \in X : Gf' \in X\}$. Taking closures and bearing in mind the fact from the general theory of operator semigroups that $D(\Gamma)$ is dense in $[\varphi_t, X]$ (see for example [10, Lemma 8, p. 620]) we get the desired inclusion.

For the converse inclusion let $f \in \mathcal{D}(\varphi_t, X)$ and write $m(z) = G(z)f'(z) \in X$. The argument in the proof of the converse of [4, Theorem 2.2] can be repeated here to show

$$(f \circ \varphi_t)'(z) - f'(z) = \int_0^t (m \circ \varphi_t)'(z)\, ds, \quad t \geq 0,\ z \in \mathbf{D},$$

from which we obtain

$$\begin{aligned}(f \circ \varphi_t)(z) - f(z) &= f(\varphi_t(0)) - f(0) + \int_0^z \int_0^t (m \circ \varphi_s)'(\zeta)\, ds\, d\zeta \\ &= f(\varphi_t(0)) - f(0) + \int_0^t \int_0^z (m \circ \varphi_s)'(\zeta)\, d\zeta\, ds \\ &= f(\varphi_t(0)) - f(0) + \int_0^t [(m \circ \varphi_s)(z) - m(\varphi_s(0))]\, ds.\end{aligned}$$

It follows that for $t < 1$ we have

$$\begin{aligned}\|f \circ \varphi_t - f\|_X &\leq C|f(\varphi_t(0)) - f(0)| + \int_0^t \|m \circ \varphi_s - m(\varphi_s(0))\|_X\, ds \\ &\leq C|f(\varphi_t(0)) - f(0)| + \int_0^t \|m \circ \varphi_s\|_X ds + C \int_0^t |m(\varphi_s(0))|\, ds \\ &\leq C|f(\varphi_t(0)) - f(0)| + \left(M\|m\|_X + C \sup_{|z|\leq \rho} |m(z)| \right) t,\end{aligned}$$

where $C = \|1\|_X$ and $\rho = \sup_{s\in[0,1]} |\varphi_s(0)| < 1$. Taking $t \to 0$ we obtain $\|f \circ \varphi_t - f\|_X \to 0$, therefore $\mathcal{D}(\varphi_t, X) \subset [\varphi_t, X]$. Taking closures we get the desired inclusion and this finishes the proof. □

Theorem 2. *Let (φ_t) be a semigroup with generator G and X a Banach space of analytic functions such that (C_t) is strongly continuous on X. Then the infinitesimal generator Γ of (C_t) is given by $\Gamma(f)(z) = G(z)f'(z)$ with domain $D(\Gamma) = \mathcal{D}(\varphi_t, X)$.*

Proof. An argument similar to the one in the first part of the proof of Theorem 1 shows that if $f \in D(\Gamma)$ then $\Gamma(f)(z) = G(z)f'(z) \in X$ so that $D(\Gamma) \subset \mathcal{D}(\varphi_t, X)$.

On the other hand for λ in the resolvent set $\rho(\Gamma)$ of Γ we have

$$\begin{aligned}\mathcal{D}(\varphi_t, X) &= \{f \in X : Gf' \in X\} = \{f \in X : Gf' - \lambda f \in X\} \\ &= \{f \in X : \text{there is } m \in X \text{ such that } m = Gf' - \lambda f\} \\ &= \{f \in X : \text{there is } m \in X \text{ such that } f = \mathcal{R}(\lambda, \Gamma)(m)\} \\ &= \mathcal{R}(\lambda, \Gamma)(X).\end{aligned}$$

Since $\mathcal{R}(\lambda, \Gamma)(X) \subseteq D(\Gamma)$ this gives the conclusion. □

4. The maximal subspace for $BMOA$ and $\mathcal{B}$

For any given semigroup (φ_t), the induced operator semigroup (C_t) is known to be strongly continuous on the little Bloch space $\mathcal{B}_0$ and on $VMOA$. A proof for both spaces is contained in [25, Theorem 4.1] and an alternative proof for $VMOA$ can be found in [4, Theorem 2.4]. For the sake of completeness we recount a short proof of the strong continuity on these two spaces. Since the polynomials are dense in $\mathcal{B}_0$ and in $VMOA$ and since $\sup_{0\le t\le 1} \|C_t\|_{X\to X} < \infty$ for $X = \mathcal{B}_0$ and $X = VMOA$, by a use of the triangle inequality, the strong continuity requirement $\lim_{t\to 0} \|f \circ \varphi_t - f\|_X = 0$ for $f \in X$ reduces to the same requirement for f a polynomial. Now if f is a polynomial then $f \circ \varphi_t - f$ is a function in the Dirichlet space $\mathcal{D}$ and we have from (7) and [17, p. 592],

$$\|f \circ \varphi_t - f\|_{\mathcal{B}} \le C_1 \|f \circ \varphi_t - f\|_{BMOA} \le C_2 \|f \circ \varphi_t - f\|_{\mathcal{D}},$$

with C_1 and C_2 absolute constants. But composition semigroups are strongly continuous on $\mathcal{D}$, [22, Theorem 1], so $\lim_{t\to 0} \|f \circ \varphi_t - f\|_X = 0$ for f polynomial and the argument is complete.

Thus we have $[\varphi_t, \mathcal{B}_0] = \mathcal{B}_0$ and $[\varphi_t, VMOA] = VMOA$ so that for every semigroup (φ_t),

$$\mathcal{B}_0 \subseteq [\varphi_t, \mathcal{B}] \subseteq \mathcal{B},$$

and

$$VMOA \subseteq [\varphi_t, BMOA] \subseteq BMOA.$$

The question arises whether there are cases of semigroups for which equality holds at one or the other end of these inclusions. In the case of $BMOA$ it was proved by Sarason [19] that for each of the semigroups $\varphi_t(z) = e^{it}z$ or $\varphi_t(z) = e^{-t}z$ we have $VMOA = [\varphi_t, BMOA]$. In [4] a whole class of semigroups was identified for which this equality holds. It is easy to see that for the above semigroups of Sarason we also have $\mathcal{B}_0 = [\varphi_t, \mathcal{B}]$.

For the right hand side equalities it is unknown if there are semigroups such that $BMOA = [\varphi_t, BMOA]$. For the Bloch space however we show below that there are no non-trivial semigroups such that $[\varphi_t, \mathcal{B}] = \mathcal{B}$, answering a relevant question from [23, page 237].

We state the result for the more general class of Bloch spaces $\mathcal{B}_\alpha$, $\alpha > 0$, defined by

$$\mathcal{B}_\alpha = \left\{ f \in \mathcal{H}(\mathbf{D})\colon \sup_{z\in\mathbf{D}} \left(1 - |z|^2\right)^\alpha |f'(z)| < \infty \right\},$$

endowed with the norm $\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup_{z\in\mathbf{D}} \left(1 - |z|^2\right)^\alpha |f'(z)|$. The basic properties of these spaces can be found in [29].

Theorem 3. *Suppose $\alpha > 0$ and (φ_t) is a non-trivial semigroup of analytic functions. Then $[\varphi_t, \mathcal{B}_\alpha] \subsetneq \mathcal{B}_\alpha$.*

Proof. The result will be proved in two steps. First we show that any strongly continuous operator semigroup in $\mathcal{B}_\alpha$ is uniformly continuous, therefore its infinitesimal generator is a bounded operator. In the second step we show that for composition semigroups (C_t), this implies that the infinitesimal generator is the null function hence the semigroup must be the trivial one.

