Theoretical analysis and control results for the FitzHugh-Nagumo equation

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Abstract

In this paper we are concerned with some theoretical questions for the FitzHugh-Nagumo equation. First, we present a simple proof of the existence and uniqueness of strong solution. We also consider an optimal control problem for this system. We prove the existence of optimal state-control pairs and, as an application of the Dubovitski-Milyoutin formalism, we deduce the corresponding optimality system. We also connect the optimal control problem with a controllability question and we construct a sequence of controls that produce solutions that converge strongly to a desired global state. Finally, we present some open questions related to the control of this equation.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$ ($N = 1, 2$ or 3) and let $T > 0$ be a finite number. We will set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $|\cdot|$ (resp. $(\cdot\,,\cdot)$) the usual norm (resp. scalar product) in $L^2(\Omega)$. In the sequel, C denotes a generic positive constant.

Let ψ_1, ψ_2 and ψ_3 be three given functions in $L^\infty(Q)$. We will consider the FitzHugh-Nagumo equation

$$\begin{cases} u_t - \Delta u + v + F_0(x,t;u) = g, \\ v_t - \sigma u + \gamma v = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = 0, \end{cases} \tag{1}$$

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where $g \in L^2(Q)$, $\sigma > 0$ and $\gamma \geq 0$ are constants, $u_0 \in L^2(\Omega)$ (at least) and $F_0(x,t;u)$ is given by

$$F_0(x,t,u) = (u + \psi_1(x,t))(u + \psi_2(x,t))(u + \psi_3(x,t)).$$

In this system, g is the control, which is constraint to belong to a nonempty closed convex set $\mathcal{G}_{ad} \subset L^2(Q)$ and u and v are the state variables.

The FitzHugh-Nagumo system is a simplified version of the Hodgkin-Huxley model, which seems to reproduce most of its qualitative features. The variable u is the electrical potential across the axonal membrane; v is a recovery variable, associated to the permeability of the membrane to the principal ionic components of the transmembrane current; g is the medicine actuator (the control variable), see [11, 10] for more details. Taking into account the role that can be played by actuators in this context (by inhibiting in the case of calmant medicines and by exciting in the case of anti-depressive products), it is natural to consider control questions for this model.

An equivalent formulation to (1) is easily obtained by solving the second equation, which gives

$$v(x,t) = \sigma \int_0^t e^{-\gamma(t-s)} u(x,s)\, ds. \tag{2}$$

We obtain:

$$\begin{cases} u_t - \Delta u + \sigma \displaystyle\int_0^t e^{-\gamma(t-s)} u(s)\, ds + F_0(x,t;u) = g, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{3}$$

In the sequel, unless otherwise specified, we will always prefer this shorter formulation of the problem. Accordingly, we will work with couples (u,g) which *a posteriori* give the secondary variable v through (2).

This paper deals with several questions concerning systems (1) and (3). First, we will deal with existence, uniqueness and regularity results. In this context, we will provide a simple proof of a known result; a previous proof was given in [12].

The result is the following:

Theorem 1 *Assume that one has $g \in L^2(Q)$ and $u^0 \in H_0^1(\Omega)$. Then* (1) *possesses exactly one solution (u,v), with*

$$u \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \quad u_t \in L^2(Q), \tag{4}$$

$$v \in C^0([0,T];H^2(\Omega)), \quad v_t \in L^2(0,T;H^2(\Omega)). \tag{5}$$

In the sequel, $H^{1,2}(Q)$ stands for the Hilbert space

$$H^{1,2}(Q) = \{\, w \in L^2(0,T;H^2(\Omega)) : w = 0 \ \text{ on } \ \Sigma, \quad w_t \in L^2(Q) \,\}.$$

Our second goal in this paper is to study an optimal control problem for (3). We will mainly deal with the cost functional

$$\mathcal{J}(u,g) = \frac{1}{2}\iint_Q |u-u_d|^2\,dx\,dt + \frac{a}{2}\iint_Q |g|^2\,dx\,dt, \tag{6}$$

where $a > 0$. In particular, we will deduce the optimality system for (3), (6) following the Dubovistky-Milyutin formalism (see [8]).

Definition 1 *Let $\mathcal{Q}$ be the set*

$$\mathcal{Q} = \{\,(u,g) \in H^{1,2}(Q) \times L^2(Q) : (3) \quad \textit{is satisfied}\,\}, \tag{7}$$

Then the admissibility set for (3), (6) is

$$\mathcal{U}_{ad} = \{\,(u,g) : (u,g) \in \mathcal{Q}, \quad g \in \mathcal{G}_{ad}\,\}. \tag{8}$$

It will be said that $(\hat{u},\hat{g})$ is a global optimal state-control if $(\hat{u},\hat{g}) \in \mathcal{U}_{ad}$ and

$$\mathcal{J}(\hat{u},\hat{g}) \le \mathcal{J}(u,g) \quad \forall (u,g) \in \mathcal{U}_{ad}.$$

It will be said that $(\hat{u},\hat{g})$ is a local optimal state-control if $(\hat{u},\hat{g}) \in \mathcal{U}_{ad}$ and there exists $\varepsilon > 0$ such that, whenever $(u,g) \in \mathcal{U}_{ad}$ and $\|u-\hat{u}\|_{H^{1,2}(Q)} + \|g-\hat{g}\|_{L^2(Q)} \le \varepsilon$, one has

$$\mathcal{J}(\hat{u},\hat{g}) \le \mathcal{J}(u,g).$$

Let us state our second main result:

Theorem 2 *Assume that $u^0 \in H_0^1(\Omega)$ and $\mathcal{G}_{ad} \subset L^2(Q)$ is a nonempty closed convex set. Then there exists at least one global optimal state-control $(\hat{u},\hat{g})$. Furthermore, if $(\hat{u},\hat{g})$ is a local optimal state-control of (3), (6) and we assume that $\mathcal{G}_{ad}$ has nonempty interior and $\mathcal{J}'(\hat{u},\hat{g})$ does not vanish, there exists $\hat{p} \in H^{1,2}(Q)$ such that the triplet $(\hat{u},\hat{p},\hat{g})$ satisfies (3) with g replaced by $\hat{g}$, the linear backwards system*

$$\begin{cases} -\hat{p}_t - \Delta\hat{p} + \sigma\displaystyle\int_t^T e^{-\gamma(s-t)}\hat{p}(s)\,ds + D_uF_0(x,t;\hat{u})\,\hat{p} = \hat{u} - u_d, \\ \hat{p}(x,t)|_\Sigma = 0, \\ \hat{p}(x,T) = 0 \end{cases} \tag{9}$$

and the additional inequalities

$$\iint_Q (\hat{p} + a\hat{g})(g - \hat{g})\,dx\,dt \ge 0 \quad \forall g \in \mathcal{G}_{ad}, \quad \hat{g} \in \mathcal{G}_{ad}. \tag{10}$$

