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ESTIMATES FOR APPROXIMATION NUMBERS OF SOME CLASSES OF COMPOSITION OPERATORS ON THE HARDY SPACE

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Abstract. We give estimates for the approximation numbers of composition operators on H^2 , in terms of some modulus of continuity. For symbols whose image is contained in a polygon, we get that these approximation numbers are dominated by $e^{-c\sqrt{n}}$. When the symbol is continuous on the closed unit disk and has a domain touching the boundary non-tangentially at a finite number of points, with a good behavior at the boundary around those points, we can improve this upper estimate. A lower estimate is given when this symbol has a good radial behavior at some point. As an application we get that, for the cusp map, the approximation numbers are equivalent, up to constants, to $e^{-cn/\log n}$, very near to the minimal value e^{-cn} . We also see the limitations of our methods. To finish, we improve a result of El-Fallah, Kellay, Shabankhah and Youssfi, in showing that for every compact set K of the unit circle \mathbf{T} with Lebesgue measure 0, there exists a compact composition operator $C_\varphi: H^2 \rightarrow H^2$, which is in all Schatten classes, and such that $\varphi = 1$ on K and $|\varphi| < 1$ outside K .

1. Introduction and notation

If the approximation numbers of some classes of operators on Hilbert spaces are well understood (for example, those of Hankel operators: see [17]), it is not the case of those of composition operators. Though their behavior remains mysterious, some recent results are obtained in [15] and [13] for approximation numbers of composition operators on the Hardy space H^2 . In [15], it is proved that one always has $a_n(C_\varphi) \gtrsim e^{-cn}$ for some $c > 0$ [15, Theorem 3.1] and that this speed of decay can only be attained when the symbol φ maps the unit disk \mathbf{D} into a disk centered at 0 of radius strictly less than 1, i.e. $\|\varphi\|_\infty < 1$ [15, Theorem 3.4].

In this paper, we give estimates which are somewhat general, in terms of some modulus of continuity. In Section 2, we obtain an upper estimate when the symbol φ is continuous on the closed unit disk and has an image touching non-tangentially the unit circle at a finite number of points, with a good behavior on the boundary around

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this point. As an application, we show that for symbols φ whose image is contained in a polygon $a_n(C_\varphi) \leq ae^{-b\sqrt{n}}$, for some constants $a, b > 0$; this has to be compared with [13], Proposition 2.7, where it is shown that if φ is a univalent symbol such that $\varphi(\mathbf{D})$ contains an angular sector centered on the unit circle and with opening $\theta\pi$, $0 < \theta < 1$, then $a_n(C_\varphi) \geq ae^{-b\sqrt{n}}$, for some (other) positive constants a and b , depending only on θ . In Section 3, we obtain a lower bound when φ has a good radial behavior at the contact point. Both proofs use Blaschke products. This allows to recover the estimation $a_n(C_{\lambda_\theta}) \approx e^{-c\sqrt{n}}$ obtained in [15], Proposition 6.3, and [13], Theorem 2.1 for the lens map λ_θ . In Section 4.1, we give another example, the cusp map, for which $a_n(C_\varphi) \approx e^{-cn/\log n}$, very near the minimum value e^{-cn} . We end that section by considering a one-parameter class of symbols, first studied by Shapiro and Taylor [23] and seeing the limitations of our methods. In Section 5, we improve a result of Gallardo-Gutiérrez and González (later on generalized by El-Fallah, Kellay, Shabankhah and Youssfi [5, Theorem 3.1]). It is known that for every compact composition operator $C_\varphi: H^2 \rightarrow H^2$, the set $E_\varphi = \{e^{i\theta}; |\varphi^*(e^{i\theta})| = 1\}$ has Lebesgue measure 0. These authors showed ([6]), with a rather difficult construction, that there exists a compact composition operator $C'_\varphi: H^2 \rightarrow H^2$ such that the Hausdorff dimension of E_φ is equal to 1 (and in [5], it is shown that for any negligible compact set K , there is a Hilbert–Schmidt operator C_φ such that $E_\varphi = K$). We improve this result in showing that for every compact set K of the unit circle \mathbf{T} with Lebesgue measure 0, there exists a compact composition operator $C_\varphi: H^2 \rightarrow H^2$, which is even in all Schatten classes, and such that $E_\varphi = K$.

Notation. We denote by \mathbf{D} the open unit disk and by $\mathbf{T} = \partial\mathbf{D}$ the unit circle; m is the normalized Lebesgue measure on \mathbf{T} : $dm(t) = dt/2\pi$. The disk algebra $A(\mathbf{D})$ is the space of functions which are continuous on the closed unit disk $\overline{\mathbf{D}}$ and analytic in the open unit disk. If H^2 is the usual Hardy space on \mathbf{D} , every analytic self-map $\varphi: \mathbf{D} \rightarrow \mathbf{D}$ (also called *Schur function*) defines, by Littlewood’s subordination principle, a bounded operator $C_\varphi: H^2 \rightarrow H^2$ by $C_\varphi(f) = f \circ \varphi$, called the *composition operator of symbol φ* .

Recall that if $T: E \rightarrow F$ is a bounded operator between two Banach spaces, the *approximation numbers* $a_n(T)$ of T are defined by:

$$a_n(T) = \inf\{\|T - R\|; \text{rank}(R) < n\}, \quad n = 1, 2, \dots$$

The sequence $(a_n(T))_n$ is non-increasing and, when F has the Approximation Property, T is compact if and only if $a_n(T)$ tends to 0.

The *Gelfand numbers* $c_n(T)$ are defined by $c_n(T) = \inf\{\|T|_G\|; \text{codim } G < n\}$. For compact operators T on Hilbert spaces, one has $c_n(T) = a_n(T)$ (see [9]).

Definition 1.1. A *modulus of continuity* ω is a continuous function

$$\omega: [0, A] \rightarrow \mathbf{R}^+,$$

which is increasing, sub-additive, and vanishes at zero.

Some examples are:

$$\omega(h) = h^\alpha, \quad 0 < \alpha \leq 1; \quad \omega(h) = h \log \frac{1}{h}; \quad \omega(h) = \frac{1}{\log \frac{1}{h}}.$$

For any modulus of continuity ω , there is a concave modulus of continuity ω' such that $\omega \leq \omega' \leq 2\omega$ (see [18] for example); therefore we may and shall assume that ω

is concave on $[0, A]$. In that case, ω^{-1} is convex, and

$$(1.1) \quad r_\omega(x) := \frac{\omega^{-1}(x)}{x}$$

is non-decreasing.

The notation $u(t) \lesssim v(t)$ means that $u(t) \leq Av(t)$ for some constant $A > 0$ and $u(t) \approx v(t)$ means that both $u(t) \lesssim v(t)$ and $v(t) \lesssim u(t)$.

2. Upper bound and boundary behavior

Definition 2.1. Let ω be a modulus of continuity and φ a symbol in the disk algebra $A(\mathbf{D})$. Let $\xi_0 \in \partial\mathbf{D} \cap \varphi(\overline{\mathbf{D}})$. We say that the symbol φ has an ω -regular behavior at ξ_0 if, setting

$$(2.1) \quad \gamma(t) = \varphi(e^{it}),$$

and $E_{\xi_0} = \{t; \gamma(t) = \xi_0\}$, there exists $r_0 > 0$ such that

1) for some positive constant $C > 0$, one has, for every $t_0 \in E_{\xi_0}$ and $|t - t_0| \leq r_0$:

$$(2.2) \quad |\gamma(t) - \gamma(t_0)| \leq C(1 - |\gamma(t)|).$$

2) for some positive constant $c > 0$, one has, for for every $t_0 \in E_{\xi_0}$ and $|t - t_0| \leq r_0$:

$$(2.3) \quad c\omega(|t - t_0|) \leq |\gamma(t) - \gamma(t_0)|.$$

The first condition implies that the image of φ touches $\partial\mathbf{D}$ at the point ξ_0 , and non-tangentially. The second one implies that φ does not stay long near $\xi_0 = \gamma(t_0)$.

