Hiroshima Math. J.
34 (2004), 57–80

Singular boundary value problems of a porous media logistic equation

Manuel Delgado[1], Julián López-Gómez[2] and Antonio Suárez[1]

(Received May 20, 2003)
(Revised November 6, 2003)

Abstract. In this paper we characterize the existence of large solutions for a general class of porous medium logistic equations in the presence of a vanishing carrying capacity. The decay rate of the carrying capacity at the boundary of the underlying domain determines the exact blow-up rate of the large solutions. Its explicit knowledge allows us to obtain a general uniqueness result.

1. Introduction

In this work we study the existence, the blow up rate and the uniqueness of the classical positive solutions to the singular boundary value problem

$$\begin{cases} -\Delta u = W(x)u^q - a(x)f(u) & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbf{R}^N$, $N \geq 1$, with boundary $\partial\Omega$ of class $\mathscr{C}^2$, $W \in L^\infty(\Omega)$, $0 < q < 1$, and, for some $\alpha \in (0,1)$, $a \in \mathscr{C}^\alpha(\bar{\Omega}; \mathbf{R}_+)$, where $\mathbf{R}_+ := [0, +\infty)$ satisfies the following structural assumption:

(H1) The open set

$$\Omega_+ := \{x \in \Omega : a(x) > 0\}$$

is connected with boundary $\partial\Omega_+$ of class $\mathscr{C}^2$, and the open set

$$\Omega_0 := \Omega \setminus \bar{\Omega}_+$$

satisfies $\bar{\Omega}_0 \subset \Omega$; thus, a can vanish in some region of Ω, as well as on some piece of $\partial\Omega$.

The function f is assumed to satisfy the following:

2000 *Mathematics Subject Classification*. 35J25, 35J65, 35K57.
Key words and phrases. Large solutions, vanishing weight at variable rates, blowing-up rates.
[1] Supported by the Spanish Ministry of Science and Technology under Grants BFM2000-0797 and BFM2003-06446.
[2] Supported by the Spanish Ministry of Science and Technology under Grant BFM2000-0797 and REN2003-00707.

(H2) $f \in \mathscr{C}^1(\mathbf{R}_+;\mathbf{R}_+)$ satisfies $f(0)=0$, $f(s)>0$ for each $s>0$, f is increasing, and $\lim_{t\downarrow 0} t^{-q}f(t)=0$.

(H3) $t \mapsto t^{-q}f(t)$ is increasing in $\mathring{\mathbf{R}}_+ := (0,+\infty)$.

(H4) $\int_1^\infty (\int_0^s f)^{-1/2} ds < \infty$.

The solutions of (1.1) are usually known as the *large solutions* of

$$-\Delta u = W(x)u^q - a(x)f(u) \qquad \text{in } \Omega. \tag{1.2}$$

Most precisely, by a large solution of (1.2) it is meant any classical strong solution u such that

$$\lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} u(x) = \infty.$$

Problem (1.1) arises in studying the large positive solutions of the logistic porous medium equation

$$\begin{cases} -\Delta v^m = W(x)v - a(x)v^p & \text{in } \Omega \\ v = \infty & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where $p > m > 1$. Indeed, the change of variable $u = v^m$ transforms (1.3) into (1.1) with $q = \dfrac{1}{m}$ and $f(u) = u^{p/m}$, and, in this case, f satisfies all the assumptions (H2–4).

Our main existence result reads as follows:

THEOREM 1.1. *Assume* (H1–3) *and*

$$\lim_{t\uparrow\infty} t^{-q}f(t) = \infty. \tag{1.4}$$

Then, (1.1) *possesses a non-negative solution if and only if* (H4) *holds.*

In the special case when $W=0$ and a is separated away from zero on $\partial\Omega$, Theorem 1.1 was found by A. V. Lair [13]. In the special case when $f(u)=u^p$ and $W(x)=\lambda\in\mathbf{R}$, Theorem 1.1 was obtained by M. Delgado et al. [7]. Finally, F. C. Cirstea and V. Radulescu [5], [6], found the existence in the special case when $q=1$ and $W=\lambda\in\mathbf{R}$. Theorem 1.1 substantially extends and unifies all previous existence results.

Subsequently, we denote by $\mathbf{n}:\partial\Omega\to\mathbf{R}^N$, $x\mapsto\mathbf{n}(x):=\mathbf{n}_x$, the outward unit normal vector-field of Ω, and, for each $\omega\in(0,\pi/2)$,

$$C_{x_0,\omega} := \left\{ x\in\Omega : \operatorname{angle}(x-x_0, -\mathbf{n}_{x_0}) \le \frac{\pi}{2} - \omega \right\}.$$

To state the results concerning the blow-up rate and the uniqueness of the non-negative large solutions of (1.1) we need to introduce some additional hypothesis on f. Namely, (H5) There exist $p > 1$ and $K > 0$ such that $\lim_{u \uparrow \infty} u^{-p} f(u) = K$.

(H6) There exist $x_0 \in \partial\Omega$, $\beta := \beta(x_0) > 0$ and $\gamma := \gamma(x_0) \geq 0$ such that

$$\lim_{x \to x_0} \frac{a(x)}{\beta[\text{dist}(x, \partial\Omega)]^{\gamma}} = 1. \tag{1.5}$$

(H7) There exist $\beta \in \mathscr{C}(\partial\Omega; \mathring{\mathbf{R}}_+)$ and $\gamma \in \mathscr{C}(\partial\Omega; \mathbf{R}_+)$ such that

$$\lim_{x \to x_0} \frac{a(x)}{\beta(x_0)[\text{dist}(x, \partial\Omega)]^{\gamma(x_0)}} = 1 \qquad \text{uniformly in } x_0 \in \partial\Omega. \tag{1.6}$$

(H8) $W \geq 0$ and the map $u \mapsto \dfrac{f(u)}{u}$ is increasing in $(0, \infty)$,

Then, our main result reads as follows.

Theorem 1.2. *Suppose f satisfies* (H2–3) *and* (H5–6). *Then, for each $\omega \in (0, \pi/2)$ and any positive solution u of* (1.1), *one has that*

$$\lim_{\substack{x \to x_0 \\ x \in C_{x_0, \omega}}} \frac{u(x)}{M[\text{dist}(x, \partial\Omega)]^{-\alpha}} = 1, \tag{1.7}$$

where

$$\alpha := \frac{\gamma + 2}{p - 1}, \qquad M := \left[\frac{\alpha(\alpha + 1)}{\beta K}\right]^{1/(p-1)}. \tag{1.8}$$

In particular, for any pair (u_1, u_2) of positive classical solutions of (1.1),

$$\lim_{\substack{x \to x_0 \\ x \in C_{x_0, \omega}}} \frac{u_1(x)}{u_2(x)} = 1. \tag{1.9}$$

Moreover, if, in addition, (H7) *is satisfied, then* $\lim_{x \to x_0} \dfrac{u(x)}{M(x_0)[\text{dist}(x, \partial\Omega)]^{-\alpha(x_0)}} = 1$ *uniformly in $x_0 \in \partial\Omega$, where*

$$\alpha(x_0) := \frac{\gamma(x_0) + 2}{p - 1}, \qquad M(x_0) := \left[\frac{\alpha(x_0)(\alpha(x_0) + 1)}{\beta(x_0) K}\right]^{1/(p-1)}, \qquad x_0 \in \partial\Omega.$$

Therefore, for any pair (u_1, u_2) of positive solution of (1.1), $\lim_{x \to x_0} \dfrac{u_1(x)}{u_2(x)} = 1$ *uniformly in $x_0 \in \partial\Omega$. Furthermore, if, in addition,* (H8) *holds, then,* (1.1) *possesses a unique positive solution.*

Note that if (H8) is satisfied, then $u \mapsto \dfrac{f(u)}{u^q} = \dfrac{f(u)}{u} u^{1-q}$ is increasing, since $q < 1$, and, hence, (H3) is satisfied. Also, note that (H5) implies (1.4) and, (H4). Therefore, under the assumptions of Theorem 1.2, Theorem 1.1 guarantees that (1.1) possesses a solution. Theorem 1.2 provides us with the blow-up rate of these solutions and with a sufficient condition for the uniqueness.

