

THE FERMI LIQUID AS A RENORMALIZATION GROUP FIXED POINT

par

Guennadi Chitov

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date

Président-rapporteur: M. Claude Bourbonnais
Département de physique _____

Membre: M. David Sénéchal
Département de physique _____

Membre: M. André-Marie Tremblay
Département de physique _____

Membre externe: M. Victor Yakovenko
Université de Maryland _____

SUMMARY

The renormalization-group (RG) method is applied to study interacting fermions at finite temperature. A model based on the ψ^4 -Grassmann effective action with $SU(N)$ -invariant short-range interaction and a rotationally invariant Fermi surface in spatial dimensions $d = 2, 3$ is studied. We show how the key results of the Landau Fermi liquid theory can be recovered by this finite-temperature RG technique. Applying the RG to response functions, we find the compressibility and the spin susceptibility as solutions of the RG flow equations.

We discuss subtleties associated with the symmetry properties of the four-point vertex (the implications of the Pauli principle). We point out distinctions between three quantities: the bare interaction of the low-energy effective action, the Landau function and the forward scattering vertex. The bare interaction of the effective action is not a RG fixed point, but a common starting point of the flow trajectories of two limiting forms of the four-point vertex. We have derived RG equations for the Landau channel that take into account both contributions of the direct (ZS) and the exchange (ZS') particle-hole graphs at one-loop level. The basic quantities of Fermi Liquid theory, the Landau interaction function and the forward scattering vertex, are calculated as fixed points of these flows in terms of the effective action's interaction function.

The classic derivations of Fermi Liquid theory applying the Bethe-Salpeter equation and other analogous approaches, tantamount to some sort of RPA-type (decoupled) approximation, neglect the zero-angle singularity in the ZS' graph. As a consequence, the antisymmetry of the forward scattering vertex is not guaranteed in the final result, and the RPA sum rule must be imposed by hand on the components of the Landau function to satisfy the Pauli principle. This sum rule, not indispensable in the original phenomenological formulation of the Landau FLT, is equivalent, from the RG point of view, to a fine tuning of the effective interaction.

Our results show that the strong interference of the direct and exchange processes of the particle-hole scattering near zero angle invalidates the RPA

(decoupled) approximation in this region, resulting in temperature-dependent narrow-angle anomalies in the Landau function and scattering vertex, revealed by the RG analysis. In the present RG approach the Pauli principle is automatically satisfied. As follows from the RG solution, the amplitude sum rule, being an artefact of the RPA approximation, is not needed to respect statistics and, moreover, is not valid.

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TABLE OF CONTENTS

| | |
|---|--------|
| SUMMARY | ii |
| ACKNOWLEDGEMENTS | iv |
| TABLE OF CONTENTS | v |
| LIST OF FIGURES | vii |
| CHAPTER 1: Introduction | 1 |
| CHAPTER 2: RG Preliminaries | 13 |
| 2.1 The RG method | 13 |
| 2.2 The model | 22 |
| 2.3 Tree-level RG analysis | 29 |
| 2.4 Coupling functions and vertices | 34 |
| CHAPTER 3: RG without interference | 40 |
| 3.1 The Landau channel | 40 |
| 3.2 RG equations in the BCS channel | 48 |
| 3.3 RG equations in three dimensions | 50 |
| 3.4 Response functions | 54 |
| CHAPTER 4: Role of the interference in the Landau channel | 58 |
| 4.1 Coupled RG equations in the Landau channel | 58 |
| 4.2 Deficiencies of the decoupled approximations in the Landau channel | 63 |
| 4.3 Solution of the coupled RG equations | 66 |
| 4.3.1 Exact numerical solution | 66 |
| 4.3.2 Approximate analytical solution | 67 |

| | |
|--|----|
| 4.3.3 Extension of the effective action | 71 |
| 4.4 Analysis and discussion of the RG results | 72 |
| 4.5 Contact with the Landau FLT and discussion | 76 |
| Conclusion | 82 |
| REFERENCES | 85 |

LIST OF FIGURES

| | |
|---|----|
| 1. The three diagrams contributing to the RG flow at one-loop . . . | 41 |
| 2. Conic momentum configuration in the 3D case | 50 |
| 3. Diagrams for the calculation of response functions | 57 |
| 4. Phase space constraints | 60 |
| 5. Numerical solution of the RG equations. | 73 |
| 6. Numerical vs approximate solutions of the RG equations | 79 |

CHAPTER I

Introduction

In 1956-1957 L.D. Landau formulated his theory of Fermi liquids.¹ Let us first recall briefly the crucial premises of Landau's phenomenology. It is assumed that the ground state of the interacting fermion system at $T = 0$ is in one-to-one correspondence with that of the ideal Fermi gas, i.e., the Fermi sea is filled up to the Fermi momentum k_F , which is related to the fermion density in the same fashion as for non-interacting fermions ($d = 3$):

$$\frac{N}{V} = 2 \int n_0(\mathbf{k}) \frac{d^3\mathbf{k}}{(2\pi)^3} = \frac{k_F^3}{3\pi^2} . \quad (1.1)$$

Here N is the number of fermions and V is the volume of the system. We will set from now on $k_B = \hbar = 1$. The ground-state distribution function $n_0(\mathbf{k})$ at $T = 0$ is just the Heaviside step function:

$$n_0(\mathbf{k}) = \Theta(k_F - k) . \quad (1.2)$$

The excitations of this system are created when particles from the filled Fermi sphere pass to the available states with $k > k_F$. Landau's idea was to describe the low-lying excitations in the interacting system (i.e., the excitations into the states with k such that $|k - k_F| \ll k_F$) in terms of fermion quasiparticles which have a spectrum like that of free particles, but renormalized because of interactions. The invariance of the Fermi sphere's volume under switching on the interaction, expressed by Eq. (1.1), can be interpreted as the following statement: the number of quasiparticles equals the number of real fermions constituting the system, i.e., $N_{qp} = N$. Notice that the total energy of the interacting fermion system (the Fermi Liquid) is not an additive quantity of the quasiparticle energies $\varepsilon(\mathbf{k})$, contrary to the ideal Fermi gas

$$\mathcal{E} \equiv \frac{E}{V} \neq 2 \int n(\mathbf{k}) \varepsilon(\mathbf{k}) \frac{d^3\mathbf{k}}{(2\pi)^3} .$$

Instead, Landau proposed the following expansion for the total energy

$$\mathcal{E}[\delta\hat{n}(\mathbf{k})] = \mathcal{E}_0 + \int_k \hat{\varepsilon}_0(\mathbf{k}) \delta\hat{n}(\mathbf{k}) + \frac{1}{2} \int_{k,k'} \hat{f}(\mathbf{k}, \mathbf{k}') \delta\hat{n}(\mathbf{k}) \delta\hat{n}(\mathbf{k}') \quad (1.3)$$

in terms of the variations $\delta\hat{n}$ of the quasiparticle's distribution function $\hat{n}(\mathbf{k})$ over the ground-state distribution $\hat{n}_0(\mathbf{k})$

$$\delta\hat{n}(\mathbf{k}) \equiv \hat{n}(\mathbf{k}) - \hat{n}_0(\mathbf{k}) . \quad (1.4)$$

\mathcal{E}_0 stands for the ground-state energy at zero temperature and the following short-hand notations are used:

$$\int_k \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3} . \quad (1.5)$$

Unless necessary, we will not write down explicitly the spin dependence of the parameters, indicating it by hats only, e.g., $\hat{n}(\mathbf{k}) \leftrightarrow n_{\alpha\beta}(\mathbf{k})$; etc. The energy of the quasiparticle is given by

$$\hat{\varepsilon}(\mathbf{k}) \equiv \frac{\delta\mathcal{E}}{\delta\hat{n}(\mathbf{k})} = \hat{\varepsilon}_0(\mathbf{k}) + \int_{k'} \hat{f}(\mathbf{k}, \mathbf{k}') \delta\hat{n}(\mathbf{k}') , \quad (1.6)$$

wherein $\hat{\varepsilon}_0(\mathbf{k})$ gives the equilibrium energy of the quasiparticle at zero temperature. The \hat{f} -function, accounting for the interactions between quasiparticles in the Fermi Liquid (the Landau interaction function), is defined as follows:

$$\hat{f}(\mathbf{k}, \mathbf{k}') = \frac{\delta^2\mathcal{E}}{\delta\hat{n}(\mathbf{k})\delta\hat{n}(\mathbf{k}')} . \quad (1.7)$$

Being the second derivative of the energy, which is invariant under the exchange $\delta\hat{n}(\mathbf{k}) \leftrightarrow \delta\hat{n}(\mathbf{k}')$, the Landau function satisfies the symmetry properties

$$f_{\alpha\beta,\gamma\delta}(\mathbf{k}, \mathbf{k}') = f_{\gamma\delta,\alpha\beta}(\mathbf{k}', \mathbf{k}) . \quad (1.8)$$

The Landau picture of quasiparticles is limited to the close vicinity of the Fermi sphere when the quasiparticle's momenta satisfy the condition $|k - k_F| \ll k_F$, and to the low temperatures $T \ll E_F$, when the equilibrium quasiparticle's distribution function differs from the step function (1.2) only in a narrow neighborhood of width T around the Fermi energy. Then the quasiparticle energy $\hat{\varepsilon}_0(\mathbf{k})$, measured from the zero-temperature value of the chemical potential $\mu(T=0) = E_F = k_F^2/2m^*$, can be written as $\hat{\varepsilon}_0(\mathbf{k}) \approx \frac{k_F}{m^*}(k - k_F)$ wherein m^* is the quasiparticle effective mass. The quasiparticle at low temperature is a well-defined excitation, since its life-time τ

increases as $\tau \propto T^{-2}$ when $T \rightarrow 0$. Since $\delta\hat{n}(\mathbf{k})$ is appreciable only in the immediate proximity of the Fermi surface, the quasiparticle momenta \mathbf{k} and \mathbf{k}' entering the Landau function \hat{f} can be put at the Fermi surface, so \hat{f} will depend on the directions of those vectors only. If the system is spin-rotationally invariant (e.g., there is no external magnetic field), the Landau function simplified in such manner depends only on the relative angle θ between the vectors \mathbf{k} and \mathbf{k}' , where $|\mathbf{k}| = |\mathbf{k}'| = k_F$. In such case the Landau function can be represented by two dimensionless angular functions F and G as follows:

$$\nu_F \cdot f_{\alpha\beta,\gamma\delta}(\mathbf{k}, \mathbf{k}') = \nu_F \cdot f_{\alpha\beta,\gamma\delta}(\theta) = F(\theta)\delta_{\alpha\beta}\delta_{\gamma\delta} + G(\theta)\sigma_{\alpha\beta} \cdot \sigma_{\gamma\delta} , \quad (1.9)$$

wherein ν_F is the density of electron states on the Fermi level and σ are the Pauli matrices. It is convenient to use the coefficients $\{F_l, G_l\}$ of the Legendre polynomial expansion of $\{F(\theta), G(\theta)\}$ (cf. definition of this expansion (3.39) below). To emphasize the significance of the Landau function, characterizing quantitatively the Fermi Liquid state, we recall some exact results of the Fermi Liquid Theory (FLT). The components of the \hat{f} -function enter into formulas for the physical observables, resulting in a renormalization of the free Fermi gas results, due to the interaction effects. The interaction coefficient F_1 provides the exact relationship between the quasiparticle effective mass and the (bare) mass of the real interacting particles:

$$\frac{m^*}{m} = 1 + F_1 . \quad (1.10)$$

The above relationship is exact since it follows from Galilean invariance. In turn, the renormalized effective mass enters the Fermi-gas-type linear temperature dependence of the specific heat $C = \frac{1}{3}m^*k_F T$. The interaction effects in the Fermi Liquid result in renormalization factors $(1 + F_0)^{-1}$ and $(1 + G_0)^{-1}$ in the Fermi-gas formulas for the compressibility (K) and the spin susceptibility (χ), respectively:

$$K = \frac{1}{n^2} \frac{\nu_F}{1 + F_0} , \quad \chi = \frac{1}{4} g^2 \frac{\nu_F}{1 + G_0} ,$$

wherein n is the particle density and g is the gyromagnetic ratio. For the thermodynamic stability of the energy functional (1.3) Pomeranchuk derived the conditions for the components of the Landau interaction function:²

$$\{F_l, G_l\} > -1, \quad \forall l . \quad (1.11)$$

Shortly after the appearance of the Fermi Liquid phenomenology, much effort has been dedicated, including by Landau himself,³ to vindicate some intuitive assumptions of Landau and elucidate the foundations of the phenomenological theory. The original phenomenological formulation of this theory is formulated in terms of bosonic variables [variations of the distribution function $\hat{n}(\mathbf{k})$]. The field-theoretic interpretation of the Landau FLT has reformulated the key notions and basic results of the phenomenological theory entirely in terms of the fermionic Green functions technique.^{3,4,5,6} The demonstration of the equivalence of the field-theoretic results obtained from the solution of the Bethe-Salpeter equation with the results obtained from the functional expansion (1.3) and from the Boltzmann transport equation describing the collective modes, has become a textbook topic.^{5,6,7,8,9} The field-theoretic approach provided not only a solid basis to phenomenology, but also a potentially efficient method to calculate the phenomenological parameters of FLT from first principles. Silin's extension of the FLT, which incorporates the long-range Coulomb interaction between particles, made this theory applicable to charged Fermi liquids as well.¹⁰

The conditions under which the FLT breaks down have also been well known for a long time. If the Pomeranchuk stability conditions (1.11) or the Landau theorem for the stability of Fermi Liquid against Cooper pairing at arbitrary angular momentum⁷ are not satisfied, then a phase transition towards a phase different from the Fermi Liquid could occur. (This theorem demands, roughly speaking, the absence of effective attraction for all components of the interaction function. For the exact conditions of this theorem see Eqs (3.27) below.) For instance, the attraction between fermions violates the conditions of Landau's theorem, and the superconductive phase transition takes place. Another classical example is provided by interacting fermions in one spatial dimension, wherein the FLT never works, even without spontaneously broken symmetry (phase transition). Instead, the operational notion in 1D is what was called by Haldane the Luttinger Liquid.¹¹ (For a recent review on 1D systems see, e.g., the paper by Voit.¹²) However, the discovery of the fractional quantum Hall effect^{13,14} and of high- T_c superconductors¹⁵ revealed the existence in Nature of completely new phases of fermion systems in $d > 1$ with unbroken

symmetry, which do not fit in the description provided by the FLT. Those two extraordinary discoveries engendered a new branch of condensed matter physics, the physics of strongly correlated fermion systems. [For reviews on the recent developments in this rapidly advancing field see, for example, Refs. [16,17] and more references therein.]