Note that, due to (2.3), the intervals $[t - r_0/2, t + r_0/2]$, for $t \in E_{\xi_0}$ are pairwise disjoint and therefore the set E_{ξ_0} must be finite.

We shall make the following assumption (to avoid the Lipschitz class):

$$(2.4) \quad \lim_{h \rightarrow 0^+} \frac{\omega(h)}{h} = \infty; \quad \text{equivalently} \quad \lim_{h \rightarrow 0^+} \frac{\omega^{-1}(h)}{h} = 0.$$

Indeed, assume that γ is K -Lipschitz at some point $t_0 \in [0, 2\pi]$, namely $|\varphi(e^{it}) - \varphi(e^{it_0})| \leq K|t - t_0|$, with $|\varphi(e^{it_0})| = 1$; then

$$m(\{t \in [0, 2\pi]; |\varphi(e^{it}) - \varphi(e^{it_0})| \leq h\}) \geq m(\{t \in [0, 2\pi]; |t - t_0| \leq h/K\}) = h/2\pi K;$$

hence this measure is not $o(h)$ and the composition operator C_φ is not compact ([16], or [3, Theorem 3.12]).

In order to treat the case where the image of φ is a polygon, we need to generalize the above definition. We ask not only that φ is ω -regular at the points ξ_1, \dots, ξ_p of contact of $\varphi(\overline{\mathbf{D}})$ with $\partial\mathbf{D}$, but a little bit more.

Definition 2.2. Assume that $\varphi(\overline{\mathbf{D}}) \cap \partial\mathbf{D} = \{\xi_1, \dots, \xi_p\}$. We say that φ is globally-regular if there exists a modulus of continuity ω such that, writing $E_{\xi_j} = \{t; \gamma(t) = \xi_j\}$, one has, for some $r_1, \dots, r_p > 0$

$$\mathbf{T} = \bigcup_{j=1}^p (E_{\xi_j} + [-r_j, r_j])$$

and for some positive constants $C, c > 0$,

1') one has, for $j = 1, \dots, p$, every $t_j \in E_{\xi_j}$ and $|t - t_j| \leq r_j$:

$$(2.5) \quad |\gamma(t) - \gamma(t_j)| \leq C(1 - |\gamma(t)|).$$

2') one has, for $j = 1, \dots, p$, every $t_j \in E_{\xi_j}$ and $|t - t_j| \leq r_j$:

$$(2.6) \quad c\omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)|.$$

Let us note that condition 1') is equivalent to say that $\varphi(\overline{\mathbf{D}})$ is contained in a polygon inside $\overline{\mathbf{D}}$ whose vertices contain ξ_1, \dots, ξ_p , and these are the only vertices in the boundary $\partial\mathbf{D}$. Of course, we may assume that (2.5) and (2.6) hold only when t is in a neighborhood of t_j , since they will then hold for $|t - t_j| \leq r_j$, provided we change the constants C, c .

Before stating our theorem, let us introduce a notation. If φ is as in Definition 2.2 and $\sigma, \kappa > 0$ are some constants, we set

$$(2.7) \quad d_N = \left\lceil \sigma \log \frac{\kappa 2^{-N}}{\omega^{-1}(\kappa 2^{-N})} \right\rceil + 1,$$

where $\lceil \cdot \rceil$ stands for the integer part. For every integer $q \geq 1$, we denote by

$$(2.8) \quad N = N_q \quad \text{the largest integer such that } pNd_N < q$$

($N_q = 1$ if no such N exists).

We then have the following result.

Theorem 2.3. *Let φ be a symbol in $A(\mathbf{D})$ whose image touches $\partial\mathbf{D}$ at the points ξ_1, \dots, ξ_p , and nowhere else. Assume that φ is globally-regular. Then, there are constants $\kappa, K, L > 0$, depending only on φ , such that, using the notation (2.7) and (2.8), one has, for every $q \geq 1$:*

$$(2.9) \quad a_q(C_\varphi) \leq K \sqrt{\frac{\omega^{-1}(\kappa 2^{-N_q})}{\kappa 2^{-N_q}}}.$$

Before proving this theorem, let us indicate two applications. In these examples, we can give an upper estimate for all approximation numbers $a_n(C_\varphi)$, $n \geq 1$, because we can interpolate between the integers Nd_N and $(N + 1)d_{N+1}$, which is not the case in general.

1) $\omega(h) = h^\theta$, $0 < \theta < 1$. This is the case for inscribed polygons (see the proof of the foregoing Theorem 2.4; here $\theta = \max\{\theta_1, \dots, \theta_p\}$, where $\theta_1\pi, \dots, \theta_p\pi$ are the values of the angles of the polygon). This is also the case, with $p = 2$, of lens maps λ_θ (see [22], page 27, for the definition; see also [13]). We have here $\omega^{-1}(h) = h^{1/\theta}$. Hence $d_N \approx N$, $N_q \approx \sqrt{q}$, and we then get from (2.9) that $a_q(C_\varphi) \leq \alpha 2^{-\delta N}$ for $q \gtrsim N^2$, with $\delta > 0$. Equivalently, for suitable constants $\alpha, \beta > 0$,

$$(2.10) \quad a_n(C_\varphi) \leq \alpha e^{-\beta\sqrt{n}},$$

which is the result obtained in [13, Theorem 2.1].

2) $\omega(h) = \frac{1}{(\log 1/h)^\alpha}$, $0 < \alpha \leq 1$, as this is the case, when $\alpha = 1$, for the cusp map, defined below in Section 4.1 (with $p = 1$). Then, we have $\omega^{-1}(h) = e^{-h^{-1/\alpha}}$ and $d_N \approx 2^{N/\alpha}$, so that $N_q \approx \log q$ and $2^{N_q/\alpha} \approx q/\log q$. Now, a simple computation gives

$$(2.11) \quad a_n(C_\varphi) \leq \alpha e^{-\beta n/\log n}.$$

Without assuming some regularity, one has the following general upper estimate.

Theorem 2.4. *Let $\varphi: \mathbf{D} \rightarrow \mathbf{D}$ be an analytic self-map whose image is contained in a polygon \mathbf{P} with vertices on the unit circle. Then, there exist constants $\alpha, \beta > 0$, β depending only on \mathbf{P} , such that*

$$(2.12) \quad a_n(C_\varphi) \leq \alpha e^{-\beta\sqrt{n}}.$$

In [13, Proposition 2.7], it is shown that if φ is a univalent symbol such that $\varphi(\mathbf{D})$ contains an angular sector centered on the unit circle and with opening $\theta\pi$, $0 < \theta < 1$, then $a_n(C_\varphi) \geq \alpha e^{-\beta\sqrt{n}}$, for some (other) positive constants α and β , depending only on θ . Note that the injectivity of the symbol is there necessary, since there exists (see the proof of Corollary 5.4 in [15]), for every sequence (ε_n) of positive numbers tending to 0, a symbol φ whose image is $\mathbf{D} \setminus \{0\}$, and hence contains polygons, which is 2-valent, and for which $a_n(C_\varphi) \lesssim e^{-\varepsilon_n n}$. This bound may be much smaller than $e^{-\beta\sqrt{n}}$.