Theorem 1.2 is a substantial extension of the main result of J. López-Gómez [16], obtained for the very special case when $q = 1$ and $f(u) = u^p$. Even in this special situation, Theorem 1.2 is a very sharp improvement of Y. Du & Q. Huang [8, Theorem 2.8] and J. García-Melián et al. [9, Theorem 1], where it was assumed that

$$a(x) = \beta[\mathrm{dist}(x, \partial\Omega)]^\gamma[1 + \rho\ \mathrm{dist}(x, \partial\Omega) + o(\mathrm{dist}(x, \partial\Omega))] \qquad \text{as} \quad \mathrm{dist}(x, \partial\Omega) \downarrow 0$$

for some constants $\beta > 0$, $\gamma \geq 0$, and $\rho \in \mathbf{R}$, and, hence,

$$\lim_{x \to x_0} \frac{a(x)}{\beta[\mathrm{dist}(x, \partial\Omega)]^\gamma} = 1 \qquad \text{uniformly in } x_0 \in \partial\Omega,$$

while, in the present paper, the weight function $a(x)$ is allowed to decay towards zero on $\partial\Omega$ with arbitrary rates, depending upon the particular point, or region, of $\partial\Omega$. Hence, $a(x)$ might exhibit several different decays at $\partial\Omega$.

Some pioneering results were given by J. B. Keller [11], R. Osserman [17], C. Loewner & L. Nirenberg [14], C. Bandle & M. Marcus [2], [3], [4], V. A. Kondratiev & V. A. Nikishin [10], L. Véron [18], and M. Marcus & L. Véron [12], although most of them were found for very special cases where $q = 1$ and a is a positive constant.

The distribution of this paper is as follows. In Section 2 we collect some preliminary results of a technical nature that are going to be used later. In Section 3 we give the proof of Theorem 1.1. In Section 4 we prove Theorem 1.2.

2. Some preliminary results

In this section we collect two results of technical character that are going to be very useful for proving Theorem 1.1. Throughout, we assume f to satisfy (H2–4).

LEMMA 2.1. *Suppose f satisfies* (H2–4). *Then,* (1.4) *is satisfied. In particular,* $\lim_{t \uparrow \infty} f(t) = \infty$.

PROOF. Thanks to (H3),

$$\lim_{t \uparrow \infty} t^{-q} f(t) = L \in (0, \infty].$$

Suppose $L \in \mathbf{R}$. Then, there exists $B_1 > 0$ such that $f(t) \leq (L+1)t^q$ for each $t \geq B_1$ and, hence, for each $s \geq B_1$,

$$\int_0^s f = C + \int_{B_1}^s f \le C + \frac{L+1}{q+1}(s^{q+1} - B_1^{q+1}) = C_1 + \frac{L+1}{q+1}s^{q+1}.$$

for some constants $C, C_1 > 0$, whose explicit knowledge is not important. Thus,

$$\int_1^\infty \left(\int_0^s f\right)^{-1/2} ds \ge \int_1^\infty \left(C_1 + \frac{L+1}{q+1}s^{q+1}\right)^{-1/2} ds = \infty$$

since $\frac{q+1}{2} < 1$. As this relation again contradicts (H4), necessarily $L = \infty$. This concludes the proof. $\square$

Subsequently, we consider the function h defined by

$$h(t) := Af(t) - \lambda t^q = t^q(At^{-q}f(t) - \lambda) \tag{2.1}$$

for certain constants $A > 0$ and $\lambda > 0$ to be chosen later. Due to (H2), $h \in \mathscr{C}^1(\mathring{\mathbf{R}}_+; \mathbf{R}_+)$, $h(0) = 0$, $\lim_{t \downarrow 0} h'(t) = -\infty$. Actually,

$$\lim_{t \downarrow 0} t^{-q}h(t) = -\lambda.$$

Moreover, thanks to (H3), $t^{-q}h(t)$ is increasing in $t > 0$, if $0 < t < t_0$. On the other hand, thanks to Lemma 2.1, $\lim_{t \uparrow \infty} t^{-q}f(t) = \infty$. Hence, there exists a unique $t_0 > 0$ such that $h(t) < 0$ if $0 < t < t_0$, $h(t_0) = 0$, and $h(t) > 0$ for each $t > t_0$. In particular,

$$h(t) > 0 \qquad \text{and} \qquad h'(t) > 0 \qquad \text{for each } t > t_0. \tag{2.2}$$

It should be noted that t_0 depends on A and λ. The value t_0 that we have just constructed satisfies the following result.

Proposition 2.2. *Suppose f satisfies* (H2–4). *Then, for each $z > t_0$,*

$$I(z) := \int_z^\infty \left[\int_z^s h(t)dt\right]^{-1/2} ds < \infty,$$

and

$$\lim_{z \downarrow t_0} I(z) = \infty.$$

Proof. Setting

$$g(s) := \int_z^s h(t)dt, \qquad s > z,$$

$I(z)$, $z > t_0$, can be expressed as

$$I(z) = \int_z^\infty [g(s)]^{-1/2} ds.$$

Note that $g(z) = 0$ and $g'(z) = h(z) > 0$, since $z > t_0$, and, hence,

$$\lim_{s \downarrow z} \frac{[g(s)]^{-1/2}}{(s-z)^{-1/2}} = \lim_{s \downarrow z} \left(\frac{g(z) + g'(z)(s-z) + o(s-z)}{s-z} \right)^{-1/2} = [h(z)]^{-1/2}. \tag{2.3}$$

Moreover, by l'Hôpital rule and Lemma 2.1,

$$\lim_{s \uparrow \infty} \frac{g(s)}{\int_0^s f(t)dt} = \lim_{s \uparrow \infty} \frac{g'(s)}{f(s)} = \lim_{s \uparrow \infty} \frac{h(s)}{f(s)} = A,$$

and, hence,

$$\lim_{s \uparrow \infty} \frac{[g(s)]^{-1/2}}{[\int_0^s f(t)dt]^{-1/2}} = A^{-1/2}. \tag{2.4}$$

Thanks to (H4), (2.3) and (2.4), it is apparent, by the asymptotic comparison test for improper integrals, that $I(z) < \infty$.

Finally, setting

$$G(u) := \int_{t_0}^u h(s) ds, \qquad u \geq t_0,$$

we have

$$G(t_0) = 0, \qquad G'(t_0) = h(t_0) = 0.$$

For which one can easily obtain $\lim_{z \downarrow t_0} I(z) = \infty$. □

3. Proof of Theorem 1.1

For each $b > 0$ we consider the following auxiliary boundary value problem

$$\begin{cases} -\Delta u = W(x)u^q - a(x)f(u) & \text{in } \Omega, \\ u = b & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The following result, whose proof can be easily adapted from [7, Theorem 3.1], is needed in proving Theorem 1.1.

Proposition 3.1. *Assume* (H2). *Then,* (3.1) *possesses a maximal nonnegative solution, denoted by* $\theta_{[W,b]}$. *If, in addition,* (H3) *holds and* $W = \lambda \in \mathbf{R}$, *or* $W \in L_+^\infty(\Omega)$, *then* $\theta_{[W,b]}$ *is the unique nonnegative solution of* (3.1). *In any circumstance, the map* $b \mapsto \theta_{[W,b]}$ *is increasing.*

The most crucial result in proving Theorem 1.1 is the next one.

