Generalized (κ, μ)-space forms and D_a-homothetic deformations

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Abstract. We study the D_a-homothetic deformations of *generalized* (κ, μ)-*space forms*. We prove that the deformed spaces are again *generalized* (κ, μ)-*space forms* in dimension 3, but not in general, although a slight change in their definition would make them so. We give infinitely many examples of *generalized* (κ, μ)-*space forms* of dimension 3.

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1 Introduction

In [1], the first named author (jointly with Pablo Alegre and David E. Blair) defined a generalized Sasakian space form as an almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor R is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3, \tag{1.1}$$

where f_1, f_2, f_3 are some differentiable functions on M and

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$
$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z,$$
$$R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi,$$

for any vector fields X, Y, Z on M. We denote it by $M(f_1, f_2, f_3)$.

P. Alegre and A. Carriazo study in [2] and [3] the generalized Sasakian space forms with contact metric structure, its submanifolds and how conformal changes of metric affects them, respectively. P. Alegre, A. Carriazo, Y. H. Kim and D. W. Yoon give results in [4] about B.-Y. Chen's inequality on submanifolds of generalized complex space forms and generalized Sasakian space forms. R. Al-Ghefari, F.R. Al-Solamy and M. H. Shahid analyse in [5] and [6] the CR-submanifolds of generalized Sasakian space forms while I. Mihai, M. H. Shahid and F. R. Al-Solamy study in

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[17] the Ricci curvature of contact CR-submanifolds of such spaces. S. Hong and M. M. Tripathi in [13] and S. S. Shukla and S. K. Tiwari in [19] also observe the Ricci curvature of some submanifolds of generalized Sasakian space forms. In [14], U. K. Kim gives results if the generalized Sasakian space forms are conformally flat or locally symmetric, while F. Gherib, F. Z. Kadi and M. Belkhelfa in [12] and F. Gherib, M. Gorine and M. Belkhelfa in [11] study them under some other symmetry properties. Lastly, D. W. Yoon and K. S. Cho consider in [21] immersions of warped products in generalized Sasakian space forms, establishing inequalities between intrinsic and extrinsic invariants and A. Olteanu provides in [18] analogous inequalities when the immersion is Legendrian.

In a recent paper, [10], the authors (jointly with M. M. Tripathi) defined a *generalized* (κ,μ)*-space form* as an almost contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ whose curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{1.2}$$

where f_1,f_2,f_3,f_4,f_5,f_6 are differentiable functions on M, R_1,R_2,R_3 are the tensors defined above and

$$\begin{aligned} R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,\\ R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,\\ R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi, \end{aligned}$$

for any vector fields X,Y,Z, where $2h = L_\xi\phi$ and L is the usual Lie derivative. This manifold was denoted by $M(f_1,\dots,f_6)$. They obviously include generalized Sasakian space forms, for $f_4 = f_5 = f_6 = 0$. Moreover, it was proved in [15] that (κ,μ)-space forms are natural examples of *generalized* (κ,μ)*-space forms* for constant functions

$$f_1 = \frac{c+3}{4},\ f_2 = \frac{c-1}{4},\ f_3 = \frac{c+3}{4} - \kappa,\ f_4 = 1,\ f_5 = \frac{1}{2},\ f_6 = 1-\mu. \tag{1.3}$$

In [10], after the formal definition of a *generalized* (κ,μ)*-space form* was given, it was checked that some results that had been true for generalized Sasakian space forms were also correct for these spaces. Then, some basic identities for *generalized* (κ,μ)*-space forms* were obtained in an analogous way to those satisfied by Sasakian manifolds.

The case of contact metric *generalized* (κ,μ)*-space forms* was deeply studied. It was proved that they are generalized (κ,μ)-spaces with $\kappa = f_1 - f_3$ and $\mu = f_4 - f_6$. Furthermore, if dimension is greater than or equal to 5, then they are $(-f_6, 1-f_6)$-spaces with constant ϕ-sectional curvature $2f_6 - 1$, where $f_4 = 1$, $f_5 = 1/2$ and f_1,f_2,f_3 depend linearly on the constant f_6.