Proof of Theorem 2.3. Recall ([13], Lemma 2.4) that for every Blaschke product B with less than N zeros (each of them being counted with its multiplicity), one has

$$(2.13) \quad [a_N(C_\varphi)]^2 \lesssim \sup_{0 < h < 1, |\xi|=1} \frac{1}{h} \int_{S(\xi, h)} |B(z)|^2 dm_\varphi(z),$$

where $S(\xi, h) = \{z \in \overline{\mathbf{D}}; |z - \xi| \leq h\}$ and m_φ is the pull-back measure by φ of the normalized Lebesgue measure m on \mathbf{T} .

The proof will come from an adequate choice of a Blaschke product. Fix a positive integer N . Set, for $j = 1, \dots, p$ and $k = 1, 2, \dots$,

$$(2.14) \quad p_{j,k} = (1 - 2^{-k})\xi_j$$

and consider the Blaschke product of length pNd (d being a positive integer, to be specified later) given by

$$(2.15) \quad B(z) = \prod_{j=1}^p \prod_{k=1}^N \left[\frac{z - p_{j,k}}{1 - \overline{p_{j,k}}z} \right]^d.$$

Recall that we have set

$$(2.16) \quad \gamma(t) = \varphi(e^{it}).$$

To use (2.13), note that if $|\gamma(t) - \xi| \leq h$, then, for some $j = 1, \dots, p$ and some $t_j \in E_{\xi_j}$, one has $|t - t_j| \leq r_j$ and, by (2.5), $|\gamma(t) - \xi_j| \leq C(1 - |\gamma(t)|) \leq C|\gamma(t) - \xi| \leq Ch$. Therefore, denoting by L_j the number of elements of E_{ξ_j} (which is finite by the remark following Definition 2.1),

$$[a_N(C_\varphi)]^2 \lesssim \sup_{0 < h < 1} \frac{1}{h} \sum_{j=1}^p L_j \int_{\{|\gamma(t) - \xi_j| \leq Ch\} \cap \{|t - t_j| \leq r_j\}} |B[\gamma(t)]|^2 \frac{dt}{2\pi},$$

and we only need to majorize the integrals

$$I_j(h) = \int_{\{|\gamma(t) - \xi_j| \leq Ch\} \cap \{|t - t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi}.$$

Moreover, it suffices, by interpolation, to do that with $h = h_n$, where $h_n = 2^{-n}$.

By (2.6), for $|t - t_j| \leq r_j$ and $|\gamma(t) - \xi_j| \leq Ch_n$, one has $c\omega(|t - t_j|) \leq |\gamma(t) - \xi_j| \leq Ch_n = C2^{-n}$, which implies that

$$(2.17) \quad |t - t_j| \leq \omega^{-1}(c^{-1}C2^{-n}).$$

Let

$$(2.18) \quad s_n = \omega^{-1}(c^{-1}C 2^{-n}).$$

One has

$$I_j(h_n) \leq \int_{\{|t-t_j| \leq s_n\} \cap \{|t-t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi}.$$

For $n \geq N$, we simply majorize $|B(\gamma(t))|$ by 1 and we get

$$\frac{1}{h_n} I_j(h_n) \leq \frac{1}{h_n} \frac{2s_n}{2\pi} = \frac{c^{-1}C}{\pi} \frac{1}{c^{-1}C 2^{-n}} \omega^{-1}(c^{-1}C 2^{-n}) \leq \frac{c^{-1}C}{\pi} \frac{\omega^{-1}(c^{-1}C 2^{-N})}{c^{-1}C 2^{-N}},$$

since the function $\omega^{-1}(x)/x$ is non-decreasing.

When $n \leq N - 1$, we write

$$I_j(h_n) \leq \int_{\{|t-t_j| \leq s_N\} \cap \{|t-t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi} + \int_{\{s_N < |t-t_j| \leq s_n\} \cap \{|t-t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi}.$$

The first integral is estimated as above. For the second one, we claim that

Claim 2.5. For some constant $\chi < 1$, one has, for $j = 1, \dots, p$ and every $t_j \in E_{\xi_j}$:

$$(2.19) \quad |B(\gamma(t))| \leq \chi^d \quad \text{when } |t - t_j| > s_N \text{ and } |t - t_j| \leq r_j.$$

To see that, we shall use [13], Lemma 2.3. Let us recall that this lemma asserts that for $w, w_0 \in \mathbf{D}$ satisfying $|w - w_0| \leq M \min(1 - |w|, 1 - |w_0|)$ for some positive constant M , one has:

$$(2.20) \quad \left| \frac{w - w_0}{1 - \overline{w_0}w} \right| \leq \frac{M}{\sqrt{M^2 + 1}}.$$

Let t such that $|t - t_j| \leq r_j$ and $|t - t_j| > s_N$. We have, on the one hand, $\omega(|t - t_j|) \geq \omega(s_N) = c^{-1}C 2^{-N}$, and, on the other hand, since $|\gamma(t_j)| = |\xi_j| = 1$

$$c\omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)| \leq C(1 - |\gamma(t)|);$$

hence $1 - |\gamma(t)| \geq 2^{-N}$.

Let $1 \leq k \leq N$ such that $2^{-k} \leq 1 - |\gamma(t)| < 2^{-k+1}$. Since $|p_{j,k}| = 1 - 2^{-k}$, we have

$$|\gamma(t) - p_{j,k}| \leq |\gamma(t) - \xi_j| + |\xi_j - p_{j,k}| \leq C(1 - |\gamma(t)|) + 2^{-k} \leq (2C + 1)2^{-k}.$$

Hence

$$|\gamma(t) - p_{j,k}| \leq M \min(1 - |\gamma(t)|, 1 - |p_{j,k}|),$$

with $M = 2C + 1$. By (2.20), we get $\left| \frac{\gamma(t) - p_{j,k}}{1 - \overline{p_{j,k}}\gamma(t)} \right| \leq \chi$, where $\chi = M/\sqrt{M^2 + 1}$ is < 1 , and therefore $|B[\gamma(t)]| \leq \chi^d$. □

We can now end the proof of Theorem 2.3. We get

$$\begin{aligned} \frac{1}{h_n} \int_{\{s_N < |t-t_j| \leq s_n\} \cap \{|t-t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi} &\leq \frac{1}{h_n} \frac{2s_n}{2\pi} \chi^{2d} = \frac{1}{h_n} \frac{\omega^{-1}(c^{-1}C 2^{-n})}{\pi} \chi^{2d} \\ &= \frac{c^{-1}C}{\pi} \frac{\omega^{-1}(c^{-1}C 2^{-n})}{c^{-1}C 2^{-n}} \chi^{2d} \\ &\leq \frac{1}{\pi} \omega^{-1}(c^{-1}C) \chi^{2d}, \end{aligned}$$

since $\omega^{-1}(x)/x$ is non-decreasing.

We therefore get, setting $\kappa = c^{-1}C$ and $L = L_1 + \dots + L_p$,

$$\frac{1}{h_n} \sum_{j=1}^p L_j \int_{\{|\gamma(t) - \xi_j| \leq Ch_n\} \cap \{|t - t_j| \leq r_j\}} |B[\gamma(t)]|^2 \frac{dt}{2\pi} \leq \frac{\kappa L}{\pi} \frac{\omega^{-1}(\kappa 2^{-N})}{\kappa 2^{-N}} + \frac{L \omega^{-1}(\kappa)}{\pi} \chi^{2d}.$$

Choose now $d = d_N$, where d_N is defined by (2.7), with $\sigma = 1/\log(\chi^{-2})$. Then $\chi^{2d} \leq \omega^{-1}(\kappa 2^{-N})/(\kappa 2^{-N})$, and, since the Blaschke product B has now pNd_N zeroes, we get, for some positive constant K

$$a_{pNd_N+1}(C_\varphi) \leq K \sqrt{\frac{\omega^{-1}(\kappa 2^{-N})}{\kappa 2^{-N}}},$$

and that ends the proof of Theorem 2.3. □

Proof of Theorem 2.4. It suffices to consider the case when φ is a conformal map from \mathbf{D} onto \mathbf{P} . Indeed, let ψ be such a conformal map. In the general case, our assumption allows to write $\varphi = \psi \circ u$, where $u = \psi^{-1} \circ \varphi: \mathbf{D} \rightarrow \mathbf{D}$ is analytic. It follows that $C_\varphi = C_u \circ C_\psi$ and that $a_n(C_\varphi) \leq \|C_u\| a_n(C_\psi)$. Therefore, we may and shall assume that φ itself is this conformal map.