Step 1. Each strongly continuous operator semigroup in $\mathcal{B}_\alpha$ is uniformly continuous. We are going to use a theorem of Lotz [14, Theorem 3] which says that if X is a Grothendieck space with the Dunford-Pettis property, then each strongly continuous semigroup of operators on X is in fact uniformly continuous (for the terminology see the above cited work). On the other hand it is well-known that $\mathcal{B}_\alpha$ is isomorphic to the space

$$H_\alpha = \left\{ f \in \mathcal{H}(\mathbf{D})\colon \sup_{z\in\mathbf{D}} \left(1-|z|^2\right)^\alpha |f(z)| < \infty \right\},$$

and this last space is isomorphic to the space l_∞ of all bounded sequences of complex numbers (see [15]). Therefore $\mathcal{B}_\alpha$ is isomorphic to l_∞ for all positive α, and it is well known that l_∞ is a Grothendieck space with the Dunford–Pettis property (see [9, Chapter VII, Exercises 1 and 12]). It follows that $\mathcal{B}_\alpha$ is a Grothendieck space with the Dunford–Pettis property and the theorem of Lotz applies.

Step 2. If (φ_t) is a semigroup with generator G and the induced semigroup of composition operators (C_t) is strongly continuous on $\mathcal{B}_\alpha$, then $G \equiv 0$. To prove this let $\Gamma\colon \mathcal{B}_\alpha \to \mathcal{B}_\alpha$ denote the infinitesimal generator of (C_t). From Step 1, Γ is a bounded operator. Now for each $f \in \mathcal{B}_\alpha$ we have $\Gamma(f) = \lim_{t\to 0} \frac{f\circ\varphi_t - f}{t}$, the convergence being in the norm of $\mathcal{B}_\alpha$. But it is easy to see that convergence in $\mathcal{B}_\alpha$ implies uniform convergence on compact subsets of the disc and in particular it implies pointwise convergence for each $z \in \mathbf{D}$. Thus for $z \in \mathbf{D}$,

$$\Gamma(f)(z) = \lim_{t\to 0} \frac{f(\varphi_t(z)) - f(z)}{t} = \frac{\partial f(\varphi_t(z))}{\partial t}|_{t=0} = G(z)f'(z),$$

therefore $\Gamma(f) = Gf'$ for each $f \in \mathcal{B}_\alpha$.

Suppose that $G \neq 0$ and recall that $G(z) = (\bar{b}z - 1)(z - b)P(z)$ where $P \in \mathcal{H}(\mathbf{D})$ with $\operatorname{Re} P(z) \geq 0$. Since $P(z)$ has boundary values almost everywhere on the unit circle (this for example follows from the fact that such a function P belongs to the Hardy spaces H^p for all $p < 1$) we can find a point $\xi \in \partial\mathbf{D}$ such that $G^*(\xi) = \lim_{r\to 1} G(r\xi)$ exists and is finite and different from zero.

Now we distinguish three cases depending on α. If $0 < \alpha < 1$, take $f(z) = (1 - \bar{\xi}z)^{1-\alpha}$, a function in $\mathcal{B}_\alpha$. Thus

$$\Gamma(f)(z) = G(z)f'(z) = G(z)(\alpha - 1)\bar{\xi}(1 - \bar{\xi}z)^{-\alpha}$$

is a function in $\mathcal{B}_\alpha$ and therefore bounded on $\mathbf{D}$ because for these values of α, $\mathcal{B}_\alpha$ is contained in the disc algebra [29, Proposition 9]. On the other hand taking $z = r\xi$, $0 < r < 1$, we have

$$\lim_{r\to 1} G(r\xi)\, f'(r\xi) = (\alpha - 1)\bar{\xi} \lim_{r\to 1} G(r\xi)(1 - r)^{-\alpha} = \infty,$$

a contradiction.

If $\alpha = 1$ then take $f(z) = \log(\frac{1}{1-\bar{\xi}z})$, a function in the Bloch space $\mathcal{B} = \mathcal{B}_1$. Thus the function

$$\Gamma(f)(z) = G(z)\frac{\bar{\xi}}{1 - \bar{\xi}z}$$

belongs to $\mathcal{B}$, and so it must satisfy the growth estimate [28, p. 82] for Bloch functions, that is for some constant C,

$$|G(z)|\frac{1}{|1 - \bar{\xi}z|} \leq |G(0)| + C\log\frac{1}{1 - |z|}, \quad z \in \mathbf{D}.$$

In particular for $z = r\xi$ we obtain

$$|G(r\xi)| \le (1-r)|G(0)| + C(1-r)\log\frac{1}{1-|r|}$$

and taking $r \to 1$ this implies $G^*(\xi) = 0$, a contradiction.

Finally if $\alpha > 1$ let $f(z) = \left(1-\overline{\xi}z\right)^{1-\alpha}$, a function in $\mathcal{B}_\alpha$. Thus the function

$$\Gamma(f)(z) = G(z)(\alpha-1)\overline{\xi}(1-\overline{\xi}z)^{-\alpha}$$

belongs to $\mathcal{B}_\alpha$ and hence it satisfies the estimate

$$|\Gamma(f)(z)| \le |\Gamma(f)(0)| + C(1-|z|)^{1-\alpha}, \quad z \in \mathbf{D},$$

for some constant C, see [29, Theorem 18 and p. 1162]. We then obtain for $z = r\xi$

$$(\alpha-1)|G(r\xi)| \le (\alpha-1)|G(0)|(1-r)^\alpha + C(1-r),$$

and letting $r \to 1$ this implies $G^*(\xi) = \lim_{r\to 1} G(r\xi) = 0$, a contradiction. This completes the proof. □

Suppose now that X is either $VMOA$ or the little Bloch space $\mathcal{B}_0$ so that the second dual X^{**} is $BMOA$ or $\mathcal{B}$ respectively. Let (φ_t) be a semigroup on $\mathbf{D}$ and let (C_t) be the induced semigroup of composition operators on X^{**}. Since each φ_t is univalent each C_t is maps X into itself, and the restriction $S_t = C_t|_X$, is a strongly continuous semigroup on X.

Lemma 1. *Using the preceding notation for the semigroups (S_t) and (C_t) on X and X^{**} respectively we have*

$$S_t^{**} = C_t$$

for each $t \ge 0$.

Proof. By the definition of the adjoint operator,

$$S_t^{**}|_X = S_t = C_t|_X.$$

But X is weak* dense in X^{**} and the conclusion follows. □

Theorem 4. *Let (φ_t) be a semigroup and X be one of the spaces $VMOA$ or $\mathcal{B}_0$. Denote by Γ the generator of the induced composition semigroup (S_t) on X and let $\lambda \in \rho(\Gamma)$. Then the following are equivalent.*

(1) $[\varphi_t, X^{**}] = X$;
(2) $\mathcal{R}(\lambda,\Gamma)$ *is weakly compact on X;*
(3) $\mathcal{R}(\lambda,\Gamma)^{**}(X^{**}) \subset X$.

Proof. Conditions (2) and (3) are well known to be equivalent for every bounded operator on every Banach space X, [10, Theorem VI.4.2]. We proceed to show that (1) and (3) are equivalent.