Moreover, it was proved that the curvature tensor of a *generalized* (κ,μ)*-space form* is not unique in the 3-dimensional case and that several properties and results must be satisfied. Examples of *generalized* (κ,μ)*-space forms* with non-constant functions f_1,f_3 and f_4 were also given.

In this paper, we continue the study of *generalized* (κ,μ)*-space forms* by analysing the behavior of such spaces under D_a-homothetic deformations. It is organized as follows. After reviewing some necessary background on almost contact metric geometry, we will see in Section 3 how the D_a-homothetic deformations affect the Riemannian curvature tensor of a *generalized (κ,μ)-space form*. We will also introduce an alternative definition of this type of space, called *generalized (κ,μ)-space form with divided R_5*, and we will prove that they remain so after a D_a-homothetic deformation, albeit with different functions $f_1,\dots,f_6$.

2 Preliminaries

In this section, we recall some general definitions and basic formulas which will be used later. For more background on almost contact metric manifolds, we recommend the reference [7].

An odd-dimensional Riemannian manifold (M,g) is said to be an *almost contact metric manifold* if there exist on M a $(1,1)$-tensor field ϕ, a vector field ξ (called the *structure vector field*) and a 1-form η such that $\eta(\xi)=1$, $\phi^2X=-X+\eta(X)\,\xi$ and $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$ for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\phi\xi=0$ and $\eta\circ\phi=0$.

Such a manifold is said to be a *contact metric manifold* if $\mathrm{d}\eta=\Phi$, where $\Phi(X,Y)=g(X,\phi Y)$ is the *fundamental* 2*-form* of M. If, in addition, ξ is a Killing vector field, then M is said to be a *K-contact manifold*. It is well-known that a contact metric manifold is a K-contact manifold if and only if

$$\nabla_X\xi=-\,\phi X \tag{2.1}$$

for all vector fields X on M. Even an almost contact metric manifold satisfying the equation (2.1) becomes a K-contact manifold.

On the other hand, the almost contact metric structure of M is said to be *normal* if the Nijenhuis torsion $[\phi,\phi]$ of ϕ equals $-2\mathrm{d}\eta\otimes\xi$. A normal contact metric manifold is called a *Sasakian manifold*. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\phi)Y=g(X,Y)\xi-\eta(Y)X \tag{2.2}$$

for any vector fields X,Y on M. Moreover, for a Sasakian manifold the following equation holds:

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y.$$

Given an almost contact metric manifold (M,ϕ,ξ,η,g), a *ϕ-section* of M at $p\in M$ is a section $\Pi\subseteq T_pM$ spanned by a unit vector X_p orthogonal to ξ_p, and ϕX_p. The *ϕ-sectional curvature of* Π is defined by $K(X,\phi X)=R(X,\phi X,\phi X,X)$. A Sasakian manifold with constant ϕ-sectional curvature c is called a *Sasakian space form*. In such a case, its Riemann curvature tensor is given by equation (1.1) with functions $f_1=(c+3)/4$, $f_2=f_3=(c-1)/4$.

It is well known that on a contact metric manifold (M,ϕ,ξ,η,g), the tensor h, defined by $2h=L_\xi\phi$, is symmetric and satisfies the following relations [7]:

$$h\xi=0,\quad \nabla_X\xi=-\phi X-\phi hX,\quad h\phi=-\phi h,\quad \mathrm{tr}h=0,\quad \eta\circ h=0. \tag{2.3}$$

Therefore, it follows from equations (2.1) and (2.3) that a contact metric manifold is K-contact if and only if $h=0$.

On the other hand, a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be a *generalized* (κ, μ)*-space* if its curvature tensor satisfies the condition

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\},$$

for some smooth functions κ and μ on M independent of the choice of vectors fields X and Y. If κ and μ are constant, the manifold is called a (κ, μ)*-space*, which were introduced in [8] under the name *contact metric manifold with* ξ *belonging to the* (κ, μ)*-nullity distribution*. T. Koufogiorgos proved in [15] that if a (κ, μ)-space M has constant ϕ-sectional curvature c and dimension greater than or equal to 5, the curvature tensor of this (κ, μ)-space form is given by equation (1.2), with functions as in (1.3).

