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The computation of abelian subalgebras in the Lie algebra of upper-triangular matrices

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Abstract

This paper deals with the computation of abelian subalgebras of the solvable Lie algebra $\mathfrak{h}_n$, of $n \times n$ upper-triangular matrices. Firstly, we construct an algorithm to find abelian Lie subalgebras in a given Lie algebra $\mathfrak{h}_n$. This algorithm allows us to compute an abelian subalgebra up to a certain dimension. Such a dimension is proved to be equal to the maximum for abelian subalgebras of $\mathfrak{h}_n$.

1 Introduction

The topic dealt in this paper is the maximal abelian dimension of a given finite-dimensional Lie algebra $\mathfrak{g}$, that is, the maximum among the dimensions of the abelian subalgebras of $\mathfrak{g}$. Although this has been studied in previous papers, most of them (for example [4, 8]) consider abelian ideals instead of abelian subalgebras, which implies that more restrictive hypothesis are needed. However, we do not assume such restrictions but our work considers all the subalgebras contained in the given Lie algebra $\mathfrak{g}$. In this way, we are using a concept which is equivalent to the one dealt in other papers like [3, 5].

Let us recall that a classical bound for the dimension of an abelian subalgebra in the matrix Lie algebra $M_n(\mathbb{K})$ of $n \times n$ square matrices over a field $\mathbb{K}$ was given by Jacobson [6]. Previously, Schur [7] obtained the same bound for $\mathbb{K} = \mathbb{C}$. Jacobson's result can be restated as follows: Let $\mathfrak{a}$ be an abelian subalgebra of the matrix algebra $M_n(\mathbb{K})$ over an arbitrary field $\mathbb{K}$.

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Then $\dim(\mathfrak{a}) \leq \left[\frac{n^2}{4}\right] + 1$, where $[x]$ denotes the integer part of x. Moreover, there exists an abelian subalgebra whose dimension is exactly this bound.

In this way, the maximal abelian dimension $\mathcal{M}(\mathfrak{g})$ of an arbitrary given subalgebra $\mathfrak{g}$ of $M_n(\mathbb{C})$ can be upper bounded by:

$$\mathcal{M}(\mathfrak{g}) \leq \left[\frac{n^2}{4}\right] + 1 = \begin{cases} k^2+1, & \text{if } n = 2k; \\ k^2+k+1, & \text{if } n = 2k+1. \end{cases}$$

At this respect, some of us have already studied in [1, 2] the maximal abelian dimension of the Lie algebra $\mathfrak{g}_n$ of $n \times n$ strictly upper-triangular matrices. More concretely, in [1] we start this study by proving some properties on these algebras and conjecturing a value for its maximal abelian dimension depending on the order n. Such a conjecture was obtained starting from an algorithmic method to compute abelian subalgebras in a given Lie algebra up to a certain dimension, which could not be increased with the method. Finally, in [2], the conjecture was proved to be true and the maximal abelian dimension was computed for the algebras $\mathfrak{g}_n$, showing that Jacobson's bound was not achieved for these algebras.

To get the proof, the vectors in a given basis of $\mathfrak{g}_n$ were distinguished between *main vectors* and *non-main* ones for a given basis of the subalgebra. Such a distinction was based on writing each vector in the basis of the subalgebra as a linear combination of the elements in the basis of $\mathfrak{g}_n$; then these coefficients were written as the rows in a matrix and the vectors corresponding to the pivot positions of its echelon form were the main vectors.

To continue this research, we are interested in computing the maximal abelian dimension for the Lie algebra $\mathfrak{h}_n$ of $n \times n$ upper-triangular matrices. Besides, we want to apply and adjust the technics and methods used in [1, 2] to this algebra. In this way, we are going to give an algorithmic procedure which computes abelian subalgebras of $\mathfrak{h}_n$ up to a certain dimension.

2 Preliminaries

Some preliminary concepts on Lie algebras are recalled in this section, bearing in mind that the reader can consult [9] for a general overview on solvable Lie algebras. In this paper, only finite-dimensional Lie algebras over the complex number field are considered.

Given a finite-dimensional Lie algebra $\mathfrak{g}$, its *commutator central series* is:

$$\mathcal{C}_1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}_2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}],\ \ldots,\ \mathcal{C}_k(\mathfrak{g}) = [\mathcal{C}_{k-1}(\mathfrak{g}), \mathcal{C}_{k-1}(\mathfrak{g})],\ \ldots$$

So, $\mathfrak{g}$ is called *solvable* if there exists $m \in \mathbb{N}$ such that $\mathcal{C}_m(\mathfrak{g}) \equiv \{0\}$.

The maximal abelian dimension of $\mathfrak{g}$ is the maximum among the dimensions of its abelian Lie subalgebras. This value will be denoted by $\mathcal{M}(\mathfrak{g})$.

