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TRANSFERRING TTP-STRUCTURES VIA CONTRACTION

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Abstract

Let $A \otimes_t C$ be a *twisted tensor product* of an algebra A and a coalgebra C, along a *twisting cochain* $t : C \to A$. By means of what is called the *tensor trick* and under some nice conditions, Gugenheim, Lambe and Stasheff proved in the early 90s that $A \otimes_t C$ is homology equivalent to the objects $M \otimes_{t'} C$ and $A \otimes_{t''} N$, where M and N are strong deformation retracts of A and C, respectively. In this paper, we attack this problem from the point of view of contractions. We find explicit contractions from $A \otimes_t C$ to $M \otimes_{t'} C$ and $A \otimes_{t''} N$. Applications to the comparison of resolutions which split off of the bar resolution, as well as to some homological models for central extensions are given.

1. Introduction

An A_∞-algebra (resp. A_∞-coalgebra) means a connected module M along with a differential ∂ which is a coderivation (resp. derivation) of the tensor coalgebra $T^c s\bar{M}$ (resp. algebra $T^a s^{-1}\bar{M}$) and a perturbation of the tensor product differential where $\bar{M}$ denotes the submodule of elements in positive degrees, and s denotes suspension. The usual notation for this module is the *tilde* construction $\tilde{B}(M)$ (resp. $\tilde{\Omega}(M)$) [**23**].

The induced maps $m_j : M^{\otimes j} \to M$ such that $m_j = \pi_1 \circ \partial \circ i_j$ (resp. $\Delta_j : M \to M^{\otimes j}$ such that $\Delta_j = \pi_j \circ \partial \circ i_1$) where $i_j \colon M^{\otimes j} \hookrightarrow \tilde{B}(M)$ and $\pi_1 \colon \tilde{B}(M) \to M$ (resp. $i_1 \colon M \hookrightarrow \tilde{\Omega}(M)$ and $\pi_j \colon \tilde{\Omega}(M) \to M^{\otimes j}$) satisfy the relations

$$\sum_{n=1}^{i} \sum_{k=0}^{i-n} (-1)^{n+k+nk} m_{i-n+1}(1^k \otimes m_n \otimes 1^{i-n-k}) = 0$$

$$(\text{resp. } \sum_{n=1}^{i} \sum_{k=0}^{i-n} (-1)^{n+k+nk} (1^{i-n-k} \otimes \Delta_n \otimes 1^k)\Delta_{i-n+1} = 0).$$

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This is, an A_∞-algebra (resp. A_∞-coalgebra) is a DG-module M endowed with morphisms $m_n\colon M^{\otimes n} \to M$ (resp. $\Delta_n\colon M \to M^{\otimes n}$) of degree $n-2$ satisfying the relations above.

Throughout this paper we follow the notation given in [**20**]. A *contraction* from N to M is a data set $c\colon \{N,\, M,\, f,\, g,\, \phi\}$ where N and M are DG-modules, $f\colon N \to M$ and $g\colon M \to N$ are morphisms of DG-modules, $\phi\colon N \to N$ is a homotopy, satisfying $fg = 1\,,\ \phi\, d + d\,\phi + g\, f = 1\,,\ \phi\, g = 0\,,\ f\,\phi = 0\,,\ \phi\,\phi = 0$.

Given a contraction from a DG-algebra (A, μ) (resp. DG-coalgebra (C, Δ)) to a DG-module, there are several apparently distinct ways of constructing an A_∞-algebra, (resp. A_∞-coalgebra) on M. For example, the obstruction method [**7, 8**], and the method based on the so-called *tensor trick* [**6, 10, 9**]: let $c : \{A,\, M,\, f,\, g,\, \phi\}$ be a *contraction* from a DG-algebra A to a DG-module M, so that the perturbation process towards $\tilde{B}(c) : \{\bar{B}(A),\, \tilde{B}(M),\, f_\infty,\, g_\infty,\, \phi_\infty\}$ is convergent (usually under the assumption of connection). In this way M is endowed with a *natural* A_∞-algebra structure from A. There is a similar diagram in the coalgebra case. However, the convergence in the dual situation involving the "coalgebra part" of the cobar construction is much more subtle. Johansson and Lambe proved in [**13**] that these methods for constructing an A_∞-structure on M are equivalent. Moreover, in [**12**] is shown that any A_∞-algebra (resp. A_∞-coalgebra) may be described as the image of a DG-algebra (resp. DG-coalgebra) through a contraction.

In this transfer of information some natural twisting cochains and twisted tensor products arise, which are canonically related to the universal twisting cochains of bar and cobar constructions.

More concretely, a classical result of E. Brown states the following [**4**]: let $t : C \to A$ be a twisting cochain and L be an A-module; there is a twisting cochain $t^* : C \to \mathrm{End}\,(H_*(L))$ such that the twisted tensor product complexes $C \otimes_t L$ and $C \otimes_{t^*} H_*(L)$ are homology equivalent.

From other hand, Kadeishvili proves in [**14**] that every algebra A induces an A_∞-structure on $H_*(A)$ and a twisting cochain $t : \tilde{B}(H_*(A)) \to A$. Moreover, given a twisting cochain $s : C \to A$ there is an A_∞*-twisting cochain* $\tilde{s} : C \to H_*(A)$, such that s is homotopic to $t\tilde{s}$ and $A \otimes_s C$ is homology equivalent to $H(A) \otimes_{\tilde{s}} C$.