Current interest in the physics of strongly correlated fermions [non-Fermi Liquids in $d > 1$] inspired a new wave of efforts aimed at clarifying the foundations of the Landau FLT and possible mechanisms of its breakdown. Let us mention only two approaches, which can be seen as sophisticated modern counterparts of the two classic formulations of the Landau FLT. Developing Luther's earlier ideas,¹⁸ Haldane put forward the method of higher-dimensional bosonization¹⁹ in order to treat the Fermi Liquid. Followed by other studies, bosonization approaches to various fermionic liquids have recently been developed.^{20,21,22,23} At about the same time, the Renormalization Group (RG) technique has been applied to interacting fermions in $d > 1$ with models based on fermionic field effective actions (see Refs. [24-34] and references therein). In both approaches it has been established, for models with reasonable fermion-fermion effective interactions, that the Fermi liquid phase is stable, whereas adding gauge-field interactions may drive the system towards a Non-Fermi-Liquid regime, or may result in a Marginal Fermi Liquid phase, like for composite fermions at the half-filled Landau level.

This work is devoted to the development of the RG approach to interacting fermions. However, we feel obliged to mention our reservations about the first method mentioned above, namely, bosonization in $d > 1$. Our concerns are, basically, two-fold. Firstly, for the simplest case of the effective action for short-range interacting fermions, the higher-dimensional bosonization approach allowed to recover most of the known results of Landau's phenomenology. But this method does not give even a conceptual clue as to what the parameters of the Fermi Liquid should be (i.e., do the coefficients of the Landau function satisfy the Pomeranchuk stability conditions?) and how they can be traced over from a microscopic Hamiltonian. Secondly, in cases of more complicated effective

actions which include, e.g., additional gauge and/or other fields and which are less well understood, the results provided by the bosonization may be controversial, since the simplifications done in order to get the approximated (Kac-Moody) algebra for the boson variables are hard to control. For instance, the authors of Ref. [36], who applied field-theoretic methods to treat interacting fermions coupled to a gauge field, claim that the bosonization results²⁰ for that problem are valid only in the unphysical limit $N \rightarrow 0$, wherein N is the number of fermion species. For a more detailed discussion of the generic bosonization algebra, its approximations and related issues, see, e.g., Ref. [22].

Until recently, the most successful applications of RG methods to interacting fermions have been achieved in the one-dimensional case, where it was known from exact solutions that non FL phases exist (e.g., the Luttinger liquid). For earlier reviews on 1D systems see Refs. [37,38]. Later, Bourbonnais and Caron³⁹ made an extensive RG study of one-dimensional and quasi-one-dimensional fermion systems at finite temperature. The necessity for theorists to understand the occurrence of Non-Fermi-Liquid phases (or regimes) of interacting fermions in “isotropic”[†] systems of dimensions greater than one, explains naturally the interest in the development of a RG theory for such systems. The RG is known to be a powerful method, well-established in other fields of physics,^{40,41,42,43} systematic in the sense of a control over approximations done in practical calculations, and capable of giving results far beyond the reach of perturbational approaches. As discussed by Shankar in a very pedagogical paper,²⁸ the RG treatment of fermions in the context of condensed matter problems is a much more complicated issue than the analogous procedure for critical phenomena or quantum field theories, because of two crucial points: *Firstly*, the existence of a Fermi surface: the low-energy modes lie in the vicinity of a continuous geometrical object (the Fermi surface) and not only around isolated points, like the origin of phase space in the bosonic case. Technically, this introduces additional phase space constraints on

[†] In the sense that the system cannot be treated as a set of weakly coupled one-dimensional chains.

the modes to be integrated out, quite a problem for an arbitrary Fermi surface. Moreover, the Fermi surface itself is a relevant parameter of theory, and its shape should renormalize under the RG procedure, except in the rotationally symmetric case. *Secondly*, instead of studying the flow of one or a few coupling constants like in other familiar physical problems, one has to treat RG flow equations for coupling *functions*, defined on the Fermi surface. Notice that the purely 1D interacting fermion system can be considered as a degenerate special case where the Fermi surface is reduced to a set of two points, with a finite number of coupling constants. It is therefore closer to the usual applications of the renormalization group. (However, in the quasi-one-dimensional approach of Ref. [39], one already witnesses the appearance of new coupling constants related to interchain hopping and one may follow the change in shape of the (open) Fermi surface under RG flow.)

Let us now more specifically discuss the RG studies of the Fermi Liquid phase. It will allow us to place our results in the context of other workers's contributions to the field, and to explain the motivation and goals of our work.

The RG studies of interacting fermions cited above^{24,25,27,28,29,30} contain general statements on the stability of the Fermi Liquid phase in $d = 2, 3$ in the case of short-range repulsion. However, the standard formulas and relationships of the Landau FLT were not recovered in those papers. To the best of our knowledge, only in the paper of Shankar were those questions addressed,²⁸ and he was closer than others to the right treatment of the FLT from the RG standpoint. Shankar correctly treated the Bardeen-Cooper-Schrieffer (BCS) interaction channel of scattering quasiparticles with opposite incoming (outgoing) momenta and recovered the Landau theorem for the stability of the Fermi Liquid against Cooper pairing. Treating then the Landau interaction channel of nearly forward scattering quasiparticles, Shankar recovered some of the FLT results, combining the tree-level RG analysis with perturbation theory. He also tried to clarify the relationship between the parameters of the interaction appearing in the fermionic effective action and the Landau interaction function of the conventional FLT. Nonetheless, working at zero

temperature, Shankar erroneously concluded that there is no RG flow of the coupling functions (vertices) in the Landau interaction channel. (We will show below that working in the framework of the finite-temperature RG technique allows us to find results for the Landau channel that were missed by Shankar. The reason is that the β -function (i.e., the r.h.s. of the RG flow equation) for the running couplings in that channel becomes singular at $T = 0$, and it is easy to get the wrong result in the zero-temperature limit.) As a consequence, Shankar was not able to distinguish between the forward scattering amplitudes (scattering vertex) and the Landau interaction function, known from the field-theoretic version of the FLT to be two different zero-transfer limits of the non-analytic four-point vertex.^{5,6,7} Notice that the mistake of neglecting the RG flow *for the forward scattering vertex* was partially circumvented in Shankar's paper by applying perturbation theory (or, more exactly, by summing an infinite series of particle-hole ladder diagrams) for the calculation of the physical observables involving that vertex, e.g., the compressibility. Eventually it gave the standard RPA-type result which could have been directly provided by the RG at the one (particle-hole) loop level.³³ However, Shankar's neglect of the RG flow in the Landau channel also resulted in another problem, since by doing so he mistakenly identified the (bare) coupling function of the effective interaction with the Landau interaction function. Indeed, the latter has to be identified with a fixed point of the RG equations, but when the corresponding RG flow given by the β -function of those equations is found to be identically zero (Shankar's case), then the bare value (initial point of the flow) and the fixed-point value of the coupling function are the same.

Let us give two arguments showing, even without explicit calculation of the RG flow equation whose solution gives the Landau interaction function, why this flow is non-zero and, consequently, why the bare interaction function of the low-energy fermion effective action cannot be identified with the Landau interaction function. *First*, identifying the Landau function with the effective action's bare interaction is inconsistent with other standard FLT results, because of the role of Fermi statistics (the Pauli principle). Let us consider for simplicity two-dimensional spinless fermions. Then the Landau interaction

function defined as in Eq. (1.9) is totally determined by a single function $F(\theta)$ or, equivalently, by the set of its Fourier components F_l . The forward scattering vertex constructed and decomposed in the same fashion as the \hat{f} -function in Eq. (1.9), is given for the case in hand by a single “charge” component, which we will denote $\Gamma(\theta)$. The Pauli principle for the scattering vertex demands that $\Gamma(\theta = 0) = 0$. Then *the statement* is that in a stable Fermi liquid, the standard relationship between Fourier components of the scattering amplitude (Γ_l) and of the Landau interaction function (F_l), i.e., $\Gamma_l = F_l/(1 + F_l)$, cannot satisfy the Pauli principle for the amplitude (the amplitude sum rule)

$$\sum_l \frac{F_l}{1 + F_l} = 0 \quad (1.12)$$

if F has the symmetry properties of the action’s bare interaction i.e., if $F(\theta = 0) = \sum_l F_l = 0$. (For the explanation and rigorous proof of this point see Sec. 4.2 below). *Second*, identifying the Landau function with the bare interaction is inconsistent with the premise of low-energy effective action method itself, in the way it is applied to condensed matter problems. Namely, at the starting point of the analysis, the bare parameters of the effective action, including the interaction, are regular functions of their variables.^{27,28} It is known, however, that this is not the case even for parameters of a normal Fermi liquid. The scattering amplitude and the Landau function are two distinct limits of the four-point vertex when energy-momentum transfer goes to zero. This non-analyticity of the vertex appears in its dependence *both* on the small energy-momentum transfer and, due to the crossing symmetry, on the small angles between incoming (outgoing) particles lying near the Fermi surface. This contradiction between the analytical properties of the two functions in question (i.e., the bare interaction and the Landau function) becomes flagrant in the case of more exotic fermion systems. For instance, the Landau function of the marginal Fermi Liquid of composite fermions at the half-filled Landau level (interacting fermions coupled with gauge field) is shown by other (non-RG) methods⁴⁴ to develop a δ -function singularity in the forward direction ($\theta = 0$). Such behavior of the Landau function is related to the divergence of the quasiparticle’s effective mass, according to the theory of

Halperin, Lee and Read for the half-filled Landau level⁴⁵ (see also Ref. [46]). So, our two arguments show that the Landau function cannot be a regular interaction in the effective action at the starting point of the RG analysis.

The preceding comments on the earlier RG approaches to Fermi Liquids show that they are insufficient on many points. Let us now give a general outlook of the key components of our study which presents the quantitative RG theory of the Fermi-Liquid fixed point.

The first part of our RG results for the FLT, presented in Chapter 3, like other such analyses already published, are obtained from a low-energy effective action^{27,28,29} with a marginal (in the RG sense) short-range interaction at the starting point of the RG procedure. However, contrary to other works on the subject, our finite-temperature RG approach reveals that, in the Landau channel of nearly forward scattering quasiparticles, the effective action's coupling functions flow with successive mode eliminations towards the Fermi surface, even in the absence of singular or gauge interactions. In other words, the coupling functions do not stay purely marginal under the RG transformation, since their β -functions are not identically zero. From the RG flow equations which *explicitly* take into account the direct particle-hole loop (*ZS*) *only*, the standard FLT formulas for the susceptibilities and the relationships between the scattering amplitudes and the components of the Landau function are recovered. The formulas of the conventional FLT, obtained at that stage of the study, are strictly equivalent to the results of the classic diagrammatic approach to Fermi Liquid.

The aim of the next stage, presented in Chapter 4, is twofold. Once the classic FLT results have been recovered by the RG approach in the form of relationships between fixed-point values of different coupling functions (running vertices), the approach itself would lose its appeal if it did not provide a constructive method for calculating the Fermi liquid's parameters. This is especially important goal in the long-term perspective of applying this powerful method to more complex (strongly correlated) fermion systems. In order to

provide that kind of quantitative RG theory, we explicitly calculate the Landau function and the forward scattering vertex, starting from the short-range effective bare interaction. We do it in a one-loop RG approximation which takes into account contributions of the direct (ZS) and exchange (ZS') particle-hole graphs. In particular, this enables us to reveal singular temperature-dependent features of the Landau function and scattering vertex in the forward direction ($\theta = 0$).

An equally important goal of that second part of the RG analysis is to resolve the old problem of FLT with the Pauli principle. In its treatment of FLT, the field-theoretic approach encountered a very subtle problem caused by Fermi statistics and by the necessity to provide *both* stability for the Fermi liquid and a solution for the two-particle vertex (scattering amplitude) that meets the Pauli principle.^{47,7} (We remind, that according to the requirements of Fermi statistics, the two-particle vertex is antisymmetric under exchange of the incoming (outgoing) particles.) The problem was “settled” by imposing (practically by hand) the amplitude sum rule on the components of the Landau quasiparticle’s interaction function. [Cf. the example of such sum rule for the case of fermions without spin given above by Eq. (1.12).] It is worth noting that the phenomenological FLT is spared from this problem partially because it is formulated in terms of bosonic variables [see, e.g., Eqs. (1.3-1.7)], partially because it says nothing about the two-(quasi)particle fermion vertex. Landau’s phenomenology only provides us with the condition (1.8) for the symmetry properties of the \hat{f} -function. As we can easily see from Eq. (1.9), in the absence of $SU(2)$ symmetry breaking fields (interactions) any function of the relative angle between quasiparticle momenta meets the requirement (1.8). In the phenomenological theory no more constraints (sum rules) on the functions F , G are implied. Seen from the RG standpoint, the same problem of constraints manifests itself in the form of a “naturalness problem”²⁷ of the effective action: the low-energy effective action has to be “fine tuned” in order for the scattering amplitude to meet the Pauli principle. More detailed discussion of this problem is postponed until Sec. 4.2, where it will be put in contact with the present RG approach. It will be shown in Chapter 4 that if quantum

interference of the direct and exchange processes is taken into account in the RG equations, this problem is eliminated in a natural manner.

The thesis is organized as follows. In Chapter 2 we give a short description of the RG method, we define the model to be studied and the quantities to be calculated by the present RG technique. In Chapter 3 we develop the (decoupled) one-loop RG approach to the model based on the fermionic low-energy effective action. We demonstrate that such an approach is equivalent to the classic field-theoretic treatment of the Fermi liquid. In Chapter 4 we derive and solve the RG equations which explicitly preserve the exchange symmetry of the four-point vertex. From a more physical point of view, the novelty of those equations is the handling of the interference between the direct and exchange particle-hole processes in the Landau channel of the (nearly) forward scattering particles. The consequences of the RG corrections on FLT results are discussed.

In the beginning of each chapter we give a short description of its content. The main results of this thesis have been presented in Refs. [33,34,35]

CHAPTER II

RG Preliminaries

This chapter is mostly introductory. It contains auxiliary information and the results of a scaling analysis, necessary in order to venture beyond tree-level in the RG calculations. In Sec. 2.1 we give a short summary of the RG method. In Sections 2.2 we define the effective action of the model. At the technical level, the model is a straightforward extension of that considered earlier by Shankar²⁸ for spinless fermions at zero temperature. The extension incorporates spin and a finite temperature. For the sake of generality, we study a model of N -component fermions with an $SU(N)$ -invariant short-range interaction. We study the effective action defined for a circular Fermi surface in spatial dimension two, spherical in dimension three. The scaling analysis of Sec. 2.3 allows us to single out two interaction channels (Landau and BCS) wherein the coupling functions of the effective action's interaction are marginal at tree level. The Landau interaction channel corresponds to the (nearly) forward scattering processes of (quasi)particles in the vicinity of the Fermi surface. The Bardeen-Cooper-Schrieffer (BCS) interactions channel corresponds to the scattering of the (quasi)particles with opposite incoming (outgoing) momenta near the Fermi surface. In Sec. 2.4 we define the coupling functions (running vertices) to be calculated in the RG framework in both channels.