(1) ⇒ (3). Take $\lambda \in \rho(\Gamma)$ a big real number and $f \in X$. Writing the resolvent as a Laplace transform [10, Theorem 11, p. 622],

$$\mathcal{R}(\lambda,\Gamma)(f) = \int_0^\infty e^{-\lambda u} S_u(f)\,du,$$

we have

$$S_t \circ \mathcal{R}(\lambda,\Gamma)(f) = \int_0^\infty e^{-\lambda u} S_{t+u}(f)\,du = e^{\lambda t}\int_t^\infty e^{-\lambda u} S_u(f)\,du.$$

From this we obtain

$$S_t \circ \mathcal{R}(\lambda,\Gamma)(f) - \mathcal{R}(\lambda,\Gamma)(f) = (e^{\lambda t}-1)\int_t^\infty e^{-\lambda u} S_u(f)\,du - \int_0^t e^{-\lambda u} S_u(f)\,du,$$

thus

$$\|S_t \circ \mathcal{R}(\lambda,\Gamma)(f) - \mathcal{R}(\lambda,\Gamma)(f)\| \le \left(|e^{\lambda t}-1|\int_t^\infty e^{-\lambda u}\|S_u\|\,du + \int_0^t e^{-\lambda u}\|S_u\|\,du\right)\|f\|,$$

see [10, Corollary 5, p. 619] for justification of the finiteness of the integrals. Consequently

$$\lim_{t\to 0} ||S_t \circ \mathcal{R}(\lambda,\Gamma) - \mathcal{R}(\lambda,\Gamma)|| = 0$$

and so, recalling that $S_t^{**} = C_t$ and that S_t commutes with $\mathcal{R}(\lambda,\Gamma)$ we have

$$\lim_{t\to 0} ||C_t \circ \mathcal{R}(\lambda,\Gamma)^{**} - \mathcal{R}(\lambda,\Gamma)^{**}|| = 0.$$

Thus if $f \in X^{**}$, for the function $F = \mathcal{R}(\lambda,\Gamma)^{**}(f)$ we have

$$\lim_{t\to 0} ||C_t(F) - F|| = 0,$$

which says that $\mathcal{R}(\lambda,\Gamma)^{**}(f) = F \in [\varphi_t, X^{**}] = X$, i.e. $\mathcal{R}(\lambda,\Gamma)^{**}(X^{**}) \subset X$. This gives the result for big real values of λ and the resolvent equation gives the inclusion for any other $\lambda \in \rho(\Gamma)$.

(3) $\Rightarrow$ (1). To show this put $Y = [\varphi_t, X^{**}]$ then $X \subseteq Y \subseteq X^{**}$. The restriction of (C_t) on Y is a strongly continuous semigroup with generator

$$\Delta(f) = Gf', \quad D(\Delta) = \{f \in Y : Gf' \in Y\}.$$

It is clear that $D(\Gamma) \subset D(\Delta)$ so that Δ is an extension of Γ. Let λ be a big real number such that $\lambda \in \rho(\Gamma) \cap \rho(\Delta)$. An argument similar to the one in the proof of Lemma 1 shows that

$$\mathcal{R}(\lambda,\Gamma)^{**}|_X = \mathcal{R}(\lambda,\Gamma), \quad \mathcal{R}(\lambda,\Gamma)^{**}|_Y = \mathcal{R}(\lambda,\Delta)$$

We have

$$D(\Delta) = \mathcal{R}(\lambda,\Delta)(Y) = \mathcal{R}(\lambda,\Gamma)^{**}|_Y(Y) \subset \mathcal{R}(\lambda,\Gamma)^{**}(X^{**}) \subset X.$$

Recalling that $D(\Delta)$ is dense in Y we have $[\varphi_t, X^{**}] = Y = \overline{D(\Delta)} \subset X$, and this finishes the proof. $\square$

Corollary 1. *Let (φ_t) be a semigroup of functions in $\mathbf{D}$, let Γ be the generator of (C_t) on the space $VMOA$ or on the space $\mathcal{B}_0$ and let $\lambda \in \rho(\Gamma)$. Then*

(1) *$[\varphi_t, BMOA] = VMOA$ if and only if $\mathcal{R}(\lambda,\Gamma)$ is weakly compact on $VMOA$.*
(2) *$[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ if and only if $\mathcal{R}(\lambda,\Gamma)$ is weakly compact on $\mathcal{B}_0$.*

5. The integral operator T_g

We are going to use a certain integral operator T_g defined on analytic functions by

$$T_g(f)(z) = \int_0^z f(\xi)g'(\xi)d\xi, \quad f \in \mathcal{H}(\mathbf{D}),$$

where its symbol g is an analytic function on $\mathbf{D}$. This operator (also called in the literature the Volterra operator or the generalized Cesàro operator) was first considered by Pommerenke [17] and has been widely studied in several recent papers

[2, 1, 24, 27]. We will use properties of this operator for a particular choice of the symbol g in order to better describe information for the maximal space of strong continuity in various cases. Before doing so we present some facts for T_g acting on the spaces of our concern.

To describe the symbols g for which the integral operator T_g is bounded or compact on spaces like $BMOA$ or $\mathcal{B}$ we need to consider the logarithmically weighted versions of those spaces. They are defined as follows.

Definition 3. Let $f\colon \mathbf{D} \to \mathbf{C}$ an analytic function.

(1) We say that f belongs to $BMOA_{\log}$ if

$$\|f\|_{**}^2 := \sup_{I \in \partial\mathbf{D}} \left\{ \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{R(I)} |f'(z)|^2 (1-|z|^2)\, dA(z) \right\} < \infty.$$

The subspace $VMOA_{\log}$ of $BMOA_{\log}$ contains by definition all the functions f such that

$$\lim_{|I|\to 0} \left\{ \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{R(I)} |f'(z)|^2 (1-|z|^2)\, dA(z) \right\} = 0.$$

(2) We say that f belongs to the weighted Bloch space $\mathcal{B}_{\log}$ if

$$\sup_{z\in\mathbf{D}} (1-|z|^2) \log\left(\frac{1}{1-|z|^2}\right) |f'(z)| < \infty.$$

The subspace $\mathcal{B}_{\log,0}$ contains all functions f that satisfy

$$\lim_{|z|\to 1} (1-|z|^2) \log\left(\frac{1}{1-|z|^2}\right) |f'(z)| = 0.$$

It is clear that $BMOA_{\log} \subset VMOA$ and $\mathcal{B}_{\log} \subset \mathcal{B}_0$. The following theorem characterizes boundedness of the operators T_g on $BMOA$ and Bloch spaces.