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. We recall that a D_a-homothetic deformation is defined by

$$\overline{\phi} = \phi, \quad \overline{\xi} = \frac{1}{a}\xi, \quad \overline{\eta} = a\eta, \quad \overline{g} = ag + a(a-1)\eta \otimes \eta, \tag{2.4}$$

where a is a positive constant (see [20]). It is clear that the D_a-deformed manifold $(M, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is also an almost contact metric manifold and that

$$\overline{h} = \frac{1}{a}h. \tag{2.5}$$

Furthermore, it is well known ([8], [16]) that a D_a-homothetic deformation of a generalized (κ, μ)-space yields a new generalized $(\overline{\kappa}, \overline{\mu})$-space with

$$\overline{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \qquad \overline{\mu} = \frac{\mu + 2a - 2}{a}, \tag{2.6}$$

for some $a > 0$.

Finally, we assume that all the functions considered in this paper will be differentiable functions on the corresponding manifolds.

3 Generalized (κ, μ)-space forms and D_a-homothetic deformations

In this section we will study how D_a-homothetic deformations affect *generalized* (κ, μ)*-space forms*. We will see that the manifold obtained by this deformation is not always a *generalized* (κ, μ)*-space form* (except in dimension 3) but that a little change in the definition of the curvature tensor of the original manifold would make it so.

We will first study how a D_a-homothetic deformation affects the curvature tensor of an almost contact metric manifold. A direct computation proves:

Lemma 3.1. *If* (M, ϕ, ξ, η, g) *is a contact metric manifold with Riemannian connection* ∇*, the connection* $\overline{\nabla}$ *of the* D_a*-deformed manifold,* $(M, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$*, is given by*

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{a-1}{a} g(hX, \phi Y)\xi - (a-1)\{\eta(X)\phi Y + \eta(Y)\phi X\}, \tag{3.1}$$

for any X, Y *on* M.

We will now use (3.1) to prove the next proposition.

Proposition 3.2. *Let (M,ϕ,ξ,η,g) be a contact metric manifold with Riemannian curvature R. Then the Riemannian curvature $\overline{R}$ of the D_a-deformed manifold is given by*

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z \\ &\quad + (a-1)\{g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y - 2g(X,\phi Y)\phi Z \\ &\qquad + \eta(X)(\nabla_Y\phi)Z - \eta(Y)(\nabla_X\phi)Z + \eta(Z)((\nabla_Y\phi)X - (\nabla_X\phi)Y)\} \\ &\quad + \frac{a-1}{a}\{g((\nabla_Y\phi h)X - (\nabla_X\phi h)Y, Z)\xi + g(\phi hY,Z)\phi hX - g(\phi hX,Z)\phi hY\} \\ &\quad + (a-1)^2\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &\quad + \frac{(a-1)^2}{a}\{\eta(Y)g(hX,Z)\xi - \eta(X)g(hY,Z)\xi\}, \end{aligned} \tag{3.2}$$

for any X,Y,Z on M.

Proof. If we substitute equation (3.1) in the definition of the Riemannian curvature tensor

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z,$$

after long calculations we obtain:

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z \\ &\quad + (a-1)\{-\eta(X)\phi(\nabla_Y Z) - \eta(\nabla_Y Z)\phi X + \eta(Y)\phi(\nabla_X Z) + \eta(\nabla_X Z)\phi Y \\ &\qquad - g(hY,\phi Z)\phi X) + g(hX,\phi Z)\phi Y - 2g(X,\phi Y)\phi Z \\ &\qquad + \eta(Z)(\nabla_Y\phi X - \nabla_X\phi Y + \phi(\nabla_X Y - \nabla_Y X)) \\ &\qquad - X(\eta(Z))\phi Y + \eta(X)\nabla_Y\phi Z + Y(\eta(Z))\phi X - \eta(Y)\nabla_X\phi Z\} \\ &\quad + \frac{a-1}{a}\{g(hX,\phi\nabla_Y Z)\xi - g(hY,\phi\nabla_X Z)\xi \\ &\qquad + X(g(hY,\phi Z))\xi - Y(g(hX,\phi Z))\xi \\ &\qquad - g(h[X,Y],\phi Z)\xi - g(hY,\phi Z)\phi hX + g(hX,\phi Z)\phi hY\} \\ &\quad + (a-1)^2\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &\quad + \frac{(a-1)^2}{a}\{\eta(Y)g(hX,Z)\xi - \eta(X)g(hY,Z)\xi\}.\end{aligned}$$

Using now the properties of h and the fact that ∇ is the Levi-Civita connection of g, we conclude that equation (3.2) is satisfied. □

We will now give a couple of lemmas that will be useful to prove the next theorem.

Lemma 3.3. *Let (M,ϕ,ξ,η,g) be a contact metric manifold with Riemannian curvature R and tensors $R_1,\dots,R_6$ defined as in (1.2). Then the tensors $\overline{R}_1,\dots,\overline{R}_6$*

defined analogously on the manifold $(M,\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ *satisfy*

$$\begin{aligned}
R_1(X,Y)Z &= \frac{1}{a}\overline{R}_1(X,Y)Z-(a-1)(\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y),\\
R_2(X,Y)Z &= \frac{1}{a}\overline{R}_2(X,Y)Z,\\
R_3(X,Y)Z &= \frac{1}{a}\overline{R}_3(X,Y)Z+(a-1)(\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y),\\
R_4(X,Y)Z &= \overline{R}_4(X,Y)Z-(a-1)(\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY),\\
R_5(X,Y)Z &= a\overline{R}_5(X,Y)Z,\\
R_6(X,Y)Z &= \overline{R}_6(X,Y)Z+(a-1)(\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY),
\end{aligned}$$

for any X,Y,Z *on* M.

Proof. We only need to substitute (2.4) and (2.5) in $\overline{R}_1,\ldots,\overline{R}_6$. □

Lemma 3.4. *Let* (M,ϕ,ξ,η,g) *be a generalized* (κ,μ)*-space. Then*

$$(\nabla_X\phi)Y = g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{3.3}$$

$$\begin{aligned}(\nabla_X\phi h)Y-(\nabla_Y\phi h)X &= \phi((\nabla_X h)Y-(\nabla_Y h)X)=\\ &=(1-\kappa)\{\eta(Y)X-\eta(X)Y\}+(1-\mu)\{\eta(Y)hX-\eta(X)Y\},\end{aligned} \tag{3.4}$$

for any X,Y *on* M.

Proof. Similar to the one given for (κ,μ)-spaces in Lemma 3.1 of [8]. □

Theorem 3.5. *If* $M(f_1,\ldots,f_6)$ *is a* generalized (κ,μ)-space form *with contact metric structure* (ϕ,ξ,η,g), *a* D_a*-homothetic deformation transforms the Riemannian curvature tensor* R *into* $\overline{R}$ *in the following way:*

(3.5)

$$\begin{aligned}
\overline{R}(X,Y)Z = &\left(\frac{f_1}{a}\overline{R}_1+\frac{f_2-a+1}{a}\overline{R}_2+\frac{1}{a^2}((a-1)f_1+f_3+1-a^2)\overline{R}_3\right)(X,Y)Z\\
&+\left(f_4\overline{R}_4+af_5\overline{R}_5+\frac{1}{a}((a-1)f_4+f_6-2(a-1))\overline{R}_6\right)(X,Y)Z\\
&+(a-1)\{\overline{g}(\overline{\phi h}Y,Z)\overline{\phi h}X-\overline{g}(\overline{\phi h}X,Z)\overline{\phi h}Y\},
\end{aligned}$$

for any X,Y,Z *vector fields on* M.