Indeed an A_∞-twisting cochain is a linear map of degree -1, $t\colon C \to A$, for C being a DGA-coalgebra and A an A_∞-algebra (resp. A being a DGA-algebra and C an A_∞-coalgebra), such that

$$td + \sum_{i=1}^{\infty} m_i t^{\otimes i} \Delta^{(i)} = 0$$

$$(\text{resp.,}\quad dt + \sum_{i=1}^{\infty} \mu^{(i)} t^{\otimes i} \Delta_i = 0),$$

where $\Delta^{(1)} = 1$, $\mu^{(1)} = 1$, $\Delta^{(2)} = \Delta$, $\mu^{(2)} = \mu$ and generally $\Delta^{(k)} = (1 \otimes \Delta^{(k-1)})\Delta$ and $\mu^{(k)} = \mu(1 \otimes \mu^{(k-1)})$.

From another point of view, $t : C \to A$ is an A_∞-twisting cochain if and only if there is an elevation $\hat{t} : C \to \tilde{B}(A)$ (resp. $\hat{t} : \tilde{\Omega}(C) \to A$) which is a morphism of DGA-coalgebras (resp. DGA-algebras), with $t = \theta\hat{t}$ (resp. $t = \hat{t}\theta$) θ being the universal cochain in $\bar{\Omega}(C)$ (resp. in $\bar{B}(A)$). An analogous condition for proper twisting cochains holds.

Let us recall that every A_∞-twisting cochain $t : C \to A$ gives rise to the A_∞-twisted tensor product $A \otimes_t C$ endowed with the differential

$$d_t = 1 \otimes d + \sum_{i=1}^{\infty} (m_i \otimes 1)(1 \otimes t^{\otimes i-1} \otimes 1)(1 \otimes \Delta^{(i)})$$

$$(\text{resp. } d_t = d \otimes 1 + \sum_{i=1}^{\infty} (\mu^{(i)} \otimes 1)(1 \otimes t^{\otimes i-1} \otimes 1)(1 \otimes \Delta_i)).$$

References on A_∞-twisted tensor product are [**15, 16**].

In [**9**] Gugenheim, Lambe and Stasheff explain the relationship between above Brown and Kadeishvili's results and the tensor trick. Let us briefly recall this. Given an algebra M, the map $\rho : M \to \mathrm{End}\,(M)$ defined as $\rho(a)(b) = ab$ becomes a morphism of algebras, since the product in M is associative. In case that M is an A_∞-algebra, ρ may no longer be a morphism of algebra because of the lack of associativity. However, it extends to a DG-coalgebra map $\tilde{\rho} : \tilde{B}(M) \to \bar{B}(\mathrm{End}(M))$, such that $\tilde{\rho} = \rho\tilde{\theta}$, $\tilde{\theta}$ being the universal cochain in $\tilde{B}(M)$. This way every A_∞-twisting cochain $t : C \to M$ lifts to a proper twisting cochain $\bar{t} = \rho t : C \to \mathrm{End}(M)$.

In these circumstances, the last theorem in [**9**] states that given a twisting cochain $t\colon C \to A$ and a contraction $(f, g, \phi) : A \to M$, there is an A_∞-structure on M (which comes from the tensor trick), an A_∞-twisting cochain $\tilde{t} : C \to M$, a proper twisting cochain $t^* : C \to \mathrm{End}(M)$ and a DG-coalgebra morphism $\hat{t} : C \to \bar{B}(A)$ (the elevation of t) such that $\tilde{t} = \tilde{\theta} f_\infty \hat{t}$ and $t^* = \rho\tilde{t}$. Futhermore, $f_\infty : \bar{B}(A) \to \tilde{B}(M)$ is a coalgebra homology equivalence, and therefore $f_\infty\, \hat{t}\colon C \to \tilde{B}(M)$ is a DG-coalgebra morphism (the elevation of $\tilde{t}$). Moreover the twisted tensor products $C \otimes_t A$ and $C \otimes_{t^*} M$ and the A_∞-twisted tensor product $C \otimes_{\tilde{t}} M$ are homology equivalent.

We are concerned here with the problem of establishing a contraction from the twisted tensor products $C \otimes_t A$ to the A_∞-twisted tensor product $C \otimes_{\tilde{t}} M$.

We organize the paper as follows. The main theorems of the paper are stated and proved in Section 2. Applications to the comparison of resolutions which split off of the bar construction, so as to the study of the structure of homological models for central extensions of abelian groups are included in Section 3. The last section is devoted to related questions and future work.

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2. Main theorems

Let (C, Δ) be a DG-coalgebra, C' be a DG-module and

$$c\colon \{C,\, C',\, f,\, g,\, \phi\}$$

be a contraction of DG-modules. Let us recall that the tensor trick induces on C' a structure of A_∞-coalgebra $(C', \{\Delta_i\})$ and a contraction of DG-algebras

$$c_{TT}\colon \{\bar{\Omega}(C),\, \tilde{\Omega}(C'),\, f_\infty,\, g_\infty,\, \phi_\infty\}.$$

Also, suppose that a twisting cochain $t\colon C \to A$ is given so that we have a twisted tensor product $A \otimes_t C$. There are two ways for constructing a differential on the tensor product $A \otimes C'$.