Theorem 5. *Let $X = VMOA$ or respectively, $X = \mathcal{B}_0$. Then the following are equivalent:*

(1) *T_g is bounded on X;*
(2) *T_g is bounded on X^{**};*
(3) *$g \in BMOA_{\log}$, respectively $g \in \mathcal{B}_{\log}$.*

Proof. For the space $VMOA$, this result is contained in [24, Corollary 3.3]. For the space $\mathcal{B}_0$, the proof is almost contained in [27, Theorem 2.1]. The only thing we have to check is that if $g \in \mathcal{B}_{\log}$, then T_g is bounded on $\mathcal{B}_0$. To show this recall that if $f \in \mathcal{B}_0$ then, [28, p. 102],

$$\lim_{|z|\to 1} \frac{f(z)}{\log\left(\frac{1}{1-|z|^2}\right)} = 0.$$

Write

$$M = \sup_{z\in\mathbf{D}} (1-|z|^2) \log\left(\frac{1}{1-|z|^2}\right) |g'(z)|.$$

Then

$$\begin{aligned}(1-|z|^2)|T_g(f)'(z)| &= (1-|z|^2)|f(z)||g'(z)| \\ &= (1-|z|^2)\log\left(\frac{1}{1-|z|^2}\right)|g'(z)|\frac{|f(z)|}{\log\left(\frac{1}{1-|z|^2}\right)} \\ &\leq M\frac{|f(z)|}{\log\left(\frac{1}{1-|z|^2}\right)} \longrightarrow 0 \qquad \text{as } |z|\to 1.\end{aligned}$$

That is $T_g(f)$ is also in $\mathcal{B}_0$. □

The next theorem is about the compactness and weak compactness of T_g on the spaces of our interest. The case $X = BMOA, VMOA$ of it was obtained by different methods in [13].

Theorem 6. *Let $X = VMOA$ or respectively $X = \mathcal{B}_0$. Suppose that T_g is bounded on X, that is, $g \in BMOA_{\log}$, respectively $g \in \mathcal{B}_{\log}$. Then the following are equivalent:*

(1) *T_g is weakly compact on X;*
(2) *T_g is compact on X;*
(3) *T_g is weakly compact on X^{**};*
(4) *T_g is compact on X^{**};*
(5) *$T_g(X^{**}) \subset X$;*
(6) *$g \in VMOA_{\log}$, respectively $g \in \mathcal{B}_{\log,0}$.*

Proof. For the little Bloch space, this result follows from general theory of weakly compact operators, the fact that $\mathcal{B}_0$ is isomorphic to c_0 and [27, Theorem 2.3].

We have to address the case $X = VMOA$. The equivalence between statements (2), (4), and (6) is included in [24, Theorem 3.6 and p. 310]. From general theory of weakly compact operators we know that (1), (3), and (5) are equivalent and (2) implies (1). Therefore we only have to prove that (5) implies (6).

To do this suppose that $T_g(BMOA) \subset VMOA$ and $g \notin VMOA_{\log}$. This implies that there is some $\delta > 0$ and some sequence I_n of intervals in $\partial\mathbf{D}$ such that $|I_n| \to 0$ and

$$\frac{\log^2\frac{4\pi}{|I_n|}}{|I_n|}\int_{R(I_n)}|g'(z)|^2\,dA(z) \geq \delta$$

for all n. Now let z_{I_n} denote the point in the middle of the internal side of $R(I_n)$ and take $f \in BMOA$. Clearly, for every $z \in \mathbf{D}$,

$$T_g(f)'(z) - f(z_{I_n})g'(z) = (f(z) - f(z_{I_n}))g'(z).$$

Then, integrating and following literally the argument in [16, p. 581, lines 7–14 from above], we deduce that, for some constant c,

$$\frac{1}{|I_n|}\int_{R(I_n)}|T_g(f)'(z) - f(z_{I_n})g'(z)|^2\,dA(z) \leq \frac{c}{\log^2\frac{4\pi}{|I_n|}}\|f\|_{BMOA}^2\|g\|_{**}^2.$$

These estimates imply that

$$\begin{aligned}
&\frac{1}{|I_n|}\int_{R(I_n)} |T_g(f)'(z)|^2\, dA(z) \\
&\geq \frac{1}{2}|f(z_{I_n})|^2 \frac{1}{|I_n|}\int_{R(I_n)} |g'(z)|^2\, dA(z) - \frac{c}{\log^2 \frac{4\pi}{|I_n|}} \|f\|^2_{BMOA}\|g\|^2_{**} \\
&\geq \frac{1}{2}\frac{|f(z_{I_n})|^2}{\log^2 \frac{4\pi}{|I_n|}}\delta - \frac{c}{\log^2 \frac{4\pi}{|I_n|}} \|f\|^2_{BMOA}\|g\|^2_{**}.
\end{aligned}$$

Now, we notice that if we are able to construct some $f \in BMOA$ such that

$$\limsup_{n\to\infty} \frac{|f(z_{I_n})|}{\log \frac{4\pi}{|I_n|}} > 0,$$

this would imply that

$$\limsup_{n\to\infty} \frac{1}{|I_n|}\int_{R(I_n)} |T_g(f)'(z)|^2\, dA(z) > 0$$

and, hence, $T_g(f) \notin VMOA$ obtaining in this way a contradiction. The rest of the proof is devoted to the construction of such function f.

Taking a subsequence, we may assume that the intervals I_n accumulate to some point of $\partial\mathbf{D}$ and, without loss of generality, we may take this point to be the point 1. Consider the function

$$p(z) = \log \frac{(1-z)^2}{(1+z)^2 + (1-z)^2}$$

which is the conformal mapping of the unit disc $\mathbf{D}$ onto the open set $E = \{w\colon |\mathrm{Im}(w)| < \pi\} \setminus \{w\colon \mathrm{Re}(w) \geq 0,\ \mathrm{Im}(w) = 0\}$. It is easy to prove the following properties of p:

- $|p(z)| \leq c|z+1|^2$ whenever $|z+1| \leq 1$, $z \in \mathbf{D}$;
- $|p(z)| \geq c\log \frac{2}{|z-1|}$ whenever $|z-1| \leq 1$, $z \in \mathbf{D}$;
- $\mathrm{Im}(p(\zeta)) = \pm\pi$ whenever $0 < |\arg\zeta| < \frac{\pi}{2}$, $\zeta \in \partial\mathbf{D}$;
- $\mathrm{Im}(p(\zeta)) = 0$ whenever $\frac{\pi}{2} < |\arg\zeta| < \pi$, $\zeta \in \partial\mathbf{D}$.

We write

$$z_n = z_{I_n}$$

and

$$\delta_n = 1 - |z_n| \asymp |I_n|, \quad z_n = |z_n|e^{i\theta_n}, \quad |\theta_n| < \pi.$$

By our assumption, we have that

$$\delta_n \to 0, \quad \theta_n \to 0.$$

There are exactly two cases.

Case 1. There is some positive constant c and a subsequence of (z_n) such that

$$\delta_n \geq c\theta_n^2.$$

In this case, and restricting to this subsequence, we have $|1-z_n| \leq c(\delta_n + |\theta_n|) \leq c\sqrt{\delta_n}$ and, hence,

$$|p(z_n)| \geq c\log\frac{2}{|z_n-1|} \geq c\log\frac{1}{\delta_n} \geq c\log\frac{4\pi}{|I_n|}.$$

Since p has bounded imaginary part, it belongs to BMOA and we are done just taking $f = p$.

Case 2. The case $\frac{\delta_n}{\theta_n^2} \to 0$.