THEOREM 3.2. *Suppose* $W \in L^\infty(\Omega)$, $a \in C^\alpha(\bar{\Omega}, \mathring{\mathbf{R}}_+)$, $0 < \alpha < 1$, *and* (H2–4). *Then, for each compact subset* $K \subset \Omega_+$, *there exists a constant* $M := M(K)$ *such that any nonnegative regular solution,* v, *of*

$$-\Delta u = W(x)u^q - a(x)f(u) \tag{3.2}$$

satisfies

$$\|v\|_{C(K)} \le M.$$

PROOF. Let $K \subset \Omega_+$ be compact, and pick $x_0 \in K$. It suffices to prove that there exist $\rho(x_0) > 0$ and $M(x_0) > 0$ such that $\bar{B} := \bar{B}_{\rho(x_0)}(x_0) \subset \Omega_+$ and

$$\|v\|_{C(\bar{B})} \le M(x_0)$$

for any nonnegative regular solution v of (3.2).

Consider $\rho(x_0) > 0$ such that $\bar{B} := \bar{B}_{\rho(x_0)}(x_0) \subset \Omega_+$ and a nonnegative regular solution of (3.2), say v. Then,

$$-\Delta v = W(x)v^q - a(x)f(v) \le \lambda v^q - Af(v),$$

where we have denoted

$$\lambda := \|W\|_{L^\infty(\Omega)}, \qquad A = \min_{\bar{B}} a > 0. \tag{3.3}$$

Let h be the function defined in (2.1) with the choice (3.3) and t_0 the unique positive zero of h. Now, for each

$$b > \max\left\{\max_{\partial B} v, t_0 + 1\right\}$$

consider the auxiliary problem

$$\begin{cases} -\Delta u = \lambda u^q - Af(u) & \text{in } B \\ u = b & \text{on } \partial B. \end{cases} \tag{3.4}$$

Thanks to Proposition 3.1, (3.4) possesses a unique nonnegative regular solution, $\theta_{[\lambda, b]}$. Moreover, due to Lemma 2.1 sufficiently large positive constants provide us with positive supersolutions of (3.4). Thus, since v is a subsolution, it is apparent, from the uniqueness, that

$$v \le \theta_{[\lambda, b]}.$$

By the uniqueness of the positive solution of (3.4) and the rotational invariance of the Laplacian, for each $x \in B$,

$$\theta_{[\lambda, b]}(x) = \Psi_b(r), \qquad r := |x - x_0|,$$

where Ψ_b is the unique positive solution of

$$\begin{cases} \Psi_b''(r) + \dfrac{N-1}{r}\Psi_b'(r) = h(\Psi_b(r)), & r \in (0, \rho(x_0)), \\ \Psi_b'(0) = 0, \quad \Psi_b(\rho(x_0)) = b. \end{cases} \tag{3.5}$$

Since $b \geq t_0 + 1$, adapting the proof of [7, Theorem 4.1], it is easy to see that $\Psi_b \geq t_0$, $h(\Psi_b) > 0$, $h'(\Psi_b) > 0$. The functions Ψ_b satisfy

$$(r^{N-1}\Psi_b'(r))' = r^{N-1}h(\Psi_b(r)) \tag{3.6}$$

and, hence, integrating (3.6) from 0 to r yields

$$\Psi_b'(r) = r^{1-N}\int_0^r s^{N-1}h(\Psi_b(s))ds > 0 \tag{3.7}$$

This shows that $r \to \Psi_b(r)$ is increasing, as well as $r \to h(\Psi_b(r))$. Thus we find from (3.7) that

$$\Psi_b'(r) \leq r^{1-N}h(\Psi_b(r))\int_0^r s^{N-1}\, ds = \frac{r}{N}h(\Psi_b(r)). \tag{3.8}$$

Now, substituting (3.8) into (3.5) gives $\Psi_b'' \geq \dfrac{h(\Psi_b)}{N}$; moreover, since $\Psi_b \geq 0$, (3.5) gives $\Psi_b'' \leq h(\Psi_b)$ and hence,

$$h(\Psi_b) \geq \Psi_b'' \geq \frac{h(\Psi_b)}{N}. \tag{3.9}$$

We now multiply (3.9) by Ψ_b' and integrate from 0 to r to obtain

$$2\int_{\Psi_b(0)}^{\Psi_b(r)} h(z)dz \geq [\Psi_b'(r)]^2 \geq \frac{2}{N}\int_{\Psi_b(0)}^{\Psi_b(r)} h(z)dz. \tag{3.10}$$

Now, taking the square root of the reciprocal of (3.10) and integrating again gives

$$\frac{1}{\sqrt{2}}\int_{\Psi_b(0)}^{\Psi_b(r)}\left[\int_{\Psi_b(0)}^{u} h(s)ds\right]^{-1/2} du \leq r \leq \sqrt{\frac{N}{2}}\int_{\Psi_b(0)}^{\Psi_b(r)}\left[\int_{\Psi_b(0)}^{u} h(s)ds\right]^{-1/2} du \tag{3.11}$$

and in particular

$$\rho(x_0) \leq \sqrt{\frac{N}{2}}\int_{\Psi_b(0)}^{b}\left[\int_{\Psi_b(0)}^{s} h(t)dt\right]^{-1/2} ds.$$

Note that $\Psi_b(0) > t_0$ and $h(t) > 0$ for each $t > \Psi_b(0)$. Thus, since $h' > 0$,

$$\rho(x_0) \leq \sqrt{\frac{N}{2}}\int_{\Psi_b(0)}^{\infty}\left[\int_{\Psi_b(0)}^{s} h(t)dt\right]^{-1/2} ds$$

and, thanks to Proposition 2.2, $\Psi_b(0)$ must be bounded above by a universal constant—independent of b—. Finally, arguing as in the proof of [7, Theorem 4.1] concludes the proof (see also [15]). □

Now, we establish the sufficiency part of Theorem 1.1.

Proposition 3.3. *Suppose* (H1–4). *Then,* (1.1) *possesses a solution.*

Proof. The proof follows the general scheme of the proof of [7, Theorem 5.1]. Considering the point-wise limit

$$\Theta_{[W]} := \lim_{b\uparrow\infty} \theta_{[W,b]},$$

it suffices to show that $\Theta_{[W]}$ solves (1.1).

Thanks to (H1), for each sufficiently small $\delta > 0$,

$$K_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le \delta\} \subset \Omega_+, \qquad D_\delta := \Omega \backslash K_\delta,$$

and, for each of those δ's, there exists an open set $\mathcal{O}_\delta$ satisfying

$$\partial D_\delta \subset \mathcal{O}_\delta \subset \bar{\mathcal{O}}_\delta \subset \Omega_+.$$

Fix one of those δ's. Then, thanks to Theorem 3.2, there exists a constant $M > 0$ such that, for each $b > 0$,

$$\|\theta_{[W,b]}\|_{C(\partial D_\delta)} \le \|\theta_{[W,b]}\|_{C(\bar{\mathcal{O}}_\delta)} \le M \tag{3.12}$$

and, hence,

$$\theta_{[W,b]} \le \theta_{[\|W\|_{L^\infty(\Omega)}, M]} \qquad \text{in } D_\delta,$$

where $\theta_{[\|W\|_{L^\infty(\Omega)}, M]}$ stands for the unique solution of

$$\begin{cases} -\Delta u = \|W\|_{L^\infty(\Omega)} u^q - a(x) f(u) & \text{in } D_\delta, \\ u = M & \text{on } \partial D_\delta. \end{cases} \tag{3.13}$$

This shows that the point-wise limit $\Theta_{[W]}$ is well defined. Now, we take two open sets $\mathcal{O}, \mathcal{O}_1$ and a sufficiently small $\delta > 0$ so that

$$\mathcal{O}_1 \subset \bar{\mathcal{O}}_1 \subset \mathcal{O} \subset \bar{\mathcal{O}} \subset D_\delta \subset \bar{D}_\delta \subset \Omega.$$

By the elliptic L^p-estimates and Morrey's theorem, there exists a constant $C > 0$ such that, for each $b > 0$,

$$\|\theta_{[W,b]}\|_{C^1(\bar{\mathcal{O}}_1)} \le C.$$

From these estimates the details of the proof can be easily completed by using a rather standard compactness argument and the uniqueness of the point-wise limit. □

To complete the proof of Theorem 1.1 it remains to show that, under conditions (H1–3) and (1.4), (H4) is necessary for the existence of a large solution. We begin by establishing the following result.