1. The method described in [**9**], where $t\colon C \to A$ is transported across the contraction c in order to obtain an A_∞-twisting cochain $\bar{t}\colon C' \to A$,
$$\bar{t} = \hat{t}\, g_\infty\, \tilde{\theta} \tag{1}$$
where $\tilde{\theta}\colon C' \to \tilde{\Omega}(C')$ is the universal twisting cochain and $\hat{t}\colon \bar{\Omega}(C) \to A$ is a DG-algebra morphism. This way a twisted tensor product $A \otimes_{\bar{t}} C'$ is obtained which is homological equivalent to $A \otimes_t C$.
2. The second way is to establish the contraction $1 \otimes c$
$$(1 \otimes f,\, 1 \otimes g,\, 1 \otimes \phi)\colon A \otimes C \Rightarrow A \otimes C'$$
and to use the Basic Perturbation Lemma [**5, 22**] with $t\cap = (\mu \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta)$ as perturbation datum. As a result, the following contraction arises
$$A \otimes_t C \Rightarrow (A \otimes C',\, d^\infty)$$
It is not clear whether $(A \otimes C',\, d^\infty)$ is a twisted tensor product.

In the theorem below we prove that both ways coincide.

Theorem 2.1. *Let $t : C \to A$ be a twisting cochain and $c(f, g, \phi) : C \Rightarrow C'$ be a contraction such that c induces on C' an A_∞-coalgebra structure. Additionally, assume that $t\phi = 0$ and $(1 \otimes \phi)t\cap$ is pointwise nilpotent. There is a contraction*

$$A \otimes_t C \Rightarrow A \otimes_{\bar{t}} C',$$

where $\bar{t} = tg$ is an A_∞-twisting cochain and $A \otimes_{\bar{t}} C'$ is an A_∞-twisted tensor product.

Proof.

Since $(1 \otimes \phi)t\cap$ is pointwise nilpotent, the perturbation of

$$A \otimes C \Rightarrow A \otimes C'$$

by means of $t\cap$ converges to give

$$A \otimes_t C \Rightarrow (A \otimes C', 1 \otimes d_{C'} + d_A \otimes 1 + d_{t\cap}),$$

with

$$d_{t\cap} = (1 \otimes f) \sum_{n \geqslant 0} (-1)^n [(\mu \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta_C)(1 \otimes \phi)]^n (\mu \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta_C g).$$

From other hand, taking into account the formula (1) of the A_∞-twisting cochain $\bar{t}$ and the hypothesis $t\phi = 0$, it is straightforward to verify that $\bar{t} = tg$.

Our aim is to prove that, under the assumption $t\phi = 0$, the differential $1 \otimes d_C + d_A \otimes 1 + d_{t\cap}$ equals to the differential $d_{\bar{t}}$ which $\bar{t}$ induces on the A_∞-twisted tensor product $A \otimes_{\bar{t}} C'$,

$$d_{\bar{t}} = d_A \otimes 1 + \sum_{i=1}^{\infty} (\mu^{(i)} \otimes 1)(1 \otimes \bar{t}^{i-1} \otimes 1)(1 \otimes \Delta_i),$$

with $\mu^{(1)} = 1$, $\mu^{(2)} = \mu$, $\mu^{(i)} = \mu(1 \otimes \mu^{(i-1)})$, $\Delta_1 = \pi_1 \Theta$ and $\Delta_i : C' \to C' \otimes \overset{i}{\cdots} \otimes C'$, $i \geqslant 2$ defined as $-\pi_i \Theta$, $\Theta \colon C' \to T(C')$ such that

$$d_{\tilde{\Omega}} = (-s^{-1} d_{C'} s)^{[\,]} + d_{\partial_{alg}} = -(T(s^{-1})\Theta s)^{[\,]},$$

$$d_{\partial_{alg}} = T(s^{-1} f s)\partial_{alg}(\sum_{i \geqslant 0} (-1)^i (T(s^{-1}\phi s)\partial_{alg})^i) T(s^{-1} g s)$$

and

$$\partial_{alg}|_{|\,|_s = n} = \sum_{k=1}^{n} 1^{k-1} \otimes (s^{-1} \otimes s^{-1}) \Delta_C s \otimes 1^{n-k}.$$

This way, for $i \geqslant 2$,

$$\Delta_i = (-1)^{\frac{i(i-1)}{2}} \pi_i T(s) d_{\tilde{\Omega}} s^{-1}.$$

Well now, among the morphisms in which $d_{\tilde{\Omega}} = (-s^{-1} d_{C'} s)^{[\,]} + d_{\partial_{alg}}$ decomposes, the first of them respects the number of factors on the input element, whereas the second one increases it at least by one. Since the input data on Δ_i is an element of C' in $T(C')$, it follows that $(-s^{-1} d_{C'} s)^{[\,]}$ is the only term that involved in the calculation of Δ_1, whereas $d_{\partial_{alg}}$ is the only one involved in the computation of Δ_i, for $i \geqslant 2$.

