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GENERALIZED S-SPACE-FORMS

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Abstract. We introduce and study generalized S-space-forms. Moreover, we investigate generalized S-space-forms endowed with an additional structure and we obtain some obstructions for them to be S-manifolds.

1. Introduction

It is an interesting problem to analyze what kind of Riemannian manifolds may be determined by special pointwise expressions for their curvatures. For instance, it is well known that the sectional curvatures of a Riemannian manifold determine the curvature tensor field completely. So, if (M, g) is a connected Riemannian manifold with dimension greater than 2 and its curvature tensor field R has the pointwise expression

$$R(X,Y)Z = \lambda \{g(X,Z)Y - g(Y,Z)X\},$$

where λ is a differentiable function on M, then M is a space of constant sectional curvature, that is, a real-space-form and λ is a constant function.

Further, when the manifold is equipped with some additional structure, it is sometimes possible to obtain conclusions from the special form of the curvature tensor field for this structure too. Thus, an almost-Hermitian manifold (M, J, g) is said to be a *generalized complex-space-form* [**9**] if its curvature tensor satisfies

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}, \tag{1.1}$$

where f_1 and f_2 are differentiable functions on M. This name derives from the fact that, when M is a complex-space-form, that is, a Kaehlerian manifold of constant holomorphic curvature equal to c, the curvature tensor field of M satisfies (1.1) with $f_1 = f_2 = c/4$.

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Since Sasakian-spaces-forms play a similar role in contact metric geometry to that of complex-space-forms in complex geometry, Alegre, Blair and Carriazo have defined and studied *generalized Sasakian-space forms* [**1**] as those almost-contact metric manifolds (M, ϕ, ξ, η, g) whose curvature tensor field satisfies

$$\begin{aligned} R(X,Y)Z = f_1 \{g(Y,Z)X - g(X,Z)Y\} + \\ + f_2 \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ + f_3 \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

f_1, f_2, f_3 being differentiable functions on M. If M is actually a Sasakian-space-form, that is a Sasakian manifold with constant ϕ-sectional curvature equal to c, then $f_1 = \frac{1}{4}(c+3)$, $f_2 = f_3 = \frac{1}{4}(c-1)$.

More in general, Yano [**10**] introduced the notion of f-structure on a $(2n+s)$ -dimensional manifold as a tensor field f of type (1,1) and rank $2n$ satisfying $f^3 + f = 0$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of f-structures. In this context, Blair [**2**] defined K-manifolds (and particular cases of S-manifolds and C-manifolds) as the analogue of Kaehlerian manifolds in the almost complex geometry and of quasi-Sasakian manifolds in the almost contact geometry and he showed that the curvature of either S-manifolds or C-manifolds is completely determined by their f-sectional curvatures. Later, Kobayashi and Tsuchiya [**8**] got expressions of the curvature tensor field of S-manifolds and C-manifolds when their f-sectional curvature is constant depending on such a constant.

For these reasons, we consider that it is interesting to introduce a notion of generalized S-space-form on metric f-manifolds (see Section 2 for a precise definition of these manifolds). We observe that this work was made in [**5**] for metric f-manifolds with two structure vector fields, giving some interesting examples. Now, we present the definition for any number of structure vector fields. To this end, we have followed the same procedure as in almost complex and almost contact cases, that is, we have substituted the constants in the expression of the curvature tensor field of an S-space-form (an S-manifold of constant f-sectional curvature) obtained in [**8**] by certain differentiable functions on the manifold. So, S-space-forms are natural examples of generalized S-space-forms. Furthermore, we check that C-space-forms are also generalized S-space-forms.

We have organized the communication in the following way. In Section 2 we review definitions and formulas concerning metric f-manifolds which we shall use later. In Section 3 we define generalized S-space-forms and study the sectional curvatures of such manifolds. Moreover, we establish that the writing of the curvature tensor field is unique in terms of a family of differentiable functions on the manifold if and only if the dimension of the manifold is greater than $2+s$, s being the number of structure vector fields. In Section 4, we consider a different definition given by Falcitelli and Pastore in [**6**], comparing both definitions. Finally, in Section 5, we study generalized S-space-forms endowed with an additional structure and the relationships between the functions in such a case. Thus, we prove that any generalized S-space-form with a metric f-K-contact structure is actually an S-manifold and we deduce an obstruction for a generalized S-space-form to be an S-manifold, depending on the functions. The same result holds for a metric f-contact structure with some additional conditions on the functions. We also study generalized S-space-forms with an underlying C-structure and, more in general, with a K-structure.