PROPOSITION 3.4. *Suppose* (H1–3), (1.4), $\mu < 0$, $A > 0$, *and the problem*

$$\begin{cases} -\Delta u = \mu u^q - Af(u) & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

possesses a solution. Then, (H4) *is satisfied.*

PROOF. Suppose, in addition, that $\Omega = B_R(x_0)$ is the ball of radius $R > 0$ centered at $x_0 \in \mathbf{R}^N$, and let u be any solution of (3.14) in this special case. Then, due to the theory developed in pages 506 and 507 of [11] (cf. [17] as well), u must be radially symmetric, $u(x) = \varphi(r)$, $r = |x - x_0|$, and, setting

$$h(t) := Af(t) - \mu t^q,$$

gives

$$\frac{1}{\sqrt{2}} \int_{u(x_0)}^{\infty} \left[\int_{u(x_0)}^{z} h(s) ds \right]^{-1/2} dz \le R, \tag{3.15}$$

because of (3.11).

Since

$$\int_1^{\infty} \left(\int_0^z f \right)^{-1/2} dz = \int_1^{u(x_0)} \left(\int_0^z f \right)^{-1/2} dz + \int_{u(x_0)}^{\infty} \left(\int_0^z f \right)^{-1/2} dz$$

and the first term of the right hand side of this identity is finite, to prove (H4) it suffices to show that

$$\int_{u(x_0)}^{\infty} \left(\int_0^z f \right)^{-1/2} dz < \infty. \tag{3.16}$$

Since $f > 0$, for each $M > 0$ and $z > M$ we have

$$\int_0^z f \ge \int_M^z f = \int_M^z s^{-q} f(s) s^q \, ds \ge \frac{f(M)}{M^q} \int_M^z s^q \, ds = \frac{f(M)}{(q+1)M^q} (z^{q+1} - M^{q+1}),$$

because $s \mapsto s^{-q} f(s)$ is increasing in $(0, \infty)$. Thus, for each $M > 0$,

$$\lim_{z \uparrow \infty} \frac{\int_0^z f}{z^{q+1}} \ge \frac{f(M)}{(q+1)M^q}$$

and, hence, due to (1.4),

$$\lim_{z \uparrow \infty} \frac{\int_0^z f}{z^{q+1}} \ge \frac{1}{q+1} \lim_{M \uparrow \infty} \frac{f(M)}{M^q} = \infty.$$

Consequently,

$$\lim_{z\uparrow\infty}\frac{\int_{u(x_0)}^z h}{\int_0^z f}=\lim_{z\uparrow\infty}\frac{A\int_0^z f-A\int_0^{u(x_0)} f-\frac{\mu}{q+1}(z^{q+1}-[u(x_0)]^{q+1})}{\int_0^z f}=A$$

and, therefore, (3.16) follows straight ahead from (3.15). This concludes the proof of (H4).

Now, suppose Ω is a general open set for which (3.14) has a solution v. Pick $x_0\in\Omega$, choose a sufficiently large $R>0$ so that $\bar{\Omega}\subset B_R(x_0)$, and consider the auxiliary problems

$$\begin{cases}-\Delta u=\mu u^q-Af(u) & \text{in } B_R(x_0)\\ u=b & \text{on } \partial B_R(x_0)\end{cases}\tag{3.17}$$

for sufficiently large $b>0$. Thanks to Proposition 3.1 and the proof of [7, Theorem 4.1], (3.17) has a unique solution which is radially symmetric $u_b(x)=\Psi_b(r)$, $r=|x-x_0|$, and it satisfies

$$\frac{1}{\sqrt{2}}\int_{\Psi_b(0)}^b\left(\int_{\Psi_b(0)}^z h\right)^{-1/2}dz\le R.\tag{3.18}$$

We already know that $b\mapsto\Psi_b(0)$ is increasing. Thus, by passing to the limit as $b\uparrow\infty$ in (3.18), it is apparent that (H4) holds if

$$\lim_{b\uparrow\infty}\Psi_b(0)<\infty.\tag{3.19}$$

To show (3.19) one can argue as follows. Set, for each sufficiently small $\delta>0$,

$$\Omega_\delta:=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\delta\}.$$

Then, for any sufficiently large $b>0$ there exists $\delta>0$ such that $u_b\le v$ on $\partial\Omega_\delta$. Thus, $u_b\le v$ in Ω_δ, since v is a supersolution of (3.17) and, hence, $u_b(x_0)\le v(x_0)$. Taking $b\to+\infty$, concludes the proof of (3.19). □

The following result concludes the proof of Theorem 1.1.

Proposition 3.5. *Suppose* (H1–3), (1.4), *and* (1.1) *possesses a solution. Then,* (H4) *holds.*

Proof. Pick

$$\mu\in\left(-\infty,\min\left\{\inf_{\Omega}\ W,0\right\}\right)$$

and set

$$A:=\max_{\bar{\Omega}}\ a.$$

Let u be a solution of (1.1). Then, for each $b > 0$, u provides us with a supersolution of

$$\begin{cases} -\Delta u = \mu u^q - Af(u) & \text{in } \Omega \\ u = b & \text{on } \partial\Omega \end{cases} \tag{3.20}$$

and, hence, $\theta_{[\mu,b]} \le u$, where we have denoted by $\theta_{[\mu,b]}$ the unique solution of (3.20). Passing to the limit as $b \uparrow \infty$, and using a very well known compactness argument, shows that (3.14) possesses a solution. Therefore, thanks to Proposition 3.4, condition (H4) is satisfied. □

4. Proof of Theorem 1.2

4.1. Two auxiliary radially symmetric problems

In this subsection we include some useful preliminary results. The first one is an extension of [9, Lemma 4], whose proof easily follows from [1, Theorem A]; so, we will omit it.

THEOREM 4.1. *Suppose* $\underline{u}$ *and* $\bar{u}$ *satisfy*

$$-\Delta \underline{u} \le W(x)\underline{u}^q - a(x)f(\underline{u}), \qquad -\Delta \bar{u} \ge W(x)\bar{u}^q - a(x)f(\bar{u}), \qquad \text{in } \Omega,$$

$$\lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} \underline{u}(x) = \infty, \qquad \lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} \bar{u}(x) = \infty,$$

and

$$\underline{u} \le \bar{u} \qquad \text{in } \Omega.$$

Then, (1.1) *possesses a solution* u *in between* $\underline{u}$ *and* $\bar{u}$.

The main result of this subsection is the following theorem. It will be crucial in proving Theorem 1.2.