2.1 The RG method

In many fundamental domains of physics it is necessary to handle a problem of interacting (coupled) fields, i.e., a problem containing a very large number of interacting degrees of freedom (infinite in the continuum limit). This is the usual situation in Quantum Field Physics and in Condensed Matter Physics. What particularly complicates the analysis in most cases is that interactions between fields cannot be considered as weak, nor can a problem be satisfactorily solved by singling out some characteristic mode from the ensemble

of interacting fields. The latter means that if, say, we have fields $\varphi(k)$ wherein k is wavenumber, then we cannot obtain a good approximation by separately treating a single degree of freedom $\varphi(k_0)$ specified by the wavenumber k_0 . The scale k_0 can be usually chosen on physical grounds. In many situations such kind of treatment, which is tantamount to some sort of mean-field approximation, fails.

Unfortunately, there are only very few physically interesting exactly solvable models in field theory or statistical physics and, moreover, their exact solutions can be found only in low spatial dimensions.^{48,49} (Examples of such exactly solvable problems are provided by the Tomonaga-Luttinger model of interacting fermions in $d = 1$, the Ising model in $d = 1, 2$.) However oversimplified those models might appear, much can be learned from their solutions. The complexity of the problem with interacting fields, classical or quantum, which precludes the use of a naive perturbation calculation around the non-interacting or mean-field solution, is due not only to – and in some cases not necessarily to – the strength of interactions, but also to the large number of degrees of freedom involved in the relevant physics and correlations between them. For instance, for 1D fermions, it is known that any interaction, however weak, destroys the Fermi liquid phase. The latter is the mean-field approximation for interacting fermions. From a technical point of view, the absence of a single momentum (energy) scale and, instead, the involvement of a continuum of modes below some characteristic scale ($\sim E_F$), is signalled in perturbation (diagrammatic) calculations by appearance of the logarithmically divergent terms ($\propto \ln(E_F/E)$, $E \rightarrow 0$).

The problem of phase transitions provides another well-known example of physics essentially involving a continuum of degrees of freedom. For the sake of simplicity, let us consider the transition at the Curie temperature (T_c) from a paramagnetic state to a uniaxial ferromagnet. This type of transition can be described by studying a one-component classical fluctuating field $\varphi(k)$. It is known that the Landau mean-field theory, which is formulated in terms of the average of this field *only* (i.e., the magnetization $m = \langle \varphi(0) \rangle$) for the

example chosen), gives wrong predictions for the temperature behavior of thermodynamic parameters (e.g., magnetization, specific heat, susceptibility) near T_c in spatial dimensions $d \leq 3$. In the parlance of the theory of critical phenomena, the mean-field theory predicts wrong values of critical exponents in $d \leq 3$. (For reviews on this subject see, e.g., Ref. [40,50], as well as Ref. [51] for the original pedagogical formulation of Landau's theory of phase transitions. Interestingly enough, the non-mean-field critical exponents near T_c are found not only from various experimental data, but the theoretical proof of their existence is also provided by the available exact solution of the 2D Ising model.^{49,51}) Efforts aimed at obtaining the correct critical exponents by plunging into straightforward perturbation calculations are fruitless, since one obtains divergent (as $T_c \rightarrow 0$) corrections invalidating the perturbation approach itself. To understand the main reason of the failure of the mean-field theory near T_c , it was important to realize the role of fluctuations. As pointed out by Wilson,⁵² taking into account interacting fluctuations of the field $\varphi(k)$ occurring in the *whole range of scales* $k < \Lambda$ (wherein $\Lambda \sim a^{-1}$ and a is atomic or lattice spacing) is crucial for the correct description of physics near the critical point.

The renormalization group (RG) approach, in its most general and enlightening formulation due to Wilson,[†] is the theory designed to handle fields (quantum or classical) fluctuating over range of momentum (energy) scales. Let us consider as an example the (effective) Hamiltonian $\mathcal{H}(\varphi)$ with a one-component classical field $\varphi(k)$. This example is taken from the context of critical phenomena. The specific form of the Hamiltonian is not important for

[†] There exist several versions of RG developed in Quantum Field Theory and in Condensed Matter Physics. For reviews and textbooks on the RG in different contexts and its history, see Refs. [40,53,54,55]. We will present a conception of the RG which was put forward by Wilson. For reviews on this approach see Refs. [40,52,56]. The references on Wilson's original papers are in there. A very pedagogical review on the Wilson RG can be found in Ref. [50]. Since the Wilson RG theory was strongly motivated by Kadanoff's more intuitive approach of successive averaging of the spin Hamiltonian in real space,⁵⁷ it is often called the Kadanoff-Wilson RG.

what follows. Without loss of generality it can be written as the following expansion in the disordered phase ($T > T_c$):

$$\mathcal{H} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int_{\mathbf{k}_1, \dots, \mathbf{k}_{2n}} u_{2n}(\mathbf{k}_1, \dots, \mathbf{k}_{2n}) \delta\left(\sum_{i=1}^{2n} \mathbf{k}_i\right) \prod_{i=1}^{2n} \varphi(k_i) \quad (2.1)$$

For instance, the particular form of \mathcal{H} with only the first two terms retained on the r.h.s. of (2.1) with

$$u_2(k) = r_0 + k^2 \quad [r_0 \propto (T - T_c)] , \quad u_4(k_1, k_2, k_3) = u = \text{const} , \quad (2.2)$$

(i.e., the Ginzburg-Landau effective Hamiltonian) can be interpreted as the continuous (soft) spin limit of the d -dimensional Ising model. (For discussion on the soft model see Refs. [40,50].) The soft model, which is also called φ^4 -model, serves, e.g., to describe the aforementioned uniaxial ferromagnetic transition.

As is known from statistical mechanics,⁵¹ to solve the problem with a given Hamiltonian is to calculate the partition function (Z). The latter can be written as the following functional integral:⁵⁸

$$Z = \int_{0 < k < \Lambda} \mathcal{D}\varphi e^{-\mathcal{H}_\Lambda(\varphi)} . \quad (2.3)$$

Following the tradition in the theory of critical phenomena, we absorbed the temperature factor ($1/T$) in the exponential of Eq. (2.3) into the definition of the Hamiltonian. The subscripts in the path integral and the Hamiltonian indicate that the fields to be integrated out have their momenta in the range $[0, \Lambda]$.[‡] Only if the Hamiltonian corresponds to noninteracting fields (i.e., only the first term on the r.h.s. of (2.1) is present), does the calculation of the partition function reduce to performing Gaussian integrations and thus, can be done exactly.

[‡] Notice that in the case of quantum fields as, e.g., the fermionic fields considered in the following sections, one must add an extra coordinate (i.e., time for the system at zero temperature, or the “imaginary time” at $T > 0$). Also, one has to work with the effective action rather than the effective Hamiltonian (cf. next section). However, these technical particularities are not important in explaining the basic ideas of the RG theory.

The strategy underlying the RG calculation of the partition function, is to divide the range of momenta from zero up to the ultraviolet cutoff Λ , into a set of subranges, and to integrate fields inside each subrange successively. More precisely, we integrate out the fields $\varphi(k)$ with their momenta lying in the range $[\Lambda - \delta\Lambda, \Lambda]$. Afterwards, the partition function can be written again in the form of Eq. (2.3), but with the new Hamiltonian $\mathcal{H}_{\Lambda-\delta\Lambda}(\varphi)$, wherein

$$e^{\mathcal{H}_{\Lambda-\delta\Lambda}(\varphi)} = \int_{\Lambda-\delta\Lambda < k < \Lambda} \mathcal{D}\varphi e^{-\mathcal{H}_{\Lambda}(\varphi)} , \quad (2.4)$$

and the fields in the new Hamiltonian $\mathcal{H}_{\Lambda-\delta\Lambda}(\varphi)$ have momenta $k \in [0, \Lambda - \delta\Lambda]$. By doing so infinitesimally, i.e., $\delta\Lambda \rightarrow 0$, calculating the partition function (2.3) for the ensemble of fields $\varphi(k)$ ($k \in [0, \Lambda]$) can be seen as a smooth thinning of the momentum scale of the fields to be integrated out (smooth lowering of the cutoff Λ). After each (infinitesimal) step is done, the effective Hamiltonian changes $\mathcal{H}_{\Lambda} \mapsto \mathcal{H}_{\Lambda-\delta\Lambda}$. Thus, RG is an approach that maps the problem (2.3) of calculating the partition function to the problem of studying the evolution of the effective Hamiltonian \mathcal{H}_{Λ} while lowering the cutoff Λ . The evolution of the Hamiltonian is determined by the *RG equation*:

$$\Lambda \frac{\partial \mathcal{H}_{\Lambda}}{\partial \Lambda} = \mathfrak{R}\{\mathcal{H}_{\Lambda}\} , \quad (2.5)$$

wherein $\mathfrak{R}\{\mathcal{H}_{\Lambda}\}$ is a functional of \mathcal{H}_{Λ} . In such a formulation, the RG, as a mapping of the problem (2.3) to the functional differential equation (2.5), is exact. The explicit form of the exact RG equation (2.5), i.e., the explicit expression of the functional $\mathfrak{R}\{\mathcal{H}_{\Lambda}\}$ for the Hamiltonian with the classical field, was derived by Wilson,⁴⁰ and, in a slightly different RG scheme, by Wegner and Houghton.⁵⁹ The particular form of the functional $\mathfrak{R}\{\mathcal{H}_{\Lambda}\}$ depends on the concrete realization of the (infinitesimal) RG transformation resulting in Eq. (2.5). In more formal terms, this transformation preserving invariance of the partition function (2.3), is given by its generator $\hat{\mathfrak{R}}$ which acts on the Hamiltonian as defined by the r.h.s. of Eq. (2.5):

$$\hat{\mathfrak{R}} \mathcal{H}_{\Lambda} = \mathfrak{R}\{\mathcal{H}_{\Lambda}\} . \quad (2.6)$$

Formal aspects of RG transformations and their possible realizations were thoroughly discussed by Wegner.^{60,61} The Wilson RG transformation^{40,52} consists

not only in the elimination of modes ($\hat{\mathfrak{R}}_{\text{el}}$), but it is also accompanied by the following rescaling of momenta back to the initial value of the cutoff ($\Lambda - \delta\Lambda \mapsto \Lambda$), and of the remaining fields ($\varphi \mapsto \varphi/\zeta$). This element of the RG transformation ($\hat{\mathfrak{R}}_{\text{resc}}$), i.e., rescaling, is done, roughly speaking, in order to make the Hamiltonian after the complete RG transformation look more like the Hamiltonian before.[†] Then the operator of the Wilson RG transformation can be formally written as:

$$\hat{\mathfrak{R}} = \hat{\mathfrak{R}}_{\text{el}} + \hat{\mathfrak{R}}_{\text{resc}} \quad (2.7)$$

In the case of noninteracting (Gaussian) fields, the Wilson RG transformation is chosen to leave the Hamiltonian invariant at the critical point ($T = T_c$), up to constant (field-independent) terms.[‡]

A general RG transformation ($\hat{\mathfrak{R}}$) preserving the partition function, may contain what was called by Wegner⁶⁰ *redundant operators*. The contribution of those operators to the flow of the Hamiltonian [i.e., to the r.h.s. of Eq. (2.5)] generates “superfluous” flows between points of the Hamiltonian manifold[‡] corresponding to equivalent physics. (For a more rigorous discussion of redundant operators see Refs. [61,62].) The rescaling of variables as a component of the Wilson RG transformation, is an effective way to remove the redundant

[†] A concrete example of such (three-step) RG transformation is given in Sec. 2.3 for the fermionic effective action.

[‡] These constant terms are always generated by RG, even if at the beginning they are absent, as in Eq. (2.1). They are important for calculation of the free energy, but not for the correlation functions (vertices), and they are not specifically discussed in this study. For interacting (non-Gaussian) fields, the sequence of RG transformations [or the RG flow, according to Eq. (2.5)] generates an infinite series of interaction terms (as in the r.h.s. of Eq. (2.1), even if at the beginning the Hamiltonian had only few of them, or even one, e.g., $u\varphi^4$).

[‡] For example, the Hamiltonian (2.1) can be specified by the set of couplings $\{u_2, u_4, \dots\}$. The latter can be thought as a point in the manifold (space) of couplings. Thus, the evolution of the Hamiltonian according to Eq. (2.5) is represented as a trajectory on this manifold.

flows between the Hamiltonians, distinct only by a change of the normalization of the fields φ and thus, describing the same physics. This invariance of physics under arbitrarily change (ζ) of the field normalization $\varphi \mapsto \varphi/\zeta$ is called *reparametrization invariance*. In the Wilson RG transformation, in order to suppress the redundant (reparametrization) flows and to obtain well-defined critical exponents,⁵² the rescaling of the field $\varphi \mapsto \varphi/\zeta$ accompanying each step of the mode elimination is chosen to keep constant the coefficient $\frac{1}{2}k^2$ in the Gaussian term of the Hamiltonian [cf. Eqs (2.1,2.2)].

However, the exact RG equation based on the RG transformation (2.7) combining the mode elimination *and* rescaling, is not the only possible way to do RG. In the context of quantum field theory, Polchinski derived the (exact) RG equation of the type (2.5) for the effective action (S_Λ).⁶³ The Polchinski equation defines the evolution of S_Λ under a lowering of the cutoff Λ , i.e., under the elimination of fields. In other words, Polchinski's RG transformation contains only the element $\hat{\mathfrak{R}}_{\text{el}}$. (For the further development of this approach and its relation to the Wilson RG theory, see Refs. [64,65] and more references therein. In the RG approach based on the Polchinski equation, the reparametrization invariance is handled differently⁶⁵, but we will not discuss those questions here, since it would take us too far afield.)

Once the RG equation is derived, instead of the initial problem (2.3) one has to solve (2.5). The complete integration of fields $\varphi(k)$ in the range $k \in [0, \Lambda]$ is equivalent to finding the fixed point of the effective Hamiltonian (\mathcal{H}^*), which we define following Wegner⁶¹ as

$$\mathcal{H}^* \equiv \lim_{t \rightarrow \infty} \mathcal{H}_{\Lambda(t)} , \quad (2.8)$$

wherein $\mathcal{H}_{\Lambda(t)}$ is a solution of Eq. (2.5). We parametrize the lowering cutoff as $\Lambda(t) = \Lambda e^{-t}$, where $t = 0$ corresponds to the initial cutoff before the RG procedure is applied, and $t = \infty$ corresponds to fields completely integrated out of the partition function (2.3).[†]

[†] In critical phenomena, a fixed point of the RG equation is more often defined as $\mathfrak{R}\{\mathcal{H}^*\} = 0$, but the definition (2.8) is more general, and this is the one we will use in the following sections.