2. Metric f-manifolds

A Riemannian manifold (M, g) of dimension $2n + s$ and endowed with an f-structure f (that is, a tensor field of type (1,1) and rank $2n$ satisfying $f^3 + f = 0$ [**10**]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1, \ldots, \xi_s$ on M (called *structure vector fields*) such that, if $\eta_1, \ldots, \eta_s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$, then

$$f\xi_\alpha = 0; \;\; \eta_\alpha \circ f = 0; \;\; f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha;$$

$$g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y),$$

for any $X, Y \in \mathcal{X}(M)$ and $\alpha = 1, \ldots, s$. The distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$ and its complementary orthogonal distribution is denoted by $\mathcal{L}$. Consequently, $TM = \mathcal{L} \oplus \mathcal{M}$. Moreover, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha = 1, \ldots, s$ and if $X \in \mathcal{M}$, then $fX = 0$.

Let F be the 2-form on M defined by $F(X, Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(M)$. Since f is of rank $2n$, then $\eta_1 \wedge \cdots \wedge \eta_s \wedge F^n \neq 0$ and, particularly, M is orientable. A metric f-manifold is said to be a *metric f-contact manifold* if $F = \mathrm{d}\eta_\alpha$, for any $\alpha = 1, \ldots, s$. On the other hand, a metric f-contact manifold is said to be a *metric f-K-contact manifold* if the structure vector fields are Killing vector fields. When $s = 1$, metric f-contact manifolds correspond to contact manifolds and metric f-K-contact manifolds to K-contact manifolds. Furthermore, in a metric f-K-contact manifold it easy to show that:

$$\nabla_X \xi_\alpha = -fX, \;\; X \in \mathcal{X}(M), \;\; \alpha = 1, \ldots, s. \tag{2.1}$$

The f-structure f is said to be *normal* if $[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0$, where $[f, f]$ denotes the Nijenhuis tensor of f. Then, a metric f-manifold is said to be a K-manifold [**2**] if it is normal and $\mathrm{d}F = 0$. In a K-manifold M, the structure vector fields are Killing vector fields [**2**] and:

$$\nabla_{\xi_\alpha} \xi_\beta = 0, \;\; \alpha, \beta = 1, \ldots, s. \tag{2.2}$$

A K-manifold is called an S-manifold if $F = \mathrm{d}\eta_\alpha$, for any α (that is, if it is also a metric f-K-contact manifold) and a C-manifold if $\mathrm{d}\eta_\alpha = 0$, for any α. Note that, for $s = 0$, a K-manifold is a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold, an S-manifold is a Sasakian manifold and a C-manifold is a cosymplectic manifold. When $s \geqslant 2$, non-trivial examples can be found in [**2, 3, 7**]. Moreover, a K-manifold M is an S-manifold if and only if

$$\nabla_X \xi_\alpha = -fX, \;\; X \in \mathcal{X}(M), \;\; \alpha = 1, \ldots, s,$$

and it is a C-manifold if and only if $\nabla f = 0$ [**2**].

On the other hand, the curvature tensor field R of a K-manifold M satisfies

$$R(\xi_\alpha, X, \xi_\beta, Y) = -g(\nabla_X \xi_\beta, \nabla_Y \xi_\alpha), \tag{2.3}$$

for any $X, Y \in \mathcal{X}(M)$ and $\alpha, \beta = 1, \ldots, s$ [**4**].