We now consider

$$u_n = (1 - \sqrt{\delta_n})\frac{z_n}{|z_n|}, \quad \delta_n = 1 - |z_n|, \quad \lambda_n = \frac{1 + \overline{u_n}}{1 + u_n}$$

and construct the functions

$$p_n(z) = p\left(\lambda_n \frac{z - u_n}{1 - \overline{u_n} z}\right).$$

Observe that $\arg u_n = \arg z_n = \theta_n$ and that $1 - |u_n| = \sqrt{\delta_n} = \sqrt{1 - |z_n|}$. Also, the function $\lambda_n \frac{z-u_n}{1-\overline{u_n}z}$ maps u_n to 0 and -1 to -1. Furthermore, there is an interval J_n of $\partial\mathbf{D}$ of length $|J_n| \asymp 1 - |u_n| = \sqrt{\delta_n}$ which is centered at a point $e^{i\phi_n}$ such that $|\phi_n - \theta_n| \leq c\delta_n$ with the property

$$\begin{aligned} \operatorname{Im}(p_n(\zeta)) &= \pm\pi, \quad \zeta \in J_n, \\ \operatorname{Im}(p_n(\zeta)) &= 0, \quad \zeta \in \mathbf{T} \setminus J_n. \end{aligned}$$

Since $\frac{|J_n|}{\theta_n} \to 0$, by taking a subsequence, we may assume that the intervals J_n are disjoint, that J_{n+1} is closer to 1 than J_n and that

$$\sum_{k=1}^{\infty} \frac{\delta_k}{\theta_k^2} < +\infty.$$

Note that

$$\lambda_n \frac{z - u_n}{1 - \bar{u}_n z} + 1 = \frac{1 + z}{1 + u_n} \frac{1 - |u_n|^2}{1 - \bar{u}_n z}. \tag{8}$$

This shows that

$$|\lambda_n \frac{z - u_n}{1 - \bar{u}_n z} + 1| \leq C \frac{\sqrt{\delta_n}}{1 - |z|}$$

and hence $|p_n(z)| \leq C \frac{\sqrt{\delta_n}}{(1-|z|)^2}$ and we can define the analytic function in the unit disc given by

$$f(z) = \sum_{n=1}^{\infty} p_n(z).$$

Notice that we have that the boundary values of the harmonic function $\operatorname{Im} f(z)$ are absolutely bounded by π and, hence, that f belongs to $BMOA$.

A calculation shows that

$$\lambda_n \frac{z_n - u_n}{1 - \overline{u_n} z_n} = \frac{1}{1 + \sqrt{\delta_n} - \delta_n}\left(1 - \frac{2\sqrt{\delta_n} - \delta_n}{1 + u_n}\right).$$

Therefore

$$\left|\lambda_n \frac{z_n - u_n}{1 - \overline{u_n} z_n} - 1\right| \leq c\sqrt{\delta_n}$$

and, hence,

$$|p_n(z_n)| \geq c \log \frac{1}{\delta_n} \geq c \log \frac{4\pi}{|J_n|}.$$

On the other hand, using

$$|1-\overline{u_k}z_n| \geq ||1-z_n|-|1-u_k||,$$

and

$$|1-z_n| \asymp |\theta_n|, \quad |1-u_k| \asymp |\theta_k|,$$

one easily gets from (8) that

$$\left|\lambda_k \frac{z_n-u_k}{1-\overline{u_k}z_n}+1\right| \leq c\frac{1-|u_k|}{|1-\overline{u_k}z_n|} \leq \begin{cases} c\frac{\sqrt{\delta_k}}{\theta_k}, & k<n, \\ c\frac{\sqrt{\delta_k}}{\theta_n}, & k>n, \end{cases}$$

and, hence,

$$|p_k(z_n)| \leq \begin{cases} c\frac{\delta_k}{\theta_k^2}, & k<n, \\ c\frac{\delta_k}{\theta_n^2}, & k>n. \end{cases}$$

From these estimates we get

$$\begin{aligned} |f(z_n)| &\geq |p_n(z_n)| - \sum_{k<n}|p_k(z_n)| - \sum_{k>n}|p_k(z_n)| \\ &\geq c\log\frac{4\pi}{|I_n|} - c\sum_{k<n}\frac{\delta_k}{\theta_k^2} - c\sum_{k>n}\frac{\delta_k}{\theta_n^2} \geq c\log\frac{4\pi}{|I_n|} - c\sum_{k\neq n}\frac{\delta_k}{\theta_k^2} \geq c\log\frac{4\pi}{|I_n|} \end{aligned}$$

and this finishes the proof. □

The above result and in particular the implication (5) ⇒ (6) answers in the negative a question from [24] where it was asked if there are functions g such that T_g is weakly compact but not compact in $VMOA$.

6. Applications

We are going now to apply these properties of T_g for a special choice of the symbol g.

Definition 4. Given a semigroup (φ_t) with generator G and Denjoy–Wolff point b, we define the function $\gamma(z)\colon \mathbf{D}\to\mathbf{C}$ as follows

(1) If $b\in\mathbf{D}$, let

$$\gamma(z)=\int_b^z \frac{\xi-b}{G(\xi)}\,d\xi,$$

(2) If $b\in\partial\mathbf{D}$, let

$$\gamma(z)=\int_0^z \frac{1}{G(\xi)}\,d\xi.$$

Notice that $\gamma(z)$ is analytic on $\mathbf{D}$ and that when $b\in\partial\mathbf{D}$ then $\gamma(z)=h(z)$, the Koenigs function for (φ_t). We call $\gamma(z)$ *the associated g-symbol of* (φ_t). The following proposition shows the connection between $[\varphi_t, X]$ and integral operators.

Proposition 2. *Let (φ_t) be a semigroup with associated g-symbol $\gamma(z)$. Let also X be a Banach space of analytic functions with the properties:*

(i) *X contains the constant functions,*
(ii) *For each $b\in\mathbf{D}$, $f\in X \iff \frac{f(z)-f(b)}{z-b}\in X$,*
(iii) *If (C_t) is the induced semigroup on X then $\sup_{t\in[0,1]}\|C_t\|_X<\infty$.*

Then

$$[\varphi_t, X] = \overline{X \cap (T_\gamma(X) \oplus \mathbf{C})}.$$

Proof. Let G be the generator of (φ_t). First observe that

$$\{f \in X \colon Gf' \in X\} = \{f \in X \colon \frac{f'}{\gamma'} \in X\}.$$

This is clear in the case $b \in \partial\mathbf{D}$ because then $G\gamma' = Gh' = 1$. In the case $b \in \mathbf{D}$ we have $G(b) = 0$ and the assumption (ii) on X gives

$$G(z)f'(z) \in X \iff \frac{G(z)}{z-b} f'(z) = \frac{f'(z)}{\gamma'(z)} \in X.$$

Now write $m(z) = \frac{f'(z)}{\gamma'(z)} \in X$, then

$$f(z) = \int_0^z m(\xi)\gamma'(\xi)\, d\xi + c = T_\gamma(m)(z) + c,$$

where $c \in \mathbf{C}$. Thus

$$\begin{aligned}
\mathcal{D}(\varphi_t, X) &= \{f \in X \colon Gf' \in X\} = \{f \in X \colon \frac{f'}{\gamma'} = m \text{ for some } m \in X\} \\
&= \{f \in X \colon f' = m\gamma' \text{ for some } m \in X\} \\
&= \{f \in X \colon f(z) = \int_0^z m(\xi)\gamma'(\xi)\, d\xi + c \text{ for some } m \in X \text{ and } c \in \mathbf{C}\} \\
&= X \cap (T_\gamma(X) \oplus \mathbf{C}),
\end{aligned}$$

and the conclusion follows by taking closures and using Theorem 1. □

In view of the Proposition 2 it would be desirable to have the operator T_γ bounded on X. The following proposition says that this is not always the case.