THEOREM 4.2. *Suppose* f *satisfies* (H5) *and consider the singular problem*

$$\begin{cases} -\psi'' - \dfrac{N-1}{r}\psi' = \lambda\psi^q - b(r)(R-r)^\gamma f(\psi) & \text{in } (0,R) \\ \lim_{r\uparrow R} \psi(r) = \infty \\ \psi'(0) = 0 \end{cases} \tag{4.1}$$

where $R > 0$, $\lambda \in \mathbf{R}$, $\gamma \ge 0$, *and* $b \in \mathscr{C}([0,R]; \mathring{\mathbf{R}}_+)$. *Then, for each* $\varepsilon > 0$, (4.1) *possesses a positive solution* ψ_ε *such that*

$$1 - \varepsilon \le \liminf_{r\uparrow R} \frac{\psi_\varepsilon(r)}{M(R-r)^{-\alpha}} \le \limsup_{r\uparrow R} \frac{\psi_\varepsilon(r)}{M(R-r)^{-\alpha}} \le 1 + \varepsilon \tag{4.2}$$

where α *and* M *are defined in* (1.8) *with* $\beta := b(R)$. *Therefore, for each* $x_0 \in \mathbf{R}^N$, *the function*

$$u_\varepsilon(x) := \psi_\varepsilon(r), \qquad r := |x - x_0|,$$

provides us with a radially symmetric positive large solution of

$$\begin{cases} -\Delta u = \lambda u^q - b(r)[d(x)]^\gamma f(u) & \text{in } B_R(x_0) \\ u = \infty & \text{on } \partial B_R(x_0) \end{cases} \tag{4.3}$$

satisfying

$$1 - \varepsilon \le \liminf_{d(x)\downarrow 0} \frac{u_\varepsilon(x)}{M[d(x)]^{-\alpha}} \le \limsup_{d(x)\downarrow 0} \frac{u_\varepsilon(x)}{M[d(x)]^{-\alpha}} \le 1 + \varepsilon \tag{4.4}$$

where

$$d(x) := \operatorname{dist}(x, \partial B_R(x_0)) = R - |x - x_0| = R - r.$$

Proof. First, we claim that, for each $\varepsilon > 0$ sufficiently small, there exists a constant $A_\varepsilon > 0$ such that for $A > A_\varepsilon$

$$\bar{\psi}_\varepsilon(r) = A + B_+ \left(\frac{r}{R}\right)^2 (R - r)^{-\alpha}$$

is a positive supersolution of (4.1) where

$$B_+ = (1 + \varepsilon)\left[\frac{\alpha(\alpha + 1)}{K\beta}\right]^{1/(p-1)}. \tag{4.5}$$

Indeed, taking into account that $\alpha + 2 + \gamma - \alpha p = 0$, we find that $\bar{\psi}_\varepsilon$ is a supersolution of (4.1) if, and only if,

$$\begin{aligned} -2N\frac{B_+}{R^2}(R - r)^2 - \alpha(N + 3)\frac{B_+}{R^2}r(R - r) - \alpha(\alpha + 1)B_+\left(\frac{r}{R}\right)^2 \\ \ge \lambda(R - r)^{\alpha(1-q)+2}\left[A(R - r)^\alpha + B_+\left(\frac{r}{R}\right)^2\right]^q \\ - b(r)\left[A(R - r)^\alpha + B_+\left(\frac{r}{R}\right)^2\right]^p \frac{f(\bar{\psi}_\varepsilon)}{\bar{\psi}_\varepsilon^p}. \end{aligned} \tag{4.6}$$

Since $q < 1$, by (H5) the inequality (4.6) at the value $r = R$ becomes into

$$B_+^{p-1} \ge \frac{\alpha(\alpha + 1)}{K\beta}.$$

Therefore, by making the choice (4.5), the inequality (4.6) is satisfied in $(R-\delta, R]$, for some $\delta = \delta(\varepsilon) > 0$. Finally, by choosing A sufficiently large it is clear that the inequality is satisfied in the whole interval $[0, R]$, since $p > 1 > q$ and b is bounded away from zero. This concludes the proof of the claim above.

For each sufficiently small $\varepsilon > 0$, there exists $C < 0$ for which the function

$$\underline{\psi}_\varepsilon(r) := \max\left\{0, C + B_-\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha}\right\}$$

provides us with a non-negative subsolution of (4.1) if

$$B_- = (1-\varepsilon)\left[\frac{\alpha(\alpha+1)}{K\beta}\right]^{1/(p-1)}. \tag{4.7}$$

Indeed, it is easy to see that $\underline{\psi}_\varepsilon$ is a subsolution of (4.1) if in the region where

$$C + B_-\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha} \geq 0 \tag{4.8}$$

the following inequality is satisfied

$$\begin{aligned}
&-2N\frac{B_-}{R^2}(R-r)^2 - \alpha(N+3)\frac{B_-}{R^2}r(R-r) - \alpha(\alpha+1)B_-\left(\frac{r}{R}\right)^2 \\
&\quad \leq \lambda(R-r)^{2+\alpha(1-q)}\left[C(R-r)^\alpha + B_-\left(\frac{r}{R}\right)^2\right]^q \\
&\qquad - b(r)\left[C(R-r)^\alpha + B_-\left(\frac{r}{R}\right)^2\right]^p \frac{f(C + B_-(\frac{r}{R})^2(R-r)^{-\alpha})}{(C + B_-(\frac{r}{R})^2(R-r)^{-\alpha})^p}.
\end{aligned} \tag{4.9}$$

Making the choice (4.7) and using the continuity of $b(r)$, it is easy to see that there exists a constant $\delta = \delta(\varepsilon) > 0$ for which (4.9) is satisfied in $[R-\delta, R)$. Moreover, for each $C < 0$ there exists a constant $z = z(C) \in (0, R)$ such that

$$C + B_-\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha} < 0 \qquad \text{if } r \in [0, z(C)),$$

while

$$C + B_-\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha} \geq 0 \qquad \text{if } r \in [z(C), R).$$

Actually, $z(C)$ is decreasing and

$$\lim_{C\downarrow-\infty} z(C) = R, \qquad \lim_{C\uparrow 0} z(C) = 0. \tag{4.10}$$

Thus, thanks to (4.10), there exists $C < 0$ such that

$$z(C) = R - \delta(\varepsilon).$$

For this choice of C, it readily follows that $\underline{\psi}_\varepsilon$ provides us with a subsolution of (4.1).

Finally, since

$$\lim_{r \uparrow R} \frac{\bar{\psi}_\varepsilon(r)}{B_+(R-r)^{-\alpha}} = \lim_{r \uparrow R} \frac{\underline{\psi}_\varepsilon(r)}{B_-(R-r)^{-\alpha}} = 1,$$

it follows the existence of a solution of (4.1), denoted by ψ_ε, satisfying (4.2). The remaining assertions of the theorem are easy consequences from these features. □

As an immediate consequence from Theorem 4.2, combining a translation together with a reflection around $r_0 := \dfrac{\rho + R}{2}$ it readily follows the corresponding result in each of the annuli

$$A_{\rho,R}(x_0) := \{x \in \mathbf{R}^N : 0 < \rho < |x - x_0| < R\}.$$

Corollary 4.3. *Consider the problem*

$$\begin{cases} -\Delta u = \lambda u^q - b(r)[\operatorname{dist}(x, \partial A_{\rho,R}(x_0))]^\gamma f(u) & \text{in } A_{\rho,R}(x_0) \\ u = \infty & \text{on } \partial A_{\rho,R}(x_0) \end{cases} \tag{4.11}$$

where $\lambda \in \mathbf{R}$, $\gamma \geq 0$, $0 < \rho < R$, *and* $b \in \mathscr{C}([\rho, R]; \mathring{\mathbf{R}}_+)$ *is the reflection around* $r = r_0$ *of some function*

$$\tilde{b} \in \mathscr{C}([r_0, R]; \mathring{\mathbf{R}}_+).$$

Then, for each $\varepsilon > 0$ *the problem* (4.11) *possesses a positive solution* $v_\varepsilon(x)$ *satisfying*

$$1 - \varepsilon \leq \liminf_{\delta(x) \downarrow 0} \frac{v_\varepsilon(x)}{M[\delta(x)]^{-\alpha}} \leq \limsup_{\delta(x) \downarrow 0} \frac{v_\varepsilon(x)}{M[\delta(x)]^{-\alpha}} \leq 1 + \varepsilon \tag{4.12}$$

where α, β *and* M *are defined through* (1.8) *and*

$$\delta(x) := \operatorname{dist}(x, \partial A_{\rho,R}(x_0)) = \begin{cases} R - |x - x_0|, & \text{if } r_0 \leq |x - x_0| < R, \\ |x - x_0| - \rho, & \text{if } \rho < |x - x_0| < r_0. \end{cases}$$