A plane section π on a metric f-manifold M is said to be an *f-section* if it is determined by a unit vector $X \in \mathcal{L}$ and fX. The sectional curvature $K(\pi)$ of π is called an *f-sectional curvature*. An S-manifold (resp., a C-manifold) is said to be an *S-space-form* (resp., a *C-space-form*) if it has a constant f-sectional curvature c and then, it is denoted by $M(c)$. In such cases, the curvature tensor field R of $M(c)$ satisfies

$$\begin{aligned} R(X,Y,Z,W) = & \sum_{\alpha,\beta} \big(g(fX,fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX,fZ)\eta_\alpha(Y)\eta_\beta(W) \\ & \quad + g(fY,fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY,fW)\eta_\alpha(X)\eta_\beta(Z) \big) \\ & + \frac{c+3s}{4}\big(g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW) \big) \\ & + \frac{c-s}{4}\big(F(X,W)F(Y,Z) - F(X,Z)F(Y,W) \\ & \qquad - 2F(X,Y)F(Z,W) \big), \end{aligned} \tag{2.4}$$

(resp.,

$$\begin{aligned} R(X,Y,Z,W) = & \frac{c}{4}(g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW)) \\ & + F(X,W)F(Y,Z) - F(X,Z)F(Y,W) \\ & - 2F(X,Y)F(Z,W))), \end{aligned} \tag{2.5}$$

for any $X, Y, Z, W \in \mathcal{X}(M)$ [**8**].

3. Generalized S-space-forms

A metric f-manifold $(M, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ is said to be a *generalized S-space-form* if there exists a family of differentiable functions on M,

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\},$$

such that the curvature tensor field R of M satisfies

$$R = F_1 R_1 + F_2 R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta} R_{\alpha\beta} + \sum_{1 \leqslant \alpha < \beta \leqslant s} G_{\alpha\beta} \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1, \\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma} R_{\alpha\beta\gamma}, \tag{3.1}$$

where

(3.2)

$$\begin{aligned}
R_1(X,Y,Z,W) &= g(X,W)g(Y,Z) - g(X,Z)g(Y,W);\\
R_2(X,Y,Z,W) &= F(X,W)F(Y,Z) - F(X,Z)F(Y,W)\\
&\quad - 2F(X,Y)F(Z,W);\\
R_{\alpha\beta}(X,Y,Z,W) &= g(Y,W)\eta_\alpha(X)\eta_\beta(Z) - g(X,W)\eta_\alpha(Y)\eta_\beta(Z)\\
&\quad + g(X,Z)\eta_\alpha(Y)\eta_\beta(W) - g(Y,Z)\eta_\alpha(X)\eta_\beta(W);\\
\widetilde{R}_{\alpha\beta}(X,Y,Z,W) &= \eta_\alpha(X)\eta_\beta(Y)\eta_\beta(Z)\eta_\alpha(W) - \eta_\beta(X)\eta_\alpha(Y)\eta_\beta(Z)\eta_\alpha(W)\\
&\quad + \eta_\beta(X)\eta_\alpha(Y)\eta_\alpha(Z)\eta_\beta(W) - \eta_\alpha(X)\eta_\beta(Y)\eta_\alpha(Z)\eta_\beta(W);\\
R_{\alpha\beta\gamma}(X,Y,Z,W) &= \eta_\alpha(X)\eta_\beta(Y)\eta_\gamma(Z)\eta_\alpha(W) - \eta_\beta(X)\eta_\alpha(Y)\eta_\gamma(Z)\eta_\alpha(W)\\
&\quad + \eta_\beta(X)\eta_\alpha(Y)\eta_\alpha(Z)\eta_\gamma(W) - \eta_\alpha(X)\eta_\beta(Y)\eta_\alpha(Z)\eta_\gamma(W),
\end{aligned}$$

for any $X,Y,Z,W \in \mathcal{X}(M)$.