Let us denote by ξ_1, \dots, ξ_p the vertices of \mathbf{P} . Let $0 < \pi\mu_j < \pi$ be the exterior angle of \mathbf{P} at ξ_j , namely the complement to π of the interior angle; so that

$$\sum_{j=1}^p \mu_j = 2, \quad \text{and} \quad 0 < \mu_j < 1.$$

If one sets $\theta_j = 1 - \mu_j$, one has $0 < \theta_j < 1$.

We then use the explicit form of φ given by the Schwarz–Christoffel formula [19, page 193]:

$$(2.21) \quad \varphi(z) = A \int_0^z \frac{dw}{(a_1 - w)^{\mu_1} \dots (a_p - w)^{\mu_p}} + B,$$

for some constants $A \neq 0$ and $B \in \mathbf{C}$ and where $a_1, \dots, a_p \in \partial\mathbf{D}$ are such that $\xi_j = \varphi(a_j)$, $j = 1, \dots, p$. If, as before, we write $\gamma(t) = \varphi(e^{it})$, we have $\xi_j = \gamma(t_j)$, with $a_j = e^{it_j}$ (note that here $E_{\xi_j} = \{t_j\}$).

As we already said, condition (2.5) is trivially satisfied for a polygon.

To end the proof, we use Theorem 2.3 and its Example 1. For that it suffices to show that, for $|t - t_j|$ small enough, we have

$$(2.22) \quad |\gamma(t) - \xi_j| \approx |t - t_j|^{\theta_j}.$$

If $z \in \mathbf{D}$ is close to a_j , it follows from (2.21) that we can write

$$\varphi(z) = A \int_0^z f_j(w) \frac{dw}{(a_j - w)^{\mu_j}} + B,$$

where f_j is holomorphic near a_j and $f_j(a_j) \neq 0$ since

$$|f_j(a_j)| = \prod_{k \neq j, 1 \leq k \leq p} |a_j - a_k|^{-\mu_k}.$$

Write $f_j(w) = f_j(a_j) + (a_j - w)g_j(w)$ where g_j is holomorphic near a_j . We get

$$\begin{aligned} \varphi(z) &= Af_j(a_j) \int_0^z \frac{dw}{(a_j - w)^{\mu_j}} + B + \int_0^z g_j(w)(a_j - w)^{\theta_j} dw \\ &:= Af_j(a_j) \int_0^z \frac{dw}{(a_j - w)^{\mu_j}} + B + \psi_j(z), \end{aligned}$$

which can still be written (since $\theta_j > 0$)

$$(2.23) \quad \varphi(z) = \lambda_j(a_j - z)^{\theta_j} + c_j + \psi_j(z),$$

where $\lambda_j \neq 0$, $c_j \in \mathbf{C}$, ψ_j is Lipschitz near a_j and $\xi_j = \varphi(a_j) = c_j + \psi_j(a_j)$. Now, we easily get (2.22). Indeed, for t near t_j , it follows from (2.23) that (recall that $\gamma(t) = \varphi(e^{it})$ and $\gamma(t_j) = \xi_j$)

$$|\gamma(t) - \gamma(t_j)| = |\lambda_j| |e^{it} - e^{it_j}|^{\theta_j} + O(|t - t_j|),$$

which the claimed estimate (2.22) since $\lambda_j \neq 0$ and $|t - t_j|$ is negligible compared to $|t - t_j|^{\theta_j} \approx |e^{it} - e^{it_j}|^{\theta_j}$. □

3. Lower bound and radial behavior

We shall consider symbols φ taking real values in the real axis (i.e. its Taylor series has real coefficients) and such that $\lim_{r \rightarrow 1^-} \varphi(r) = 1$, with a given speed.

Definition 3.1. We say that an analytic map $\varphi: \mathbf{D} \rightarrow \mathbf{D}$ is *real* if it takes real values on $] - 1, 1[$, and that φ is an ω -*radial symbol* if it is real and there is a modulus of continuity $\omega: [0, 1] \rightarrow [0, 2]$ such that

$$(3.1) \quad 1 - \varphi(r) \leq \omega(1 - r), \quad 0 \leq r < 1.$$

With those definitions and notations, one has:

Theorem 3.2. *Let φ be a real and ω -radial symbol. Then, for the approximation numbers $a_n(C_\varphi)$ of the composition operator C_φ of symbol φ , one has the following lower bound:*

$$(3.2) \quad a_n(C_\varphi) \geq c \sup_{0 < \sigma < 1} \sqrt{\frac{\omega^{-1}(a \sigma^n)}{a \sigma^n}} \exp \left[- \frac{20}{1 - \sigma} \right],$$

where $a = 1 - \varphi(0) > 0$ and c is another constant depending only on φ .

Observe that, for the lens map λ_θ (see [13, Lemma 2.5]), we have $\omega^{-1}(h) \approx h^{1/\theta}$, so that adjusting $\sigma = 1 - 1/\sqrt{n}$, we get

$$(3.3) \quad a_n(C_{\lambda_\theta}) \geq c \exp(-C\sqrt{n}),$$

which is the result of [15, Proposition 6.3].

For the cusp map φ (see Section 4.1), we have $\omega^{-1}(h) \approx e^{-C'/h}$, so that taking $\sigma = \exp(-\log n/2n)$, we get

$$(3.4) \quad a_n(C_\varphi) \geq c \exp(-C n / \log n).$$

We shall use the same methods as for lens maps (see [15, Proposition 6.3]).

We need a lemma. Recall (see [8, pages 194–195] or [20, pages 302–303]) that if (z_j) is a Blaschke sequence, its Carleson constant δ is defined as $\delta = \inf_{j \geq 1} (1 - |z_j|^2) |B'(z_j)|$, where B is the Blaschke product whose zeros are the z_j 's. Now (see [7, Chapter VII, Theorem 1.1]), every H^∞ -interpolation sequence (z_j) is a Blaschke

sequence and its Carleson constant δ is connected to its interpolation constant C by the inequalities

$$(3.5) \quad 1/\delta \leq C \leq \kappa/\delta^2$$

where κ is an absolute constant (actually $C \leq \kappa_1(1/\delta)(1 + \log 1/\delta)$). Now, if (z_j) is a H^∞ -interpolation sequence with constant C , the sequence of the normalized reproducing kernels $f_j = K_{z_j}/\|K_{z_j}\|$ satisfies

$$(3.6) \quad C^{-1} \left(\sum |\lambda_j|^2 \right)^{1/2} \leq \left\| \sum \lambda_j f_j \right\|_{H^2} \leq C \left(\sum |\lambda_j|^2 \right)^{1/2}$$

(see [15, Lemma 2.2]).