Proof. If we apply Lemma 3.4 to equation (3.2) and use the definition of tensors

$R_1,\dots,R_6$, we obtain:

$$\begin{aligned}\overline{R}(X,Y)Z &= f_1R_1(X,Y)Z+(f_2-a+1)R_2(X,Y)Z\\ &\quad+\frac{1}{a}((a-1)f_1+f_3+1-a^2)R_3(X,Y)Z+f_4R_4(X,Y)Z+f_5R_5(X,Y)Z\\ &\quad+\frac{1}{a}((a-1)f_4+f_6-2(a-1))R_6(X,Y)Z\\ &\quad+\frac{a-1}{a}(f_1-f_3+a^2-1)\{\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\\ &\quad+\frac{a-1}{a}(f_4-f_6+2(a-1))\{\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY\}\\ &\quad+\frac{a-1}{a}\{g(\phi hY,Z)\phi hX-g(\phi hX,Z)\phi hY\}. \end{aligned} \tag{3.6}$$

If we now use Lemma 3.3 in (3.6) and reorder, then

$$\begin{aligned}\overline{R}(X,Y)Z &= \left(\frac{f_1}{a}\overline{R}_1+\frac{f_2-a+1}{a}\overline{R}_2+\frac{1}{a^2}((a-1)f_1+f_3+1-a^2)\overline{R}_3\right)(X,Y)Z\\ &\quad+\left(f_4\overline{R}_4+af_5\overline{R}_5+\frac{1}{a}((a-1)f_4+f_6-2(a-1))\overline{R}_6\right)(X,Y)Z\\ &\quad+\frac{a-1}{a}\{g(\phi hY,Z)\phi hX-g(\phi hX,Z)\phi hY)\}.\end{aligned}$$

To obtain equation (3.5), it is enough to observe that

$$g(\phi hY,Z)\phi hX-g(\phi hX,Z)\phi hY=a(\overline{g}(\overline{\phi h}Y,Z)\overline{\phi h}X-\overline{g}(\overline{\phi h}X,Z)\overline{\phi h}Y)$$

by equations (2.4) and (2.5). □

The previous theorem suggests that it would be useful to redefine *generalized (κ,μ)-space forms* $M(f_1,\dots,f_6)$ with the tensor field

$$R_5(X,Y)Z=g(hY,Z)hX-g(hX,Z)hY+g(\phi hX,Z)\phi hY-g(\phi hY,Z)\phi hX$$

divided in two:

$$\begin{aligned}R_{5,1}(X,Y)Z&=g(hY,Z)hX-g(hX,Z)hY,\\ R_{5,2}(X,Y)Z&=g(\phi hY,Z)\phi hX-g(\phi hX,Z)\phi hY.\end{aligned}$$

It follows that $R_5=R_{5,1}-R_{5,2}$.

It is obvious that the manifolds defined this way, which we will call *generalized (κ,μ)-space forms with divided* R_5 will include *generalized (κ,μ)-space forms* because

$$\begin{aligned}R&=f_1R_1+f_2R_2+f_3R_3+f_4R_4+f_5R_5+f_6R_6\\ &=f_1R_1+f_2R_2+f_3R_3+f_4R_4+f_5R_{5,1}-f_5R_{5,2}+f_6R_6,\end{aligned}$$

and it would be enough to take $f_{5,1}=f_5$ and $f_{5,2}=-f_5$. We would then obtain:

Theorem 3.6. *Let $M(f_1, \ldots, f_6)$ be a contact metric* generalized (κ, μ)-space form with divided R_5. *Then the Riemannian curvature tensor $\overline{R}$ of the D_a-deformed manifold has the form*

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \frac{f_1}{a}\overline{R}_1 + \frac{f_2 - a + 1}{a}\overline{R}_2 + \frac{1}{a^2}((a-1)f_1 + f_3 + 1 - a^2)\overline{R}_3 \\ & + f_4\overline{R}_4 + af_{5,1}\overline{R}_{5,1} + (af_{5,2} + a - 1)\overline{R}_{5,2} + \frac{1}{a}((a-1)f_4 + f_6 - 2(a-1))\overline{R}_6,\end{aligned} \tag{3.7}$$

for any X, Y, Z on M. Therefore, $(M, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is also a generalized (κ, μ)-space form with divided R_5.