Roughly speaking, in order to apply the Dubovistky-Milyutin formalism, we first reformulate the control problem in the form

$$\begin{cases} \text{Minimize} \;\; \mathcal{J}(u,g) \\ \text{subject to} \;\; (u,g) \in \mathcal{Q}, \quad g \in \mathcal{G}_{ad}, \end{cases} \tag{11}$$

where we recall that the set $\mathcal{G}_{ad}$ is a nonempty closed convex subset of $L^2(Q)$ (the control constraint set) and $\mathcal{Q}$ is given by an equality constraint:

$$\mathcal{Q} = \{(u, g) \in H^{1,2}(Q) \times L^2(Q) : M(u, g) = 0\}$$

for a suitable operator M.

Assume that $(\hat{u}, \hat{g})$ is a local minimizer of (11). Then we associate to $(\hat{u}, \hat{g})$ the cone K_0 of decreasing directions of $\mathcal{J}$, the cone K_1 of feasible directions of $\mathcal{G}_{ad}$ and the tangent subspace K_2 to the constraint set $\mathcal{Q}$. We have the following (geometrical) necessary condition of optimality:

$$K_0 \cap K_1 \cap K_2 = \phi.$$

Accordingly, there exist continuous linear functionals f_0, f_1 and f_2, not simultaneously zero, such that $f_i \in K_i^*$ for $i = 1, 2, 3$ and

$$f_0 + f_1 + f_2 = 0$$

(this is the Euler-Lagrange equation for the previous extremal problem). From this equation we obtain the optimality system (3) (with g replaced by $\hat{g}$), (9), (10).

A large family of control problems involving partial differential equations can be solved by this method. In particular, several interesting generalizations and modified versions of (3), (6) can be considered: other non-quadratic functionals, control problems with constraints on the state, multi-objective control problems, etc.

Remark 1 When int $\mathcal{G}_{ad}$ is the empty set, to determine the cone K_1 is more complicated. In this case, we can argue as in [7] to obtain a similar result. For simplicity, we will not give the details.

Remark 2 When $\mathcal{J}'(\hat{u}, \hat{g}) = (0, 0)$, it is natural to look for second-order optimality conditions. This can be made following the results in [1] for this and many other problems. An analysis of this situation will be given in a next paper.

Our third goal in this paper is to analyze the behavior of the solutions to problems of the kind (3), (6) as $a \to 0^+$. It is well known that this is a way to pass from the optimal control to a controllability approach. More precisely, if $\mathcal{G}_{ad} = L^2(\Omega)$, it is expected that the solutions $(\hat{u}, \hat{g})$ of (3), (6) satisfy $\hat{u} \to u_d$ as $a \to 0^+$.

A result of this kind in established in our next theorem. In order to give the statement, we have to introduce a new function:

$$H_0(x, t; s) = \begin{cases} \dfrac{F_0(x, t; s) - F_0(x, t; 0)}{s} & \text{if } \ s \neq 0, \\ D_u F_0(x, t; 0) & \text{otherwise.} \end{cases}$$

Then we have:

Theorem 3 *Assume that* $u^0 = 0$*,* $\mathcal{G}_{ad} = L^2(Q)$ *and* $u_d \in L^r(Q)$*, where* $r \geq 4$*. For each* $n = 1, 2, \ldots$*, let* (u^n, p^n, g^n) *be a solution of the coupled problem*

$$\begin{cases} u^n_t - \Delta u^n + \sigma \displaystyle\int_t^T e^{-\gamma(s-t)} u^n(s)\, ds + F_0(x,t;u^n)\, p^n = g^n, \\ -p^n_t - \Delta p^n + \sigma \displaystyle\int_t^T e^{-\gamma(s-t)} p^n(s)\, ds + H_0(x,t;u^n)\, p^n = |u^n - u_d|^{r-2}(u^n - u_d), \\ u^n(x,t)|_\Sigma = p^n(x,t)|_\Sigma = 0, \\ u^n(x,0) = 0, \quad p^n(x,T) = 0, \\ p^n + \dfrac{1}{n} g^n = 0. \end{cases} \tag{12}$$

Then $u^n \to u_d$ *strongly in* $L^r(Q)$ *as* $n \to \infty$*.*

In this way, for any target $u_d \in L^r(Q)$ we can construct a sequence of (possibly unbounded) controls g^n and associated states u^n that converge to u_d.

The proof of this theorem will be given below. It relies on some estimates for u^n in $L^r(Q)$ and p^n in $L^2(Q)$.

Remark 3 This result is inspired by the ideas of J.-L. Lions in the context of the approximate controllability of linear parabolic equations; see [13, 9].

Remark 4 The equation satisfied by p^n in (12) is not exactly the same satisfied by $\hat{p}$ in (9). First, we have a different right hand side. This is motivated by the search of a good estimate for u^n. Indeed, it will be seen in Section 4 that the term $|u^n - u_d|^{r-2}(u^n - u_d)$ with $r \geq 4$ is needed to bound u^n in $L^r(Q)$ and then $H_0(\cdot; u^n)$ in $L^2(Q)$. The second difference is that the coefficient of p^n in (12) is $H_0(x,t;u^n)$ and not $D_u F_0(x,t;u^n)$. This is also needed to estimate u^n.

This paper is organized as follows. Sections 2 to 4 are respectively devoted to the proofs of theorems 1, 2 and 3. Then, in Section 5 we present several additional remarks and open questions. Among other things, we will address some controllability questions. It will be seen there that, unfortunately, very few is known on the subject.