Lemma 3.3. *Let $\varphi: \mathbf{D} \rightarrow \mathbf{D}$ be an analytic self-map. Let $u = (u_1, \dots, u_n)$ be a finite sequence in \mathbf{D} and set $v_j = \varphi(u_j)$, $v = (v_1, \dots, v_n)$. Denote by δ_v the Carleson constant of the finite sequence v and set*

$$\mu_n^2 = \inf_{1 \leq j \leq n} \frac{1 - |u_j|^2}{1 - |\varphi(u_j)|^2}.$$

Then, for some constant $c' > 0$, we have the lower bound

$$(3.7) \quad a_n(C_\varphi) \geq c' \delta_v^4 \mu_n.$$

Proof. Recall first that the Carleson constant δ of a Blaschke sequence (z_j) is also equal to

$$\delta = \inf_{k \geq 1} \prod_{j \neq k} \rho(z_k, z_j),$$

where $\rho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{z}\zeta} \right|$ is the pseudo-hyperbolic distance between z and ζ . Now, the Schwarz–Pick Lemma (see [1, Theorem 3.2]) asserts that every analytic self-map of \mathbf{D} contracts the pseudo-hyperbolic distance. Hence $\rho(\varphi(u_j), \varphi(u_k)) \leq \rho(u_j, u_k)$ and so, if δ_u and δ_v denote the Carleson constants of u and v :

$$\delta_u \geq \delta_v.$$

Let now R be an operator of rank $< n$. There exists a function $f = \sum_{j=1}^n \lambda_j K_{u_j} \in H^2 \cap \ker R$ with $\|f\| = 1$. We thus have

$$\begin{aligned} \|C_\varphi^* - R\|^2 &\geq \|C_\varphi^*(f) - R(f)\|_2^2 = \|C_\varphi^*(f)\|_2^2 = \left\| \sum_{j=1}^n \lambda_j K_{v_j} \right\|_2^2 \\ &\geq C_v^{-2} \sum_{j=1}^n |\lambda_j|^2 \|K_{v_j}\|_2^2 = C_v^{-2} \sum_{j=1}^n \frac{|\lambda_j|^2}{1 - |v_j|^2} \geq C_v^{-2} \mu_n^2 \sum_{j=1}^n \frac{|\lambda_j|^2}{1 - |u_j|^2} \\ &\geq C_u^{-2} C_v^{-2} \mu_n^2 \|f\|_2^2 = C_u^{-2} C_v^{-2} \mu_n^2 \geq \kappa^{-4} \delta_u^4 \delta_v^4 \mu_n^2 \geq \kappa^{-4} \delta_v^8 \mu_n^2, \end{aligned}$$

and hence $a_n(C_\varphi) \geq \kappa^{-2} \delta_v^4 \mu_n$. □

Remark. This lemma allows to give, in the Hardy case, a simpler proof of Theorem 4.1 in [15], avoiding the use of Lemma 2.3 and Lemma 2.4 (concerning the backward shift) in that paper. Recall that this theorem says that for every non-increasing sequence $(\varepsilon_n)_{n \geq 1}$ of positive real numbers tending to 0, there exists a univalent symbol φ such that $\varphi(0) = 0$ and $C_\varphi: H^2 \rightarrow H^2$ is compact, but $a_n(C_\varphi) \gtrsim \varepsilon_n$ for every $n \geq 1$. Let us sketch briefly the argument.

First of all, we may assume that $\varepsilon_n \leq 1/2$, for all n , and that the sequence $(\varepsilon_n)_n$ is log-convex and decreasing to 0 (see [15, Lemma 2.6]). The symbol φ is defined as $\varphi(z) = \sigma^{-1}(e^{-1}\sigma(z))$, where σ is the Riemann map $\sigma: \mathbf{D} \rightarrow \Omega$, with $\sigma(0) = 0$ and $\sigma'(0) > 0$ from \mathbf{D} onto some domain Ω . This domain Ω is defined as follows. Let A be the piecewise linear function on the intervals (e^{n-1}, e^n) with $A(e^{n-1}) = (1/C_0) \log(1/\varepsilon_n)$ (C_0 is a suitable numerical positive constant). Define $\psi: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ to be linear on $[0, 1]$, with $\psi(0) = (1/2C_0) \log(1/\varepsilon_1)$, and $\psi(t) = t/A(t)$ for $t \geq 1$. Then $\Omega = \{w \in \mathbf{C}; |\Im w| < \psi(|\Re w|)\}$.

If $A_j = (1/C_0) \log(1/\varepsilon_{j+1})$, then the numbers $r_j = \sigma^{-1}(e^j)$ satisfy $\varphi(r_{j+1}) = r_j$ and (see [15, pages 444–447]):

$$\frac{1 - r_{j+1}}{1 - r_j} \geq \exp(-2C_0 A_j).$$

We apply the above Lemma 3.3 with $u_j = r_j$. Then $v_j = \varphi(u_j) = r_{j-1}$. Hence

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \frac{1}{2} \frac{1 - |u_j|}{1 - |v_j|} = \frac{1}{2} \frac{1 - r_j}{1 - r_{j-1}} \geq \frac{1}{2} \exp(-2C_0 A_{j-1}) = \frac{1}{2} \varepsilon_j^2 \geq \frac{1}{2} \varepsilon_n^2.$$

It follows that $\mu_n \geq \varepsilon_n/\sqrt{2}$.

On the other hand, $(r_j)_{j \geq 1}$ is an interpolating sequence (see [15, Lemma 4.6]); hence there is a constant $\delta > 0$ (which does not depend on $n \geq 1$) such that $\delta_v \geq \delta$. Therefore Lemma 3.3 gives

$$a_n(C_\varphi) \geq c \delta^4 \varepsilon_n,$$

which gives Theorem 4.1 of [15]. □

Proof of Theorem 3.2. Fix $0 < \sigma < 1$ and define inductively $u_j \in [0, 1)$ by $u_0 = 0$ and the relation

$$1 - \varphi(u_{j+1}) = \sigma[1 - \varphi(u_j)] \quad \text{with } 1 > u_{j+1} > u_j$$

(using the intermediate value theorem).

Setting $v_j = \varphi(u_j)$, we have $-1 < v_j < 1$,

$$(3.8) \quad \frac{1 - v_{j+1}}{1 - v_j} = \sigma,$$

and

$$(3.9) \quad 1 - v_n = a \sigma^n, \quad \text{with } a = 1 - \varphi(0).$$

Now observe that, for $1 \leq j \leq n$, one has, due to the positivity of u_j , to (3.1), and the fact that $r_\omega(x) = \omega^{-1}(x)/x$ is increasing,

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \frac{1 - u_j}{2(1 - v_j)} \geq \frac{1}{2} \frac{\omega^{-1}(1 - v_j)}{1 - v_j} = \frac{1}{2} r_\omega(1 - v_j) \geq \frac{1}{2} r_\omega(1 - v_n) = \frac{1}{2} r_\omega(a \sigma^n),$$

which proves that $\mu_n^2 \geq r_\omega(a \sigma^n)/2$. Furthermore, the sequence (v_j) satisfies, by (3.8), a condition very similar to Newman’s condition with parameter σ . In fact, for $k > j$, we have

$$\frac{|v_k - v_j|}{|1 - v_k v_j|} = \frac{(1 - v_j) - (1 - v_k)}{(1 - v_j) + v_j(1 - v_k)} \geq \frac{(1 - v_j) - (1 - v_k)}{(1 - v_j) + (1 - v_k)} = \frac{1 - \sigma^{k-j}}{1 + \sigma^{k-j}}.$$

Analogously, for $j > k$, we have $\frac{|v_k - v_j|}{|1 - v_k v_j|} \geq \frac{1 - \sigma^{j-k}}{1 + \sigma^{j-k}}$. Thus, as in the proof of [4, Theorem 9.2], we have, for every k ,

$$\prod_{j \neq k} \rho(v_j, v_k) = \prod_{j \neq k} \frac{|v_k - v_j|}{|1 - v_k v_j|} \geq \prod_{l=1}^{\infty} \left(\frac{1 - \sigma^l}{1 + \sigma^l} \right)^2.$$