This kind of manifold appears as a natural generalization of S-space-forms because a straightforward computation from (2.4) gives that any S-space-form $M(c)$ is a generalized S-space-form with functions

$$\begin{gathered}
F_1 = \tfrac{1}{4}(c+3s);\quad F_2 = \tfrac{1}{4}(c-s);\quad F_{\alpha\alpha} = \tfrac{1}{4}(c+3s) - 1;\\
F_{\alpha\beta} = -1 \ (\alpha \neq \beta);\quad G_{\alpha\beta} = \tfrac{1}{4}(c+3s) - 2 \ (\alpha < \beta);\\
H_{\alpha\beta\gamma} = -1 \ (\alpha \neq \beta \neq \gamma \neq \alpha),
\end{gathered}$$

where $\alpha,\beta,\gamma \in \{1,\dots,s\}$. Moreover, any C-space-form $M(c)$ is also a generalized S-space-form. In fact, from (2.5), we only have to take

$$\begin{gathered}
F_1 = F_2 = F_{\alpha\alpha} = G_{\alpha\beta} = \tfrac{c}{4} \ (\alpha < \beta);\\
F_{\alpha\beta} = 0 \ (\alpha \neq \beta);\\
H_{\alpha\beta\gamma} = 0 \ (\alpha \neq \beta \neq \gamma \neq \alpha),
\end{gathered}$$

where $\alpha,\beta,\gamma \in \{1,\dots,s\}$.

From (3.2) we easily deduce that $\widetilde{R}_{\alpha\alpha} = 0$; $\widetilde{R}_{\alpha\beta} = \widetilde{R}_{\beta\alpha}$; $R_{\alpha\beta\beta} = \widetilde{R}_{\alpha\beta}$; $R_{\alpha\alpha\alpha} = R_{\alpha\alpha\beta} = 0$, for any $\alpha,\beta = 1,\dots,s$. Furthermore, from (3.1) we get that

$$R(X,\xi_\alpha,X,\xi_\beta) = F_{\alpha\beta}, \tag{3.3}$$

$$R(\xi_\alpha,\xi_\beta,\xi_\gamma,\xi_\alpha) = H_{\alpha\beta\gamma} - F_{\beta\gamma}, \tag{3.4}$$

for any unit vector field $X \in \mathcal{L}$ and any $\alpha,\beta,\gamma = 1,\dots,s$, $\alpha \neq \beta \neq \gamma \neq \alpha$. Then, by using the symmetries of the curvature tensor field R, from (3.3) and (3.4) together, we obtain $F_{\alpha\beta} = F_{\beta\alpha}$ and $H_{\alpha\beta\gamma} = H_{\alpha\gamma\beta}$,$\alpha,\beta,\gamma = 1,\dots,s$, $\alpha \neq \beta \neq \gamma \neq \alpha$.

Now, we observe that, if $s = 2$, (3.1) agrees with (3.1) of [**5**]. In that paper, more examples of generalized S-space-forms with two structure vector fields were given and they can be generalized to any s. Thus, pseudo-umbilical, totally contact-umbilical, totally contact-geodesic, totally umbilical and totally geodesic hypersurfaces of a generalized S-space-form are also generalized S-space-forms and, moreover, the bundle space of a principal toroidal bundle over a Kaehlerian manifold and the warped product of $\mathbb{R}$ times a generalized S-space-form are generalized S-space-forms too.

Next, for the sectional curvatures of a generalized S-space form and by using (3.1) and (3.2), we can prove the following proposition.

PROPOSITION 3.1. *Let M be a generalized S-space-form with functions:*

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}.$$

Then, for any orthonormal vector fields $X, Y \in \mathcal{L}$ and $\alpha, \beta \in \{1, \ldots, s\}$, we have

(i) $K(X, Y) = R(X, Y, Y, X) = F_1 + 3F_2 g(X, fY)^2$.
(ii) $H(X) = K(X, fX) = F_1 + 3F_2$.
(iii) $K(X, \xi_\alpha) = F_1 - F_{\alpha\alpha}$.
(iv) $K(\xi_\alpha, \xi_\beta) = F_1 - F_{\alpha\alpha} - F_{\beta\beta} + G_{\alpha\beta}$, $(\alpha < \beta)$.