Remark 3.7. *Using the new notation, the theorem that Boeckx presented in [9] based in [8] would say that a non-Sasakian (κ, μ)-space M has curvature tensor R given by*

$$\begin{aligned}R = {} & \left(1 - \frac{\mu}{2}\right) R_1 - \frac{\mu}{2} R_2 + \left(1 - \frac{\mu}{2} - \kappa\right) R_3 \\ & + R_4 + \frac{1 - \frac{\mu}{2}}{1-\kappa} R_{5,1} + \frac{\kappa - \frac{\mu}{2}}{1 - \kappa} R_{5,2} + (1-\mu) R_6.\end{aligned}$$

This result means that every non-Sasakian (κ, μ)-space is a contact metric generalized (κ, μ)-space form with divided R_5

$$M\left(1 - \frac{\mu}{2}, -\frac{\mu}{2}, 1 - \frac{\mu}{2} - \kappa, 1, \frac{1-\mu/2}{1-\kappa}, -\frac{\mu/2 - \kappa}{1-\kappa}, 1 - \mu\right).$$

Both the expressions (3.5) and (3.7) can be simplified if the *generalized (κ, μ)-space form* is of dimension 3 as shown by the following lemma and proposition.

Lemma 3.8. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost contact metric manifold. Then the following equation holds:*

$$R_2 = 3(R_1 + R_3). \tag{3.8}$$

If M is also contact metric, then it is satisfied:

$$R_6 = -R_4. \tag{3.9}$$

If M is also a generalized (κ, μ)-space*, then:*

$$R_{5,1} = R_{5,2} = (\kappa - 1)(R_1 + R_3). \tag{3.10}$$

Proof. Equation (3.8) can be easily proved using a ϕ-basis.

For equation (3.9), we also use that h is symmetric and satisfies $h\phi = -\phi h$ because the structure is contact metric.

We know from [16] that $h^2 = (\kappa - 1)\phi^2$. Therefore, if $\kappa = 1$, then $h = 0$ and (3.10) is trivial. If $\kappa < 1$, then there exists a ϕ-basis $\{E, \phi E, \xi\}$ satisfying $hE = \lambda E$, where $\lambda = \sqrt{1-\kappa} > 0$. If we calculate $R_2(X,Y)Z$ and $R_{5,2}(X,Y)Z$ for all $X, Y, Z \in \{E, \phi E, \xi\}$, it is easy to check that the only non-zero values are:

$$\begin{aligned} & R_2(E, \phi E)e = -3\phi E, && R_2(E, \phi E)\phi e = 3E, \\ & R_{5,2}(E, \phi E)e = -(\kappa - 1)\phi E, && R_{5,2}(E, \phi E)\phi e = (\kappa - 1)E. \end{aligned}$$

Therefore, $R_{5,2}(X,Y)Z = \dfrac{\kappa-1}{3}R_2(X,Y)Z$, for every X, Y, Z. We only need to use (3.8) to obtain (3.10). □

Using the previous lemma, we obtain a generalization of a result obtained in [10]:

Proposition 3.9. *Let $M^3(f_1,\ldots,f_6)$ be a contact metric* generalized (κ,μ)-space form with divided R_5. *Then its curvature tensor can be written as follows:*

$$R = f_1^*R_1 + f_3^*R_3 + f_4^*R_4,$$

where

$$\begin{aligned}
f_1^* &= f_1 + 3f_2 + (f_{5,1}+f_{5,2})(f_1 - f_3 - 1),\\
f_3^* &= f_3 + 3f_2 + (f_{5,1}+f_{5,2})(f_1 - f_3 - 1)\\
f_4^* &= f_4 - f_6.
\end{aligned}$$

Applying the previous result, Theorem 3.5 can be simplified to:

Theorem 3.10. *Let $M^3(f_1,0,f_3,f_4,0,0)$ be a contact metric* generalized (κ,μ)-space form. *Then the Riemannian curvature tensor $\overline{R}$ of the D_a-deformed manifold can be written as*

$$\overline{R} = \overline{f}_1\overline{R}_1 + \overline{f}_3\overline{R}_3 + \overline{f}_4\overline{R}_4,$$

where

$$\begin{aligned}
\overline{f}_1 &= \frac{1}{a^2}((2a-1)f_1 - (a-1)f_3 - 3a^2 + 2a + a),\\
\overline{f}_3 &= \frac{1}{a^2}(2(a-1)f_1 + (2-a)f_3 - 4a^2 + 2a + 2),\\
\overline{f}_4 &= \frac{1}{a}(f_4 + 2a - 2).
\end{aligned}$$

Therefore, $(M,\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is a generalized (κ,μ)-space form $M^3(\overline{f}_1,0,\overline{f}_3,\overline{f}_4,0,0)$.

Proof. Using Theorem 3.5 with $f_2 = f_5 = f_6 = 0$, we know that the deformed manifold would be a *generalized (κ,μ)-space form with divided R_5 $M(\overline{f}_1,\ldots,\overline{f}_6)$* with functions:

$$\begin{aligned}
&\overline{f}_1 = \frac{f_1}{a}, \qquad \overline{f}_2 = \frac{-a+1}{a},\\
&\overline{f}_3 = \frac{1}{a^2}((a-1)f_1 + f_3 + 1 - a^2), \qquad \overline{f}_4 = f_4,\\
&\overline{f}_{5,1} = 0, \qquad \overline{f}_{5,2} = a-1,\\
&\overline{f}_6 = \frac{1}{a}((a-1)f_4 - 2(a-1)).
\end{aligned}$$

Applying now Proposition 3.9, we obtain that the deformed manifold has curvature tensor as follows:

$$\overline{R} = \overline{f}_1^*\overline{R}_1 + \overline{f}_3^*\overline{R}_3 + \overline{f}_4^*\overline{R}_4,$$

where

$$\begin{aligned}\overline{f}_1^* =&\overline{f}_1 + 3\overline{f}_2 + (\overline{f}_{5,1} + \overline{f}_{5,2})(\overline{f}_1 - \overline{f}_3 - 1)\\ =&\frac{f_1}{a} + 3\frac{-a+1}{a} + (a-1)\left(\frac{f_1}{a} - \frac{1}{a^2}((a-1)f_1 + f_3 + 1 - a^2) - 1\right),\\ \overline{f}_3^* =&\overline{f}_3 + 3\overline{f}_2 + (\overline{f}_{5,1} + \overline{f}_{5,2})(\overline{f}_1 - \overline{f}_3 - 1)\\ =&\frac{1}{a^2}((a-1)f_1 + f_3 + 1 - a^2) + 3\frac{-a+1}{a}\\ &+ (a-1)\left(\frac{f_1}{a} - \frac{1}{a^2}((a-1)f_1 + f_3 + 1 - a^2) - 1\right),\\ \overline{f}_4^* =&\overline{f}_4 - \overline{f}_6 = f_4 - \frac{1}{a}((a-1)f_4 - 2(a-1)).\end{aligned}$$

A simple calculation would produce the desired result. □

We proved in [10] that Example 1 of [16] is a contact metric *generalized* (κ, μ)*-space form* $M^3(f_1, 0, f_3, f_4, 0, 0)$ with non-constant functions:

$$f_1 = -3 + \frac{2}{x_3^2} + \frac{1}{x_3^4} + \frac{2}{x_3^6}, \qquad f_3 = -4 + \frac{2}{x_3^2} + \frac{2}{x_3^4} + \frac{2}{x_3^6}, \qquad f_4 = 2\left(1 - \frac{1}{x_3^2}\right).$$