Consequently, $\delta_v \geq \prod_{l=1}^{\infty} \left(\frac{1 - \sigma^l}{1 + \sigma^l} \right)^2 \geq \exp \left(- \frac{5}{1 - \sigma} \right)$, by [15, Lemma 6.4]. Finally, use (3.7) to get

$$a_n(C_\varphi) \geq c' \delta_v^4 \mu_n \geq c \exp \left(- \frac{20}{1 - \sigma} \right) \sqrt{r_\omega(a \sigma^n)}.$$

Taking the supremum over σ , that ends the proof of Theorem 3.2. □

Remark. The proof shows that

$$(3.10) \quad a_n(C_\varphi) \geq \sup_{u_1, \dots, u_n \in (0,1)} \inf_{\substack{f \in \langle K_{u_1}, \dots, K_{u_n} \rangle \\ \|f\|=1}} \|C_\varphi^* f\|,$$

where $\langle K_{u_1}, \dots, K_{u_n} \rangle$ is the linear space generated by n distinct reproducing kernels K_{u_1}, \dots, K_{u_n} . But if B is the Blaschke product with zeros u_1, \dots, u_n , then $\langle K_{u_1}, \dots, K_{u_n} \rangle = (BH^2)^\perp$, the *model space* associated to B . Hence

$$(3.11) \quad a_n(C_\varphi) \geq \sup_B \inf_{\substack{f \in (BH^2)^\perp \\ \|f\|=1}} \|C_\varphi^* f\|,$$

where the supremum is taken over all Blaschke products with n zeros on the real axis $(0, 1)$. This has to be compared with the upper bound (which gives (2.13), see [13, proof of Lemma 2.4])

$$(3.12) \quad a_n(C_\varphi) \leq \inf_B \|C_\varphi|_{BH^2}\| = \inf_B \sup_{\substack{f \in BH^2 \\ \|f\|=1}} \|C_\varphi f\|,$$

where the infimum is over the Blaschke products with less than n zeros (in the Hilbert space H^2 , the approximation number $a_n(C_\varphi)$ is equal to the Gelfand number $c_n(C_\varphi)$, which is, by definition, less or equal to $\|C_\varphi|_{BH^2}\|$, since BH^2 is of codimension $< n$).

4. Examples

4.1. The cusp map.

Definition 4.1. The *cusp map* is the conformal mapping φ sending the unit disk \mathbf{D} onto the domain represented on Figure 1.

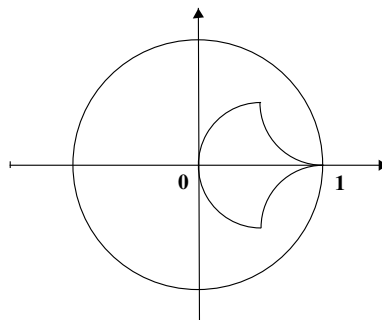


Figure 1. Cusp map domain.

This map was first introduced in [12] (see also [14]). Explicitly, φ is defined as follows.

We first map \mathbf{D} onto the half-disk $\mathbf{D}^+ = \{z \in \mathbf{D}; \Re z > 0\}$. To do that, map \mathbf{D} onto itself by $z \mapsto iz$; then map \mathbf{D} onto the upper half-plane $\mathbf{H} = \{z \in \mathbf{C}; \Im z > 0\}$ by

$$T(u) = i \frac{1 + u}{1 - u}.$$

Take the square root to map \mathbf{H} in the first quadrant $Q_1 = \{z \in \mathbf{H}; \Re z > 0\}$, and go back to the half-disk $\{z \in \mathbf{D}; \Im z < 0\}$ by T^{-1} : $T^{-1}(s) = \frac{1+is}{is-1}$; finally, make a rotation by i to go onto \mathbf{D}^+ . We get

$$(4.1) \quad \varphi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1}.$$

One has $\varphi_0(1) = 0$, $\varphi_0(-1) = 1$, $\varphi_0(i) = -i$ and $\varphi_0(-i) = i$. The half-circle $\{z \in \mathbf{T}; \Re z \geq 0\}$ is mapped onto the segment $[-i, i]$ and the segment $[-1, 1]$ onto the segment $[0, 1]$.

Set now, successively,

$$(4.2) \quad \varphi_1(z) = \log \varphi_0(z), \quad \varphi_2(z) = -\frac{2}{\pi} \varphi_1(z) + 1, \quad \varphi_3(z) = \frac{1}{\varphi_2(z)},$$

and finally

$$(4.3) \quad \varphi(z) = 1 - \varphi_3(z).$$

Hence

$$(4.4) \quad 1 - \varphi(z) = \frac{1}{1 + \frac{2}{\pi} \log(1/|\varphi_0(z)|) - i\frac{2}{\pi} \arg \varphi_0(z)}.$$

φ_2 maps \mathbf{D} onto the semiband $\{z \in \mathbf{C}; \Re z > 1 \text{ and } |\Im z| < 1\}$. One has $\varphi(1) = 1$, $\varphi(-1) = 0$, $\varphi(i) = (1+i)/2$ and $\varphi(-i) = (1-i)/2$.

The domain $\varphi(\mathbf{D})$ is edged by three circular arcs of radii $1/2$ and of respective centers $1/2$, $1+i/2$ and $1-i/2$. The real interval $] -1, 1[$ is mapped onto the real interval $]0, 1[$ and the half-circle $\{e^{i\theta}; |\theta| \leq \pi/2\}$ is sent onto the two circular arcs tangent at 1 to the real axis.

Lemma 4.2.

1) For $0 < r < 1$, let $\gamma = \frac{\pi}{4} - \arctan r = \arctan[(1-r)/(1+r)]$; then

$$(4.5) \quad \varphi_0(r) = \tan(\gamma/2).$$

Hence, when r tends to 1_- , one has

$$(4.6) \quad 1 - \varphi(r) \sim \frac{\pi}{2} \frac{1}{\log(1/\gamma)} \sim \frac{\pi}{2} \frac{1}{\log(1/(1-r))}.$$

2) For $|\theta| < \pi/2$, one has

$$(4.7) \quad \varphi_0(e^{i\theta}) = -i \frac{\tan(\theta/2)}{1 + \sqrt{1 - \tan^2(\theta/2)}}.$$

Hence, when θ tends to 0, one has

$$(4.8) \quad 1 - \varphi(e^{i\theta}) \sim \frac{\pi}{2} \frac{1}{\log(1/|\theta|)}.$$

Proof. 1) One has

$$T(ir) = \frac{r - i}{ir - 1} = -\frac{2r}{1 + r^2} + i \frac{1 - r^2}{1 + r^2} = -\sin \alpha + i \cos \alpha,$$

with $r = \tan(\alpha/2)$; hence $T(ir) = \cos(\alpha + \pi/2) + i \sin(\alpha + \pi/2) = e^{i(\alpha + \pi/2)}$. Set $\beta = \frac{\alpha}{2} + \frac{\pi}{4}$; one gets

$$\varphi_0(r) = \frac{e^{i\beta} - i}{-ie^{i\beta} + 1} = \frac{\cos \beta}{1 + \sin \beta} = \frac{\sin \gamma}{1 + \cos \gamma} = \tan(\gamma/2)$$

with $\gamma = (\pi/2) - \beta = (\pi/4) - (\alpha/2) = (\pi/4) - \tan^{-1} r$. Then (4.6) follows.