So $\Delta_1 = -T(s)\pi_1(-s^{-1} d_{C'} s) s^{-1} j = d_{C'}$, for $j : C' \to T(C')$ being the natural injection. Then

$$(\mu^{(1)} \otimes 1)(1 \otimes 1)(1 \otimes \Delta_1) = 1 \otimes d_{C'},$$

and proving that $1 \otimes d_{C'} + d_A \otimes 1 + d_{t\cap}$ equals to $d_{\bar{t}}$ reduces to prove that

$$d_{t\cap} = \sum_{i=2}^{\infty} (\mu^{(i)} \otimes 1)(1 \otimes \bar{t}^{i-1} \otimes 1)(1 \otimes \Delta_i).$$

On the other hand, for $i \geqslant 2$,

$$\Delta_i = (-1)^{\frac{i(i-1)}{2}} \pi_i T(s) d_{\partial_{alg}} s^{-1} j.$$

Since $f\phi = 0$, $\phi g = 0$ and $fgf = f$, it follows that

$$\Delta_i = (-1)^{\frac{(i-1)(i-2)}{2}} f^{\otimes i} \left[\sum_{k_2=1}^{2} \sum_{k_3=1}^{k_2+1} \cdots \sum_{k_{i-1}=1}^{k_{i-2}+1} \prod_{j=2}^{i-1} (-1)^{k_j} (1^{\otimes k_j - 1} \otimes \Delta_C \phi \otimes 1^{\otimes j - k_j}) \right] \Delta_C g,$$

where $\prod_{j=2}^{i-1} h_j$ represents the composition $h_{i-1} \circ \cdots \circ h_2$.

In short, we have to prove that $d_{t\cap}$ equals to

$$\sum_{i=2}^{\infty}(-1)^{\frac{(i-1)(i-2)}{2}}(\mu^{(i)}\otimes 1)(1\otimes (tg)^{\otimes i-1}\otimes 1)(1\otimes f^{\otimes i})\circ$$

$$\circ\left[\sum_{k_2=1}^{2}\sum_{k_3=1}^{k_2+1}\cdots\sum_{k_{i-1}=1}^{k_{i-2}+1}\prod_{j=2}^{i-1}(-1)^{k_j}(1^{\otimes k_j}\otimes\Delta_C\phi\otimes 1^{\otimes j-k_j})\right](1\otimes\Delta_C g). \qquad (2)$$

To this end, we will carry the general expression of $d_{t\cap}$,

$$d_{t\cap}=(1\otimes f)\sum_{n\geqslant 0}(-1)^n[(\mu\otimes 1)(1\otimes t\otimes 1)(1\otimes\Delta_C)(1\otimes\phi)]^n(\mu\otimes 1)(1\otimes t\otimes 1)(1\otimes\Delta_C g), \qquad (3)$$

into this form, taking into account the following identities:

1. Since t is a twisting cochain,

$$d_A t+td_C-\mu(t\otimes t)\Delta_C=0. \qquad (4)$$

2. Generalizing, it may be inductively proved that

$$(-1)^n\mu^{(n)}t^{\otimes n}d_C^{[n]}=d_A\mu^{(n)}t^{\otimes n}+\mu^{(n+1)}t^{\otimes n+1}\sum_{k=1}^{n}(-1)^k(1^{\otimes k-1}\otimes\Delta_C\otimes 1^{\otimes n-k}). \qquad (5)$$

In fact, for $n=1$ we meet (4) and for $n=2$ we find that

$$\mu t^{\otimes 2}d_C^{[2]}=\mu(t\otimes td_C-td_C\otimes t)\overset{(4)}{=}$$

$$\overset{(4)}{=}\mu(-t\otimes d_At+t\otimes\mu t^{\otimes 2}\Delta_C+d_At\otimes t-\mu t^{\otimes 2}\Delta_C\otimes t)=$$

$$=\mu d_A^{[2]}t^{\otimes 2}+\mu^{(3)}t^{\otimes 3}(-\Delta_C\otimes 1+1\otimes\Delta_C)=$$

$$=d_A\mu t^{\otimes 2}+\mu^{(3)}t^{\otimes 3}(-\Delta_C\otimes 1+1\otimes\Delta_C);$$

in general, assuming that the relation holds for $n\leqslant m-1$, setting $n=m$ we have that

$$\mu^{(m)}t^{\otimes m}d_C^{[m]}=\mu(1\otimes\mu^{(m-1)})(t\otimes t^{\otimes m-1})(1\otimes d_C^{[m-1]}+d_C\otimes 1^{\otimes m-1})=$$

$$=\mu(t\otimes\mu^{(m-1)}t^{\otimes m-1}d_C^{[m-1]})+(-1)^{m-1}\mu(td_C\otimes\mu^{(m-1)}t^{\otimes m-1})\overset{H.I.}{=}$$

$$\overset{H.I.}{=}(-1)^{m-1}\mu(t\otimes d_A\mu^{(m-1)}t^{\otimes m-1})+$$

$$+(-1)^{m-1}\mu(t\otimes\mu^{(m)}t^{\otimes m}\sum_{k=1}^{m-1}(-1)^k(1^{\otimes k-1}\otimes\Delta_C\otimes 1^{\otimes m-k-1}))+$$

$$+(-1)^m\mu(d_At\otimes\mu^{(m-1)}t^{\otimes m-1})+(-1)^{m-1}\mu(\mu t^{\otimes 2}\Delta_C\otimes\mu^{(m-1)}t^{\otimes m-1})=$$

$$=(-1)^m d_A\mu^{(m)}t^{\otimes m}+(-1)^m\mu^{(m+1)}t^{\otimes m+1}\sum_{k=1}^{m}(-1)^k(1^{\otimes k-1}\otimes\Delta_C\otimes 1^{\otimes m-k}).$$