2 Existence, uniqueness and regularity results

Assume that $g \in L^2(Q)$ and $u^0 \in H^1_0(\Omega)$ in (3). Notice that (3) can be written in the form

$$\begin{cases} u_t - \Delta u + G(u) + F(u) = g, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{13}$$

where we have set

$$G(u)(x,t) \equiv \sigma \int_0^t e^{-\gamma(t-s)} u(x,s)\, ds \tag{14}$$

and

$$F(u)(x,t) \equiv F_0(x,t;u(x,t)). \tag{15}$$

We will first prove that (13) possesses at least one solution $u \in H^{1,2}(Q)$ with the help of the Leray-Schauder's principle.

Thus, let us consider for each $\lambda \in [0,1]$ the auxiliary problem

$$\begin{cases} u_t - \Delta u = \lambda(g - G(u) - F(u)), \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{16}$$

Also, let us introduce the mapping $\Lambda : L^6(Q) \times [0,1] \mapsto L^6(Q)$, with $u = \Lambda(w,\lambda)$ if and only if u is the unique solution to

$$\begin{cases} u_t - \Delta u = \lambda(g - G(w) - F(w)), \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{17}$$

We will prove the following:

Lemma 1 *The mapping $\Lambda : L^6(Q) \times [0,1] \mapsto L^6(Q)$ is well-defined, continuous and compact.*

Lemma 2 *All functions u such that $\Lambda(u,\lambda) = u$ for some λ are uniformly bounded in $L^6(Q)$.*

In view of the Leray-Schauder's principle, this suffices to affirm that (3) possesses at least one solution.

Proof of lemma 1: First, notice that for any $w \in L^6(Q)$ we have $F(w) \in L^2(Q)$ and $G(w) \in L^\infty(0,T;L^6(\Omega))$. Furthermore, the mappings $w \mapsto F(w)$ and $w \mapsto G(w)$ are continuous. Consequently, it is obvious that $(w,\lambda) \mapsto \Lambda(w,\lambda)$ is well-defined and continuous from $L^6(Q) \times [0,1]$ into $L^6(Q)$.

The compactness of Λ is a consequence of parabolic regularity. Indeed, if $(w,\lambda) \in L^6(Q) \times [0,1]$ and $u = \Lambda(w,\lambda)$, we have

$$u \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \quad u_t \in L^2(Q),$$

i.e. $u \in H^{1,2}(Q)$ (we are using here that $u_0 \in H_0^1(\Omega)$).

Moreover, the estimates we will prove in lemma 2 show that, whenever (w,λ) belongs to a bounded set of $L^6(Q) \times [0,1]$, the associated u belongs to a bounded set in $H^{1,2}(Q)$. Since this space is compactly embedded in $L^6(Q)$ for $N = 1,2$ or 3, we deduce that $\Lambda : L^6(Q) \times [0,1] \mapsto L^6(Q)$ is compact.

Proof of lemma 2: Let us assume that $\lambda \in [0,1]$, $u \in L^6(Q)$ and $\Lambda(u,\lambda) = u$, i.e. u solves (16). We will prove that, for some constant $C > 0$ independent of λ and u, one has

$$\|u\|_{L^6(Q)} \le C. \tag{18}$$

In fact, we will directly prove much more: that u is uniformly bounded in $H^{1,2}(Q)$.

Let us rewrite (16) in the form

$$\begin{cases} u_t - \Delta u + \lambda v + \lambda F(u) = \lambda g, \\ v_t + \gamma v - \sigma u = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = 0. \end{cases} \tag{19}$$

Then, by multiplying by u (resp. $\frac{\lambda}{\sigma} v$) the first equation (resp. the second equation), integrating in Ω and adding the resulting identities, we get

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{\lambda}{2\sigma}\frac{d}{dt}|v|^2 + |\nabla u|^2 + \frac{\lambda\gamma}{\sigma}|v|^2 + \lambda(F(u), u) = \lambda(g, u) \tag{20}$$

in $(0, T)$.

Notice that, for any $\varepsilon > 0$, there exists C_ε such that

$$(F(u), u) \geq (1-\varepsilon)\|u\|_{L^4}^4 - C_\varepsilon. \tag{21}$$

Indeed, we have for instance

$$\left| \int_\Omega \psi_j \, u^3 \, dx \right| \leq C\|\psi_j\|_{L^\infty}\|u\|_{L^4}^3 \leq \frac{\varepsilon}{8}\|u\|_{L^4}^4 + C_\varepsilon$$

for any $j = 1, 2, 3$. In view of (20) and (21), we have:

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{\lambda}{2\sigma}\frac{d}{dt}|v|^2 + |\nabla u|^2 + \frac{\lambda\gamma}{\sigma}|v|^2 + \lambda(1-\varepsilon)\|u\|_{L^4}^4 \leq \frac{1}{2}|\nabla u|^2 + \lambda C_\varepsilon.$$

Now, from Gronwall's Lemma, the following is obtained:

$$\begin{cases} \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H_0^1(\Omega))} \leq C, \\ \lambda\|v\|_{L^\infty(0,T;H_0^1(\Omega))} + \lambda\|u\|_{L^4(Q)} \leq C. \end{cases} \tag{22}$$

Let us now multiply by u_t the first equation in (19) and let us integrate in Ω. We get:

$$\frac{1}{2}|u_t|^2 + \frac{1}{2}\frac{d}{dt}|\nabla u|^2 + \lambda(v, u_t) + \lambda(F(u), u_t) = \lambda(g, u_t). \tag{23}$$

Now, we have

$$(F(u), u_t) \geq \frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 - \varepsilon|u_t|^2 - C_\varepsilon(1 + |\nabla u|^2) - C_\varepsilon\|u\|_{L^4}^4 \tag{24}$$

since, for instance,

$$\left| \int_\Omega \psi_j \, u^2 u_t \, dx \right| \leq C\|\psi_j\|_{L^\infty}\|u\|_{L^4}^2|u_t| \leq \frac{\varepsilon}{8}|u_t|^4 + C_\varepsilon\|u\|_{L^4}^4$$

for any $j = 1, 2, 3$. On the other hand,

$$|(v, u_t)| \leq \varepsilon |u_t|^2 + C_\varepsilon |v|^2 \leq \varepsilon |u_t|^2 + C_\varepsilon. \tag{25}$$

From (23)–(25), we obtain the inequality

$$(1 - 2\varepsilon)|u_t|^2 + \frac{1}{2}\frac{d}{dt}|\nabla u|^2 + \frac{\lambda}{4}\frac{d}{dt}\|u\|_{L^4}^4 \leq C_\varepsilon(1 + |\nabla u|^2) + \lambda C_\varepsilon \|u\|_{L^4}^4$$

and, from Gronwall's Lemma, we find:

$$\|u_t\|_{L^2(Q)} + \|u\|_{L^\infty(0,T;H_0^1(\Omega))} + \lambda \|u\|_{L^\infty(0,T;L^4(\Omega))} \leq C. \tag{26}$$

Notice that we have used here again the fact that $u_0 \in H_0^1(\Omega)$.