2) Let $\tau = \frac{\pi}{2} - \theta$; one has

$$T(ie^{i\theta}) = \frac{e^{i\theta} - i}{ie^{i\theta} - 1} = \frac{-\cos \theta}{1 + \sin \theta} = \frac{-\sin \tau}{1 + \cos \tau} = -\tan(\tau/2).$$

Note that $0 < \tau/2 < \pi/2$ since $|\theta| < \pi/2$; hence $\tan(\tau/2) > 0$. Therefore

$$\varphi_0(e^{i\theta}) = \frac{i\sqrt{\tan(\tau/2)} - i}{-i \cdot i\sqrt{\tan(\tau/2)} + 1} = i \frac{\sqrt{\tan(\tau/2)} - 1}{\sqrt{\tan(\tau/2)} + 1}.$$

But

$$\tan(\tau/2) = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1 - \tan(\theta/2)}{1 + \tan(\theta/2)};$$

it follows that

$$\begin{aligned} \varphi_0(e^{i\theta}) &= i \frac{\sqrt{1 - \tan(\theta/2)} - \sqrt{1 + \tan(\theta/2)}}{\sqrt{1 - \tan(\theta/2)} + \sqrt{1 + \tan(\theta/2)}} \\ &= i \frac{(1 - \tan(\theta/2)) - (1 + \tan(\theta/2))}{(\sqrt{1 - \tan(\theta/2)} + \sqrt{1 + \tan(\theta/2)})^2} = -i \frac{\tan(\theta/2)}{1 + \sqrt{1 - \tan^2(\theta/2)}}. \end{aligned}$$

Now, since $\varphi_0(e^{i\theta}) \sim -i\theta/4$ as θ tends to 0, we get that

$$1 + \frac{2}{\pi} \log(1/|\varphi_0(e^{i\theta})|) - i \frac{2}{\pi} \arg \varphi_0(e^{i\theta}) \sim \frac{2}{\pi} \log(1/|\theta|)$$

and hence (4.8). □

It follows from this lemma and from Theorem 2.3 and Theorem 3.2 that one has the following estimate.

Theorem 4.3. *For the approximation numbers $a_n(C_\varphi)$ of the composition operator $C_\varphi: H^2 \rightarrow H^2$ of symbol the cusp map φ , we have*

$$(4.9) \quad e^{-c_1 n/\log n} \lesssim a_n(C_\varphi) \lesssim e^{-c_2 n/\log n}, \quad n = 2, 3, \dots,$$

for some constants $c_1 > c_2 > 0$.

Proof. 1) *Upper estimate.* Note first that, since the domain $\varphi(\mathbf{D})$ is contained in the right half-plane and in the symmetric angular sector of vertex 1 and opening $\pi/2$, there is a constant $C > 0$ such that $|1 - \gamma(t)| \leq C(1 - |\gamma(t)|)$ and we have (2.2). Then (4.8) in Lemma 4.2 gives (2.3). The upper estimate is hence given in Theorem 2.3 and (2.11).

2) *Lower estimate.* By Lemma 4.2, (4.6), one has (3.1). Since φ is a real symbol, the upper estimate follows from Theorem 3.2, and (3.4). \square

4.2. The Shapiro–Taylor map. This one-parameter map ς_θ , $\theta > 0$, was introduced by Shapiro and Taylor in 1973 [23] and was further studied, with a slightly different definition, in [10, Section 5]. Shapiro and Taylor proved that $C_{\varsigma_\theta} : H^2 \rightarrow H^2$ is always compact, but is Hilbert–Schmidt if and only if $\theta > 2$. It is proved in [10, Theorem 5.1] that C_{ς_θ} is in the Schatten class S_p if and only if $p > 4/\theta$.

Here, we shall use these maps ς_θ to see the limitations of our previous methods. We first recall their definition.

For $\varepsilon > 0$, we set $V_\varepsilon = \{z \in \mathbf{C}; \Re z > 0 \text{ and } |z| < \varepsilon\}$. For $\varepsilon = \varepsilon_\theta > 0$ small enough, one can define

$$(4.10) \quad f_\theta(z) = z(-\log z)^\theta,$$

for $z \in V_\varepsilon$, where $\log z$ will be the principal determination of the logarithm. Let now g_θ be the conformal mapping from \mathbf{D} onto V_ε , which maps $\mathbf{T} = \partial\mathbf{D}$ onto ∂V_ε , defined by $g_\theta(z) = \varepsilon \varphi_0(z)$, where φ_0 is given in (4.1).

Then, we define

$$(4.11) \quad \varsigma_\theta = \exp(-f_\theta \circ g_\theta).$$

One has $\varsigma_\theta(1) = 1$ and $g_\theta(e^{it}) \sim -it/4$ as t tends to 0, by Lemma 4.2; hence, when t is near 0,

$$|1 - \varsigma_\theta(e^{it})| \approx |f_\theta[g_\theta(e^{it})]| \approx |t| [\log(1/|t|)]^\theta.$$

If we were allowed to apply Theorem 2.3, we would get that $a_n(C_{\varsigma_\theta}) \lesssim 1/n^{\theta/4}$, which would be in accordance with the fact that C_{ς_θ} is in the Schatten class S_p if and only if $p > 4/\theta$. However, condition (2.2) is not satisfied: by [10], equations (5.5) and (5.6), one has $1 - |\varsigma_\theta(e^{it})| \approx |t|(\log 1/|t|)^{\theta-1}$, whereas $|1 - \varsigma_\theta(e^{it})| \approx |t|(\log 1/|t|)^\theta$.

On the other hand, by the Lemma 4.2 again, $g_\theta(r) \sim \varepsilon(1 - r)/4$ as r tends to 1; hence, when r is near to 1,

$$1 - \varsigma_\theta(r) \approx (1 - r)(\log 1/(1 - r))^\theta,$$

so ς_θ is a real ω -radial symbol with $\omega(t) = t(\log 1/t)^\theta$. Hence, we get from Theorem 3.2

$$a_n(C_{\varsigma_\theta}) \gtrsim \frac{1}{n^{\theta/2}},$$

taking $\sigma = 1/e$ in (3.2). However, this lower estimate is not the right one, since C_{ς_θ} is in S_p if and only if $p > 4/\theta$.

5. Contact points

It is well-known (and easy to prove) that for every compact composition operator $C_\varphi : H^2 \rightarrow H^2$, the set of contact points

$$E_\varphi = \{e^{i\theta}; |\varphi^*(e^{i\theta})| = 1\}$$

has Lebesgue measure 0. A natural question is: to what extent is this negligible set arbitrary? The following partial answer was given by Gallardo-Gutiérrez and González in [6].

Theorem 5.1. (Gallardo-Gutiérrez and González) *There is a compact composition operator C_φ on H^2 such that the Hausdorff dimension of E_φ is one.*

This was generalized by El-Fallah, Kellay, Shabankhah, and Youssfi [5, Theorem 3.1]:

Theorem 5.2. (El-Fallah, Kellay, Shabankhah, and Youssfi) *For every compact set K of measure 0 in \mathbf{T} , there exists a Schur function $\varphi \in A(\mathbf{D})$, the disk algebra, such that the associated composition operator C_φ is Hilbert–Schmidt on H^2 and $E_\varphi = K$.*

As an application of our previous results, we shall extend these results, with a very simple proof. Our composition operator will not only be compact, or Hilbert–Schmidt, but in all Schatten classes S_p , and moreover its approximation numbers will be as small as possible.

Theorem 5.3. *Let K be a Lebesgue-negligible compact set of the circle \mathbf{T} . Then, there exists a Schur function $\psi \in A(\mathbf{D})$, the disk algebra, such that $E_\psi = K$, $\psi(e^{i\theta}) = 1$ for all $e^{i\theta} \in K$, and*

$$(5.1) \quad a_n(C_\psi) \leq a \exp(-bn / \log n).$$

In particular, $C_\psi \in \bigcap_{p>0} S_p$.