There are different ways to represent the exact RG equation (2.5), as, e.g., the functional differential equation for the Hamiltonian in terms of its variational derivatives on the r.h.s., or in terms of operators and scaling fields (see Refs. [61,62]). Whatever the representation of the RG equation, in practice, one can handle it by doing some sort of approximation. Following Wilson,^{40,52} one can recast the RG equation (2.5) into an infinite set of ordinary differential equations for the couplings u_n . (As we said above, the set of couplings uniquely defines the Hamiltonian.) In this approach, which we will use in the following with some modifications for the fermionic action, the partial elimination of modes, as a first step of the RG transformation, can be translated into diagrammatic language (i.e., Feynman graphs). For details on the diagrammatic calculations see Ref. [40]. By combining use of the small parameter ϵ ($\epsilon \equiv 4 - d$, wherein d is spatial dimension) and discarding the irrelevant couplings (defined below), the infinite set of the RG equations for the momentum-dependent couplings can be reduced to the two-parameter equations for the couplings r and u of the $u\phi^4$ (Ginzburg-Landau) Hamiltonian (2.2). Those equations solve the problem of the phase transition, providing non-classical critical exponents, unattainable by previous perturbational approaches. The accuracy of the diagrammatic solution of the RG equations is controlled by the number of loops taken into account in the calculation of β -functions (i.e., the r.h.s. of the flow equations for couplings, cf. Eq. (2.9) below).

To conclude this review on RG, we will also recall the important notions of relevance, irrelevance and marginality, which will be often invoked in what follows. Those notions are more often defined in terms of operators⁶¹, but we will introduce them by carrying out the linear analysis near the fixed point of the RG equations for the couplings, following Weinberg.⁵⁵ This presentation is closer to the way we derive the RG equations in the next sections. Suppose we have a set of RG equations

$$\frac{\partial u_n}{\partial t} = \beta_n(\mathbf{u}) , \quad (2.9)$$

wherein the beta-function of the n -th coupling depends on the set of all couplings, denoted as a vector in coupling space, i.e., $\mathbf{u} \equiv (u_2, u_4, \dots)$. [In

general, this space is of infinite dimension. See the footnote after Eq. (2.7).] Let $\mathbf{u}^* \equiv (u_2^*, u_4^*, \dots)$ be a fixed-point of the RG flow equations (2.9), which defines uniquely the fixed-point Hamiltonian (\mathcal{H}^*). We want to check how a Hamiltonian ($\mathcal{H} \Leftrightarrow \mathbf{u}$) lying sufficiently close[†] to the fixed-point Hamiltonian ($\mathcal{H}^* \Leftrightarrow \mathbf{u}^*$) behaves as $t \rightarrow \infty$, i.e., whether it approaches the fixed point ($\mathcal{H}^* \Leftrightarrow \mathbf{u}^*$) or runs away. Writing the couplings of the Hamiltonian \mathbf{u} as

$$u_n = u_n^* + \Delta_n \text{ , wherein } \Delta_n \equiv u_n - u_n^* \text{ ,} \quad (2.10)$$

we obtain from Eqs (2.9) the following RG equations in the linear approximation

$$\frac{\partial(u_n^* + \Delta_n)}{\partial t} \approx \beta_n(\mathbf{u}^*) + \sum_k \left. \frac{\partial \beta_n}{\partial u_k} \right|_{\mathbf{u}^*} \Delta_k \text{ .} \quad (2.11)$$

From the above equations we have

$$\frac{\partial \Delta_n}{\partial t} = \sum_k M_{nk} \Delta_k \text{ ,} \quad (2.12)$$

wherein

$$M_{nk} \equiv \left. \frac{\partial \beta_n}{\partial u_k} \right|_{\mathbf{u}^*} \quad (2.13)$$

Using compact notations for operators and vectors, Eqs (2.12) can be rewritten as follows:

$$\frac{\partial \Delta(t)}{\partial t} = \hat{M} \Delta(t) \text{ .} \quad (2.14)$$

Let us denote by $\{\mathbf{V}^s\}$ a set of eigenvectors of the operator \hat{M} with eigenvalues λ_s , i.e.,

$$\hat{M} \mathbf{V}^s = \lambda_s \mathbf{V}^s \text{ ,} \quad (2.15)$$

and suppose that this set is complete, so the vector Δ can be expanded over the eigenvectors \mathbf{V}^s :

$$\Delta(t) = \sum_s c_s(t) \mathbf{V}^s \text{ .} \quad (2.16)$$

Introducing the above expansion into both sides of Eq. (2.14) and using Eq. (2.15), we obtain the following equations for the coefficients $c_s(t)$:

$$\frac{\partial c_s(t)}{\partial t} = \lambda_s c_s(t) \text{ ,} \quad (2.17)$$

[†] Close enough to apply a linear analysis near the fixed point.

which give us the solutions

$$c_s(t) = c_s e^{\lambda_s t} . \quad (2.18)$$

Thus for the vector $\Delta(t)$ we have

$$\Delta(t) = \sum_s c_s e^{\lambda_s t} \mathbf{V}^s . \quad (2.19)$$

Coming back to the notations for the couplings of the Hamiltonian $\mathcal{H}(t)$, we obtain:

$$u_n(t) = u_n^* + \sum_s c_s e^{\lambda_s t} V_n^s \quad (2.20)$$

We can classify, according to the eigenvalues of the matrix M_{nk} (2.13), how the flow of Hamiltonians $[\mathcal{H}_A(t)]$ (understood as vectors $\mathbf{u}_A(t)$ in the couplings space) behaves as $t \rightarrow \infty$ regarding the point \mathbf{u}^* . The latter is a known fixed point of some given Hamiltonian $[\mathcal{H}(t)]$, i.e. $\mathcal{H}^* = \lim_{t \rightarrow \infty} \mathcal{H}(t)$. If the Hamiltonian \mathcal{H}_A contains couplings in the “direction” of the eigenvector(s) \mathbf{V}^s with the eigenvalue(s) $\lambda_s < 0$, then, according to Eq. (2.20), such a Hamiltonian ($\mathcal{H}_A \Leftrightarrow \mathbf{u}_A$) flows towards the fixed-point ($\mathcal{H}^* \Leftrightarrow \mathbf{u}^*$), i.e.,

$$\lim_{t \rightarrow \infty} \mathcal{H}_A(t) = \lim_{t \rightarrow \infty} \mathcal{H}(t) = \mathcal{H}^* , \quad (2.21)$$

and those eigenvectors in the coupling space are called *irrelevant*. If $\lambda_s > 0$, then the Hamiltonian \mathcal{H}_A flows away from the fixed point ($\mathcal{H}^* \Leftrightarrow \mathbf{u}^*$), and the corresponding eigenvectors are called *relevant*. The situation when $\lambda_s = 0$ is called *marginal*, and (at least) a second-order analysis is needed in order to describe the flow of the Hamiltonian \mathcal{H}_A .

2.2 The model

We treat the problem of interacting fermions at finite temperature in the standard path integral formalism⁵⁸ using Grassmann variables for Fermi fields. The partition function is given by the path integral

$$Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{S_0 + S_{\text{int}}} , \quad (2.22)$$

wherein the free part of the action is

$$S_0 = \int_{(1)} \bar{\psi}_\alpha(\mathbf{1}) [i\omega_1 + \mu - \epsilon_0(\mathbf{K}_1)] \psi_\alpha(\mathbf{1}) . \quad (2.23)$$

We introduced the following notation:

$$\int_{(i)} \equiv \frac{1}{\beta} \int \frac{d\mathbf{K}_i}{(2\pi)^d} \sum_{\omega_i} , \quad (2.24a)$$

$$(i) \equiv (\mathbf{K}_i, \omega_i) , \quad (2.24b)$$

where β is the inverse temperature, μ the chemical potential, ω_i the fermion Matsubara frequencies and $\psi_\alpha(i)$ an N -component Grassmann field with a “flavor” index α . Summation over repeated flavor indices is implicit throughout this paper. We set $k_B = 1$ and $\hbar = 1$. The more physically interesting case of spin- $\frac{1}{2}$ fermions (electrons) corresponds to $N = 2$, but the generalization to $N \neq 2$ is straightforward, and it incorporates automatically the simpler case of spinless fermions ($N = 1$).

The general $SU(N)$ -invariant quartic interaction may be written as follows:

$$S_{\text{int}} = -\frac{1}{4} \int_{(1,2,3,4)} \bar{\psi}_\alpha(\mathbf{1}) \bar{\psi}_\beta(\mathbf{2}) \psi_\gamma(\mathbf{3}) \psi_\delta(\mathbf{4}) U_{\gamma\delta}^{\alpha\beta}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4}) \delta^{(d+1)}(\mathbf{1} + \mathbf{2} - \mathbf{3} - \mathbf{4}) . \quad (2.25)$$

Here the conservation of energy and momentum is enforced by the symbolic delta function

$$\delta^{(d+1)}(\mathbf{1} + \mathbf{2} - \mathbf{3} - \mathbf{4}) \equiv \beta(2\pi)^d \delta(\mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{K}_4) \Delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) , \quad (2.26)$$

where the discrete delta function Δ is equal to 1 if its argument is zero, and equal to zero otherwise. We presume that the density of particles in the system is kept fixed.

It can be shown in group theory (see, for instance, Refs. [66,67]) that a representation of the potential as

$$U_{\gamma\delta}^{\alpha\beta} = U_1 \delta_{\alpha\delta} \delta_{\beta\gamma} + U_2 \sum_{a=1}^{N^2-1} \lambda_{\alpha\delta}^a \lambda_{\beta\gamma}^a \quad (2.27)$$

supplies us with the most general $SU(N)$ -invariant form with two independent scalar functions U_1 and U_2 . The $N^2 - 1$ Hermitian traceless matrices $\hat{\lambda}^a$ are the generators of $SU(N)$. We need not write down explicit expressions of the commutations or other relations for those matrices. The only identity used in the following is

$$\sum_{a=1}^{N^2-1} \lambda_{\alpha\beta}^a \lambda_{\gamma\delta}^a = 2 \left(\delta_{\alpha\delta} \delta_{\beta\gamma} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\gamma\delta} \right). \quad (2.28)$$

In the $SU(2)$ case, the three generators λ^a are the usual Pauli matrices and the identity (2.28) reduces to a well-known relation involving these matrices. The decomposition (2.27) then has the same form as Eq. (1.9) used in the Landau theory for the \hat{f} -function. Using the relation (2.28), one readily checks that the form (2.27) of the interaction is indeed $SU(N)$ invariant. The fact that Eq. (2.27) is the most general $SU(N)$ invariant may be verified by counting the number of singlets in the tensor product $\bar{N} \otimes \bar{N} \otimes N \otimes N$, wherein N stands for the fundamental representation of $SU(N)$ (acting on the N -component field ψ) and \bar{N} for its conjugate. This number is indeed two, meaning that only two $SU(N)$ -invariant scalars may be constructed in this way.

One should bear in mind the difference between the $SU(N)$ -invariant interaction considered here and a rotation-invariant interaction for particles of spin s : In the absence of symmetry-breaking interactions (e.g. spin-orbit, dipole-dipole, or external fields) there will be rotational invariance in spin space, but the corresponding symmetry operations are still obtained from the three generators of $SU(2)$, although in a $(2s + 1)$ -dimensional representation. The $SU(N)$ -invariance imposed here is more stringent. The particular form of Eqs (2.27,2.28), and consequently of Eq. (2.32), for any N , is the artefact of such a symmetry enlargement. A generic, $SU(2)$ -invariant interaction with spin- s fermions would contain a greater variety of terms. Accordingly, the different components of ψ are called “flavors” if $N = 2s + 1 > 2$, in order to avoid misunderstandings.

The flavor dependence of the potential (2.27) may be factorized and expressed via two independent flavor operators. It is convenient to introduce two operators \hat{I} and \hat{T} , respectively symmetric and antisymmetric, as follows:

$$I_{\gamma\delta}^{\alpha\beta} \equiv \delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta} , \quad (2.29a)$$

$$T_{\gamma\delta}^{\alpha\beta} \equiv \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta} . \quad (2.29b)$$

These operators satisfy the properties

$$I_{\gamma\delta}^{\alpha\beta} = I_{\gamma\delta}^{\beta\alpha} = I_{\delta\gamma}^{\alpha\beta} , \quad (2.30a)$$

$$T_{\gamma\delta}^{\alpha\beta} = -T_{\gamma\delta}^{\beta\alpha} = -T_{\delta\gamma}^{\alpha\beta} , \quad (2.30b)$$

and the convolution relations

$$I_{\mu\nu}^{\alpha\beta} I_{\gamma\delta}^{\nu\mu} = 2I_{\gamma\delta}^{\alpha\beta} , \quad (2.31a)$$

$$T_{\mu\nu}^{\alpha\beta} T_{\gamma\delta}^{\nu\mu} = 2T_{\gamma\delta}^{\alpha\beta} , \quad (2.31b)$$

$$T_{\mu\nu}^{\alpha\beta} I_{\gamma\delta}^{\nu\mu} = 0 , \quad (2.31c)$$

$$I_{\gamma\nu}^{\alpha\mu} I_{\mu\delta}^{\nu\beta} = \frac{N+3}{2} I_{\gamma\delta}^{\alpha\beta} - \frac{N+1}{2} T_{\gamma\delta}^{\alpha\beta} , \quad (2.31d)$$

$$I_{\gamma\nu}^{\alpha\mu} T_{\mu\delta}^{\nu\beta} = -\frac{N-1}{2} I_{\gamma\delta}^{\alpha\beta} + \frac{N+1}{2} T_{\gamma\delta}^{\alpha\beta} , \quad (2.31e)$$

$$T_{\gamma\nu}^{\alpha\mu} T_{\mu\delta}^{\nu\beta} = \frac{N-1}{2} I_{\gamma\delta}^{\alpha\beta} - \frac{N-3}{2} T_{\gamma\delta}^{\alpha\beta} . \quad (2.31f)$$

Instead of Eq. (2.27), we may decompose the potential as follows:

$$U_{\gamma\delta}^{\alpha\beta} = U^A I_{\gamma\delta}^{\alpha\beta} + U^S T_{\gamma\delta}^{\alpha\beta} , \quad (2.32)$$

where the functions U^S and U^A have the symmetry properties

$$U^A(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4}) = -U^A(\mathbf{2}, \mathbf{1}; \mathbf{3}, \mathbf{4}) = -U^A(\mathbf{1}, \mathbf{2}; \mathbf{4}, \mathbf{3}) , \quad (2.33a)$$

$$U^S(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4}) = U^S(\mathbf{2}, \mathbf{1}; \mathbf{3}, \mathbf{4}) = U^S(\mathbf{1}, \mathbf{2}; \mathbf{4}, \mathbf{3}) . \quad (2.33b)$$

The general form (2.25) of the interaction allows us to easily recover various special cases. Spinless fermions correspond to $N=1$: matrices have only one component and $\hat{I} \equiv 2$, $\hat{T} \equiv 0$. Thus there is only one interaction function. For instance, in the spinless Hubbard model with nearest-neighbor interaction constant U^{nn} , the function U^A is expressed in terms of U^{nn} and some

combination of trigonometric functions, depending on the spatial dimension.²⁸ In the electron Hubbard model with on-site interaction constant U^{os} , the functions are $U^A = 0$ and $U^S \propto U^{\text{os}}$. Switching on a constant interaction U^{nn} between nearest neighbors, we come up with two independent functions U^S and U^A in the Hamiltonian. [Cf. Eq. (3.25) below].