We are going now to study if the writing of the curvature tensor field of a generalized S-space-form is unique. First, we can prove:

PROPOSITION 3.2. *Let M be a $(2n+s)$-dimensional generalized S-space-form. If $n \geqslant 2$, the writing of the curvature tensor field R of M in terms of a family of functions is unique.*

PROOF. Let us suppose that there exist two families of differentiable functions, $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$ and $\{F_1^*, F_2^*, F_{\alpha\beta}^*, G_{\alpha\beta}^*, H_{\alpha\beta\gamma}^*\}$, such that

$$\begin{aligned} R &= F_1 R_1 + F_2 R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta} R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta}\widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma} R_{\alpha\beta\gamma} \\ &= F_1^* R_1 + F_2^* R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta}^* R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta}^*\widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma}^* R_{\alpha\beta\gamma}. \end{aligned} \tag{3.5}$$

Since $n \geqslant 2$, we can consider a pair of orthonormal vector fields $X, Y \in \mathcal{L}$ such that $g(X, fY) = 0$. From (3.5) we get that $R(X, Y, fX, fY) = F_2 = F_2^*$ and so, $R(X, Y, Y, X) = F_1 = F_1^*$. From (iii) and (iv) of Proposition 3.1 we deduce that $F_{\alpha\alpha} = F_{\alpha\alpha}^*$, for any $\alpha = 1, \ldots, s$ and $G_{\alpha\beta} = G_{\alpha\beta}^*$, for any $\alpha, \beta = 1, \ldots, s$, $\alpha < \beta$.

Finally, if $X \in \mathcal{L}$ is a unit vector field and $\alpha, \beta = 1, \ldots, s$, $\alpha \neq \beta$, from (3.5) again, we get that $R(X, \xi_\alpha, X, \xi_\beta) = F_{\alpha\beta} = F_{\alpha\beta}^*$ and, by using (3.4), $H_{\alpha\beta\gamma} = H_{\alpha\beta\gamma}^*$, for any $\alpha, \beta, \gamma \in \{1, \ldots, s\}$, $\alpha \neq \beta \neq \gamma \neq \alpha$. □

Next, what about $(2+s)$-dimensional generalized S-space-forms? In this case, the writing of the curvature tensor field is not unique. Actually, if M is a generalized S-space-form of dimension $2+s$ such that its curvature tensor field R can be simultaneously written as

$$R = F_1 R_1 + F_2 R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta} R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta}\widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma} R_{\alpha\beta\gamma}$$

and

$$R = F_1^* R_1 + F_2^* R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta}^* R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta}^* \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma}^* R_{\alpha\beta\gamma},$$

then, given a unit vector field $X \in \mathcal{L}$ and $\alpha, \beta, \gamma \in \{1, \ldots, s\}$, from (3.3), (3.4) and Proposition 3.1, we obtain the system

$$\begin{aligned}
F_1 - F_1^* &= 3(F_2^* - F_2); \\
F_1 - F_1^* &= F_{\alpha\alpha} - F_{\alpha\alpha}^*; \\
F_{\alpha\beta} - F_{\alpha\beta}^* &= 0; && (\alpha \neq \beta) \\
F_{\alpha\alpha} - F_{\alpha\alpha}^* &= G_{\alpha\beta} - G_{\alpha\beta}^*; && (\alpha < \beta) \\
F_{\beta\beta} - F_{\beta\beta}^* &= G_{\alpha\beta} - G_{\alpha\beta}^*; && (\alpha < \beta) \\
H_{\alpha\beta\gamma} - H_{\alpha\beta\gamma}^* &= 0, && (\alpha \neq \beta \neq \gamma \neq \alpha)
\end{aligned}$$

whose general solution is given by

$$\begin{aligned}
F_1^* &= F_1 + h, \qquad F_2^* = F_2 - \tfrac{1}{3}h, \qquad F_{\alpha\alpha}^* = F_{\alpha\alpha} + h, \\
G_{\alpha\beta}^* &= G_{\alpha\beta} + h, \quad F_{\alpha\beta}^* = F_{\alpha\beta}, \qquad H_{\alpha\beta\gamma}^* = H_{\alpha\beta\gamma},
\end{aligned} \tag{3.6}$$

where h is a differentiable function on M. Consequently, if $h \neq 0$, the writing of R in not unique and the functions of two different writings are related by (3.6).