3. Taking into account that $t\phi = 0$, now

$$\mu^{(n)} t^{\otimes n} d_C^{[n]} \phi^{[n]} = \mu^{(n+1)} t^{\otimes n+1} \sum_{k=1}^{n} (-1)^{k+n} (1^{\otimes k-1} \otimes \Delta_C \phi \otimes (gf)^{\otimes n-k}). \quad (6)$$

4. From the morphisms of the contraction $c^{\otimes n}$ and the relation $t\phi = 0$, we may deduce that

$$t^{\otimes n} = (tg)^{\otimes n} f^{\otimes n} + t^{\otimes n} d_C^{[n]} \phi^{[n]}. \quad (7)$$

5. Finally, the identities (6) and (7) may be combined in order to get

$$\mu^{(n)} t^{\otimes n} = \mu^{(n)} (tg)^{\otimes n} f^{\otimes n} +$$

$$+ t^{\otimes n} d_C^{[n]} \phi^{[n]} \mu^{(n+1)} t^{\otimes n+1} \sum_{k=1}^{n} (-1)^{k+n} (1^{\otimes k-1} \otimes \Delta_C \phi \otimes (gf)^{\otimes n-k}). \quad (8)$$

These relations affect (3) in the following way:

$\boxed{n = 0}$

$$(\mu \otimes 1)(1 \otimes t \otimes f)(1 \otimes \Delta_C)(1 \otimes g) \overset{(7)}{=} (\mu \otimes 1)(1 \otimes tg \otimes 1)(1 \otimes f^{\otimes 2})(1 \otimes \Delta_C g) +$$

$$+ (\mu \otimes 1)(1 \otimes t d_C \phi \otimes 1)(1 \otimes 1 \otimes f)(1 \otimes \Delta_C g) \overset{(4)}{=}$$

$$\overset{(4)}{=} \boxed{i = 2} + (\mu \otimes 1)(1 \otimes \mu t^{\otimes 2} \Delta_C \phi \otimes 1)(1^{\otimes 2} \otimes f)(1 \otimes \Delta_C g) =$$

$$= \boxed{i = 2} + (\mu \otimes 1)(1 \otimes \mu t^{\otimes 2} \otimes 1)(1^{\otimes 3} \otimes f)(1 \otimes \Delta_C \phi \otimes 1)(1 \otimes \Delta_C g) \overset{(8)}{=}$$

$$\overset{(8)}{=} \boxed{i = 2} + (\mu^{(3)} \otimes 1)(1 \otimes (tg)^{\otimes 2} \otimes 1)(1 \otimes f^{\otimes 3})(1 \otimes \Delta_C \phi \otimes 1)(1 \otimes \Delta_C g) +$$

$$+ (\mu \otimes 1)(1 \otimes \mu^{(3)} t^{\otimes 3} \sum_{k_3=1}^{2} (-1)^{k_3+2} (1^{k_3-1} \otimes \Delta_C \phi \otimes (gf)^{\otimes 2-k_3}) \otimes 1) \circ$$

$$\circ (1^{\otimes 3} \otimes f)(1 \otimes \Delta_C \phi \otimes 1)(1 \otimes \Delta_C g) \overset{(8)}{=} \cdots$$

$$\sum_{j=2}^{\infty} \sum_{k_2=1}^{1} \sum_{k_3=1}^{k_2+1} \cdots \sum_{k_{j-1}=1}^{k_{j-2}+1} (\mu^{(j)} \otimes 1)(1 \otimes (tg)^{\otimes j-1} \otimes 1)(1 \otimes f^{\otimes j})$$

$$\left[\prod_{l=2}^{j-1} (-1)^{l+k_l-1} (1^{\otimes k_l} \otimes \Delta_{C'} \phi \otimes 1^{\otimes l-k_l}) \right] (1 \otimes \Delta_{C'} g),$$

so that it gives raise, for $j \geqslant 2$, to the terms

$$\boxed{i = j, k_2 = 1, 1 \leqslant k_3 \leqslant 2, 1 \leqslant k_4 \leqslant k_3 + 1, \ldots, 1 \leqslant k_{j-1} \leqslant k_{j-2} + 1.}$$

$\boxed{n = m}$

$$(-1)^m (1 \otimes f) \left[(\mu \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta_{C'} \phi) \right]^m (\mu \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta_{C'} g) =$$

$$= (1\otimes f)(\mu^{(m+2)}\otimes 1)(1\otimes t^{\otimes m+1}\otimes 1)(1^{\otimes m+1}\otimes \Delta_{C'}\phi)\cdots(1^{\otimes 2}\otimes\Delta_{C'}\phi)(1\otimes\Delta_{C'}g) \overset{(8)}{=} \cdots$$

$$\cdots \overset{(8)}{=} \sum_{j=m+2}^{\infty}\sum_{k_2=2}^{2}\cdots\sum_{k_{m+1}=m+1}^{m+1}\sum_{k_{m+2}=1}^{m+1}\sum_{k_{m+3}=1}^{k_{m+2}+1}\cdots\sum_{k_{j-1}=1}^{k_{j-2}+1}(\mu^{(j)}\otimes 1)(1\otimes(tg)^{\otimes j-1}\otimes 1)\circ$$

$$(1\otimes f^{\otimes j})\circ\left[\prod_{l=2}^{j-1}(-1)^{l+k_l-1}(1^{\otimes k_l}\otimes\Delta_{C'}\phi\otimes 1^{\otimes l-k_l})\right](1\otimes\Delta_{C'}g),$$

so that it gives raise, for $j \geqslant m+2$, to the terms

$$\boxed{i = j, k_2 = 2, \ldots, k_{m+1} = m+1, 1 \leqslant k_{m+2} \leqslant m+1,}$$

$$\boxed{1 \leqslant k_{m+3} \leqslant k_{m+2}+1, \ldots, 1 \leqslant k_{j-1} \leqslant k_{j-2}+1.}$$

This way it is proved that both the differential expressions are the same, and the result follows.