From now on, we will deal with the complex solvable Lie algebra $\mathfrak{h}_n$, whose vectors are all the $n \times n$ upper-triangular matrices with the following form:

$$h_n(x_{r,s}) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{nn} \end{pmatrix}$$

where $n \in \mathbb{N}$ and $x_{ij} \in \mathbb{C}$, for all $i, j \in \mathbb{N}$, with $1 \leq i \leq j \leq n$.

It is easy to prove that a basis of the algebra $\mathfrak{h}_n$ is formed by the vectors $X_{ij} = h_n(x_{r,s})$, which verify:

$$x_{r,s} = \begin{cases} 1, & \text{if } (r,s) = (i,j), \\ 0, & \text{if } (r,s) \neq (i,j), \end{cases}$$

where $1 \leq i \leq j \leq n$. Hence, the dimension of $\mathfrak{h}_n$ is $d_{\mathfrak{h}_n} = \frac{n(n+1)}{2}$ and the nonzero brackets with respect to this basis are the following:

$$\begin{array}{ll} [X_{i,j}, X_{j,k}] = X_{i,k}, & \forall\, i = 1, \ldots, n-2,\ \forall\, j = i+1, \ldots, n-1,\ \forall\, k = j+1, \ldots, n; \\ [X_{i,i}, X_{i,j}] = X_{i,j}, & \forall\, i = 1, \ldots, n-1,\ \forall\, j = i+1, \ldots, n; \\ [X_{k,i}, X_{i,i}] = X_{k,i}, & \forall\, k = 1, \ldots, n-1,\ \forall\, i = k+1, \ldots, n. \end{array}$$

Let us note that the center of $\mathfrak{h}_n$ is generated by the vector $\sum_{i=1}^{n} X_{i,i}$, coming from the main diagonal. This vector is the only one which commutes with every vector in $\mathfrak{h}_n$. Therefore this vector has to belong to any abelian subalgebra which is not contained in another.

3 Algorithm to obtain abelian subalgebras

Next, we show an algorithmic procedure which allows us to obtain abelian subalgebras of the Lie algebra $\mathfrak{h}_n$. Before giving the general structure of this algorithm, we study the obtainment of abelian subalgebras for low-dimensional Lie algebras $\mathfrak{h}_n$; that is, for $n \leq 5$. Starting from the results obtained for these algebras, we can give a general algorithm for an arbitrary $\mathfrak{h}_n$.

3.1 Lie algebras $\mathfrak{h}_n$ with $n \leq 5$

Case $n = 2$: $\mathfrak{h}_2$ is generated by the basis $\{X_{1,1}, X_{1,2}, X_{2,2}\}$, whose nonzero brackets are the following:

$$[X_{1,1}, X_{1,2}] = X_{1,2}; \qquad [X_{1,2}, X_{2,2}] = X_{1,2}.$$

A 1-dimensional abelian subalgebra is obtained by taking any of the three vectors in the basis of $\mathfrak{h}_n$. To obtain a 2-dimensional abelian subalgebra, it is sufficient to consider the subalgebra $\langle X_{1,1}, X_{2,2}\rangle$, corresponding to the elements in the main diagonal. In this way, the maximal abelian dimension of $\mathfrak{h}_2$ is 2.

Case $n = 3$**:** $\mathfrak{h}_3$ is generated by the basis $\{X_{1,1}, X_{1,2}, X_{1,3}, X_{2,2}, X_{2,3}, X_{3,3}\}$, whose nonzero brackets are the following:

$$\begin{array}{lll} [X_{1,2}, X_{2,3}] = X_{1,3}; & [X_{1,1}, X_{1,2}] = X_{1,2}; & [X_{1,1}, X_{1,3}] = X_{1,3}; \\ [X_{2,2}, X_{2,3}] = X_{2,3}; & [X_{1,2}, X_{2,2}] = X_{1,2}; & [X_{1,3}, X_{3,3}] = X_{1,3}; \\ [X_{2,3}, X_{3,3}] = X_{2,3}. & & \end{array}$$

Now, the following two steps allow us to obtain abelian subalgebras of $\mathfrak{h}_3$, giving a first explanation for our general algorithmic method, which will be generalized later for an arbitrary $\mathfrak{h}_n$:

Step 1: Take the three vectors coming from the 3$^{\text{rd}}$ column and remove the one coming from the 3$^{\text{rd}}$ row. So, we obtain the abelian subalgebra $\langle X_{1,3}, X_{2,3}\rangle$.

Step 2: Add the vectors coming from the 2$^{\text{nd}}$ column and remove the ones coming from the 2$^{\text{nd}}$ row (to avoid nonzero brackets). Consequently, the 2-dimensional abelian subalgebra $\langle X_{1,2}, X_{1,3}\rangle$ is obtained.