On the other hand, if M is a $(2+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$ and we define the functions

$$\{F_1^*, F_2^*, F_{\alpha\beta}^*, G_{\alpha\beta}^*, H_{\alpha\beta\gamma}^*\}$$

as in (3.6), for any differentiable function h on M, then we deduce:

$$\begin{aligned}
R &= F_1 R_1 + F_2 R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta} R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta} \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma} R_{\alpha\beta\gamma}, \\
&= F_1^* R_1 + F_2^* R_2 + \sum_{\alpha,\beta=1}^{s} F_{\alpha\beta}^* R_{\alpha\beta} + \sum_{1\leqslant\alpha<\beta\leqslant s} G_{\alpha\beta}^* \widetilde{R}_{\alpha\beta} + \sum_{\substack{\alpha,\beta,\gamma=1,\\ \alpha\neq\beta\neq\gamma\neq\alpha}}^{s} H_{\alpha\beta\gamma}^* R_{\alpha\beta\gamma} \\
&\qquad - hR_1 + \frac{h}{3} R_2 - h \sum_{\alpha=1}^{s} R_{\alpha\alpha} - h \sum_{1\leqslant\alpha<\beta\leqslant s} \widetilde{R}_{\alpha\beta}.
\end{aligned}$$

But it is straightforward to check that

$$hR_1 - \frac{h}{3} R_2 + h \sum_{\alpha=1}^{s} R_{\alpha\alpha} + h \sum_{1\leqslant\alpha<\beta\leqslant s} \widetilde{R}_{\alpha\beta} = 0$$

and, consequently, M is also a generalized S-space-form with functions

$$\{F_1^*, F_2^*, F_{\alpha\beta}^*, G_{\alpha\beta}^*, H_{\alpha\beta\gamma}^*\}.$$

4. A different definition

In [**6**], Falcitelli and Pastore defined a generalized $f.pk$-space-form as a metric $f.pk$-manifold M of dimension $2n+s$ (actually, a metric f-manifold) endowed with a family of differentiable functions $\{\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_{\alpha\beta},\ \alpha,\beta = 1,\dots,s\}$, such that $\widetilde{F}_{\alpha\beta} = \widetilde{F}_{\beta\alpha}$, for any $\alpha,\beta \in \{1,\dots,s\}$ and such that the curvature tensor field R of M can be written as

$$\begin{aligned} R(X,Y)Z &= \widetilde{F}_1\big\{g(fX,fZ)f^2Y - g(fY,fZ)f^2X\big\} \\ &\quad + \widetilde{F}_2\big\{g(X,fZ)fY + g(Y,fZ)fX + 2g(X,fY)fZ\big\} \\ &\quad + \sum_{\alpha,\beta=1}^{s} \widetilde{F}_{\alpha\beta}\big\{\eta_\alpha(X)\eta_b(Z)f^2Y - \eta_\alpha(Y)\eta_b(Z)f^2X \\ &\qquad + g(fY,fZ)\eta_\alpha(X)\xi_\beta - g(fX,fZ)\eta_\alpha(Y)\xi_\beta\big\}, \end{aligned} \tag{4.1}$$

for any $X,Y,Z \in \mathcal{X}(M)$. This definition is more restrictive than the one concerning generalized S-space-form. In fact, we observe that, from (4.1), $R(\xi_\alpha,\xi_\beta)\xi_\gamma = 0$, for any $\alpha,\beta,\gamma \in \{1,\dots,s\}$ (this means that the distribution $\mathcal{M}$ is flat), but some examples of generalized S-space-forms not satisfying this condition were presented in [**5**].