Finally, let us multiply by $-\Delta u$ the first equation in (19) and let us integrate in Ω. This time, we obtain the following identity:

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 + |\Delta u|^2 + \lambda(v, -\Delta u) + \lambda(F(u), -\Delta u) = \lambda(g, -\Delta u). \tag{27}$$

It is not difficult to check that

$$(F(u), -\Delta u) \geq \frac{3}{4}|\nabla(u^2)|^2 - \varepsilon|\Delta u|^2 - C_\varepsilon(1 + |u|^2 + \|u\|_{L^4}^4). \tag{28}$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 + |\Delta u|^2 + \frac{3\lambda}{4}\frac{d}{dt}|\nabla(u^2)|^2 \leq \varepsilon|\Delta u|^2 + \lambda C_\varepsilon$$

and using again Gronwall's Lemma we find that

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|u\|_{L^\infty(0,T;H_0^1(\Omega))} \leq C. \tag{29}$$

This ends the proof of the Lemma.

Let us now see that the solution we have found is unique. Thus, let u^1 and u^2 be two solutions (in $H^{1,2}(Q)$) of (3) and let us set $u = u^1 - u^2$. Let us also introduce

$$v = v^1 - v^2 = \sigma \int_0^t e^{-\gamma(t-s)}(u^1(s) - u^2(s))\, ds.$$

Then the following holds:

$$\begin{cases} u_t - \Delta u + v + F(u^1) - F(u^2) = 0, \\ v_t + \gamma v - \sigma u = 0, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = 0, \quad v(x,0) = 0. \end{cases}$$

Consequently, by multiplying the first and second equations respectively by u and $\frac{1}{\sigma}v$ and integrating in Ω, we get:

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2\sigma}\frac{d}{dt}|v|^2 + |\nabla u|^2 + \frac{\gamma}{\sigma}|v|^2 + (F(u^1) - F(u^2), u) = 0. \tag{30}$$

We have

$$(F(u^1) - F(u^2), u) =$$

$$\int_\Omega \left[(u^1+\psi_1)(u^1+\psi_2)(u^1+\psi_3) - (u^2+\psi_1)(u^2+\psi_2)(u^2+\psi_3)\right] u\,dx$$

$$= I_0 + \sum_{j=1}^{3} I_j + \sum_{1\le j<k\le 3} I_{j,k},$$

where we have used the notation

$$I_0 = \int_\Omega \left((u^1)^3 - (u^2)^3\right)(u^1 - u^2)\,dx,$$

$$I_j = \int_\Omega \psi_j (u^1+u^2)|u^1-u^2|^2\,dx$$

for $1 \le j \le 3$ and

$$I_{j,k} = \int_\Omega \psi_j\psi_k |u^1-u^2|^2\,dx$$

for $1 \le j < k \le 3$. Since $I_0 \ge 0$, we find that

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2\sigma}\frac{d}{dt}|v|^2 + |\nabla u|^2 + \frac{\gamma}{\sigma}|v|^2$$
$$\le C\int_\Omega (1+|u^1|+|u^2|)\,|u|^2\,dx \le \|\beta(t)\|_{L^\infty}|u|^2,$$

where the function β belongs to $L^2(0,T;L^\infty(\Omega))$. Since $u(x,0)\equiv 0$ and $v(x,0)\equiv 0$, we deduce that u vanishes identically, whence $u^1 = u^2$.

Hence, (3) possesses exactly one solution in $H^{1,2}(Q)$.

Remark 5 Instead of (1), we could have started from the more general system

$$\begin{cases} u_t - \Delta u + v + F_0(x,t;u) = g, \\ v_t - \sigma u + \gamma v = \tilde g, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{31}$$

where $\tilde g \in L^1(0,T;L^2(\Omega))$, $v_0 \in L^2(\Omega)$. Then, the problem is reduced again to a system of the form (3), with g replaced by

$$\overline{g} = g - v_0(x)e^{-\gamma t} - \int_0^t e^{-\gamma(t-s)}\tilde g(s)\,ds.$$

Indeed, the solution of (31) is given by the couple (u,v), where u is the solution of (3) with this right hand side and

$$v = v_0(x)e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}\tilde g(s)\,ds + \sigma\int_0^t e^{-\gamma(t-s)}u(s)\,ds.$$

3 An optimal control problem. The Dubovitski-Milyoutin formalism

Let us write the optimal control problem (3), (6) in the form

$$\begin{cases} \text{Minimize } \ \mathcal{J}(u,g) \\ \text{subject to } \ g \in \mathcal{G}_{ad}, \quad (u,g) \in \mathcal{Q}, \end{cases} \tag{32}$$

where $\mathcal{G}_{ad} \subset L^2(Q)$ is a nonempty closed convex set and $\mathcal{Q}$ is given by (7).

The proof of the existence of at least one (global) optimal state-control $(\hat{u}, \hat{g})$ is completely standard. For completeness, let us sketch the argument.