4.2. Proof of Theorem 1.2

Let u be a positive strong solution of (1.1) and consider $x_0 \in \partial\Omega$, $\beta = \beta(x_0) > 0$ and $\gamma = \gamma(x_0) \geq 0$ satisfying (1.5). Since Ω is of class $\mathscr{C}^2$, there exist $R > 0$ and $\delta_0 > 0$ such that

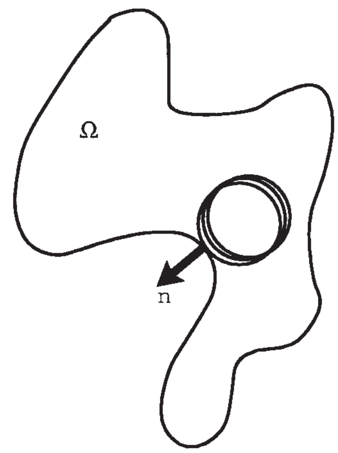

Fig. 4.1. The balls where the supersolutions are supported

$$B_R(x_0 - (R+\delta)\mathbf{n}_{x_0}) \subset \Omega \qquad \text{for each } \delta \in [0, \delta_0] \tag{4.13}$$

and

$$\bar{B}_R(x_0 - R\mathbf{n}_{x_0}) \cap \partial\Omega = \{x_0\}.$$

In Figure 4.1 we have represented this one-parameter-dependent family of balls.

Observe that, in $B_R(x_0 - (R+\delta)\mathbf{n}_{x_0})$,

$$\begin{aligned} \operatorname{dist}(x, \partial\Omega) &\geq \operatorname{dist}(x, \partial B_R(x_0 - (R+\delta)\mathbf{n}_{x_0})) \\ &= R - \operatorname{dist}(x, x_0 - (R+\delta)\mathbf{n}_{x_0}) = R - r \end{aligned}$$

where

$$r := |x - [x_0 - (R+\delta)\mathbf{n}_{x_0}]|.$$

Fix a sufficiently small $\eta > 0$. Thanks to (1.5), $R > 0$ can be shortened, if necessary, so that, for each $\delta \in [0, \delta_0]$,

$$a \geq (\beta - \eta)(R - r)^{\gamma} \qquad \text{in } B_R(x_0 - (R+\delta)\mathbf{n}_{x_0}). \tag{4.14}$$

Thanks to (4.14), for any $\delta \in (0, \delta_0]$, the restriction

$$\underline{u}_\delta := u|_{B_R(x_0 - (R+\delta)\mathbf{n}_{x_0})}$$

provides us with a positive smooth subsolution of

$$\begin{cases} -\Delta u = \lambda u^q - (\beta - \eta)(R - r)^{\gamma} f(u) & \text{in } B_R(x_0 - (R+\delta)\mathbf{n}_{x_0}) \\ u = \infty & \text{on } \partial B_R(x_0 - (R+\delta)\mathbf{n}_{x_0}) \end{cases} \tag{4.15}$$

where

$$\lambda := \sup_{\Omega} W.$$

Thus, any positive solution of (4.15) is a supersolution of the equation that u verifies in $B_r(x_0 - (R+\delta)\mathbf{n}_{x_0})$. So, thanks to the uniqueness (cf. Proposition 3.1), it follows from the strong maximum principle that

$$\underline{u}_\delta = u|_{B_R(x_0-(R+\delta)\mathbf{n}_{x_0})} \leq \Phi_\delta. \tag{4.16}$$

Now, for each sufficiently small $\varepsilon > 0$, let Ψ_ε be any positive radially symmetric solution of

$$\begin{cases} -\Delta u = \lambda u^q - (\beta - \eta)(R-r)^\gamma f(u) & \text{in } B_R(x_0 - R\mathbf{n}_{x_0}) \\ u = \infty & \text{on } \partial B_R(x_0 - R\mathbf{n}_{x_0}) \end{cases} \tag{4.17}$$

satisfying

$$\limsup_{r\uparrow R} \frac{\psi_\varepsilon(r)}{N_\eta (R-r)^{-\alpha}} \leq 1 + \varepsilon \tag{4.18}$$

where

$$\alpha := \frac{\gamma+2}{p-1}, \qquad \Psi_\varepsilon(x) := \psi_\varepsilon(r),$$

$$r := |x - [x_0 - R\mathbf{n}_{x_0}]|, \qquad N_\eta := \left[\frac{\alpha(\alpha+1)}{K(\beta-\eta)}\right]^{1/(p-1)}.$$

It should be noted that its existence is guaranteed by Theorem 4.2. Fix one of those ε's and for each sufficiently small $\delta > 0$ consider the function Φ_δ defined by

$$\Phi_\delta(x) := \Psi_\varepsilon(x + \delta\mathbf{n}_{x_0}), \qquad x \in B_R(x_0 - (R+\delta)\mathbf{n}_{x_0}).$$

By construction, for each sufficiently small $\delta > 0$, Φ_δ provides us with a large positive solution of (4.15) and, hence, (4.16) implies

$$u(x) \leq \Psi_\varepsilon(x + \delta\mathbf{n}_{x_0}) \qquad \text{for each } x \in B_R(x_0 - (R+\delta)\mathbf{n}_{x_0}) \text{ and } \delta \in (0, \delta_0].$$

Thus, passing to the limit as $\delta \downarrow 0$ gives

$$u \leq \Psi_\varepsilon \qquad \text{in } B_R(x_0 - R\mathbf{n}_{x_0})$$

and, hence for each $\omega \in (0, \pi/2)$, (4.18) implies

$$\limsup_{\substack{x\to x_0 \\ x \in C_{x_0,\omega}}} \frac{u(x)}{N_\eta [\operatorname{dist}(x, \partial\Omega)]^{-\alpha}} \leq 1 + \varepsilon, \tag{4.19}$$

where $C_{x_0,\omega}$ is the wedge defined in the statement of Theorem 1.2. In obtaining (4.19) we have used

$$\lim_{\substack{x\to x_0\\ x\in C_{x_0,\omega}}} \frac{\operatorname{dist}(x,\partial\Omega)}{R-r} = \lim_{\substack{x\to x_0\\ x\in C_{x_0,\omega}}} \frac{\operatorname{dist}(x,\partial\Omega)}{\operatorname{dist}(x,\partial B_R(x_0 - R\mathbf{n}_{x_0}))} = 1.$$

As the estimate (4.19) is valid for any sufficiently small $\varepsilon > 0$ and $\eta > 0$, for proving (1.7) it remains to show that

$$1 \le \liminf_{\substack{x\to x_0\\ x\in C_{x_0,\omega}}} \frac{u(x)}{M[\operatorname{dist}(x,\partial\Omega)]^{-\alpha}}. \tag{4.20}$$

Since Ω is of class $\mathscr{C}^2$, there exist $R_2 > R_1 > 0$ and $\delta_0 > 0$ such that

$$\Omega \subset \bigcap_{\delta\in[0,\delta_0]} A_{R_1,R_2}(x_0 + (R_1+\delta)\mathbf{n}_{x_0})$$

and

$$\partial\Omega \cap \partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}) = \{x_0\}.$$

Moreover, R_2 can be taken arbitrarily large. In Figure 4.2 we have represented these annuli.