Let us note that Step 2 does not increase the dimension of the abelian subalgebra obtained in Step 1. This will be the largest dimension which can be obtained with this procedure.

Step 3: Add the vector $X_{1,1}+X_{2,2}+X_{3,3}$ belonging to the center of $\mathfrak{h}_3$. Hence, the 3-dimensional abelian subalgebra $\langle X_{1,2}, X_{1,3}, X_{1,1}+X_{2,2}+X_{3,3}\rangle$ is obtained.

Case $n = 4$**:** $\mathfrak{h}_4$ is generated by the basis $\{X_{1,1}, X_{1,2}, X_{1,3}, X_{1,4}, X_{2,2}, X_{2,3}, X_{2,4}, X_{3,3}, X_{3,4}, X_{4,4}\}$, having the following nonzero brackets:

$$\begin{array}{lll} [X_{1,2}, X_{2,3}] = X_{1,3}; & [X_{1,2}, X_{2,4}] = X_{1,4}; & [X_{1,3}, X_{3,4}] = X_{1,4}; \\ [X_{2,3}, X_{3,4}] = X_{2,4}; & [X_{1,1}, X_{1,2}] = X_{1,2}; & [X_{1,1}, X_{1,3}] = X_{1,3}; \\ [X_{1,1}, X_{1,4}] = X_{1,4}; & [X_{2,2}, X_{2,3}] = X_{2,3}; & [X_{2,2}, X_{2,4}] = X_{2,4}; \\ [X_{3,3}, X_{3,4}] = X_{3,4}; & [X_{1,2}, X_{2,2}] = X_{1,2}; & [X_{2,3}, X_{3,3}] = X_{2,3}; \\ [X_{1,3}, X_{3,3}] = X_{1,3}; & [X_{3,4}, X_{4,4}] = X_{3,4}; & [X_{2,4}, X_{4,4}] = X_{2,4}; \\ [X_{1,4}, X_{4,4}] = X_{1,4}. & & \end{array}$$

By considering the adding-removing procedure which was commented for the previous case, we are going to obtain an abelian subalgebra of dimension 4:

Step 1: Take the four vectors coming from the 4^{th} column and remove the one coming from the 4^{th} row, obtaining the 3-abelian subalgebra $\langle X_{1,4}, X_{2,4}, X_{3,4}\rangle$.

Step 2: Add the vectors coming for the 3^{rd} column and remove the ones coming from the 3^{rd} row, obtaining the 4-dimensional abelian subalgebra $\langle X_{1,3}, X_{1,4}, X_{2,3}, X_{2,4}\rangle$.

Step 3: Add the vectors coming from the 2^{nd} column and remove the ones coming from the 2^{nd} row. A 4-dimensional abelian Lie subalgebra is computed again.

Step 4: Add the vector $X_{1,1}+X_{2,2}+X_{3,3}+X_{4,4}$ belonging to the center of $\mathfrak{h}_4$. So, the 5-dimensional abelian subalgebra $\langle X_{1,3}, X_{1,4}, X_{2,3}, X_{2,4}, X_{1,1}+X_{2,2}+X_{3,3}+X_{4,4}\rangle$ is obtained.

3.2 The general case

This subsection is devoted to explain an algorithmic method to obtain abelian subalgebras in an arbitrary Lie algebra $\mathfrak{h}_n$, with $n \geq 4$. This method is achieved by generalizing the one shown before for $n = 3$ and $n = 4$. Depending on the parity of n, two possible cases have to be considered:

Case 1: n is even and $n \geq 4$ (i.e., $n = 2\,k$, with $k \in \mathbb{N} \setminus \{1\}$).

The general reasoning consists on considering the vectors in the basis of $\mathfrak{h}_n$, corresponding to the columns in the matrix expression of $\mathfrak{h}_n$. Let us remember that, when the vectors corresponding to the i^{th} column are chosen, all the vectors corresponding to the i^{th} row have to be removed. In this way, all the nonzero brackets are avoided.

Step 1: $(2k)^{\text{th}}$ column.

Let us consider the $2k$ vectors corresponding to the $(2k)^{\text{th}}$ column. Now, the unique vector coming from the $(2k)^{\text{th}}$ row has to be removed because it does not commute with the rest of the vectors coming from the $(2k)^{\text{th}}$ column. In this way, the abelian subalgebra $\langle X_{1,2k}, \ldots, X_{2k-1,2k}\rangle$ is obtained.

Step $2k-i+1$**:** i^{th} column, with $2k > i > k+1$.