Proof. According to the Rudin–Carleson theorem [2], we can find $\chi \in A(\mathbf{D})$ such that

$$\chi = 1 \text{ on } K \quad \text{and} \quad |\chi| < 1 \text{ on } \overline{\mathbf{D}} \setminus K.$$

Consider now the cusp map φ , defined in Section 4.1. One has $\varphi \in A(\mathbf{D})$, $\varphi(1) = 1$ and

$$a_n(C_\varphi) \leq a' \exp(-bn / \log n).$$

We now spread the point 1 by composing with the function χ , which is equal to 1 on the whole of K . We check that the composed map $\psi = \varphi \circ \chi$ has the required properties.

That $\psi \in A(\mathbf{D})$ is clear. For $z \in K$, one has $\psi(z) = \varphi(1) = 1$, and for $z \in \overline{\mathbf{D}} \setminus K$, one has $|\chi(z)| < 1$; hence $|\psi(z)| < 1$.

To finish, since $C_\psi = C_\chi \circ C_\varphi$, we have

$$a_n(C_\psi) \leq \|C_\chi\| a_n(C_\varphi) \leq a' \|C_\chi\| \exp(-bn / \log n) := \sigma_n,$$

proving the result (with $a = a' \|C_\chi\|$), since clearly $\sum_{n=1}^\infty \sigma_n^p < \infty$ for each $p > 0$. \square

Actually, we can improve on the previous theorem by proving the following result. This result is optimal because if $\|\psi\|_\infty = 1$, we know (see [15, Theorem 3.4]) that $\liminf_{n \rightarrow \infty} [a_n(C_\psi)]^{1/n} = 1$, so we cannot hope to get rid with the forthcoming vanishing sequence $(\varepsilon_n)_n$.

Theorem 5.4. *Let K be a Lebesgue-negligible compact set of the circle \mathbf{T} and $(\varepsilon_n)_n$ a sequence of positive real numbers with limit zero. Then, there exists a Schur function $\varphi \in A(\mathbf{D})$ such that $E_\varphi = K$, $\varphi(e^{i\theta}) = 1$ for all $e^{i\theta} \in K$, and*

$$(5.2) \quad a_n(C_\varphi) \leq C \exp(-n \varepsilon_n),$$

where C is a positive constant.

This theorem is a straightforward consequence of the following lemma. Recall that the Carleson function of the Schur function $\psi: \mathbf{D} \rightarrow \mathbf{D}$ is defined by

$$\rho_\psi(h) = \sup_{|\xi|=1} m(\{t \in \mathbf{T}; |\psi(e^{it})| \geq 1 - h \text{ and } |\arg(\psi(e^{it}) \bar{\xi})| \leq \pi h\}).$$

Lemma 5.5. *Let δ be a nondecreasing positive function on $(0, 1]$ tending to 0 as $h \rightarrow 0$. Then, there exists a Schur function $\psi \in A(\mathbf{D})$ such that $\psi(1) = 1$, $|\psi(\xi)| < 1$ for $\xi \in \mathbf{T} \setminus \{1\}$, and such that $\rho_\psi(h) \leq \delta(h)$, for $h > 0$ small enough.*

Once we have the lemma, in view of the upper bound in [15, Theorem 5.1] for approximation numbers

$$(5.3) \quad a_n(C_\psi) \lesssim \inf_{0 < h < 1} \left[(1 - h)^n + \sqrt{\rho_\psi(h)/h} \right], \quad n = 1, 2, \dots,$$

we can adjust the function δ so as to have $a_n(C_\psi) \leq Ke^{-n\varepsilon_n}$. Then, we compose ψ with a peaking function χ as in the previous section and the map $\varphi = \psi \circ \chi$ fulfills the requirements of Theorem 5.4, with $C = K\|C_\chi\|$. \square

Proof of Lemma 5.5. Consider the domain Ω represented on the Figure 2. This domain is limited by the two hyperbolas $y = 1/x$ and $y = (1/x) + 4\pi$ and to the right-hand side by, say, a semicircle. This limiting semicircle is chosen in order that $\Im w \geq 1$ for $w \in \Omega$. The lower parts of the “saw-teeth” have an imaginary part equal to $4\pi n$. If $a \in \Omega$ is fixed, with $\Im a < 4\pi$, and Ω_n is the part of the domain Ω such that $\Im w < 4\pi n$, the horizontal sizes of the “saw-teeth” are chosen in order that the harmonic measure $\omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n)$ is $\leq \delta_n := \delta(1/16\pi(n + 1))$. Note that $\partial\Omega \setminus \partial\Omega_n \supseteq \{w \in \partial\Omega; \Im w > 4\pi n\}$ (see [11, Lemma 4.2]).

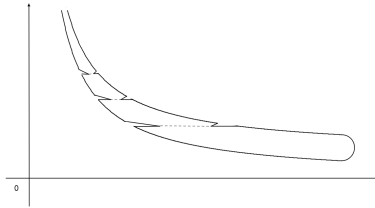


Figure 2. Domain Ω .

By Carathéodory–Osgood’s Theorem (see [21], Theorem IX.4.9), there is a unique homeomorphism g from $\overline{\mathbf{D}}$ onto $\overline{\Omega} \cup \{\infty\}$ which maps conformally \mathbf{D} onto Ω and such that $g(0) = a$ and $g(1) = \infty$ (we may choose these two values because if $h: \overline{\mathbf{D}} \rightarrow \overline{\Omega} \cup \{\infty\}$ is such a map, and u is the automorphism of $\overline{\mathbf{D}}$ such that $u(0) = h^{-1}(a)$ and $u(1) = h^{-1}(\infty)$, then $g = h \circ u$ suits—alternatively, having chosen $h(0) = a$, then, if $h(e^{i\theta_0}) = \infty$, we take $g(z) = h(e^{i\theta_0}z)$).

We define $\psi = (g - i)/(g + i)$. Then $\psi: \mathbf{D} \rightarrow \mathbf{D}$ is a Schur function and $\psi \in A(\mathbf{D})$. Moreover, since the domain Ω is bounded horizontally, we have $\psi(1) = 1$ and $|\psi(e^{it})| < 1$ for $0 < t < 2\pi$.

Now, $\rho_\psi(h) \leq m(\{z \in \mathbf{T}; |\psi(z)| > 1 - h\})$. Writing $g = u + iv$, one has

$$|\psi|^2 = \frac{u^2 + (v - 1)^2}{u^2 + (v + 1)^2} = 1 - \frac{4v}{u^2 + (v + 1)^2}.$$

Since $(1 - h)^2 \geq 1 - 2h$, the condition $|\psi(z)| > 1 - h$ implies that $\frac{2v}{u^2 + (v + 1)^2} \leq h$. But $0 < u \leq 1 + 2\pi \leq 8$ and $(v + 1)^2 \leq 4v^2$ (since $v \geq 1$); we get hence $\frac{v}{32 + 2v^2} \leq h$, or $\frac{32}{v} + 2v \geq \frac{1}{h}$. Using again the fact that $v \geq 1$, one obtains $2v \geq \frac{1}{h} - 32$, and hence $2v \geq \frac{1}{2h}$ for $0 < h \leq 1/64$. Therefore, for $0 < h \leq 1/64$,

$$\rho_\psi(h) \leq m(\{z \in \mathbf{T}; \Im g(z) \geq 1/4h\}).$$

Now, for $n \geq 2$ and $1/16\pi(n+1) \leq h < 1/16\pi n$, one gets hence

$$\begin{aligned} \rho_\psi(h) &\leq m(\{z \in \mathbf{T}; \Im g(z) > 4\pi n\}) \\ &= \omega_\Omega(a, \{w \in \partial\Omega; \Im w > 4\pi n\}) \leq \omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n) \leq \delta_n \leq \delta(h), \end{aligned}$$

proving Lemma 5.5. □

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