Proposition 3. *Let $X = BMOA$ or $X = \mathcal{B}$, and (φ_t) be a semigroup with Denjoy–Wolff point on $\partial\mathbf{D}$ and associated g-symbol γ. Then T_γ is not bounded on X.*

Proof. Denote by h the Koenigs function of (φ_t). Then h is a univalent function on $\mathbf{D}$ with $h(0) = 0$ and such that the range of h has the following geometrical property:

$$w \in h(\mathbf{D}) \implies w + t \in h(\mathbf{D}), \quad \text{for all } t \geq 0.$$

Recall that a univalent function belongs to $\mathcal{B}_0$ if and only if it belongs to $VMOA$ and that such a function belongs $\mathcal{B}_0$ if and only if whenever $D(w, r)$ are discs contained in its range with the centers w moving to the boundary of the range, then the radii tend to zero. It follows from above geometric property of the range of h that h is not in $\mathcal{B}_0$ and therefore also it is not in $VMOA$.

On the other hand the associated g-symbol of the semigroup is $\gamma(z) = h(z)$. Assume T_γ is bounded on $BMOA$, then $\gamma = h \in BMOA_{\log} \subset VMOA$ and this is a contradiction. Similarly if we assume T_γ is bounded on $\mathcal{B}$ then $\gamma = h \in \mathcal{B}_{\log} \subset \mathcal{B}_0$ and we obtain again a contradiction. □

Remark 1. If the Denjoy–Wolff point of (φ_t) is inside $\mathbf{D}$ and there is a regular boundary fixed point for some (and then for all) φ_t, then we can show a similar result on $\mathcal{B}$. Recall that a regular boundary fixed point is a point $a \in \partial\mathbf{D}$ such that

$\varphi_s(a) = a$ and $\varphi_s'(a) \neq \infty$ for some $s > 0$ in the sense of angular limits and angular derivative. It then follows that there is a $\beta \in (0, \infty)$ such that

$$\varphi_t(a) = a \quad \text{and} \quad \varphi_t'(a) = e^{\beta t}$$

for all t, and that for the generator G of (φ_t) we have

$$\lim_{z \to a} G(z) = 0 \quad \text{and} \quad \lim_{z \to a} \frac{G(z)}{z - a} = \beta,$$

where the limits are taken non-tangentially, see [8]. In particular taking $r < 1$ real we obtain $\lim_{r \to 1} \frac{G(ra)}{(r-1)a} = \beta$.

Let γ be the associated g-symbol for (φ_t), then $\gamma'(z) = \frac{z-b}{G(z)}$ where $b \in \mathbf{D}$ is the Denjoy–Wolff point. We have,

$$\begin{aligned} \sup_{z \in \mathbf{D}} (1 - |z|^2) \log(\frac{1}{1 - |z|^2}) |\gamma'(z)| &\geq \sup_{r \in (0,1)} (1 - |r|^2) \log(\frac{1}{1 - |r|^2}) \left| \frac{ra - b}{G(ra)} \right| \\ &= \sup_{r \in (0,1)} |ra - b|(1 + r) \log(\frac{1}{1 - |r|^2}) \left| \frac{1 - r}{G(ra)} \right| = \infty, \end{aligned}$$

therefore T_γ is not bounded on $\mathcal{B}$.

Undoubtedly, the above proposition and remark are a handicap for our applications because the boundedness of T_γ is desired. However, in many families of examples our techniques work to give a description of the maximal space of strong continuity.

Corollary 2. *Let $X = VMOA$, or respectively $X = \mathcal{B}_0$. Suppose (φ_t) is a semigroup with associated g-symbol $\gamma(z)$ and suppose $\gamma \in BMOA_{\log}$, respectively $\gamma \in \mathcal{B}_{\log}$. Then $[\varphi_t, X^{**}] = X$ if and only if $\gamma \in VMOA_{\log}$, respectively $\gamma \in \mathcal{B}_{\log, 0}$.*

Proof. If $\gamma \in VMOA_{\log}$, respectively $\gamma \in \mathcal{B}_{\log, 0}$, then by Theorem 6 we have $T_\gamma(X^{**}) \subset X$. It then follows from Proposition 2 that

$$[\varphi_t, X^{**}] = \overline{X^{**} \cap (T_\gamma(X^{**}) \oplus \mathbf{C})} \subset \overline{X^{**} \cap X} = X$$

and we have equality because $X \subset [\varphi_t, X^{**}]$.

Conversely suppose $[\varphi_t, X^{**}] = X$. Then from Proposition 2 we must have $\overline{X^{**} \cap (T_\gamma(X^{**}) \oplus \mathbf{C})} = X$ and in particular

$$X^{**} \cap (T_\gamma(X^{**}) \oplus \mathbf{C}) \subset X. \tag{9}$$

The hypothesis on γ implies that T_γ is bounded on X^{**} and X^{**} contains the constants therefore $T_\gamma(X^{**}) \oplus \mathbf{C} \subset X^{**}$. This in view of (9) implies $T_\gamma(X^{**}) \oplus \mathbf{C} \subset X$ and then $T_\gamma(X^{**}) \subset X$ because X contains the constants. Using Theorem 6 again we conclude $\gamma \in VMOA_{\log}$, respectively $\gamma \in \mathcal{B}_{\log, 0}$. □

Corollary 3. (see [4, Theorem 3.1]) *Let (φ_t) be a semigroup with generator $G(z)$. Assume that for some $0 < \alpha < 1$,*

$$\frac{(1 - |z|)^\alpha}{G(z)} = O(1), \quad |z| \to 1.$$

Then

$$[\varphi_t, BMOA] = VMOA \quad \text{and} \quad [\varphi_t, \mathcal{B}] = \mathcal{B}_0.$$

Proof. Let γ be the associated g-symbol for (φ_t). The assumption implies that there is a constant C such that $(1-|z|)^\alpha|\gamma'(z)| \le C$ for $z \in \mathbf{D}$. Therefore γ has boundary values $(1-\alpha)$-Hölder continuous, in particular $\gamma \in VMOA_{\log}$ and $\gamma \in \mathcal{B}_{\log,0}$. The conclusion follows by applying Corollary 2. □

7. Final remarks and open questions

In this section we obtain some additional information, mostly on the Koenigs function and the maximal space of strong continuity for semigroups acting on $BMOA$ or on $\mathcal{B}$. We also present some questions which we could not answer in this article.

Recall that for any semigroup (φ_t) with generator

$$G(z) = (\bar{b}z - 1)(z-b)P(z),$$

the associated g-symbol γ is a univalent function when $b \in \partial\mathbf{D}$ because it coincides with the Koenigs map h. We observe that γ is also univalent when $b \in \mathbf{D}$. The easiest way to see this is to show that γ is in this case a close-to-convex function (see [18, p. 68] for the definition). Indeed the function $g(z) = (1/\bar{b})\log(1-\bar{b}z)$ is a convex univalent function for $b \in \mathbf{D}$ and it can be checked easily that $\mathrm{Re}(\frac{\gamma'}{g'}) = \mathrm{Re}(\frac{1}{P}) \ge 0$.