If we transform it by a D_a-homothetic deformation, we get a contact metric *generalized* (κ, μ)*-space form* $M^3(\overline{f}_1, 0, \overline{f}_3, \overline{f}_4, 0, 0)$ with

$$\begin{aligned}\overline{f}_1 &= \frac{1}{a^2}\left(-3a^2 + \frac{2a}{x_3^2} + \frac{1}{x_3^4} + \frac{2a}{x_3^6}\right),\\ \overline{f}_3 &= \frac{2}{a^2}\left(-2a^2 + \frac{a}{x_3^2} + \frac{1}{x_3^4} + \frac{a}{x_3^6}\right), \qquad \overline{f}_4 = \frac{2}{a}\left(a - \frac{1}{x_3^2}\right),\end{aligned}$$

for every constant $a > 0$.

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References

[1] P. Alegre, D. E. Blair and A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math. 141 (2004), 157-183.

[2] P. Alegre and A. Carriazo, *Structures on generalized Sasakian-space-forms*, Differential Geom. Appl. 26 (2008), 656-666.

[3] P. Alegre and A. Carriazo, *Submanifolds of generalized Sasakian space forms*, Taiwanese J. Math. 13 (2009), 923-941,

[4] P. Alegre, A. Carriazo, Y. H. Kim and D. W. Yoon, *B.-Y. Chen's inequality for submanifolds of generalized space forms*, Indian J. Pure Appl. Math. 38 (2007), 185-201.

[5] R. Al-Ghefari, F. R. Al-Solamy and M. H. Shahid, *CR-submanifolds of generalized Sasakian space forms*, JP J. Geom. Topol. 6 (2006), 151-166.

[6] R. Al-Ghefari, F. R. Al-Solamy and M. H. Shahid, *Contact CR-warped product submanifolds in generalized Sasakian space forms*, Balkan J. Geom. Appl. 11, 2 (2006), 1-10.

[7] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston, 2002.

[8] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. 91 (1995), 189-214.

[9] E. Boeckx, *A class of locally ϕ-symmetric contact metric spaces*, Arch. Math. 2 (1999), 466-472.

[10] A. Carriazo, V. Martín-Molina and M. M. Tripathi, *Generalized (κ, μ)-space forms*, Submitted for publication. Available in arXiv:0812.2605v1.

[11] F. Gherib, M. Gorine and M. Belkhelfa, *Parallel and semi symmetry of some tensors in generalized Sasakian space forms*, Bull. Transilv. Univ. Braşov Ser. III 1(50) (2008), 139-148.

[12] F. Gherib, F. Z. Kadi and M. Belkhelfa, *Symmetry properties of generalized Sasakian space forms*, Bull. Transilv. Univ. Braşov B (N.S.) 14(49) (2007), 107-114.

[13] S. Hong and M. M. Tripathi, *On Ricci curvature of submanifolds of generalized Sasakian space forms*, Int. J. Pure Appl. Math. Sci. 2 (2005), 173-201.

[14] U. K. Kim, *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms*, Note Mat. 26 (2006), 55-67.

[15] T. Koufogiorgos, *Contact Riemannian manifolds with constant ϕ-sectional curvature*, Tokyo J. Math. 20 (1997), 55-67.

[16] T. Koufogiorgos and C. Tsichlias, *On the existence of a new class of contact metric manifolds*, Canad. Math. Bull. 43 (2000), 400-447.

[17] I. Mihai, M. H. Shahid and F. R. Al-Solamy, *Ricci curvature of a contact CR-submanifold in a generalised Sasakian space form*, Rev. Bull. Calcutta Math. Soc. 13 (2005), 89-94.

[18] A. Olteanu, *Legendrian warped product submanifolds in generalized Sasakian space forms*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 25 (2009), 137-144.

[19] S. S. Shukla and S. K. Tiwari, *Ricci curvature of slant submanifolds in generalized Sasakian space forms*, Bull. Allahabad Math. Soc. 23 (2008), 405-417.

[20] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. 12 (1968), 700-717.

[21] D. W. Yoon and K. S. Cho, *Inequality for warped products in generalized Sasakian space forms*, Int. Math. J. 5 (2004), 225-235.

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