There are i vectors corresponding to the i^{th} column. These are added to the generators of the subalgebra obtained in the previous step. Now, we remove the $2k-(i-1)$ vectors corresponding to i^{th} row. In this way, we obtain an abelian Lie subalgebra whose dimension increases $i-(2k-(i-1)) = 2i-2k-1$ with respect to the one already obtained in the previous step.

Let us note that the dimension of the subalgebra really increases because the following condition is verified: $2i-2k-1>0$. Since the previous inequality is equivalent to $i > k+1/2$, k will be the last step in which the number of vectors generating the obtained abelian subalgebra is greater than the one obtained in the previous step.

Step k: $(k+1)^{\text{th}}$ column.

This time, the $k+1$ vectors corresponding to the $(k+1)^{\text{th}}$ column are added, whereas the $2k-(k+1-1)=k$ ones corresponding to the $(k+1)^{\text{th}}$ row are removed.

Step $k+1$: Adding the vector $\sum_{i=1}^{n} X_{i,i}$ to the basis computed in Step k. So, we obtain the (k^2+1)-dimensional abelian subalgebra generated by this vector and the ones shown next:

$$\begin{array}{ccc} X_{1,k+1} & \dots & X_{1,2k} \\ X_{2,k+1} & \dots & X_{2,2k} \\ \vdots & \ddots & \vdots \\ X_{k,k+1} & \dots & X_{k,2k} \end{array}$$

Case 2: n is odd and $n \geq 4$ (i.e., $n = 2k+1$, with $k \in \mathbb{N} \setminus \{1\}$).

By arguing analogously to the Case 1, we can settle the following procedure to obtain an abelian Lie subalgebra with dimension as large as possible. Let us note that the number of steps which are necessary is different in this case, just like happening with the dimension of the computed abelian subalgebra.

Step 1: $(2k+1)^{\text{th}}$ column.

The $2k+1$ vectors corresponding to the $(2k+1)^{\text{th}}$ column are considered to generate a Lie subalgebra. Besides, the unique vector in the $(2k+1)^{\text{th}}$ row is removed in order to obtain the abelian Lie subalgebra $\langle X_{1,2k+1}, \dots, X_{2k,2k+1} \rangle$.

Step $2k-i+2$: i^{th} column, with $2k+1 > i > k+2$.

There exist i vectors corresponding to the i^{th} column, which are added to the generators of the abelian subalgebra obtained in the previous step. To obtain an abelian subalgebra in this step, the $2k-(i-1)$ vectors corresponding to the i^{th} row are removed from the generators. In this way, we obtain an abelian subalgebra whose number of generators increase $i-(2k+1-(i-1)) = 2i-2k-2$ with respect to the abelian subalgebra obtained in the previous step.

Let us note that the dimension of the abelian subalgebra which is obtained in each step increases if the inequality $2i - 2k - 2 > 0$ is verified. This is equivalent to ask wether the inequality $i > k + 1$ is satisfied. Hence, Step k will be the last step and the procedure will be stopped after it.

Step k: $(k + 2)^{\text{th}}$ column.

Now the $k + 2$ vectors corresponding to the $(k + 2)^{\text{th}}$ column are added. Once this is done, the $2k + 1 - (k + 2 - 1) = k$ vectors corresponding to the $(k + 2)^{\text{th}}$ row are removed, which allows us to obtain a $(k^2 + k)$-dimensional abelian subalgebra.

Step $k + 1$: Adding the vector $\sum_{i=1}^{n} X_{i,i}$ to the basis computed in Step $k - 1$. So, we obtain the $(k^2 + k + 1)$-dimensional abelian subalgebra generated by the previous vector and the ones shown next:

$$\begin{matrix} X_{1,k+2} & \dots & X_{1,2k+1} \\ X_{2,k+2} & \dots & X_{2,2k+1} \\ \vdots & \ddots & \vdots \\ X_{k+1,k+2} & \dots & X_{k+1,2k+1} \end{matrix}$$

According to the results obtained in this section, an abelian subalgebra of $\mathfrak{h}_n$ has been computed for all $n \in \mathbb{N}$. Indeed, the dimension of such a subalgebra is:

$$B_n = \begin{cases} n, & \text{if } n < 4; \\ k^2 + 1, & \text{if } n = 2k,\ n \geq 4; \\ k^2 + k + 1, & \text{if } n = 2\,k + 1,\ n \geq 4. \end{cases}$$

Therefore, the maximal abelian dimension $\mathcal{M}(\mathfrak{h}_n)$ is lower bounded by the value B_n, which is equal to the upper bound of $\mathcal{M}(\mathfrak{h}_n)$ given by Jacobson [6] and Schur [7]. Hence, it can be asserted that the maximal abelian dimension $\mathcal{M}(\mathfrak{h}_n)$ is equal to B_n, for all $n \in \mathbb{N}$.

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