We are going to consider also the function

$$\psi(z) = \int_0^z \frac{1}{P(\xi)}\,d\xi, \quad z \in \mathbf{D}. \tag{10}$$

Since $\mathrm{Re}(1/P) \ge 0$ this is a univalent function [11, Theorem 2.16]. The growth estimate $|1/P(z)| \le C\frac{1+|z|}{1-|z|}$ for $z \in \mathbf{D}$ for the function $1/P$ of non-negative real part says that $(1-|z|)|\psi'(z)| \le 2C$ so $\psi \in \mathcal{B}$ and since ψ is univalent it follows that $\psi \in BMOA$.

Proposition 4. *Let $X = BMOA$ or $X = \mathcal{B}$ and let (φ_t) be a semigroup with generator $G(z) = (\bar{b}z-1)(z-b)P(z)$, Koenigs function h and associated g-symbol γ and let $\psi(z)$ be given by (10). Then the following hold:*

(1) $\psi(z), \gamma(z) \in \mathcal{D}(\varphi_t, X)$,
(2) *If $h \in X$, then $h \in \mathcal{D}(\varphi_t, X)$.*
(3) (i) *If $b \in \mathbf{D}$, then $(z-b)\log\frac{h(z)}{z-b} \in [\varphi_t, X]$. Moreover, $\log\frac{h(z)}{z-b} \in \mathcal{D}(\varphi_t, X)$ if and only if $G(z) \in X$,*
(ii) *If $b \in \partial\mathbf{D}$, then $z\log\frac{h(z)}{z} \in [\varphi_t, X]$. Moreover, $\log\frac{h(z)}{z} \in \mathcal{D}(\varphi_t, X)$ if and only if $G(z) \in X$,*
(4) *If $\mathbf{C}\setminus h(\mathbf{D})$ has nonempty interior, then $\log(h(z)-c)$ belongs to $\mathcal{D}(\varphi_t, X)$ for each c in the interior of $\mathbf{C}\setminus h(\mathbf{D})$.*

Proof. (1) We have $G(z)\psi'(z) = (\bar{b}z-1)(z-b) \in X$, therefore $\psi \in \mathcal{D}(\varphi_t, X)$. For $\gamma(z)$, since $G(z)\gamma'(z) = 1$ if $b \in \partial\mathbf{D}$ while $G(z)\gamma'(z) = z-b$ if $b \in \mathbf{D}$, the conclusion follows.

(2) Since $G(z)h'(z) = 1$ if $b \in \partial\mathbf{D}$ and $G(z)h'(z) = G'(b)h(z)$ if $b \in \mathbf{D}$, the assertion is clear.

(3)(i) We observe that

$$G'(b)\gamma(z) = \int_b^z \frac{G'(b)(\xi-b)}{G(\xi)}\,d\xi = \int_b^z \frac{(\xi-b)h'(\xi)}{h(\xi)}\,d\xi$$

$$= \int_b^z \left[1 + (\xi - b)\left(\log \frac{h(\xi)}{\xi - b}\right)'\right] d\xi$$
$$= z - b + (z-b)\log\frac{h(z)}{z-b} - \int_b^z \log\frac{h(\xi)}{\xi - b}\, d\xi.$$

Now $\log \frac{h(z)}{z-b}$ belongs to $BMOA$ (and to $\mathcal{B}$) so the function $\int_b^z \log\frac{h(\xi)}{\xi-b} d\xi$ is continuous on $\overline{\mathbf{D}}$, hence it belongs to $VMOA$ (and to $\mathcal{B}_0$). Since

$$(z-b)\log\frac{h(z)}{z-b} = G'(b)\gamma(z) - (z-b) + \int_b^z \log\frac{h(\xi)}{\xi - b} d\xi,$$

with $\gamma \in \mathcal{D}(\varphi_t, X) \subset [\varphi_t, X]$ we conclude that $(z-b)\log\frac{h(z)}{z-b} \in [\varphi_t, X]$.

Finally, let $q(z) = \log\frac{h(z)}{z-b}$ and observe that

$$G(z)q'(z) = G(z)\left(\frac{h'(z)}{h(z)} - \frac{1}{z-b}\right) = G'(b) - \frac{G(z)}{z-b} = G'(b) - (\bar{b}z - 1)P(z).$$

This says that $q(z) \in \mathcal{D}(\varphi_t, X)$ if and only if $(\bar{b}z-1)P(z) \in X$ and this is equivalent to $G(z) = (z-b)(\bar{b}z-1)P(z) \in X$.

(3)(ii) Let $m(z) = \frac{z}{h(z)}$. This is a bounded function on $\mathbf{D}$ hence it belongs to $BMOA$ and to $\mathcal{B}$. Observe that

$$T_\gamma(m)(z) = \int_0^z \frac{\xi h'(\xi)}{h(\xi)} d\xi.$$

A calculation similar to the one in (3)(i) gives

$$T_\gamma(m)(z) = z\log\frac{h(z)}{z} + z - \int_0^z \log\frac{h(\xi)}{\xi} d\xi,$$

and the three terms in the right hand side are elements of X. Thus $T_\gamma(m)(z) \in X$ with $m(z) \in X$, and Proposition 2 implies that $T_\gamma(m)(z) \in [\varphi_t, X]$. By the argument of (3)(i), $z - \int_0^z \log\frac{h(\xi)}{\xi}\, d\xi$ is in $VMOA$ (and in $\mathcal{B}_0$) and we conclude $z\log\frac{h(z)}{z} \in [\varphi_t, X]$.

To show the last assertion let $q(z) = \log\frac{h(z)}{z}$. Then

$$zG(z)q'(z) = zG(z)\left(\frac{h'(z)}{h(z)} - \frac{1}{z}\right) = \frac{zG(z)h'(z)}{h(z)} - G(z) = \frac{z}{h(z)} - G(z).$$

Now $G(z)q'(z) \in X$ is equivalent to $zG(z)q'(z) \in X$. Thus $q \in \mathcal{D}(\varphi_t, X)$ if and only if $\frac{z}{h(z)} - G(z) \in X$ and since $\frac{z}{h(z)}$ is a function in X we obtain the desired conclusion.

(4) Observe that $q(z) = \log(h(z) - c)$ belongs to $BMOA$ (and to $\mathcal{B}$). By the hypothesis there is a $\delta > 0$ such that $|h(z) - c| > \delta$ for $z \in \mathbf{D}$, so for the case $b \in \partial\mathbf{D}$,

$$G(z)q'(z) = G(z)\frac{h'(z)}{h(z)-c} = \frac{1}{h(z)-c},$$

is a bounded function so it is in $BMOA$ and the conclusion follows in this case. Finally if $b \in \mathbf{D}$ then

$$G(z)q'(z) = G(z)\frac{h'(z)}{h(z)-c} = G'(b)\frac{h(z)}{h(z)-c},$$

which is also bounded on $\mathbf{D}$ so the result follows. □

Corollary 4. (see [4, Theorem 3.3]) *If* (φ_t) *is a semigroup with generator* $G(z)$ *and Denjoy–Wolff* $b \in \mathbf{D}$ *and either* $[\varphi_t, BMOA] = VMOA$ *or* $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$*, then*

$$\lim_{|z|\to 1} \frac{1-|z|}{G(z)} = 0. \tag{11}$$

Proof. By Proposition 4 and our hypothesis, we have $(z-b)\log\frac{h(z)}{z-b} \in VMOA$ or respectively $(z-b)\log\frac{h(z)}{z-b} \in \mathcal{B}_0$. This implies $\log\frac{h(z)}{z-b} \in VMOA \subset \mathcal{B}_0$, therefore

$$\begin{aligned} 0 &= \lim_{|z|\to 1}(1-|z|)\left(\log\frac{h(z)}{z-b}\right)' = \lim_{|z|\to 1}(1-|z|)\left(\frac{h'(z)}{h(z)} - \frac{1}{z}\right) \\ &= \lim_{|z|\to 1}(1-|z|)\frac{h'(z)}{h(z)} = \lim_{|z|\to 1}\frac{G'(b)(1-|z|)}{G(z)}, \end{aligned}$$

and since $G'(b) \neq 0$ the conclusion follows. □

If $F(z)$ is analytic with $\mathrm{Re}F(z) \geq 0$ on $\mathbf{D}$, the Herglotz theorem says that there is a nonnegative Borel measure μ on $\partial\mathbf{D}$ such that

$$F(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta) + i\mathrm{Im}F(0).$$