The expressions (2.23) and (2.25) for the action are adequate for a microscopic, “exact” formulation of the problem. The functions U^A and U^S may incorporate the microscopic interaction of our choice: Coulomb, on-site repulsion, and so on. The integration in Eq. (2.24) is then carried over all available phase space (the Brillouin zone), with the constraint of momentum conservation. In principle, working with the microscopic Hamiltonian allows a description of physical processes at all energy scales, up to atomic energies. Here we will consider the action defined in Eqs(2.23,2.25) as an appropriate form to construct a *low-energy effective action*,^{27,28,29} which serves to describe the physical processes occurring at energy scales below some scale provided by the cutoff Λ_0 .

The bare momentum cutoff Λ_0 of the action is defined such that each vector \mathbf{K}_i in the effective action lies in a shell of thickness $2\Lambda_0$ around the Fermi surface. We denote this shell, i.e., the support of the effective action in the d -dimensional momentum space, as $C_{\Lambda_0}^d$. In principle, such a low-energy effective action could be obtained from the microscopic action by integrating out (in the functional sense) the degrees of freedom associated with the momenta lying out of a band of width $2\Lambda_0$ around the Fermi surface. Taking as an example the partition function (2.22), this can be formally expressed as follows:

$$Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{S_0[\bar{\psi};\psi] + S_{\text{int}}[\bar{\psi};\psi]} = \int_{C_{\Lambda_0}^d} \mathcal{D}\bar{\psi}^< \mathcal{D}\psi^< e^{S_0^{\text{eff}}[\bar{\psi}^<;\psi^<] + S_{\text{int}}^{\text{eff}}[\bar{\psi}^<;\psi^<]} , \quad (2.34)$$

wherein the index under the functional integration symbol indicates that Grassmann fields $\{\bar{\psi}_\alpha^<(\mathbf{i}), \psi_\alpha^<(\mathbf{i})\}$ to be integrated out have their momenta inside the support $C_{\Lambda_0}^d$. Generally speaking, the fields $\{\bar{\psi}^<, \psi^<\}$ may differ from the “original” fields $\{\bar{\psi}, \psi\}$ by some renormalization factor, which can be absorbed

in their definition. To shorten notations we will drop the superscript “<” in the following.

In general, the effective action $S_0^{\text{eff}} + S_{\text{int}}^{\text{eff}}$ can be written as an expansion in a set of Grassmann fields, with the requirements that the symmetries of the microscopic action be satisfied (for details, see, e.g., Refs.[27,28]). Considerable simplification of the problem with such complicated effective action comes from the following physical observation. As we already discussed in the introduction, the relevant physical information in studying an interacting fermion system at low temperature (i.e., the system being slightly excited near its ground state at $T = 0$) can be obtained by considering excitations at *low-energy* scales comparatively to the scale provided, say, by the Fermi energy E_F . So, in order to study low-energy processes occurring at low temperatures $T \ll E_F$, it suffices to study the *low-energy effective action* as in Eq. (2.34) with a cutoff

$$\Lambda_0 \ll K_F . \quad (2.35)$$

By making use of the small parameter $\Lambda_0/K_F \ll 1$ in the scaling analysis²⁸ [notice that $\Lambda/K_F \rightarrow 0$ under successive mode elimination inside $C_{\Lambda_0}^d$], we can start from the effective action in the form given by Eqs (2.23) and (2.25), except that the momentum integration is restricted to the vicinity of the Fermi surface $C_{\Lambda_0}^d$. The parameters of this action, such as the one-particle energy $\epsilon(\mathbf{K})$ and the interaction function, do not coincide with those of microscopic action (i.e., $\epsilon_0(\mathbf{K})$, $U^{A,S}$). In this work we will restrict ourselves to the study of low-density electron systems, so we assume that the Fermi surfaces have rotational symmetry (i.e., they are circular or spherical). The low-lying one-particle excitations can then be linearized near the Fermi energy in a simple fashion:

$$\epsilon(\mathbf{K}) - \mu \approx v_F(K - K_F) \equiv v_F k , \quad (2.36)$$

wherein the terms of order k^2 in this expansion are irrelevant in the RG sense. We write the momentum \mathbf{K}_i lying inside the support $C_{\Lambda_0}^d$ as

$$\mathbf{K}_i \in C_{\Lambda_0}^d : \mathbf{K}_i = \mathbf{K}_F^i + \mathbf{k}_i , \quad (2.37)$$

where \mathbf{K}_F^i lies on the Fermi surface and \mathbf{k}_i ($|\mathbf{k}_i| \leq \Lambda_0$) is normal to the Fermi surface at the point \mathbf{K}_F^i . In the d -dimensional integration measure we also can keep the relevant term only:

$$\int d\mathbf{K} = \int_{-\Lambda_0}^{\Lambda_0} \int_{\Omega_d} (K_F + k)^{d-1} dk d\Omega_d \simeq K_F^{d-1} \int_{-\Lambda_0}^{\Lambda_0} \int_{\Omega_d} dk d\Omega_d . \quad (2.38)$$

After all those simplifications the $SU(N)$ -invariant low-energy effective action reads:[†]

$$\begin{aligned} S_0^{\text{eff}} + S_{\text{int}}^{\text{eff}} &= \frac{K_F^{d-1}}{(2\pi)^d \beta} \sum_{\omega_1} \int_{-\Lambda_0}^{\Lambda_0} \int_{\Omega_d} dk_1 d\Omega_d^{(1)} \bar{\psi}_\alpha(\mathbf{1}) [i\omega_1 - v_F k_1] \psi_\alpha(\mathbf{1}) \\ &- \frac{1}{4} \int_{(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}) \otimes \mathfrak{C}_0} \bar{\psi}_\alpha(\mathbf{1}) \bar{\psi}_\beta(\mathbf{2}) \psi_\gamma(\mathbf{3}) \psi_\delta(\mathbf{4}) \hat{\Gamma}^{\Lambda_0}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4}) \delta^{(d+1)}(\mathbf{1} + \mathbf{2} - \mathbf{3} - \mathbf{4}) , \end{aligned} \quad (2.39)$$

wherein \mathfrak{C}_0 stands for the condition

$$\mathfrak{C}_0 : \{ \forall i = 1 \dots 4 : \mathbf{K}_i \in C_{\Lambda_0}^d \} , \quad (2.40)$$

and $\hat{\Gamma}$ has the same form as (2.32)

$$\hat{\Gamma}^{\Lambda_0}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4}) = \Gamma^{(\Lambda_0)A}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4}) I_{\gamma\delta}^{\alpha\beta} + \Gamma^{(\Lambda_0)S}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4}) T_{\gamma\delta}^{\alpha\beta} . \quad (2.41)$$

The function $\Gamma^{(\Lambda_0)A}$ [$\Gamma^{(\Lambda_0)S}$] is antisymmetric [symmetric] under the exchange ($\mathbf{1} \leftrightarrow \mathbf{2}$) and ($\mathbf{3} \leftrightarrow \mathbf{4}$). [Cf. Eq. (2.33)]. We discarded interaction terms involving higher-order derivatives, or more powers of ψ , because such terms are irrelevant at tree-level (see next Section 2.3). The constraint \mathfrak{C}_0 indicates that momentum integrations carried out along normals and solid angles are not independent. Along with the standard condition to conserve the total momentum (imposed by the momentum delta-function), the four vectors \mathbf{K}_i must also lie in the support $C_{\Lambda_0}^d$ of the effective action in momentum space. To treat the constraint \mathfrak{C}_0 in a more formal way, we can introduce an extra factor

$$\prod_{i=1}^4 \Theta(\Lambda_0 - \|\mathbf{K}_i\| - K_F) \quad (2.42)$$

[†] For practical calculations of diagrams, etc, it is more convenient to preserve in Eq. (2.39) the symmetric form of interaction in notations (2.24), (2.26), and to use (2.37) and (2.38) after the removal of all superfluous integration variables by use of the conservation laws.

in the integration measure of the interaction term in (2.39). The Matsubara frequencies in (2.39) are allowed to run over all available values. The temperature T is restricted by the condition

$$T \ll v_F \Lambda_0 , \quad (2.43)$$

demanding the initial cutoff (arbitrarily chosen scale) to be much larger than the (physical) temperature scale. Heuristically, this condition can be understood as the one allowing to probe relevant physics by applying the RG for the effective action, without intervention of transient model-related details into the final results. The bare one-particle Green's function for the free part of action S_0 is

$$\langle \psi_\alpha(\mathbf{1}) \bar{\psi}_\beta(\mathbf{2}) \rangle_0 = G_0(\mathbf{K}_1, \omega_1) \delta^{(d+1)}(\mathbf{1} - \mathbf{2}) \delta_{\alpha\beta} , \quad (2.44a)$$

$$G_0^{-1}(\mathbf{K}_1, \omega_1) = i\omega_1 + \mu - \epsilon(\mathbf{K}_1) \simeq i\omega_1 - v_F k_1 . \quad (2.44b)$$

According to Shankar's RG analysis²⁸ of the effective action (2.39) at $N = 1$, performed at zero temperature, the action describes the Fermi liquid phase of repelling spinless fermions and the BCS superconducting phase of attracting spinless fermions. We will study here the RG fixed points of the $SU(N)$ -invariant effective action (2.39), extending the RG technique to the finite temperature case.

2.3 Tree-level RG analysis

In this section we present the tree-level RG analysis of the low-energy effective action (2.39). We will follow quite closely the ideas put forward in the papers by Polchinski and Shankar.^{27,28}

The standard (Wilson) RG transformation includes the following steps.

- (i) Firstly we integrate out modes with momenta lying between Λ and Λ/s ($s > 1$).
- (ii) Then we rescale the variables

$$\begin{aligned} \omega'_n = s\omega_n & \iff \beta' = s^{-1}\beta , \\ k' = sk & , \end{aligned}$$

(iii) and the fields

$$\begin{aligned}\psi'_\alpha(\mathbf{K}_F, k', \omega'_n) &= s^{-3/2} \psi_\alpha(\mathbf{K}_F, k, \omega_n) , \\ \bar{\psi}'_\alpha(\mathbf{K}_F, k', \omega'_n) &= s^{-3/2} \bar{\psi}_\alpha(\mathbf{K}_F, k, \omega_n) .\end{aligned}$$

At tree level, this procedure reduces to a simple scaling analysis, since after the partial mode elimination (step *i*), the “new” action has the same form (2.39) as before, with the only difference that its cutoff is now Λ/s , and it has fewer “underintegrated” degrees of freedom (ψ -fields). Notice, that no matter how the step (*i*) is carried out in practical calculations (tree-level, one-loop, etc), the rescaling of the variables (*ii*) does not change the fact that after the partial mode integration (*i*), the action has fewer degrees of freedom left.

It is easy to see the invariance of the free part of the action (2.39) under transformations (*i*)-(iii). To analyze the interaction term, let us start by considering in detail the phase space constraint \mathfrak{C}_0 (2.40). Using the decomposition (2.37) for three vectors \mathbf{K}_i ($i = 1, 2, 3$) we get, from momentum conservation,

$$\mathbf{K}_4 = K_F(\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3) + k_1 \mathbf{n}_1 + k_2 \mathbf{n}_2 - k_3 \mathbf{n}_3 , \quad (2.45)$$

wherein \mathbf{n}_i is a unit vector in the direction of \mathbf{K}_i .[†] As is readily seen from the expression (2.45), the conservation law itself does not provide automatically $\mathbf{K}_4 \in C_{\Lambda_0}^d$ even if $\mathbf{K}_i \in C_{\Lambda_0}^d \ \forall \ i = 1 \dots 3$, since for a general choice of the directions of \mathbf{K}_i ($i = 1 \dots 3$), the vector \mathbf{K}_4 may lie far outside of $C_{\Lambda_0}^d$, e.g., $|\mathbf{K}_4| \sim 3K_F$. That is why the constraint (2.40) [cf. also (2.42)] is less innocuous than it might appear. For the vector \mathbf{K}_4 to lie inside $C_{\Lambda_0}^d$, the following equation must be satisfied in the limit $\Lambda/K_F \rightarrow 0$ [cf. (2.37) and (2.45)]:[‡]

$$|\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3| = 1 . \quad (2.46)$$

[†] Notice that for the action defined on a crystal, the momentum delta function in (2.26) conserves momenta up to the reciprocal lattice momentum allowing in general the Umklapp scattering processes. However for the considered case of circular (spherical) Fermi surfaces those processes are inoperative, which justifies the simple form $\mathbf{K}_1 + \mathbf{K}_2 = \mathbf{K}_3 + \mathbf{K}_4$ of the conservation law.

[‡] At any finite cutoff Λ , Eq. (2.46) gives a condition more stringent than constraint (2.40) itself. They are equivalent only when $\Lambda \rightarrow 0$.

In the rest of this section we will restrict the analysis to the two-dimensional case with a circular Fermi surface, while the three-dimensional case will be considered in a separate section. (We will, however, preserve d -dimensional notations wherever it is possible, in order to facilitate the generalization for 3D case later.) In 2D, Eq. (2.46) has only two types of solutions.

Case 1. If $\mathbf{n}_1 \neq -\mathbf{n}_2$, then the possible solutions are:

$$\mathbf{n}_1 = \mathbf{n}_3 \quad , \quad (2.47a)$$

$$\mathbf{n}_2 = \mathbf{n}_3 \quad . \quad (2.47b)$$

Case 2. If the momenta of ingoing particles are opposite, then we have

$$\mathbf{n}_1 = -\mathbf{n}_2 \quad (\mathbf{n}_3 \text{ unrestricted}) \quad . \quad (2.48)$$

Let us analyse the action's interaction term in Eq. (2.39) at a *finite* cutoff for the *special momenta configurations* satisfying Eq. (2.46). In order to preserve in the interaction only those configurations, we introduce in the interaction term (2.39) the one-dimensional delta function as an extra factor:

$$\delta(|\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3| - 1) \quad . \quad (2.49)$$

The fourth momentum and frequency can be removed from integration due to the total momentum-energy conservation, i.e., \mathbf{K}_4 is given by Eq. (2.45) and $\omega_4 = \omega_1 + \omega_2 - \omega_3$. Let us start with the momenta configuration provided by the solution (2.47). We will consider the case (a), since the solution (b) does not result in any new independent functions because of the antisymmetry of the interaction (2.41) under the exchange of ingoing (outcoming) momenta. We can arbitrarily choose two integration vectors, say \mathbf{K}_1 and \mathbf{K}_2 , running freely in C_Λ^2 and decompose them as in Eq. (2.37). The delta function (2.49) removes the angular integration over the direction of the third (independent) vector \mathbf{K}_3 , forcing it to lie along \mathbf{K}_1 . This gives us

$$\mathbf{K}_3 = \mathbf{K}_F^1 + \mathbf{k}_3 \quad , \quad (2.50)$$

wherein \mathbf{k}_3 is normal to the Fermi surface. Accordingly, for \mathbf{K}_4 we obtain

$$\mathbf{K}_4 = \mathbf{K}_F^2 + \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 . \quad (2.51)$$

This vector also lies within the phase space during the mode elimination [up to some irrelevant terms $O(\Lambda/K_F)$].