□

Remark 2.2. *The hypothesis of the theorem above are satisfied whenever C is simply connected. For instance, this is the case of the bar construction of a connected DG-algebra.*

It is straightforward to prove a dual statement for A_∞-algebras.

Theorem 2.3. *Let $t : C \to A$ be a twisting cochain and $c(f, g, \phi) : A \Rightarrow A'$ be a contraction such that c induces on A' an A_∞-algebra structure. Additionally, assume that $\phi t = 0$ and $(\phi \otimes 1)t\cap$ is pointwise nilpotent. There is a contraction*

$$A \otimes_t C \Rightarrow A' \otimes_{\bar{t}} C,$$

where $\bar{t} = ft$ is an A_∞-twisting cochain and $A'\otimes_{\bar{t}}C$ is an A_∞-twisted tensor product.

Furthermore, the theorems above may be combined to analyze the translation of a principal twisted tensor structure $A\otimes_t C$ through the product of two contractions $A \Rightarrow A'$ and $C \Rightarrow M$.

Theorem 2.4. *Let A be a DG-algebra, C be a DG-coalgebra and $A\otimes_t C$ be a principal twisted tensor product. Let $c_A : \{A, A', f_A, g_A, \phi_A\}$ and $c_C : \{C, M, f_C, g_C, \phi_C\}$ be contractions to a DG-algebra A' and a DG-module M, respectively, such that f_A is a morphism of algebras, $f_A t\phi_C = 0$ and furthermore c' induces on M an A_∞-coalgebra structure. Then $A' \otimes_{f_A t g_C} M$ becomes an A_∞-twisted tensor product. Moreover there is a contraction from $A \otimes_t C$ to $A' \otimes_{f_A t g_C} M$.*

Proof.

From one hand, since f_A is a morphism of algebras, $A' \otimes_{f_A t} C$ acquires the structure of principal twisted tensor product. Moreover, from the proof of Theorem 2.3 a contraction $A \otimes_t C \Rightarrow A' \otimes_{f_A t} C$ may be constructed in a straightforward manner, without the need of the assumption $\phi_A t = 0$.

From another hand, since $f_A t \phi_C = 0$, Theorem 2.1 states that $A' \otimes_{f_A t g_C} M$ is an A_∞-twisted tensor product. Furthermore, there is a contraction $A \otimes_t C \Rightarrow A' \otimes_{f_A t g_C} M$.

□

Next, our interest focus on non principal twisted tensor products. The problem is determining assumptions under which a non principal twisted tensor product $M \otimes_t C$ degenerates to a non principal A_∞-twisted tensor product $N \otimes_{\bar{t}} C'$, by means of contractions $M \Rightarrow N$ and $C \Rightarrow C'$.

Theorem 2.5. *Let $M \otimes_t C$ be a non principal twisted tensor product, related to the twisting cochain $t : C \to A$ and the action $\mu : M \otimes A \to M$. Let us consider a contraction $c(f, g, \phi) : C \Rightarrow C'$ which inherits on C' an A_∞-coalgebra structure. Additionally assume that $t\phi = 0$ and $(1 \otimes \phi)t\cap$ is pointwise nilpotent. There is a contraction*

$$M \otimes_t C \Rightarrow M \otimes_{\bar{t}} C',$$

where $\bar{t} = tg$ is an A_∞-twisting cochain and $M \otimes_{\bar{t}} C'$ is a non principal A_∞-twisted tensor product.

Proof.

The proof of Theorem 2.1 may be reproduced here, taking into account that now the maps $\mu^{(i)}$ represent the action on M of the product of $i-1$ elements of A, $\mu^{(i)} = \mu(1 \otimes *_A^{(i-1)})$.

□

We need to know how to translate an action through a contraction.

Let M be a DG-module on the left of a DG-algebra A, by means of a product $*_M$, and $c : \{M, N, f, g, \phi\}$ be a contraction. There is a natural candidate to be an action on N, which is $*_N$, with $m *_N a = f(g(m) *_M a)$.

Lemma 2.6. *The map $*_N$ becomes an action of A on N if and only if*

$$f((d\phi + \phi d)(g(m) *_M a) *_M b) = 0, \qquad \forall\, a, b \in A.$$

Proof.

Since $1 - gf = d\phi + \phi d$, it is easy to check that

$$f(gf(g(m) *_M a) *_M b) = f(g(m) *_M ab),$$

is an equivalent relation for the associativity of $*_N$.