Let $\{(u^n, g^n)\}$ be a minimizing sequence for (3), (6). This means that $(u^n, g^n) \in \mathcal{U}_{ad}$ for all n and

$$\lim_{n\to\infty} \mathcal{J}(u^n, g^n) = \mathcal{J}_* := \inf_{\mathcal{U}_{ad}} \mathcal{J}$$

($\mathcal{U}_{ad}$ is given by (8)). Then, it is immediate that g^n is uniformly bounded in $L^2(Q)$. Taking into account the estimates in Section 2, we see that u^n is uniformly bounded in $H^{1,2}(Q)$ and the sequence $\{u^n\}$ is relatively compact in $L^6(Q)$. Therefore, at least for a subsequence, we have

$$g^n \to \hat{g} \ \text{ weakly in } \ L^2(Q)$$

and

$$u^n \to \hat{u} \ \text{ weakly in } \ H^{1,2}(Q) \ \text{ and strongly in } \ L^6(Q),$$

for some $(\hat{u}, \hat{g}) \in H^{1,2}(Q) \times L^2(Q)$. Obviously, $\hat{g} \in \mathcal{G}_{ad}$. Furthermore, in view of the strong convergence of u^n in $L^6(Q)$, we can take limits in the equation satisfied by u^n and deduce that $\hat{u}$ is the state associated to $\hat{g}$. This shows that $(\hat{u}, \hat{g}) \in \mathcal{U}_{ad}$.

On the other hand,

$$\mathcal{J}(\hat{u}, \hat{g}) \le \liminf_{n\to\infty} \mathcal{J}(u^n, g^n) = \mathcal{J}_*,$$

whence $(\hat{u}, \hat{g})$ is an optimal state-control.

To our knowledge, the uniqueness of optimal control is an open question.

Now, let $(\hat{u}, \hat{g})$ be a local optimal state-control. Let us prove that the optimality system (3), (9), (10) holds. As mentioned above, our approach is based on the Dubovitski-Milyoutin formalism.

Thus let us introduce the cone K_0 of decreasing directions of $\mathcal{J}$ at $(\hat{u}, \hat{g})$:

$$\begin{aligned} K_0 = \{\, (w,h) \in L^2(Q) \times L^2(Q) : \exists \delta_0 > 0 \ \text{ such that} \\ \mathcal{J}((\hat{u}, \hat{g}) + \delta(w,h)) < \mathcal{J}(\hat{u}, \hat{g}) \ \text{ for } \ 0 < \delta \le \delta_0 \,\}. \end{aligned} \tag{33}$$

Since $\mathcal{J}$ is Frechet-differentiable at any point, it is immediate that

$$K_0 = \{\, (w,h) \in L^2(Q) \times L^2(Q) : \langle \mathcal{J}'(\hat{u}, \hat{g}), (w,h) \rangle < 0 \,\}, \tag{34}$$

a nonempty set.

Let us also introduce the cone of feasible directions of $\mathcal{G}_{ad}$ at $\hat{g}$. This is the set

$$K_1 = \{\, (w,h) \in L^2(Q) \times L^2(Q) : \exists \delta_1 > 0 \text{ such that } \hat{g} + \delta h \in \mathcal{G}_{ad} \text{ for } 0 < \delta \le \delta_1 \,\}. \tag{35}$$

Since $\mathcal{G}_{ad}$ has nonempty interior, it is not difficult to check that

$$K_1 = \{\, (w, \lambda(g - \hat{g})) : w \in L^2(Q), \quad \lambda > 0, \quad g \in \operatorname{int} \mathcal{G}_{ad} \,\}. \tag{36}$$

Finally, let us consider the cone K_2 of tangent directions of $\mathcal{Q}$ at $(\hat{u}, \hat{g})$. This is given as follows:

$$\begin{aligned} K_2 = \{\, (w,h) \in H^{1,2}(Q) \times L^2(Q) : \exists \theta^n, (u^n, g^n) \text{ for } n = 1, 2, \dots \\ \text{with } \theta^n \to 0, \quad (u^n, g^n) \in \mathcal{Q} \text{ and } \\ \lim_{n\to\infty} \tfrac{1}{\theta^n} [(u^n, g^n) - (\hat{u}, \hat{g})] = (w,h) \,\}. \end{aligned} \tag{37}$$

In order to give an explicit determination of K_2, it is convenient to introduce the spaces $E_1 = H^{1,2}(Q) \times L^2(Q)$ and $E_2 = L^2(Q) \times H^1_0(\Omega)$ and the nonlinear mapping $M : E_1 \mapsto E_2$, with

$$\begin{cases} M(u,g) = (u_t - \Delta u + G(u) + F(u) - g, u|_{t=0} - u^0) \\ \forall (u,g) \in E_1. \end{cases} \tag{38}$$

Let us also set

$$F'(u)(x,t) \equiv D_u F_0(x,t; u(x,t)).$$

Then we have the following result:

Lemma 3 *The mapping M is continuously differentiable in E_1, with $M'(u,g)$ given as follows:*

$$\begin{cases} M'(u,g)(w,h) = (w_t - \Delta w + G(w) + F'(u)w - h, w|_{t=0}) \\ \forall (u,g) \in E_1, \quad (w,h) \in E_1. \end{cases} \tag{39}$$

Furthermore, for each $(u,g) \in E_1$ the linear operator $M'(u,g) : E_1 \mapsto E_2$ is onto.

Proof: There is only one nontrivial step in the proof of this lemma.

Indeed, it is clear that $M : E_1 \mapsto E_2$ is well-defined and continuously differentiable. It is also clear that its F-derivative is given by (39).

In order to see that $M'(u,g)$ is an epimorphism, let (k, w_0) be given in $L^2(Q) \times H^1_0(\Omega)$ and let us consider the linear problem

$$\begin{cases} w_t - \Delta w + G(w) + F'(u)w = k, \\ w(x,t)|_\Sigma = 0, \\ w(x,0) = w_0(x), \end{cases} \tag{40}$$

All we have to do is to prove that (40) possesses at least one solution $w \in H^{1,2}(Q)$.

Notice that, in this system, $F'(u) \in L^\infty(0,T;L^3(\Omega)) \cap L^1(0,T;L^\infty(\Omega)) \hookrightarrow L^4(Q)$. This is sufficient to prove the existence of a weak solution, i.e. a solution in $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Indeed, in order to get energy estimates, we multiply the first equation in (40) by w and we integrate in Ω. All the terms can be estimated easily except possibly $(F'(u)w,w)$. But this one satisfies

$$|(F'(u)w,w)| \le C\|F'(u)\|_{L^4}\,|w|^{5/4}|\nabla w|^{3/4} \le \varepsilon|\nabla w|^2 + C_\varepsilon\|F'(u)\|_{L^4}^{8/5}|w|^2,$$

which leads to the the usual estimates for w.