The following proposition says that the presence of point masses for the measure μ associated with $1/P$ implies automatically that $[\varphi_t, BMOA]$ is strictly larger than $VMOA$.

Corollary 5. *Let* $X = BMOA$ *or* $X = \mathcal{B}$ *and let* (φ_t) *be a semigroup with generator* $G(z) = (z-b)(\bar{b}z-1)P(z)$ *such that* $[\varphi_t, BMOA] = VMOA$ *or* $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$*. Then the Herglotz measure for* $1/P(z)$ *has no point masses on* $\partial\mathbf{D}$.

Proof. From Proposition 4 and the hypothesis we have that the function $\psi(z) = \int_0^z \frac{1}{P(\xi)}d\xi$ belongs to $VMOA$ in the first case or to $\mathcal{B}_0$ in the second case. Since this function is univalent it belongs to $VMOA$ in both cases. But then the operator

$$T_\psi(f)(z) = \int_0^z f(\xi)\psi'(\xi)d\xi = \int_0^z f(\xi)\frac{1}{P(\xi)}\, d\xi$$

is a compact operator on the Hardy space H^2, see [1]. It follows then by [21, Theorem 4] that μ has no point masses on $\partial\mathbf{D}$. □

Corollary 6. *Let* (φ_t) *be a semigroup with Denjoy-Wolff point* $b \in \partial\mathbf{D}$ *and Koenigs function* h*. If either* $[\varphi_t, BMOA] = VMOA$ *or* $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ *then* $h \in (\cap_{p<\infty}H^p) \setminus BMOA$.

Proof. First observe that the Koenigs function cannot belong to $BMOA$. Otherwise from Proposition 4(2), we would have

$$h \in \mathcal{D}(\varphi_t, BMOA) \subset [\varphi_t, BMOA]$$

so $h \in VMOA$ by our assumption, which is impossible in view of the geometric property of the range of h. On the other hand again from Proposition 4(3)(ii) and our assumption we have $z\log\frac{h(z)}{z} \in VMOA$ so in particular $\log\frac{h(z)}{z} \in VMOA$. This implies $e^{\log\frac{h(z)}{z}} \in H^p$ for all $p < \infty$, [17, p. 596], so $h \in H^p$ for all $p < \infty$.

The argument for the Bloch case is similar and is omitted. □

Remark 2. One could expect, for the case of Denjoy–Wolff point on the boundary, to obtain a result similar to (11) for the generator $G(z)$, under the assumption $[\varphi_t, BMOA] = VMOA$ or $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$. In this case, using Proposition 4 one obtains $\log \frac{h(z)}{z} \in \mathcal{B}_0$ which means

$$\lim_{|z|\to 1}(1-|z|)\frac{h'(z)}{h(z)} = 0,$$

and in terms of $G(z)$,

$$\lim_{|z|\to 1} \frac{1-|z|}{G(z)} \frac{1}{\int_0^z \frac{1}{G(\xi)}\, d\xi} = 0.$$

Note that $\int_0^z \frac{1}{G(\xi)}\, d\xi = h(z)$ is never bounded, and finer analysis is required. We can however obtain a substitute of (11) if we consider integral averages of $1/G(z)$.

Corollary 7. *Let (φ_t) be a semigroup with generator $G(z)$ and Denjoy–Wolff point $b \in \partial\mathbf{D}$. If either $[\varphi_t, BMOA] = VMOA$ or $[\varphi_t, \mathcal{B}] = \mathcal{B}_0$ then for every $0 < p < \infty$,*

$$\lim_{r\to 1}(1-r)\left(\int_0^{2\pi} \frac{1}{|G(re^{it})|^p}\, dt\right)^{1/p} = 0.$$

Proof. As observed in the previous Remark, we have $\lim_{|z|\to 1}(1-|z|)\frac{h'(z)}{h(z)} = 0$. Since from Corollary 6 the function h belongs to all Hardy spaces we have

$$\begin{aligned}\int_0^{2\pi} \frac{(1-r)^p}{|G(re^{it})|^p}\, dt &= \int_0^{2\pi} \frac{(1-r)^p|h'(re^{it})|^p}{|h(re^{it})|^p}|h(re^{it})|^p\, dt \\ &\le \sup_{|z|=r}\left(\frac{(1-|z|)^p|h'(z)|^p}{|h(z)|^p}\right)\|h\|_p,\end{aligned}$$

and the right hand side goes to 0 as $r \to 1$. □

We now present some open questions. On the basis of Theorem 3 which says that no non-trivial semigroup (φ_t) induces a strongly continuous composition operator semigroup (C_t) on the Bloch space, it is natural to expect that the same is true for $BMOA$. The space $BMOA$ however does not posses the Dunford–Pettis property (see [5]) and the method of proof for the Bloch space does not work.

Question 1. Is it true that for every non-trivial semigroup (φ_t) the space $[\varphi_t, BMOA]$ is strictly smaller than $BMOA$?

Suppose now (φ_t) has its Denjoy–Wolff point on the boundary so its generator can be written $G(z) = \bar{b}(b-z)^2P(z)$ with $\mathrm{Re}(P(z)) \ge 0$. This generator cannot satisfy the condition

$$\frac{(1-|z|)^\alpha}{G(z)} = O(1), \quad \text{as } |z| \to 1,$$

for $0 < \alpha < 1$, of Corollary 3 which implies $[\varphi_t, BMOA] = VMOA$. Indeed for $z = rb$ we have

$$\frac{(1-|z|)^\alpha}{G(z)} = \frac{(1-|r|)^\alpha}{b(1-r)^2P(rb)} = \frac{\bar{b}}{(1-r)^{2-\alpha}P(rb)},$$

and the growth estimate $|P(z)| \le C\frac{1+|z|}{1-|z|}$ for the function P of non-negative real part implies $(1-r)^{2-\alpha}P(rb) \to 0$ as $r \to 1$ for $0 < \alpha < 1$.

In addition in all examples we can work it turns out that for this case the space $[\varphi_t, BMOA]$ is strictly larger than $VMOA$. This leads to

Question 2. Suppose (φ_t) is a semigroup with Denjoy–Wolff point on the boundary. Is it true in this case that $[\varphi_t, BMOA]$ is strictly larger than $VMOA$ and $[\varphi_t, \mathcal{B}]$ is strictly larger than $\mathcal{B}_0$?

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