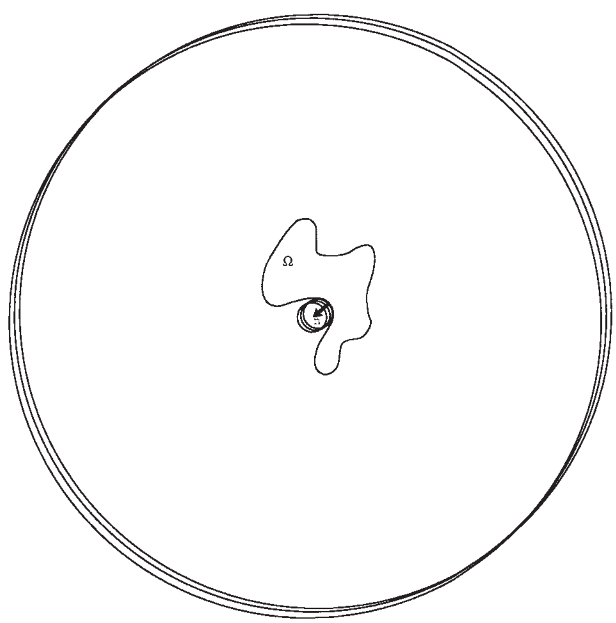

Fig. 4.2. The annuli where the subsolutions are supported

Fix a sufficiently small $\eta > 0$. Thanks to (1.5), there exists a radially symmetric function

$$\hat{a} : A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}) \to \mathring{\mathbf{R}}_+ \qquad \text{such that } \hat{a} \geq a \qquad \text{in } \Omega$$

and, for each $x \in A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0})$,

$$\hat{a}(x) = b(|x - x_0 - R_1\mathbf{n}_{x_0}|)[\operatorname{dist}(x, \partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}))]^{\gamma}$$

for some continuous function $b : [R_1, R_2] \to \mathring{\mathbf{R}}_+$ satisfying

$$b(R_1) = \beta + \eta.$$

Moreover, by enlarging R_2, if necessary, we can assume that b is the reflection around the middle point of $[R_1, R_2]$ of some continuous positive function. Indeed, it suffices assuming that

$$|x - x_0 - R_1\mathbf{n}_{x_0}| < \frac{R_1 + R_2}{2} \qquad \text{for each } x \in \Omega.$$

Furthermore, b can be chosen so that

$$\max_{\bar{A}_{R_1,R_2}(x_0+R_1\mathbf{n}_{x_0})} \hat{a} \leq \max_{\bar{\Omega}} a + 1.$$

Now, consider the auxiliary problem

$$\begin{cases} -\Delta u = \mu u^q - \hat{a} f(u) & \text{in } A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}) \\ u = \infty & \text{on } \partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}) \end{cases} \tag{4.21}$$

where

$$\mu := \inf_{\Omega} W.$$

Thanks to Corollary 4.3, for each sufficiently small $\varepsilon > 0$, (4.21) possesses a radially symmetric positive solution Ψ_ε such that

$$1 - \varepsilon \leq \liminf_{r \downarrow R_1} \frac{\psi_\varepsilon(r)}{P_\eta (r - R_1)^{-\alpha}} \tag{4.22}$$

where

$$\alpha := \frac{\gamma + 2}{p - 1}, \qquad \Psi_\varepsilon(x) := \psi_\varepsilon(r),$$

$$r := |x - [x_0 + R_1\mathbf{n}_{x_0}]|, \qquad P_\eta := \left[\frac{\alpha(\alpha + 1)}{K(\beta + \eta)}\right]^{1/(p-1)}.$$

Fix one of those ε's and for each $\delta \in (0, \delta_0]$ consider the function Φ_δ defined by

$$\Phi_\delta(x) := \Psi_\varepsilon(x - \delta \mathbf{n}_{x_0}), \qquad x \in A_{R_1, R_2}(x_0 + (R_1 + \delta)\mathbf{n}_{x_0}).$$

For each sufficiently small $\delta > 0$, Φ_δ provides us with a large positive solution of

$$\begin{cases} -\Delta u = \mu u^q - \hat{a}(\cdot + \delta \mathbf{n}_{x_0}) f(u) & \text{in } A_{R_1, R_2}(x_0 + (R_1 + \delta)\mathbf{n}_{x_0}) \\ u = \infty & \text{on } \partial A_{R_1, R_2}(x_0 + (R_1 + \delta)\mathbf{n}_{x_0}) \end{cases} \tag{4.23}$$

Moreover, by construction, the restriction $\Phi_\delta|_\Omega$ provides us with a subsolution of (1.1). Thus, thanks to Proposition 3.1, for each $\delta \in (0, \delta_0]$ we have

$$\Psi_\varepsilon(x - \delta \mathbf{n}_{x_0}) \le u(x) \qquad \text{for each } x \in A_{R_1, R_2}(x_0 + (R_1 + \delta)\mathbf{n}_{x_0}) \text{ and } \delta \in (0, \delta_0].$$

Thus, passing to the limit as $\delta \downarrow 0$ gives

$$\Psi_\varepsilon \le u \qquad \text{in } A_{R_1, R_2}(x_0 + R_1 \mathbf{n}_{x_0}),$$

and, hence, for each $\omega \in (0, \pi/2)$

$$1 - \varepsilon \le \liminf_{\substack{x \to x_0 \\ x \in C_{x_0, \omega}}} \frac{u(x)}{P_\eta (r - R_1)^{-\alpha}},$$

since

$$\lim_{\substack{x \to x_0 \\ x \in C_{x_0, \omega}}} \frac{\operatorname{dist}(x, \partial\Omega)}{r - R_1} = \lim_{\substack{x \to x_0 \\ x \in C_{x_0, \omega}}} \frac{\operatorname{dist}(x, \partial\Omega)}{\operatorname{dist}(x, \partial A_{R_1, R_2}(x_0 + R \mathbf{n}_{x_0}))} = 1.$$

This concludes the proof of (1.7). Applying (1.7) to any pair of solutions, u_1 and u_2, (1.9) readily follows.

Now, suppose there are $\beta \in \mathscr{C}(\partial\Omega; \mathring{\mathbf{R}}_+)$ and $\gamma \in \mathscr{C}(\partial\Omega; \mathbf{R}_+)$ satisfying (1.6) and fix $\eta \in (0, 1)$. Then, there exists $\delta \in (0, 1)$ such that, for each $x_0 \in \partial\Omega$,

$$a(x) \ge (1 - \eta)\beta(x_0)[\operatorname{dist}(x, \partial\Omega)]^{\gamma(x_0)} \qquad \text{if } \operatorname{dist}(x, x_0) \le \delta. \tag{4.24}$$

Fix $x_0 \in \partial\Omega$, set $\Sigma := \bar{B}_{\delta/2}(x_0) \cap \partial\Omega$ and choose $R > 0$ sufficiently small so that

$$\mathscr{K} := \bigcup_{y \in \Sigma} \bar{B}_R(y - R \mathbf{n}_y) \subset B_\delta(x_0) \cap \bar{\Omega}. \tag{4.25}$$

Then, we find from (4.24) that

$$a(x) \ge (1 - \eta)\beta(x_0)[\operatorname{dist}(x, \partial\Omega)]^{\gamma(x_0)} \qquad \forall x \in \bar{B}_\delta(x_0) \cap \Omega. \tag{4.26}$$

Subsequently, for each $x \in \mathscr{K} \cap \Omega$ with $\operatorname{dist}(x, \partial\Omega) \le R$ we denote by y_x the unique point of $B_\delta(x_0) \cap \partial\Omega$ for which

$$\operatorname{dist}(x, \partial\Omega) = |x - y_x| = R - |x - (y_x - R\mathbf{n}_{y_x})|. \tag{4.27}$$

Set

$$\lambda := \max_{\mathscr{K}} W, \qquad \beta_L := \min_{x \in \partial\Omega} \beta(x), \qquad \gamma_M := \max_{x \in \partial\Omega} \gamma(x),$$

and, for each $\varepsilon > 0$, let Ψ_ε be any positive radially symmetric solution of

$$\begin{cases} -\Delta u = \lambda u^q - (1-\eta)\beta_L (R - |x|)^{\gamma_M} f(u) & \text{in } B_R(0) \\ u = \infty & \text{on } \partial B_R(0) \end{cases} \tag{4.28}$$

satisfying

$$\limsup_{|x| \uparrow R} \frac{\Psi_\varepsilon(x)}{M_{\eta, x_0}(R - |x|)^{-\alpha(x_0)}} \le 1 + \varepsilon, \tag{4.29}$$

where

$$\alpha(x) := \frac{\gamma(x) + 2}{p - 1}, \qquad x \in \partial\Omega, \qquad M_{\eta, x_0} := \left[\frac{\alpha(x_0)[\alpha(x_0) + 1]}{K(1-\eta)\beta(x_0)}\right]^{1/(p-1)}.$$