We have

$$L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^4(0,T;L^3(\Omega)).$$

Hence, since $F'(u) \in L^4(Q)$ and $w \in L^4(0,T;L^3(\Omega))$, we get

$$F'(u)w \in L^2(0,T;L^{12/7}(\Omega)).$$

From the usual parabolic L^r estimates, we deduce that $w \in L^2(0,T;W^{2,12/7}(\Omega))$ and $w_t \in L^2(0,T;L^{12/7}(\Omega))$ and, from interpolation results, we see that $w \in L^\infty(0,T;L^4(\Omega))$.

Finally, taking into account that

$$L^\infty(0,T;L^3(\Omega)) \cap L^1(0,T;L^\infty(\Omega)) \hookrightarrow L^2(0,T;L^6(\Omega))$$

and consequently $F'(u) \in L^2(0,T;L^6(\Omega))$, we find (among other things) that $F'(u)w \in L^2(Q)$. This gives $w \in H^{1,2}(Q)$.

Notice that $\mathcal{Q}$ can be written in the form

$$\mathcal{Q} = \{\,(u,g) \in H^{1,2}(Q) \times L^2(Q) : M(u,g) = 0\,\}. \tag{41}$$

Therefore, in view of Lemma 3 and the results in [8], the tangent cone at $(\hat{u},\hat{g})$ is

$$K_2 = \{\,(w,h) \in H^{1,2}(Q) \times L^2(Q) : M'(\hat{u},\hat{g})(w,h) = 0\,\}. \tag{42}$$

In view of (34), (36) and (42), it is easy to determine the dual cones K_i^* for $i = 0,1,2$. Specifically, we have:

$$K_0^* = \{\,-\lambda\mathcal{J}'(\hat{u},\hat{g}) : \lambda \ge 0\,\}, \tag{43}$$

$$K_1^* = \{\,(0,f) : f \in L^2(Q) : \iint_Q f g\,dx\,dt \ge \iint_Q f\hat{g}\,dx\,dt \quad \forall g \in \mathcal{G}_{ad}\,\}$$

and

$$K_2^* = \{\,\Phi \in E_1' : \langle \Phi, (w,h)\rangle = 0 \quad \forall (w,h) \in E_1 \ \text{ such that } \ M'(\hat{u},\hat{g})(w,h) = 0\,\}.$$

We can now apply the main result in [8]. Thus, for some $(f_{01}, f_{02}) \in K_0^*$, $f_{12} \in K_1^*$ and $\Phi_2 \in K_2^*$ not vanishing simultaneously, one has:

$$\left\{\begin{array}{l} \displaystyle\iint_Q (f_{01}w + f_{02}h)\,dx\,dt + \iint_Q f_1 h\,dx\,dt + \langle \Phi_2, (w,h)\rangle = 0 \\ \forall (w,h) \in E_1 = H^{1,2}(Q) \times L^2(Q). \end{array}\right. \tag{44}$$

Let us now see that (44) leads to (3), (9), (10).

In view of (43), there exists $\lambda_0 \geq 0$ such that

$$(f_{01}, f_{02}) = -\lambda_0 \, (\hat{u} - u_d, a\hat{g}).$$

Let us choose $(w, h) \in E_1$ such that $M'(\hat{u}, \hat{g})(w, h) = 0$. Then

$$-\lambda_0 \iint_Q ((\hat{u} - u_d)w + a\hat{g}h) \; dx \, dt + \iint_Q f_{12} h \, dx \, dt = 0. \tag{45}$$

But this implies that $\lambda_0 > 0$; otherwise, we would have $(f_{01}, f_{02}) = (0, 0)$, $f_{12} = 0$ (by (45)) and $\Phi_2 = 0$ (by (44)). Consequently, we can assume that $\lambda_0 = 1$ and

$$\begin{cases} \displaystyle\iint_Q f_{12} h \, dx \, dt = \iint_Q ((\hat{u} - u_d)w + a\hat{g}h) \; dx \, dt \\ \forall (w, h) \in E_1 \;\; \text{such that} \;\; M'(\hat{u}, \hat{g})(w, h) = 0. \end{cases} \tag{46}$$

Let us introduce the adjoint system

$$\begin{cases} -\hat{p}_t - \Delta \hat{p} + \sigma \displaystyle\int_t^T e^{-\gamma(s-t)} \hat{p}(s) \, ds + D_u F_0(x, t; u) \, \hat{p} = \hat{u} - u_d, \\ \hat{p}(x, t)|_\Sigma = 0, \\ \hat{p}(x, T) = 0 \end{cases} \tag{47}$$

Then, for any $(w, h) \in E_1$ such that $M'(\hat{u}, \hat{g})(w, h) = 0$ one has

$$\begin{aligned} &\iint_Q (\hat{u} - u_d) w \, dx \, dt \\ &= \iint_Q \left(-\hat{p}_t - \Delta \hat{p} + \sigma \int_t^T e^{-\gamma(s-t)} \hat{p}(s) \, ds + D_u F_0(x, t; u) \, \hat{p} \right) w \, dx \, dt \\ &= \iint_Q \hat{p} \left(w_t - \Delta w + \sigma \int_0^t e^{-\gamma(t-s)} w(s) \, ds + D_u F_0(x, t; u) \, w \right) \, dx \, ds \\ &= \iint_Q \hat{p} h \, dx \, dt. \end{aligned}$$

Hence,

$$\iint_Q f_{12} h \, dx \, dt = \iint_Q (\hat{p} + a\hat{g}) h \, dx \, dt \quad \forall h \in L^2(Q).$$

From the fact that $(0, f_{12}) \in K_1^*$, we obtain that

$$\iint_Q (\hat{p} + a\hat{g})(g - \hat{g}) \, dx \, dt \geq 0 \quad \forall g \in \mathcal{G}_{ad}. \tag{48}$$

Thus, the triplet $(\hat{u}, \hat{p}, \hat{g})$ satisfies (3) (with g replaced by $\hat{g}$), (47) and (48) and this is what we wanted to prove.

4 A controllability question

Let us now prove theorem 3. Thus, let us assume that $u^0 = 0$, $\mathcal{G}_{ad} = L^2(Q)$ and $u_d \in L^r(Q)$ with $r \geq 4$.