□

Theorem 2.7. *Let $M \otimes_t C$ be the twisted tensor product according to the twisting cochain $t : C \to A$. Let $c : \{M, N, f_M, g_M, \phi_M\}$ and $c' : \{C, C', f, g, \phi\}$ be contractions, such that c induces on N an A-módulo structure on the left on N and c' induces on C' an A_∞-coalgebra structure. Additionally, let assume that $t\phi = 0$ and $(1 \otimes \phi)t\cap$ is pointwise nilpotent. There is a contraction*

$$M \otimes_t C \Rightarrow N \otimes_{\bar{t}} C',$$

where $\bar{t} = tg$ is an A_∞-twisting cochain and $N \otimes_{\bar{t}} C'$ is a non principal A_∞-twisted tensor product.

Proof.

The proof follows from Theorem 2.5 and Lemma 2.6.

□

3. Applications

In this section we apply the results above to the comparison of resolutions which split off of the bar resolution [**2**], so as to some homological models for central extensions described in [**1, 3**].

3.1. Comparison of resolutions

Let A be a connected DG-algebra and $c : (\bar{f}, \bar{g}, \bar{\phi}) : (\bar{B}(A), d_{\bar{B}}) \Rightarrow (\bar{X}, \bar{d})$ be a contraction. This contraction may be extended to give a comparison contraction from the bar resolution $B(A) = A \otimes_\theta \bar{B}(A)$ to a resolution $X = A\tilde{\otimes}\bar{X}$.

Resolutions which admit a comparison contraction with the bar resolution are termed *resolutions which split off of the bar resolution* in [**17, 18, 19**]. This split may be canonical under the assumptions of the comparison theorem for resolutions, that is, the existence of an explicit homotopy for X being contractible to the ground ring Λ. We may consider then canonical comparison contractions and non canonical ones.

In [**2**] the authors analyze the multiplicative behaviour of the differential on X. One of the main results is a straightforward consequence of the Theorem 2.1 above.

Theorem 3.1. *[2] Let A be a connected DG-algebra, (X, d) be a resolution which splits off of the bar resolution $B(A)$, by means of the canonical comparison contraction $(f, g, \phi) : B(A) \Rightarrow (X, d)$. Then, there exists an A_∞-twisting cochain $\gamma = \theta g|_{\bar{X}}$ which produces that (X, d) can be rewritten as the A_∞-twisted tensor product $A \otimes_\gamma \bar{X}$ given by the DG-algebra A, the A_∞-coalgebra $\bar{X}$ (inherited from the reduced bar contraction $\bar{B}(A)$) and $\gamma \colon \bar{X} \to A$.*

Proof.

The first step of the proof consists in to apply the functor $\Lambda \otimes_A -$ and $1 \otimes_A -$ on the complexes and morphisms involved in the canonical comparison contraction $(f, g, \phi) \colon B(A) \Rightarrow (X, d)$. Then we obtain a contraction between reduced complexes

$$(1 \otimes_A f,\ 1 \otimes_A g,\ 1 \otimes_A \phi) \colon \bar{B}(A) \Rightarrow \bar{X}$$

Let us emphasize that $1 \otimes_A g = g|_{\bar{X}}$ and $1 \otimes_A \phi = \phi|_{\bar{B}(A)}$ since g and ϕ are A-lineal and $g(\bar{X}) \subseteq \bar{B}(A)$, $\phi(\bar{B}(A)) \subseteq \bar{B}(A)$. An explicit formula for ϕ is given in [**18, 19**] which increases the simplicial degree in $\bar{B}(A)$ by one. Therefore, $\theta\, \phi|_{\bar{B}(A)} = 0$ since $\theta \colon \bar{B}(A) \to A$ is the universal twisting cochain. Now, applying theorem 2.1, we have the following A_∞-twisting cochain

$$\gamma = \theta g|_{\bar{X}} \colon \bar{X} \to A$$

The second step is to construct the tensor product contraction

$$A \otimes \bar{B}(A) \Rightarrow A \otimes \bar{X}$$

and to use the Basic Perturbation Lemma with $\theta\cap$ as perturbation datum. Then, it is straightforward to check that $(1 \otimes \phi|_{\bar{B}(A)})\, \theta\cap$ is pointwise nilpotent. Then we obtain,

$$A \otimes_\theta \bar{B}(A) \Rightarrow (A \otimes \bar{X},\, d^\infty)$$

Now, using Theorem 2.1, we have that

$$(A \otimes \bar{X},\, d^\infty) = A \otimes_\gamma \bar{X}$$

where $A \otimes_\gamma \bar{X}$ is an A_∞-twisted tensor product.

In the proof of this identity $(A\otimes\bar{X},\, d^\infty) = (A\otimes\bar{X},\, d)$ we use the special properties of the morphisms which take part in the canonical comparison contraction (see [**2**]).

□

3.2. Homological models for central extensions

In [**21**], a homological model for the central extension of groups

$$1 \to A \to A_f \ltimes G \to G \to 1$$

determined by a 2-cocycle $f\colon G \times G \to A$, is described in terms of the composition of the contractions:

$$\bar{B}(\mathbb{Z}[A_f \ltimes G]) \cong C_*(\bar{W}(A_f \ltimes G)) \overset{\varphi_1}{\cong} C_*(\bar{W}(A) \times_\tau \bar{W}(G)) \overset{EZ^\tau_{\bar{W}(A),\bar{W}(G)}}{\Rightarrow}$$

$$\overset{EZ_{\bar{W}(A),\bar{W}(G)}}{\Rightarrow} C_*(\bar{W}(A)) \otimes_t C_*(\bar{W}(G)) \overset{C_{t\cap}}{\Rightarrow} (hA\tilde{\otimes}hG, \tilde{d}),$$

where φ_1 is induced by a simplicial isomorphism and EZ refers to Eilenberg-Zilber's Theorem.