Summing all this up, we write the interacting part of the effective action in the ‘‘sector’’ governing the low-energy processes of nearly forward scattering (quasi)particles[†]

$$\begin{aligned} S_{\text{int}}^{\text{eff}} \Big|_{\text{FS}} = & -\frac{1}{4} \cdot \frac{K_F^{3(d-1)}}{(2\pi)^{3d}\beta^3} \sum_{\omega_1 \dots \omega_3} \left\{ \prod_{i=1,2} \int_{-\Lambda_0}^{\Lambda_0} dk_i \int_{\Omega_d} d\Omega_d^{(i)} \right\} \int_{-\Lambda_0}^{\Lambda_0} dk_3 \\ & \times \hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^1, \mathbf{K}_F^2 | k_n, \omega_n) \cdot [\bar{\psi}\psi]^2 . \end{aligned} \quad (2.52)$$

By following the same steps of the RG transformation (i)-(iii) as above, we readily see that at the tree level the interacting part of the action resumes its form under this transformation in terms of new (primed) variables, with the new renormalized $\hat{\Gamma}'$ given by

$$\hat{\Gamma}'(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^1, \mathbf{K}_F^2 | k'_n, \omega'_n) = \hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^1, \mathbf{K}_F^2 | k'_n/s, \omega'_n/s) . \quad (2.53)$$

Using the Taylor expansion

$$\hat{\Gamma}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^1, \mathbf{K}_F^2 | k_n, \omega_n) = \hat{\Gamma}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^1, \mathbf{K}_F^2 | 0, 0) + O(k_n, \omega_n) \quad (2.54)$$

in Eq. (2.53), we see that the first term in that expansion is marginal at tree level, i.e.,

$$\hat{\Gamma}'(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^1, \mathbf{K}_F^2 | 0, 0) = \hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^1, \mathbf{K}_F^2 | 0, 0) , \quad (2.55)$$

while the rest is irrelevant, i.e., it disappears in the limit $s \rightarrow \infty$.

To analyze then Case 2 with the solution given by Eq. (2.48), we proceed in the same manner as for Case 1. After removing the delta function according to the solution (2.48), we use the following decomposition for the momenta :

$$\begin{aligned} \mathbf{K}_1 &= \mathbf{K}_F^1 + \mathbf{k}_1 , & \mathbf{K}_3 &= \mathbf{K}_F^3 + \mathbf{k}_3 , \\ \mathbf{K}_2 &= -\mathbf{K}_F^1 + \mathbf{k}_2 , & \mathbf{K}_4 &= -\mathbf{K}_F^3 + \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 . \end{aligned} \quad (2.56)$$

[†] To shorten notations we will use a hat (^) for operators in flavor space in the sense of Eq. (2.41).

The low-energy action's interaction governing the scattering processes of (quasi)particles with nearly opposite incoming (outgoing) momenta can be written analogously to Eq. (2.52) with some minor changes:

$$S_{\text{int}}^{\text{eff}}|_{\text{BCS}} = \left((i = 1, 2) \leftrightarrow (i = 1, 3) \quad \text{and} \quad k_3 \leftrightarrow k_2 \right) \quad (2.57)$$

$$\times \hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, -\mathbf{K}_F^1; \mathbf{K}_F^3, -\mathbf{K}_F^3 | k_n, \omega_n) \cdot [\bar{\psi}\psi]^2 .$$

The same steps which lead to Eq. (2.55) allow us to conclude that the interaction function

$$\hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, -\mathbf{K}_F^1; \mathbf{K}_F^3, -\mathbf{K}_F^3 | 0, 0) \quad (2.58)$$

is marginal at tree level, while its dependence on (k_n, ω_n) is irrelevant.

It is a straightforward exercise in scaling to check the irrelevance of the terms neglected in (2.36) and (2.38), as well as of higher orders of the low-energy action's expansion over Grassmann fields, e.g., $[\bar{\psi}\psi]^3$, etc. This justifies the simple form (2.39).

To summarize this section: The tree-level scaling analysis of the effective action (2.39) shows that the relevant physical information can be obtained by studying interactions of particles scattering with small momentum and energy transfer (we will call it the *Landau (interaction) channel*), and those with nearly opposite incoming (outgoing) momenta (the *BCS (interaction) channel*). Use of the condition (2.35) allows us, instead of calculating the general function $\hat{\Gamma}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})$, to keep track of the evolution under the mode elimination of the marginal *coupling functions* [cf. (2.55, 2.58)], wherein the momenta in the set $(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})$ are chosen at special kinematic configurations and irrelevant dependences are discarded.

2.4 Coupling functions and vertices

Let us clarify the meaning of the quantities entering the effective action in terms familiar from the quantum field theory. (See, e.g., Refs. [5,42,43,58,68]). Consider the vertex function $\hat{\Gamma}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})$, constructed from the connected two-particle Green's function $\hat{G}_c^2(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4}) = -\langle \psi_\alpha(\mathbf{1}) \psi_\beta(\mathbf{2}) \bar{\psi}_\gamma(\mathbf{3}) \bar{\psi}_\delta(\mathbf{4}) \rangle_c$ by amputation of the external legs. Here $\langle \dots \rangle$ means an average with the effective action (2.39) which contains only “slow” modes, lying in the support $C_{\Lambda_0}^d$. Once auxiliary source fields (with momenta inside the shell $C_{\Lambda_0}^d$) coupled to the action's Grassmann fields $\{\psi, \bar{\psi}\}$ have been introduced, such connected n -particle Green's functions can be defined as functional derivatives of the source-dependent generating functional.⁶⁸ At tree-level, $\hat{\Gamma}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})|_{\text{tree}} = \hat{\Gamma}^{\Lambda_0}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})$. The bare vertex $\hat{\Gamma}^{\Lambda_0}$ of the effective action (2.39) can be defined in the same fashion as $\hat{\Gamma}$, with the difference that $\hat{\Gamma}^{\Lambda_0}$ is the result of averaging over the “fast” modes (those *outside* $C_{\Lambda_0}^d$) with the *microscopic* action (2.23, 2.25). Contrary to $\hat{\Gamma}$, the vertex $\hat{\Gamma}^{\Lambda_0}$ is not a physical observable, since it is not the result of an integration over all degrees of freedom.

In order to calculate physical quantities, we must perform an average with the effective action (2.39), i.e., we must integrate out the “slow” modes, which lie inside $C_{\Lambda_0}^d$, in the corresponding path integrals. This is done in Wilson's RG approach by successively integrating the high-energy modes in $C_{\Lambda_0}^d$, i.e., by progressively reducing the momentum cutoff from Λ_0 to zero. We define a RG flow parameter t such that the cutoff at an intermediate step is $\Lambda(t) = \Lambda_0 e^{-t}$. Integrating over the modes located between the cutoffs $\Lambda(t)$ and $\Lambda(t + dt)$, a recursion relation (in the form of a differential equation) can be found for the various parameters of the action.[†] This equation (or set of equations) is then solved from $t = 0$ to $t \rightarrow \infty$ and this yields the fixed-point value of the

[†] An example of such recursion formula in the *tree-level approximation* is given by Eq. (2.55). The differential form of this equation is $\partial \hat{\Gamma}^\Lambda(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^1, \mathbf{K}_F^2 | 0, 0 | t) / \partial t = 0$, wherein the parameter t (instead of s) keeps track of the mode elimination.

parameters of the action. The physical quantities are then obtained from these parameters.

A considerable simplification of this problem comes from the scaling analysis presented in the previous section. A tree-level analysis indicates that due to the smallness of the scale Λ/K_F , we need to treat the vertices at two special choices of the kinematic configurations only, and we also can discard irrelevant variables. Let us consider separately two interacting channels of the low-energy effective action.[†]

Landau Interaction Channel: Taking into account momentum and frequency conservation, we use the following (more standard) notation for the nearly forward scattering vertex:

$$\hat{\Gamma}(\mathbf{1}, \mathbf{2}; \mathbf{1} + \mathcal{Q}, \mathbf{2} - \mathcal{Q}) \equiv \hat{\Gamma}(\mathbf{1}, \mathbf{2}; \mathcal{Q}) , \quad (2.59)$$

with the transfer vector

$$\mathcal{Q} = \mathbf{3} - \mathbf{1} \equiv (\mathbf{Q}, \Omega) \quad (2.60)$$

such that $Q \ll K_F$ (Ω is a bosonic Matsubara frequency). When the initial cutoff Λ_0 satisfies condition (2.43), we can unambiguously define a *bare* coupling function which depends *only* on the angle between the incoming (or outgoing) momenta. This bare coupling function is given by the vertex $\hat{\Gamma}^{\Lambda_0}(\mathbf{1}, \mathbf{2}; \mathcal{Q})$ in the zero transfer limit ($Q = 0$) where the two external momenta are put on the Fermi surface and the external frequencies are $\omega_{\min} \equiv \pi T$ (the latter will be dropped from now on).

$$\hat{U}(\mathbf{K}_F^1, \mathbf{K}_F^2) \equiv \frac{1}{2} \nu_F \hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, \mathbf{K}_F^2; 0) , \quad (2.61)$$

[†] We find the term “channel” more conventional for the condensed matter problem under consideration, rather than the term “sector” borrowed from high-energy physics. However, in some papers the term “channel” is applied in the sense of selecting particular diagrams, while here we singled out two specific scattering (interacting) configurations, and accordingly, the low-energy action’s total interaction is a sum over two channels (sectors), i.e., $S_{\text{int}}^{\text{eff}} = S_{\text{int}}^{\text{eff}}|_{\text{FS}} + S_{\text{int}}^{\text{eff}}|_{\text{BCS}}$

where $\nu_F = 2S_d K_F^{d-1} / (2\pi)^d v_F$ is the density of electron states at the Fermi level and S_d is the area of the d -dimensional unit sphere. In 2D each vector \mathbf{K}_F^a may be specified by a plane polar angle θ_a ; because of rotation invariance, the function \hat{U} may only depend on the relative angle $\theta_1 - \theta_2$ between \mathbf{K}_F^1 and \mathbf{K}_F^2 :

$$\hat{U}(\theta_1 - \theta_2) = \mathcal{U}^A(\theta_1 - \theta_2) I_{\gamma\delta}^{\alpha\beta} + \mathcal{U}^S(\theta_1 - \theta_2) T_{\gamma\delta}^{\alpha\beta} \quad (2.62)$$

It should be pointed out that the function $\mathcal{U}^A(\theta)$ is not an antisymmetric function of its argument.²⁸ It follows from the symmetry properties of the interaction [cf. Eqs (2.30,2.41)] that

$$\mathcal{U}^{A,S}(\theta_1 - \theta_2) = \mathcal{U}^{A,S}(\theta_2 - \theta_1) \quad (2.63)$$

The only remnant of the antisymmetry of $\hat{\Gamma}^{\Lambda_0}$ (the Pauli principle) is the condition:

$$\mathcal{U}^A(0) = 0 . \quad (2.64)$$

As it will be shown below, the tree-level picture [cf. Eqs (2.53,2.55)] becomes more complicated when we carry out the mode elimination inside C_Λ^2 . It turns out that simply discarding the frequency dependence of $\hat{\Gamma}^\Lambda$ and identifying the momenta $\mathbf{K}_F^1 \rightleftharpoons \mathbf{K}_F^3$, $\mathbf{K}_F^2 \rightleftharpoons \mathbf{K}_F^4$ is an ill-defined procedure when the running cutoff Λ becomes of the order of the temperature ($v_F \Lambda \sim T$). The ambiguity arises when calculating the one loop-contribution from, say, the ZS graph (see Fig. 1), since this contribution is not an analytic function of the transfer \mathcal{Q} at $\mathcal{Q} = 0$.^{5,7,47} To describe correctly the parameters of the Fermi liquid, one should retain the dependence of the coupling function $\hat{\Gamma}^\Lambda(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathcal{Q})$ on the energy-momentum transfer \mathcal{Q} .[†] Retaining this \mathcal{Q} -dependence allows, e.g., to carry out the calculation of response functions for small \mathcal{Q} or collective modes of the Fermi liquid (see Ref. [69]). The expression for the effective action in the

[†] Expansion (2.54) is based on the assumption of the analyticity of $\hat{\Gamma}$ near $\mathcal{Q} = 0$ at any $s(t)$. Appearance of the non-analytic contributions to the renormalized $\hat{\Gamma}$ from the one-loop diagrams shows that this assumption, generally speaking, is not true.

Landau channel with the coupling function $\hat{\Gamma}^\Lambda(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathcal{Q})$, which allows us to venture calculations beyond the tree-level, can be readily written as

$$S_{\text{int}}^{\text{eff}} \Big|_{\text{L}} = -\frac{1}{4} \cdot \frac{K_F^{2(d-1)}}{(2\pi)^{2d}\beta^3} \sum_{\omega_1 \dots \omega_3} \left\{ \prod_{i=1,2} \int_{-\Lambda_0}^{\Lambda_0} dk_i \int_{\Omega_{i1}} d\Omega_d^{(i)} \right\} \int \frac{d\mathbf{Q}}{(2\pi)^d} \hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathcal{Q}) \\ \times \bar{\psi}_\alpha(\mathbf{K}_1, \omega_1) \bar{\psi}_\beta(\mathbf{K}_2, \omega_2) \psi_\gamma(\mathbf{K}_1 + \mathbf{Q}, \omega_1 + \Omega) \psi_\delta(\mathbf{K}_2 - \mathbf{Q}, \omega_2 - \Omega) . \quad (2.65)$$

For purposes of the present study we define two limiting forms of the coupling function ($\hat{\Gamma}^{\mathcal{Q}}$ and $\hat{\Gamma}^\Omega$), depending on the order in which the limits of zero momentum- (\mathbf{Q}) and energy-transfer (Ω) are taken:

$$\hat{\Gamma}^{\mathcal{Q}}(\theta, t) = \lim_{\mathcal{Q} \rightarrow 0} \left[\hat{\Gamma}(\theta; \mathcal{Q}, t) \Big|_{\Omega=0} \right] , \quad (2.66a)$$

$$\hat{\Gamma}^\Omega(\theta, t) = \lim_{\Omega \rightarrow 0} \left[\hat{\Gamma}(\theta; \mathcal{Q}, t) \Big|_{\mathcal{Q}=0} \right] . \quad (2.66b)$$

We use dimensionless vertices, by including in their definition the factor $\frac{1}{2}\nu_F$, like in Eq. (2.61). In order to unclutter notations for superscripts, we indicate the function's dependence on the running cutoff $\Lambda(t)$ by adding the parameter t to the set of variables, and even the latter will not be indicated explicitly, unless necessary. The four functions $\Gamma^{Q(A,S)}(\theta)$ and $\Gamma^{\Omega(A,S)}(\theta)$ [understood in the sense of decomposition (2.62)] are even functions of the angle θ . We will indiscriminately call these functions (running) vertices.