For each $n \geq 1$, let us consider the coupled system (12). Notice that it can be written in the form

$$\left\{\begin{array}{l} u_t^n - \Delta u^n + \sigma \displaystyle\int_t^T e^{-\gamma(s-t)} u^n(s)\, ds + F_0(x,t;u^n)\, p^n = -np^n, \\ -p_t^n - \Delta p^n + \sigma \displaystyle\int_t^T e^{-\gamma(s-t)} p^n(s)\, ds + H_0(x,t;u^n) p^n = |u^n - u_d|^{r-2}(u^n - u_d), \\ u^n(x,t)|_\Sigma = p^n(x,t)|_\Sigma = 0, \\ u^n(x,0) = 0, \quad p^n(x,T) = 0, \end{array}\right. \tag{49}$$

with $g^n = -np^n$.

Let us first show that, for each $n \geq 1$, there exists at least one solution of (49), with

$$\left\{\begin{array}{l} u^n \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \quad u_t^n \in L^2(Q), \\ p^n \in L^{r'}(0,T;W^{2,r'}(\Omega)), \quad p_t^n \in L^{r'}(Q). \end{array}\right. \tag{50}$$

To this end, we can argue as in the proof of Theorem 1. Thus, let us set

$$H(u)(x,t) \equiv H_0(x,t;u(x,t))$$

and let us introduce the space $E = L^6(Q) \times L^2(Q) \times L^2(Q)$ and the mapping $\Xi : E \times [0,1] \mapsto E$, with $(u,p,g) = \Xi(w,q,h,\lambda)$ if and only if u is the unique solution to

$$\left\{\begin{array}{l} u_t - \Delta u = \lambda \left(h - \displaystyle\int_0^t e^{-\gamma(t-s)} w(s)\, ds - F(w) \right), \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = 0 \end{array}\right. \tag{51}$$

and $g = -np$, where p is the unique solution to

$$\left\{\begin{array}{l} -p_t - \Delta p = \lambda \left(|w - u_d|^{r-2}(w - u_d) - \displaystyle\int_t^T e^{-\gamma(s-t)} q(s)\, ds - H(w) q \right), \\ p(x,t)|_\Sigma = 0, \\ p(x,T) = 0 \end{array}\right. \tag{52}$$

Then we have the following:

Lemma 4 *The mapping $\Xi : E \times [0,1] \mapsto E$ is well-defined, continuous and compact.*

Lemma 5 *All (u,p,g) such that $\Xi(u,p,g,\lambda) = (u,p,g)$ for some λ are uniformly bounded in E.*

In view of the Leray-Schauder's principle, this yields the desired existence result for (49).

Proof of lemma 4: It is very similar to the proof of Lemma 1. If $(u,p,g) \in E$ and $\lambda \in [0,1]$, then the solution of (51) is well defined and satisfies

$$u \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)), \quad u_t \in L^2(Q).$$

On the other hand, since $H(w) \in L^3(Q)$ (and consequently $H(w)q \in L^{6/5}(Q)$) and $|w-u_d|^{r-2}(w-u_d) \in L^{r'}(Q)$, (52) possesses exactly one solution p, with

$$p \in L^m(0,T;W^{2,m}(\Omega)), \quad p_t \in L^m(Q), \tag{53}$$

where $m = \min(r', 6/5)$. Notice that the space of functions satisfying (53) is compactly embedded in $L^2(Q)$. Therefore, g is also well defined through the equality $g = -np$.

Obviously, this construction shows that the mapping $(w,q,h,\lambda) \mapsto (u,p,g)$ is continuous and compact.

Proof of lemma 5: Assume that $\lambda \in [0,1]$, $(u,p,g) \in E$ and $\Xi(u,p,g,\lambda) = (u,p,g)$.

This means that u and $p = -ng$ solve the problem

$$\begin{cases} u_t - \Delta u = \lambda \left(-np - \displaystyle\int_0^t e^{-\gamma(t-s)} u(s)\, ds - F(u) \right), \\ -p_t - \Delta p = \lambda \left(|u-u_d|^{r-2}(u-u_d) - \displaystyle\int_t^T e^{-\gamma(s-t)} p(s)\, ds - H(u)p \right), \\ u(x,t)|_\Sigma = p(x,t)|_\Sigma = 0, \\ u(x,0) = 0, \quad p(x,T) = 0. \end{cases} \tag{54}$$

Let us prove that u (resp. p) is bounded in $L^6(Q)$ (resp. $L`2(Q)$) by a constant that can depend on n but is independent of λ. This will suffice to prove the lemma.

Obviously, if $\lambda = 0$, then $u \equiv 0$ and $p \equiv 0$. Consequently, it can be assumed that $\lambda > 0$.

Let us multiply the first (resp. the second) equation in (54) by p (resp. by u). Let us sum the resulting identities and let us integrate with respect to x and t in Q. After some short computations, in view of the definition of $H(u)$, and the fact that $u(x,0) = p(x,T) = 0$ in Ω, the following is found:

$$\lambda \iint_Q |u-u_d|^{r-2}(u-u_d)u\, dx\, dt + \lambda n \iint_Q |p|^2\, dx\, dt = -\lambda \iint_Q F(0)\, p\, dx\, dt.$$

Consequently,

$$\begin{cases} \displaystyle\iint_Q |u-u_d|^r\, dx\, dt + n \iint_Q |p|^2\, dx\, dt \\ \qquad = -\displaystyle\iint_Q |u-u_d|^{r-2}(u-u_d)u_d\, dx\, dt - \iint_Q F(0)\, p\, dx\, dt \end{cases} \tag{55}$$

and we have

$$\iint_Q |u-u_d|^r \,dx\,dt + n\iint_Q |p|^2\,dx\,dt \le C, \tag{56}$$

where the constant C is independent of λ and n.

From (56), arguing as in the proof of Lemma 2, we deduce that u is in fact bounded in $L^6(Q)$ by a constant that can depend on n. Obviously, we also obtain from (56) that the norm of p in $L^2(Q)$ is uniformly bounded.

Then, arguing as in the proof of Lemma 3, the same is found for p.

This ends the proof.

Let us now finish the proof of Theorem 3.