We will focus in the case that A and G are finite abelian groups. Thus the homological model $hA\tilde{\otimes}hG$ is of the type $\tilde{\otimes}_{i\in I}(E(u_i, 1)\tilde{\otimes}\Gamma(w_i, 2))$ being $E(u_i, 1)$ the exterior algebra in one generator u_i of degree 1 and $\Gamma(w_i, 2)$ the polynomial power algebra on one generator w_i of degree 2. In [**3**] it is proved that the model $hA\tilde{\otimes}hG$ is provided with a structure of A_∞-coalgebra, naturally inherited from $\bar{B}(\mathbb{Z}[A_f \ltimes G])$. Furthemore, we have the following result:

Theorem 3.2. *The model $hA\tilde{\otimes}hG$ is endowed with an A_∞-twisted tensor product structure, in terms of the DG-algebra hA and the A_∞-coalgebra hG (naturally inherited from $\bar{B}(\mathbb{Z}[G])$).*

Proof.

It suffices to prove that the hypothesis on Theorem 2.4 are satisfied. Since A is abelian, the projection f_A in $\bar{B}(\mathbb{Z}[A]) \Rightarrow hA$ is a morphism of algebras. On the other hand, $\bar{B}(\mathbb{Z}[G]) \Rightarrow hG$ induces an A_∞-coalgebra structure on hG [**11**]. Finally, it is easy to check that $f_A t \phi_G = 0$.

□

This Theorem can be extended for G being an iterated product of central extensions and semidirect products of finite abelian groups.

4. Further work

We would like to state Theorems 1 through 3 under weaker assumptions than $t\phi = 0$ or $\phi t = 0$. The following approximation may provide a solution in a near future.

Let $c(f_A, g_A, \phi_A) : A \Rightarrow A'$ and $c(f_C, g_C, \phi_C) : C \Rightarrow M$ be contractions such that f_A is a morphism of algebras and c' induces on M an A_∞-coalgebra structure. Let $t : C \to A$ be a twisting cochain, consider the following diagram:

$$\begin{array}{ccccc} \bar\Omega(C) \otimes C & \stackrel{\bar\Omega(c')\otimes 1}{\Longrightarrow} & \tilde\Omega(M) \otimes C & \stackrel{1\otimes c'}{\Longrightarrow} & \tilde\Omega(M) \otimes M \\ \bar t \otimes 1 \downarrow & & & & \\ A \otimes C & \stackrel{c\otimes 1}{\Longrightarrow} & A' \otimes C & \stackrel{1\otimes c'}{\Longrightarrow} & A' \otimes M \end{array}$$

Perturbing the top row by means of the universal twisting cochain $\theta : C \to \bar\Omega(C)$, $c \mapsto [c]$, we get

$$\bar\Omega(C) \otimes_\theta C \stackrel{(\bar\Omega(c')\otimes 1)_{\theta\cap}}{\Longrightarrow} \tilde\Omega(M) \otimes_{\bar\Omega(f)\theta} C \stackrel{(1\otimes c')_{\bar\Omega(f)\theta\cap}}{\Longrightarrow} \tilde\Omega(M) \otimes_\gamma M,$$

where $\gamma : M \to \tilde\Omega(M)$ is defined as $m \mapsto [m]$. Although γ is not an A_∞-twisting cochain in general, but surprisingly the differential $\tilde d$ on $\tilde\Omega(M) \otimes_\gamma M$ seems to be the proper differential of an A_∞-twisted tensor product,

$$\tilde d = d \otimes 1 + 1 \otimes d + \sum_{i\geqslant 2} (\mu^{(i)} \otimes 1)(1 \otimes \gamma^{\otimes i-1} \otimes 1)(1 \otimes \Delta'_i),$$

according to the maps $\Delta'_i : M \to M^{\otimes i}$,

$$\Delta'_i = f_C^{\otimes i}(1^{\otimes i-2} \otimes \Delta_C\phi_C \cdots (1 \otimes \Delta_C\phi_C)\Delta_C g_C.$$

Note that these Δ'_i are just one of the terms of the Δ_i morphisms arising from the tensor trick. From other hand the bottom row may be perturbed by means of $t\cap$, so that we get

$$A \otimes_t C \stackrel{(c\otimes 1)_{t\cap}}{\Longrightarrow} A' \otimes_{f_A t} C,$$

$f_A t$ being a twisting cochain.

An interesting task to tackle in a future work is to complete the right lower corner of the diagram

$$\begin{array}{ccccc} \bar\Omega(C) \otimes_\theta C & \stackrel{(\bar\Omega(c')\otimes 1)_{\theta\cap}}{\Longrightarrow} & \tilde\Omega(M) \otimes_{\bar\Omega(f)\theta} C & \stackrel{(1\otimes c')_{\bar\Omega(f)\theta\cap}}{\Longrightarrow} & \tilde\Omega(M) \otimes_\gamma M \\ \bar t \otimes 1 \downarrow & & & & \downarrow ?? \\ A \otimes_t C & \stackrel{(c\otimes 1)_{t\cap}}{\Longrightarrow} & A' \otimes_{f_A t} C & \stackrel{??}{\Longrightarrow} & A' \otimes M \end{array}$$

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