In the rotationally invariant 2D case it is convenient to expand the coupling functions in Fourier series:

$$\mathbf{X}(\theta) = \sum_{l=-\infty}^{\infty} e^{-il\theta} \mathbf{X}_l , \quad (2.67a)$$

$$\mathbf{X}_l = \int_0^{2\pi} \frac{d\theta}{2\pi} \mathbf{X}(\theta) e^{il\theta} , \quad (2.67b)$$

where \mathbf{X} stands for the set of all functions

$$\mathbf{X} = \{ \mathcal{U}^A, \mathcal{U}^S, \Gamma^{Q(A)}, \Gamma^{Q(S)}, \Gamma^{\Omega(A)}, \Gamma^{\Omega(S)} \} . \quad (2.68)$$

In terms of Fourier components, the symmetry property $\mathbf{X}(\theta) = \mathbf{X}(-\theta)$ becomes $\mathbf{X}_l = \mathbf{X}_{-l}$. Another consequence of the Pauli principle, namely, Eq. (2.64) becomes

$$\sum_{l=-\infty}^{\infty} \mathcal{U}_l^A = 0 . \quad (2.69)$$

BCS Interaction Channel: The two incoming (outgoing) momenta are opposite. We introduce another pair of dimensionless marginal coupling functions:

$$\begin{aligned}\hat{V}(\mathbf{K}_F^1, \mathbf{K}_F^3) &\equiv \frac{1}{2} \nu_F \hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, -\mathbf{K}_F^1; \mathbf{K}_F^3, -\mathbf{K}_F^3) \\ &= V^A(\theta_{13}) I_{\gamma\delta}^{\alpha\beta} + V^S(\theta_{13}) T_{\gamma\delta}^{\alpha\beta} ,\end{aligned}\tag{2.70}$$

wherein $\theta_{13} \equiv \theta_1 - \theta_3$. From the symmetry properties of the coupling function $\hat{\Gamma}^{\Lambda_0}$ under exchange of momenta, we find

$$V^A(\theta \pm \pi) = -V^A(\theta) ,\tag{2.71a}$$

$$V^S(\theta \pm \pi) = V^S(\theta) .\tag{2.71b}$$

In terms of Fourier components these equations become

$$\begin{aligned}V_l^A &= 0 \quad (l \text{ even}) , \\ V_l^S &= 0 \quad (l \text{ odd}) .\end{aligned}\tag{2.72}$$

Contrary to the coupling function $\hat{\Gamma}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathcal{Q}, t)$, the function $\hat{V}(\mathbf{K}_F^1, \mathbf{K}_F^3, t)$ stays single-valued after the irrelevant parameters are discarded. The only exception is the special case $\mathbf{K}_F^1 = \mathbf{K}_F^3$ when there is an overlap in the definitions of the coupling functions in two different channels. In terms of angles we have

$$\hat{\Gamma}(\theta_{12} = \pi; \mathcal{Q} = 0, t) = \hat{V}(\theta_{13} = 0, t) .\tag{2.73}$$

In the following we disregard this zero-transfer ambiguity of the \hat{V} -function, since it occurs in the domain of zero measure of its variable θ_{13} .

Let us summarize: The low-energy effective (Wilsonian) action (2.39) with the ultraviolet momentum cutoff Λ_0 is defined on the support $C_{\Lambda_0}^d$ in momentum space. From a field-theoretic point of view, the effective action's interaction $\Gamma^{\Lambda_0}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})$ is the four-point 1PI vertex calculated from a “microscopic” (“exact”) action with the infrared cutoff Λ_0 .⁶⁴ From the tree-level analysis of the effective action one can identify two interaction channels wherein coupling functions are marginal. The tree-level (bare) coupling functions are presumably analytic functions of their variables. These couplings are functions of less variables than the vertex $\Gamma^{\Lambda_0}(\mathbf{1}, \mathbf{2}; \mathbf{3}, \mathbf{4})$ since their irrelevant dependences

can be discarded. The bare couplings are constructed from the vertex Γ^{Λ_0} by a suitable choice of its variables (see how the coupling functions \hat{U} and \hat{V} are defined by Eqs (2.61,2.70), respectively). While performing the RG calculations in the phase space $C_{\Lambda_0}^d$ beyond tree level, we need to find the fixed points of the running vertices (renormalized couplings) in two interaction channels.

Landau channel: There are two limiting forms of the vertex $\hat{\Gamma}^Q$ and $\hat{\Gamma}^\Omega$ in this channel. The bare coupling \hat{U} has an unambiguous meaning only as the common initial point of the RG flow trajectories of $\hat{\Gamma}^Q$ and $\hat{\Gamma}^\Omega$. The fixed point values $\hat{\Gamma}^{Q*} \equiv \hat{\Gamma}^Q(t = \infty)$ and $\hat{\Gamma}^{\Omega*} \equiv \hat{\Gamma}^\Omega(t = \infty)$ are physical observables: the first one is the Q -limit of the vertex $\hat{\Gamma}$ (as defined at the beginning of this section) and is the scattering amplitude of quasiparticles with all four external momenta lying on the Fermi surface. The second one is the unphysical limit (Ω -limit) of the vertex $\hat{\Gamma}$ and is identified with the Landau function.³

BCS channel: The fixed point of the coupling function $\hat{V}^* \equiv \hat{V}(t = \infty)$ gives the total vertex (scattering amplitude) of quasiparticles with opposite momenta lying on the Fermi surface. The behavior of this fixed point in the BCS interaction channel controls the stability of the interacting fermion system against Cooper pairing.

CHAPTER III

RG without interference

In this chapter we present the results of the RG treatment of the Landau and Bardeen-Cooper-Schrieffer (BCS) interactions channels. The finite-temperature RG technique allows us to recover the standard results of the FLT by going one step beyond the tree-level approximation, i.e., by carrying out calculations at the one-loop RG level. This chapter is organized as follows. In Sec. 3.1 we consider the Landau interaction channel. By explicitly taking into account the direct particle-hole loop (ZS) contribution to the RG flow equations, we recover the classic relationship between the Landau function and the forward scattering amplitude. We show that such a RG treatment of the interacting fermions is tantamount to the field-theoretic approach to FLT, which applies the Bethe-Salpeter equation to the four-point vertex. In Sec. 3.2 we solve the RG equations for the marginal vertex in the BCS interaction channel. This vertex is marginally irrelevant if the bare value of the vertex satisfies Landau's theorem for the stability of the Fermi liquid against Cooper pairing at arbitrary momentum. Otherwise a BCS instability occurs at finite temperature. In Sec. 3.3 the results of the previous sections, explicitly derived for spatial dimension two, are generalized for the three-dimensional case. In Sec. 3.4 we calculate the compressibility and the spin susceptibility in the RG framework.

3.1 The Landau channel

There are three Feynman diagrams contributing to the RG flow at the one-loop level (see Fig. 1), denoted ZS (zero sound), ZS' (Peierls), and BCS .

The formal analytic expression of the ZS graph is

$$ZS = - \int_{(5)} \Gamma_{36}^{15}(\mathbf{1}, \mathbf{5}; \mathbf{1} + \mathcal{Q}, \mathbf{5} - \mathcal{Q}) \Gamma_{54}^{62}(\mathbf{5} - \mathcal{Q}, \mathbf{2}; \mathbf{5}, \mathbf{2} - \mathcal{Q}) G(\mathbf{5}) G(\mathbf{5} - \mathcal{Q}) , \quad (3.1)$$

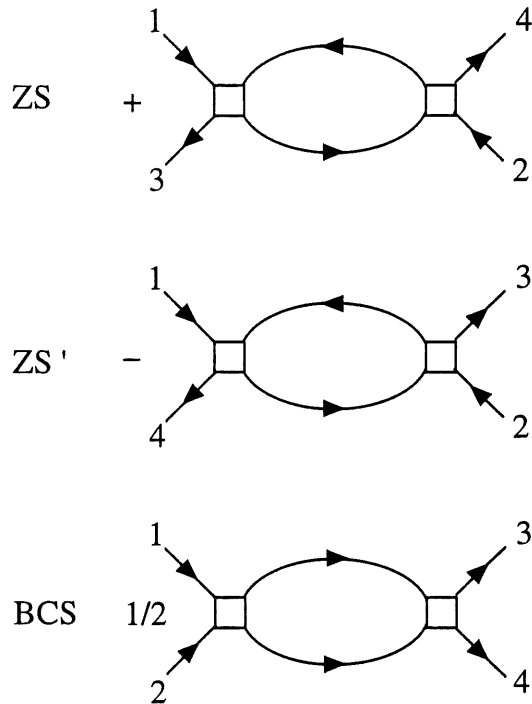


Figure 1: The three diagrams contributing to the RG flow at one-loop.

wherein the transfer vector \mathcal{Q} is given by (2.60) and for brevity the flavor Greek indices, e.g., α_i , are shown by their numbers i only. The momentum and energy conservation is already taken into account in (3.1). To calculate the contribution of this graph to the RG flow of $\Gamma^{\mathcal{Q}}$ and Γ^{Ω} , we only need to keep the dependence on the momenta \mathbf{K}_F^i and on the transfer \mathcal{Q} in the vertices on the r.h.s. of (3.1). Summations over flavor indices and Matsubara frequencies can be done easily using Eq. (2.31) and standard techniques^{5,58} for working with temperature Green's functions. In this and the next sections we will carry out explicit calculations for the 2D case, and the 3D case will be considered separately in Sec. 3.3. The phase space restrictions are satisfied automatically for any $\mathbf{K}_5 \in C_{\Lambda}^2$ in the limit $\mathcal{Q} \rightarrow 0$. When \mathbf{K}_1 and \mathbf{K}_2 lie on the Fermi surface and $\mathcal{Q} \rightarrow 0$, the r.h.s. of (3.1) contains both vertices of type (2.66) with \mathbf{K}_F^5 running freely around the Fermi surface during the angular integration. Thus, for this graph, all the phase space is available for integration. The

summation over ω_5 of the Green's functions product on the r.h.s. of (3.1) gives

$$\begin{aligned}
 & \frac{1}{\beta} \sum_{\omega_5} G(\mathbf{5})G(\mathbf{5} - \mathbf{Q}) = \\
 & = -\frac{1}{2} \frac{\tanh\left[\frac{\beta}{2}(\epsilon(\mathbf{K}_5) - \mu)\right] - \tanh\left[\frac{\beta}{2}(\epsilon(\mathbf{K}_5 - \mathbf{Q}) - \mu)\right]}{-i\Omega + \epsilon(\mathbf{K}_5) - \epsilon(\mathbf{K}_5 - \mathbf{Q})} = (\mathbf{Q} \rightarrow 0) \\
 & = \frac{1}{2} \frac{\sinh(\beta v_F \mathbf{K}_5 \cdot \mathbf{Q}/2K_F)}{i\Omega - v_F \mathbf{K}_5 \cdot \mathbf{Q}/K_F} \cosh^{-1}\left[\frac{1}{2}\beta v_F k_5\right] \cosh^{-1}\left[\frac{\beta v_F}{2K_F} \mathbf{K}_F^5 \cdot (\mathbf{k}_5 + \mathbf{q})\right] .
 \end{aligned} \tag{3.2}$$

This expression (3.2) gives zero in the Ω -limit of $\mathbf{Q} \rightarrow 0$ [cf. definition (2.66)], and thus

$$\left. \frac{\partial \hat{\Gamma}^\Omega(\theta_1 - \theta_2)}{\partial t} \right|_{ZS} = 0 . \tag{3.3}$$

The \mathbf{Q} -limit of the same product gives a factor $\frac{1}{4}\beta \cosh^{-2}(\beta v_F k_5/2)$, and accordingly

$$\left. \frac{\partial \hat{\Gamma}^Q(\theta_1 - \theta_2)}{\partial t} \right|_{ZS} = \frac{\beta_R}{\cosh^2(\beta_R)} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \hat{\Gamma}^Q(\theta_1 - \theta) \hat{\Gamma}^Q(\theta - \theta_2) , \tag{3.4}$$

where we introduced a dimensionless temperature flow parameter:

$$\beta_R(t) \equiv \frac{1}{2} v_F \beta \Lambda(t) . \tag{3.5}$$

We now turn our attention to the ZS' graph. Its analytic form is

$$ZS' = \int_{(5)} \Gamma_{46}^{15}(\mathbf{1}, \mathbf{5}; \mathbf{1} + \mathbf{Q}', \mathbf{5} - \mathbf{Q}') \Gamma_{53}^{62}(\mathbf{5} - \mathbf{Q}', \mathbf{2}; \mathbf{5}, \mathbf{2} - \mathbf{Q}') G(\mathbf{5})G(\mathbf{5} - \mathbf{Q}') , \tag{3.6}$$

wherein

$$\mathbf{Q}' \equiv \mathbf{2} - \mathbf{1} - \mathbf{Q} \tag{3.7}$$

can be thought of as an “effective” transfer vector for this graph. For $|\mathbf{K}_2 - \mathbf{K}_1| \neq 0$ the limit $\mathbf{Q} \rightarrow 0$ of the r.h.s. of (3.6) is single-valued and equivalent to the Q -limit.⁴⁷ The Green's functions contribution to this graph is

$$\left. \frac{1}{\beta} \sum_{\omega_5} G(\mathbf{5})G(\mathbf{5} - \mathbf{Q}') \right|_{\mathbf{Q}=0} = -\frac{1}{2} \frac{\tanh\left[\frac{\beta}{2}(\epsilon(\mathbf{K}_5) - \mu)\right] - \tanh\left[\frac{\beta}{2}(\epsilon(\mathbf{K}_5 - \mathbf{K}_2 - \mathbf{K}_1) - \mu)\right]}{\epsilon(\mathbf{K}_5) - \epsilon(\mathbf{K}_5 - \mathbf{K}_2 - \mathbf{K}_1)} \tag{3.8}$$

where if $|\theta_1 - \theta_2| \ll T/v_F K_F$ then the r.h.s. of Eq. (3.8) is approximately

$$-\frac{1}{4}\beta \cosh^{-2}(\beta v_F k_5/2) . \quad (3.9)$$

Thus, contrary to Eqs (3.3,3.4), for the ZS' contribution to the RG flow we have:

$$\left. \frac{\partial \hat{\Gamma}^\Omega(\theta_1 - \theta_2)}{\partial t} \right|_{ZS'} = \left. \frac{\partial \hat{\Gamma}^Q(\theta_1 - \theta_2)}{\partial t} \right|_{ZS'} \quad (3.10)$$

The explicit calculation of the ZS' contribution to the RG flow is more subtle than that for the ZS graph, since even in the zero-transfer limit $Q \rightarrow 0$ (in any order), the vector $\mathbf{Q}'|_{Q \rightarrow 0} = \mathbf{K}_2 - \mathbf{K}_1$ is free to take any modulus in the interval $[0, 2K_F]$ as the angle $\theta_1 - \theta_2$ varies. A large \mathbf{Q}' kicks the vertex momenta on the r.h.s. of (3.6) outside of C_Λ^2 , even if $\mathbf{K}_5 \in C_\Lambda^2$. In such cases the contribution of the ZS' graph is cut off,[†] except for special positions of the vector \mathbf{K}_5 . Only for angles $|\theta_1 - \theta_2| \lesssim \Lambda/K_F$ is all the phase space available for integration. If the cutoff of the low-energy effective action is chosen according to the condition (2.35), then those angles satisfy $|\theta_1 - \theta_2| \lesssim \Lambda/K_F \ll 1$. Thus, in terms of the Fourier components (2.67), the ZS' contribution to the RG flow of the vertices has the extra factor $\Lambda/K_F \ll 1$ and is inferior to the leading contribution (3.4). The same kind of analysis shows that the BCS graph also gives subleading contribution to the RG flow in the Landau interaction channel.