For each n, let (u^n, p^n, g^n) be a solution of (12). Then, the identity (55) and the estimate (56) hold for (u^n, p^n):

$$\left\{ \begin{array}{l} \displaystyle\iint_Q |u^n-u_d|^r \,dx\,dt + n\iint_Q |p^n|^2\,dx\,dt \\ \displaystyle\qquad = -\iint_Q |u^n-u_d|^{r-2}(u^n-u_d)u_d\,dx\,dt - \iint_Q F(0)\,p^n\,dx\,dt \end{array} \right. \tag{57}$$

and

$$\iint_Q |u^n-u_d|^r \,dx\,dt + n\iint_Q |p^n|^2\,dx\,dt \le C. \tag{58}$$

Accordingly, u^n is uniformly bounded in $L^r(Q)$ and $p^n \to 0$ strongly in $L^2(Q)$.

Let us look at the equation satisfied by p^n in Q:

$$-p^n_t - \Delta p^n + \sigma\int_t^T e^{-\gamma(s-t)}p^n(s)\,ds + H(u^n)\,p^n = |u^n-u_d|^{r-2}(u^n-u_d).$$

In the left hand side, the first three terms converge to zero in the distribution sense. This is also the case for the fourth one, since $H(u^n)$ is uniformly bounded in $L^2(Q)$ (it is just at this point where we use that $r \ge 4$). Consequently, the right hand side also converges to zero. Since it is bounded in $L^{r'}(Q)$, it converges weakly to zero in this space (r' is the conjugate exponent of r).

But this implies that u^n converges strongly to u_d in $L^r(Q)$. Indeed, from (57), the weak convergence of $|u^n-u_d|^{r-2}(u^n-u_d)$ and the fact that $u_d \in L^r(Q)$, we see that

$$\iint_Q |u^n-u_d|^r \,dx\,dt + n\iint_Q |p^n|^2\,dx\,dt \to 0.$$

This ends the proof.

5 Some final remarks and open problems

This Section is devoted to discuss some additional facts concerning the control of (3). Some of them lead to open problems that, in our opinion, are of considerable interest.

5.1 Other optimal control problems

There are many other optimal control problems that can be considered for systems of the kind (3). Let us mention one of them.

Thus, consider the new cost functional $\mathcal{L}$, where

$$\mathcal{L}(u,g) = \frac{1}{2}\int_\Omega |u(x,T) - u_1(x)|^2\,dx + \frac{a}{2}\iint_Q |g|^2\,dx\,dt \tag{59}$$

and $u^1 \in L^2(\Omega)$ is a given function. The following result holds:

Theorem 4 *Assume that $u^0 \in H_0^1(\Omega)$ and $\mathcal{G}_{ad} \subset L^2(Q)$ is a nonempty closed convex set. Then there exists at least one global optimal state-control $(\hat u, \hat g)$ of (3), (59). Furthermore, if $(\hat u, \hat g)$ is a local optimal state-control, $\mathcal{G}_{ad}$ has nonempty interior and $\mathcal{L}'(\hat u, \hat g)$ does not vanish, there exists $\hat p \in H^{1,2}(Q)$ such that the triplet $(\hat u, \hat p, \hat g)$ satisfies (3) with g replaced by $\hat g$, the linear backwards system*

$$\begin{cases} -\hat p_t - \Delta \hat p + \sigma \displaystyle\int_t^T e^{-\gamma(s-t)}\hat p(s)\,ds + D_u F_0(x,t;\hat u)\,\hat p = 0, \\ \hat p(x,t)|_\Sigma = 0, \\ \hat p(x,T) = \hat u(x,T) - u_1(x) \end{cases} \tag{60}$$

and the additional inequalities

$$\iint_Q (\hat p + a\hat g)(g - \hat g)\,dx\,dt \geq 0 \quad \forall g \in \mathcal{G}_{ad}, \quad \hat g \in \mathcal{G}_{ad}. \tag{61}$$

Unfortunately, it is not easy to obtain a result similar to Theorem 3 in this context. By analogy with that theorem, it is maybe reasonable to expect that, for some large r, the sequence (u^n, p^n, g^n) given by

$$\begin{cases} u_t^n - \Delta u^n + \sigma \displaystyle\int_t^T e^{-\gamma(s-t)} u^n(s)\,ds + F_0(x,t;u^n)\,p^n = g^n, \\ -p_t^n - \Delta p^n + \sigma \displaystyle\int_t^T e^{-\gamma(s-t)} p^n(s)\,ds + H_0(x,t;u^n)\,p^n = 0, \\ u^n(x,t)|_\Sigma = p^n(x,t)|_\Sigma = 0, \\ u^n(x,0) = 0, \quad p^n(x,T) = |u^n(x,T) - u_1(x)|^{r-2}(u^n(x,T) - u_1(x)), \\ p^n + \dfrac{1}{n} g^n = 0. \end{cases} \tag{62}$$

satisfies $u^n(\cdot, T) \to u_1$ in some sense as $n \to \infty$. But this is unknown.

5.2 Further comments on controllability

In general terms, the controllability approach for an evolution partial differential equation or system consists in trying to drive the system from a prescribed initial state at time $t = 0$ (u_0 in our case) to a *desired* final state (or, at least "near" a desired final state) at time $t = T$. In the interesting case, the control is supported by a set of the form $\omega \times (0, T)$, where $\omega \subset \Omega$ is a nonempty (small) open set.

At present, controllability problems are relatively well understood for linear and semilinear parabolic equations; see for instance [4, 6, 3]. Unfortunately, this is not the case for the integro-differential system (3), not even for simplified (linearized) similar problems.

For instance, consider the linear system

$$\begin{cases} u_t - \Delta u + \sigma \displaystyle\int_0^t e^{-\gamma(t-s)} u(s)\, ds + \alpha(x,t) u = g 1_\omega, \\ u(x,t)|_\Sigma = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{63}$$

where $\alpha \in L^\infty(Q)$ and 1_ω is the characteristic function of ω.

It is said that this system is *approximately controllable* in $L^2(\Omega)$ at time T if, for any $u^1 \in L^2(\Omega)$ and any $\varepsilon > 0$, there exists $g \in L^2(\omega \times (0,T))$ such that the corresponding solution satisfies

$$|u(\cdot, T) - u^1| \leq \varepsilon.$$

At present, it is unknown whether (63) is approximately controllable. This is the case if α is independent of t; see [2]. Of course, the question is completely open in the case of the nonlinear system (3).

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