Moreover, if M is a generalized $f.pk$-space-form, a straightforward computation using (3.2) gives

$$\begin{aligned} R &= \widetilde{F}_1R_1 + \widetilde{F}_2R_2 + \widetilde{F}_1\bigg\{\sum_{\alpha=1}^{s} R_{\alpha\alpha} - \sum_{1\leqslant\alpha<\beta\leqslant s} \widetilde{R}_{\alpha\beta}\bigg\} \\ &\quad - \sum_{\alpha,\beta=1}^{s} \widetilde{F}_{\alpha\beta}R_{\alpha\beta} - \sum_{\alpha,\beta=1}^{s} \widetilde{F}_{\alpha\alpha}\widetilde{R}_{\alpha\beta} - \sum_{\substack{\alpha,\beta=1,\\ \alpha\neq\beta}}^{s} \widetilde{F}_{\alpha\beta}\bigg\{\sum_{\substack{\gamma=1,\\ \alpha\neq\gamma\neq\beta}}^{s} R_{\gamma\alpha\beta}\bigg\}. \end{aligned}$$

Consequently, M is a generalized S-space form with functions

$$F_1 = \widetilde{F}_1;\ \ F_2 = \widetilde{F}_2;\ \ F_{\alpha\alpha} = \widetilde{F}_1 - \widetilde{F}_{\alpha\alpha};\ \ F_{\alpha\beta} = -\widetilde{F}_{\alpha\beta}\ \ (\alpha\neq\beta);$$
$$G_{\alpha\beta} = \widetilde{F}_1 - \widetilde{F}_{\alpha\alpha} - \widetilde{F}_{\beta\beta};\ \ H_{\alpha\beta\gamma} = -\widetilde{F}_{\beta\gamma}.$$

Conversely, if M is a generalized S-space-form with functions

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$$

such that the distribution $\mathcal{M}$ is flat, then, from (3.4) we get that $H_{\alpha\beta\gamma} = F_{\beta\gamma}$, for any $\alpha,\beta,\gamma = 1,\dots,s$, $\alpha\neq\beta\neq\gamma\neq\alpha$ and from (v) of Proposition 3.1, $G_{\alpha\beta} = F_{\alpha\alpha} + F_{\beta\beta} - F_1$, $1\leqslant\alpha<\beta\leqslant s$. Then, it is easy to check that M is a generalized $f.pk$-space-form with functions:

$$\widetilde{F}_1 = F_1;\ \ \widetilde{F}_2 = F_2;\ \ \widetilde{F}_{\alpha\alpha} = F_1 - F_{\alpha\alpha};\ \ \widetilde{F}_{\alpha\beta} = -F_{\alpha\beta}\ \ (\alpha\neq\beta).$$

5. Generalized S-space-forms with additional structures

Taking into account the results of the above section, if M is a generalized S-space-form such that the distribution $\mathcal{M}$ is flat (for instance, if M is either a metric f-K-contact manifold or a K-manifold), we can apply the results of [**6**] to it. Firstly, we can prove:

THEOREM 5.1. *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$, such that $\nabla\xi_\alpha = -f$, for any $\alpha = 1, \ldots, s$. Then, M is an S-manifold and*

$$F_1 = \tfrac{1}{4}(c+3s);\quad F_2 = \tfrac{1}{4}(c-s);\quad F_{\alpha\alpha} = \tfrac{1}{4}(c+3s) - 1;$$
$$F_{\alpha\beta} = -1\ \ (\alpha \neq \beta);\quad G_{\alpha\beta} = \tfrac{1}{4}(c+3s) - 2\ \ (\alpha < \beta);$$
$$H_{\alpha\beta\gamma} = -1\ \ (\alpha \neq \beta \neq \gamma \neq \alpha),$$

where $\alpha, \beta, \gamma \in \{1, \ldots, s\}$ and $c = F_1 + 3F_2$. In particular, any generalized S-space-form with a metric f-K-contact-structure is an S-manifold.