The existence of Ψ_ε is guaranteed by Theorem 4.2. Fix, one of those ε's. Then, arguing as in the first part of the proof, it is apparent that

$$u(x) \le \Psi_\varepsilon(x - (y_x - R\mathbf{n}_{y_x})) \qquad \text{for each } x \in \mathscr{K}. \tag{4.30}$$

Thus, for each $x \in \mathscr{K} \cap \Omega$ with $\operatorname{dist}(x, \partial\Omega) \le R$ we find from (4.27) and (4.30) that

$$\frac{u(x)}{M_{\eta, x_0}[\operatorname{dist}(x, \partial\Omega)]^{-\alpha(x_0)}} \le \frac{\Psi_\varepsilon(z_x)}{M_{\eta, x_0}[R - |z_x|]^{-\alpha(x_0)}}$$

where we have denoted

$$z_x := x - (y_x - R\mathbf{n}_{y_x}),$$

and, hence, we find from (4.29) that

$$\limsup_{\substack{\operatorname{dist}(x, \partial\Omega) \downarrow 0 \\ x \in \mathscr{K}}} \frac{u(x)}{M_{\eta, x_0}[\operatorname{dist}(x, \partial\Omega)]^{-\alpha(x_0)}} \le 1 + \varepsilon.$$

Therefore, as this inequality holds for each $\varepsilon > 0$, it is apparent that

$$\limsup_{\substack{\mathrm{dist}(x,\partial\Omega)\downarrow 0\\ x\in\mathscr{K}}} \frac{u(x)}{M_{\eta,x_0}[\mathrm{dist}(x,\partial\Omega)]^{-\alpha(x_0)}} \le 1. \tag{4.31}$$

Similarly, reducing δ, if it is necessary, one can use the large solutions on the exterior annuli of the first part of the proof to conclude that

$$1 \le \liminf_{\substack{\mathrm{dist}(x,\partial\Omega)\downarrow 0\\ x\in\mathscr{K}}} \frac{u(x)}{M_{\eta,x_0}[\mathrm{dist}(x,\partial\Omega)]^{-\alpha(x_0)}}. \tag{4.32}$$

It should be noted that $\mathscr{K}$ depends on δ and that δ depends on η; in such a way that $\lim_{\eta\downarrow 0}\delta(\eta) = 0$. Thus, $\lim_{\eta\downarrow 0}\mathscr{K} = \{x_0\}$. Moreover, δ, and, hence, $\mathscr{K}$ can be chosen independent of x_0 because (1.6) holds uniformly in x_0. Therefore, it follows from (4.31) and (4.32) that

$$\lim_{x\to x_0} \frac{u(x)}{M_{\eta,x_0}[\mathrm{dist}(x,\partial\Omega)]^{-\alpha(x_0)}} = 1 \tag{4.33}$$

uniformly in $x_0 \in \partial\Omega$, since $\partial\Omega$ is compact.

We now show the uniqueness. Suppose that (1.9) is satisfied uniformly in $\partial\Omega$ for any pair of positive solutions (u,v) of (1.1). Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1-\varepsilon)v \le u \le (1+\varepsilon)v \qquad \text{in } \Omega\backslash\bar{\Omega}_\delta,$$

where, for each small enough $\delta > 0$,

$$\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > \delta\}.$$

Now, consider the problem

$$\begin{cases} -\Delta w = W(x)w^q - a(x)f(w) & \text{in } \Omega_\delta \\ w = u & \text{on } \partial\Omega_\delta \end{cases} \tag{4.34}$$

By (H8), which implies (H3), and Proposition 3.1, (4.34) possesses a unique positive solution, necessarily u. Moreover, thanks to (H8), it is easy to see that the pair $((1-\varepsilon)v,(1+\varepsilon)v)$ provides us with an ordered sub-supersolution pair of (4.34). So, we have

$$(1-\varepsilon)v \le u \le (1+\varepsilon)v \qquad \text{in } \Omega_\delta$$

and, therefore,

$$(1-\varepsilon)v \le u \le (1+\varepsilon)v \qquad \text{in } \Omega.$$

As this is true for any $\varepsilon > 0$, we obtain that $u = v$. This concludes the proof. $\square$

Acknowledgements

We are delighted to thank to the referee for his/her careful reading of the manuscript and for several useful remarks and suggestions improving the presentation of the paper.

References

[1] H. Amann, On the existence of positive solutions of nonlinear elliptic boundary value problems, Indiana Univ. Math. J. **21** (1971), 125–146.

[2] C. Bandle & M. Marcus, Maximal solutions of quasilinear elliptic problems: isoperimetric estimates and asymptotic behavior, C. R. Acad. Sci. Paris **1, 311** (1990), 91–93.

[3] C. Bandle & M. Marcus, Large solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behavior, J. D'Analyse Math. **58** (1991), 9–24.

[4] C. Bandle & M. Marcus, On second order effects in the boundary behavior of large solutions of semilinear elliptic problems, Diff. Int. Eqns. **11** (1998), 23–34.

[5] F. C. Cirstea & V. Radulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorption, C. R. Math. Acad. Sci. Paris **335** (2002), 447–452.

[6] F. C. Cirstea & V. Radulescu, Existence and uniqueness of blow-up solutions for a class of logistic equations, Commun. Contemp. Math. **4** (2002), 559–586.

[7] M. Delgado, J. López-Gómez & A. Suárez, Characterizing the existence of large solutions for a class of sublinear problems with nonlinear diffusion, Advances in Diff. Eqns. **7** (2002), 1235–1254.

[8] Y. Du & Q. Huang, Blow-up solutions for a class of semilinear elliptic and parabolic problems, SIAM J. Math. Anal. **31** (1999), 1–18.

[9] J. García-Melián, R. Letelier-Albornoz & J. C. Sabina de Lis, Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up, Proc. Amer. Math. Soc. **129** (2001), 3593–3602.

[10] V. A. Kondratiev & V. A. Nikishin, Asymptotics, near the boundary, of a solution of a singular boundary value problem for a semilinear elliptic equation, Diff. Eqns. **26** (1990), 345–348.

[11] J. B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Maths. **X** (1957), 503–510.

[12] M. Marcus & L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, Ann. Inst. Henri Poincaré **14** (1997), 237–274.

[13] A. V. Lair, A necessary and sufficient condition for the existence of large solutions to semilinear elliptic equations, J. Math. Anal. Appl. **240** (1999), 205–218.

[14] C. Loewner & L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, Contributions to Analysis, Academic Press, New York 1974, p. 245–272.

[15] J. López-Gómez, Large solutions, metasolutions, and asymptotic behaviour of the regular positive solutions of sublinear parabolic problems, El. J. Diff. Eqns. **Conf. 05** (2000), 135–171.

[16] J. López-Gómez, The boundary blow-up rate of large solutions, J. Diff. Eqns. **195** (2003), 25–45.

[17] R. Osserman, On the inequality $\Delta u \geq f(u)$, Pacific J. Maths. **7** (1957), 1641–1647.

[18] L. Véron, Semilinear elliptic equations with uniform blow up on the boundary, J. D'Analyse Math. **59** (1992), 231–250.

Manuel Delgado[3]
Dpto. Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla
c/ Tarfia s/n, 41012-*Sevilla, Spain*
e-mail: madelgado@us.es

Julián López-Gómez
Dpto. Matemática Aplicada
Universidad Complutense de Madrid
28040-*Madrid, Spain*
e-mail: Lopez_Gomez@mat.ucm.es

Antonio Suárez
Dpto. Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla
c/ Tarfia s/n, 41012-*Sevilla, Spain*
e-mail: suarez@us.es

[3] Corresponding author