PROOF. Since, the condition of the statement implies that the distribution $\mathcal{M}$ is flat, we deduce that M is a generalized $f.pk$-space-form and we apply Proposition 7 of [**6**]. For metric f-K-contact manifolds we only have to consider (2.1). □

We point out here that, if $n \geqslant 2$, c becomes constant (see, for example, [**7**]) and M is actually an S-space-form. Moreover, we deduce:

COROLLARY 5.1. *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$. If M is an S-manifold, then $F_1 - F_2 = s$.*

For C-manifolds, we have:

THEOREM 5.2. *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$ and with an underlying C-structure. Then*

$$F_1 = F_2 = F_{\alpha\alpha} = G_{\alpha\beta} = c/4, \quad \alpha < \beta; \tag{5.1}$$

$$F_{\alpha\beta} = H_{\alpha\beta\gamma} = 0, \quad \alpha \neq \beta \neq \gamma \neq \alpha, \tag{5.2}$$

where $\alpha, \beta, \gamma \in \{1, \ldots, s\}$ and $c = F_1 + 3F_2$. Moreover, if $n > 1$, M is a C-space-form.

PROOF. Since M is a C-manifold and so, a K-manifold, from (2.2), the distribution $\mathcal{M}$ is flat and M is also a generalized $f.pk$-space.form. Furthermore, the structure vector fields are parallel and, by using Proposition 8 and Remark 2 of [**6**] and applying the relationships obtained in the above section we get the desired results. Finally, from (3.1), the Ricci tensor field S and the scalar curvature ρ of M are given by

$$S(X,Y) = \frac{(n+1)c}{2}\left(g(X,Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)\right)$$

and $\rho = n(n+1)c$. Now, from the second Bianchi identity,

$$\nabla_i \rho = 2\sum_j \nabla_j S_i^j,$$

where S_i^j denotes the components of the Ricci tensor of type (1,1). Consequently, $(n-1)dc = 0$ and hence, $dc = 0$ if $n > 1$. □

Next, we are going to study generalized S-space-forms with more general structures. First, we get

THEOREM 5.3. *Let M be a generalized S-space-form with functions*

$$\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}.$$

If M is a K-manifold, then

$$F_1 + G_{\alpha\beta} = F_{\alpha\alpha} + F_{\beta\beta}; \quad F_1 - F_{\alpha\alpha} \geqslant 0, \ \textit{with } 1 \leqslant \alpha < \beta \leqslant s;$$
$$H_{\alpha\beta\gamma} = F_{\beta\gamma}, \ \textit{for any } \alpha, \beta, \gamma = 1, \ldots, s \ \textit{such that } \alpha \neq \beta \neq \gamma \neq \alpha.$$

PROOF. Since M is a K-manifold, from (2.2) we get that the distribution $\mathcal{M}$ is flat. Thus, M is a generalized $f.pk$-space-form and by using the results of Section 4, we deduce that $G_{\alpha\beta} = F_{\alpha\alpha} + F_{\beta\beta} - F_1$, $1 \leqslant \alpha < \beta \leqslant s$ and $H_{\alpha\beta\gamma} = F_{\beta\gamma}$, $\alpha \neq \beta \neq \gamma \neq \alpha$. Now, from (2.3) together (*iii*) of Proposition 3.1, we complete the proof. ☐

Finally, for metric f-contact structures, we can prove the following theorem.

THEOREM 5.4. *Let M be a $(2n+s)$-dimensional generalized S-space-form with functions $\{F_1, F_2, F_{\alpha\beta}, G_{\alpha\beta}, H_{\alpha\beta\gamma}\}$. If M is a metric f-contact manifold and*

$$F_1 - F_{\alpha\alpha} = F_{\beta\beta} - G_{\alpha\beta} = 1, \quad 1 \leqslant \alpha < \beta \leqslant s;$$
$$F_{\alpha\alpha} = F_{\beta\beta}, \ \textit{for any } \alpha, \beta = 1, \ldots, s,$$

then M is an S-manifold.

PROOF. First, from (v) of Proposition 3.1 and the hypothesis, we deduce that $K(\xi_\alpha, \xi_\beta) = 0$. Moreover, a direct computation by using (3.1) shows that $S(\xi_\alpha, \xi_\alpha) = 2n(F_1 - F_{\alpha\alpha}) = 2n$, $\alpha = 1, \ldots, s$, where S is the Ricci curvature tensor of M. Then, by using Theorem 3.8 of [**4**], we obtain that the structure vector fields are Killing vector fields, that is, M is a metric f-K-contact manifold. Thus, from Theorem 5.1, it is an S-manifold. ☐

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