The analysis of the contributions of each of the three one-loop diagrams from the RG viewpoint (i.e., relevance vs irrelevance) is only one side of the problem. Another question is the symmetry properties of those contributions. Note that among the three one-loop contributions to RG flow of the vertex $\hat{\Gamma}$ with arbitrary incoming and outgoing external momenta and frequencies, only the BCS graph possesses the total antisymmetry of the vertex under exchange $(\mathbf{1} \leftrightarrow \mathbf{2})$ and $(\mathbf{3} \leftrightarrow \mathbf{4})$, while those of the ZS and ZS' graphs separately

[†] As for the reason for the elimination of the ZS' contribution, we refer the reader to the sharp cutoff functions (2.42). That extra factor of Θ -functions, attached to the vertex $\hat{\Gamma}^\Lambda$, was introduced in order to keep track of the condition \mathcal{C} for each momentum. When $\Lambda_0 \mapsto \Lambda$ we denote the change in the constraint by $\mathcal{C}_0 \mapsto \mathcal{C}$ [cf. (2.40)].

do not: only their combined contribution ($ZS + ZS'$) is antisymmetric under exchange of incoming (or outgoing) particles. To respect the Pauli principle, it is therefore necessary to take into account both the ZS and ZS' contributions to the RG flow. Neglecting the symmetry-preserving contribution of the BCS diagram, the RG equations for $\hat{\Gamma}^{Q,\Omega}$ which take into account the graphs ZS and ZS' on the same footing, can be written in the *implicit form*:

$$\frac{\partial \hat{\Gamma}^Q}{\partial t} = \frac{\partial \hat{\Gamma}^Q}{\partial t} \Big|_{ZS} + \frac{\partial \hat{\Gamma}^Q}{\partial t} \Big|_{ZS'} , \quad (3.11a)$$

$$\frac{\partial \hat{\Gamma}^\Omega}{\partial t} = \frac{\partial \hat{\Gamma}^\Omega}{\partial t} \Big|_{ZS'} = \frac{\partial \hat{\Gamma}^Q}{\partial t} \Big|_{ZS'} . \quad (3.11b)$$

As is known from the field-theoretic approach to FLT^{3,5}, the two limits $\hat{\Gamma}^{(Q,\Omega)}$ of the four-point vertex function with all four momenta lying on the Fermi surface have distinct symmetry properties. One of these limits ($\hat{\Gamma}^\Omega$), which does not satisfy the Pauli principle in the form $\hat{\Gamma}^A(\theta = 0) = 0$ [cf. (2.64)], yields the Landau function. The other one ($\hat{\Gamma}^Q$), which preserves the antisymmetry of the vertex, gives the total forward scattering amplitude. For details see, for instance, the especially elucidative paper by Mermin.⁴⁷ According to the analysis following Eqs (3.10), the ZS' contribution to the flows (3.11) of the functions $\hat{\Gamma}^{\Omega,Q}(\theta, t)$ is localized in the narrow vicinity of $\theta = 0$ when $\Lambda/K_K \ll 1$. In particular, the ZS' graph generates the flow for $\hat{\Gamma}^\Omega(\theta, t)$ at $\theta = 0$, breaking the condition $\Gamma^{\Omega(A)*}(\theta = 0) = 0$ for the fixed-point value of the running vertex (the Landau function). The same ZS' contribution to the flow of $\hat{\Gamma}^Q$ cancels $\partial \Gamma^{Q(A)}(\theta, t)/\partial t$ at $\theta = 0$, according to Eq. (3.4) and Eqs (3.6,3.9,3.11a).

We postpone until the next chapter the solution of the RG equations (3.11) which explicitly take into account the subleading (irrelevant) corrections from the ZS' graph and leads to results beyond those of the standard FLT. We will concentrate now on the approximation leading to the familiar results for the Landau Fermi Liquid.

The approximation consists in retaining in the RG equations the contribution of the ZS graph only. After a summation over flavor indices and

a Fourier transformation, we end up with the following equations for the components (A, S) :

$$\frac{\partial \Gamma_l^{\Omega(A,S)}}{\partial t} \approx \frac{\partial \Gamma_l^{\Omega(A,S)}}{\partial t} \Big|_{ZS} = 0 , \quad (3.12a)$$

$$\frac{\partial \Gamma_l^{Q(A,S)}}{\partial t} \approx \frac{\partial \Gamma_l^{Q(A,S)}}{\partial t} \Big|_{ZS} = \frac{\beta_R}{\cosh^2 \beta_R} P_l^{(A,S)} , \quad (3.12b)$$

$$\frac{\partial \beta_R}{\partial t} = -\beta_R . \quad (3.12c)$$

Here we introduced the following notation:

$$P_l^A \equiv \frac{N+3}{2} [\Gamma_l^{Q(A)}]^2 - (N-1) \Gamma_l^{Q(A)} \Gamma_l^{Q(S)} + \frac{N-1}{2} [\Gamma_l^{Q(S)}]^2 , \quad (3.13a)$$

$$P_l^S \equiv -\frac{N+1}{2} [\Gamma_l^{Q(A)}]^2 + (N+1) \Gamma_l^{Q(A)} \Gamma_l^{Q(S)} - \frac{N-3}{2} [\Gamma_l^{Q(S)}]^2 . \quad (3.13b)$$

As follows from Eq. (3.12a), the functions $\Gamma^{\Omega(A,S)}(\theta, t)$ are approximate RG invariants, i.e.,

$$\Gamma^{\Omega(A,S)}(\theta, t) \approx \Gamma^{\Omega(A,S)*}(\theta) . \quad (3.14)$$

The effect of the narrow-angular contribution to the RG flow from the ZS' diagram is not included in Eq. (3.12a), so we have to relax the symmetry condition (2.64), valid only for the initial point $t = 0$ where $\hat{\Gamma}^{\Omega}(\theta, t = 0) = \hat{u}(\theta)$, and consequently

$$\Gamma^{\Omega(A)*}(\theta = 0) = \sum_{l=-\infty}^{\infty} \Gamma_l^{\Omega(A)*} \neq 0 . \quad (3.15)$$

It is convenient to express the components $\Gamma^{\Omega(A)}$ and $\Gamma^{\Omega(S)}$ in terms of “charge” (F) and “flavor” (G) functions, defined as follows:

$$F \equiv (N-1) \Gamma^{\Omega(S)} - (N+1) \Gamma^{\Omega(A)} , \quad (3.16a)$$

$$G \equiv -\Gamma^{\Omega(S)} - \Gamma^{\Omega(A)} . \quad (3.16b)$$

The physical meaning of these functions becomes clear in the case of real electron spin ($N = 2$), when at the fixed point they coincide with the components of the Landau \hat{f} -function as it was defined above in Eq. (1.9) (We use the standard Russian notations of Ref. [7].)

In order to decouple Eqs (3.12b) we introduce new components of the running vertex $\hat{\Gamma}^Q$:

$$A \equiv (N - 1)\Gamma^{Q(S)} - (N + 1)\Gamma^{Q(A)} , \quad (3.17a)$$

$$B \equiv -\Gamma^{Q(S)} - \Gamma^{Q(A)} . \quad (3.17b)$$

Their RG fixed points in the case $N = 2$ coincide with the components of the total scattering amplitude in the accepted notations of FLT.⁷ (The components Γ^S and $-\Gamma^A$ are respectively called singlet and triplet amplitudes in Ref. [9]). Notice also the symmetries $A_l = A_{-l}$ and $B_l = B_{-l}$. To simplify the system of Eqs (3.12b) and (3.12c) we use an auxiliary RG parameter

$$\tau \equiv \tanh \beta_R , \quad \tau \in [0, \tau_0] , \quad (3.18)$$

wherein

$$\tau_0 \equiv \tanh \beta_0 \leq 1 , \quad \beta_0 \equiv \frac{1}{2} v_F \beta \Lambda_0 . \quad (3.19)$$

Expressed in terms of these new variables, the RG equations become much simpler:

$$\frac{\partial A_l}{\partial \tau} = A_l^2 , \quad (3.20a)$$

$$\frac{\partial B_l}{\partial \tau} = B_l^2 . \quad (3.20b)$$

From Eqs (3.20) we easily obtain the analytic solutions:

$$A_l(\tau) = \frac{A_l^{(0)}}{1 + A_l^{(0)}(\tau_0 - \tau)} , \quad (3.21a)$$

$$B_l(\tau) = \frac{B_l^{(0)}}{1 + B_l^{(0)}(\tau_0 - \tau)} , \quad (3.21b)$$

wherein zeros denote bare values of functions at $\tau = 0$ ($t = 0$).[†] Notice that, because of the condition $T \ll v_F \Lambda_0$ imposed on the low-energy effective action,

[†] Throughout the manuscript asterisks and zeros respectively denote fixed points (i.e. solutions at $t = \infty$) and bare values at the initial point of the RG flow.

we can set $\tau_0 = 1$ for all practical purposes. Eq. (3.14) gives us $A_l^{(0)} = F_l^*$ and $B_l^{(0)} = G_l^*$. Thus, for the fixed points we obtain:

$$A_l^* = \frac{F_l^*}{1 + F_l^*} , \quad (3.22a)$$

$$B_l^* = \frac{G_l^*}{1 + G_l^*} . \quad (3.22b)$$

From Eqs (3.21,3.22) it follows that if the parameters $\{F_l^*, G_l^*\}$ satisfy the Pomeranchuk conditions (1.11) then the system will remain stable in the Landau channel at any temperature. The fixed point (3.22) gives the same solution for the forward scattering vertex as the Bethe-Salpeter equation in the zero-temperature technique.^{5,7}

To preserve the Pauli principle for the scattering vertex, the condition

$$\Gamma^{Q(A)*}(\theta = 0) = \sum_{l=-\infty}^{\infty} [A_l^* + (N - 1)B_l^*] = 0$$

should be fulfilled. In the standard FLT⁷ it implies that a sum constructed from the components of the Landau function related to the scattering amplitudes by Eqs (3.22), and which has the form

$$\sum_{l=-\infty}^{\infty} \left[\frac{F_l^*}{1 + F_l^*} + (N - 1) \frac{G_l^*}{1 + G_l^*} \right] \quad (3.23)$$

must be equal to zero. This specific form of the condition is usually called the *amplitude sum rule* in FLT for electrons ($N = 2$). (See, e.g., Ref. [7,9].) However, we will not impose the amplitude sum rule, since, as will be shown in the next chapter, this sum rule is a mere artefact of the approximation (whether in the form of the RG equations (3.20), or of the Bethe-Salpeter equation) which explicitly takes into account the ZS loop contribution only.

3.2 RG equations in the BCS channel

Let us now analyze, at the one-loop RG level, the behavior of the coupling function \hat{V} defined by (2.70) above. We recall that there are three one-loop graphs contributing to the RG flow. (Cf. Fig. 1.) Proceeding in the same fashion as above while analyzing the Landau channel, we now find that the only contribution with all the phase space available for integration at arbitrary angle between vectors \mathbf{K}_F^1 and \mathbf{K}_F^3 , comes from the *BCS* graph. Since the BCS diagram preserves by itself the symmetry (2.71) of the function \hat{V} (the consequence of the Pauli principle), it makes life simpler. The RG equations are derived easily:

$$\frac{\partial V_l^A}{\partial t} = \tanh \beta_R (V_l^A)^2 \quad (l \text{ odd}) , \quad (3.24a)$$

$$\frac{\partial V_l^S}{\partial t} = -\tanh \beta_R (V_l^S)^2 \quad (l \text{ even}) . \quad (3.24b)$$

At zero temperature, Eq. (3.24a) coincides with Shankar's result²⁸ for spinless fermions.

The reader might be slightly confused by the sign of the r.h.s. of Eq. (3.24a) when comparing our result with that of Shankar for the spinless Hubbard model. In fact, this sign is just a consequence of the way we define the coupling function \hat{U} in this work. Indeed, let us consider the Hubbard Hamiltonian on a d -dimensional hypercubic lattice (with lattice constant a) at low filling, with on-site (U^{os}) and nearest-neighbor (U^{nn}) interactions. Both constants are positive for a repulsive interaction, and negative for attraction. Fourier-transforming and antisymmetrizing the interaction, we end up with the following coupling functions of the microscopic Hamiltonian:

$$U_{\text{mic}}^A(\mathbf{K}_1, \mathbf{K}_2; \mathbf{K}_3, \mathbf{K}_4) \simeq -\frac{1}{4}a^2 U^{\text{nn}} \cdot (\mathbf{K}_1 - \mathbf{K}_2) \cdot (\mathbf{K}_3 - \mathbf{K}_4) , \quad (3.25a)$$

$$U_{\text{mic}}^S(\mathbf{K}_1, \mathbf{K}_2; \mathbf{K}_3, \mathbf{K}_4) \simeq U^{\text{os}} + U^{\text{nn}} . \quad (3.25b)$$

Here, only the lowest order terms in the expansion of trigonometric functions were kept. We can use this result as the lowest-order approximation for the bare coupling functions of $S_{\text{int}}^{\text{eff}}$ in the definitions (2.61) and (2.70).

The RG equations (3.24) may be solved easily, and their fixed points are:

$$V_l^{A(*)} = \frac{V_l^{A(0)}}{1 - V_l^{A(0)} \int_0^\infty \tanh(\beta_0 e^{-t}) dt} , \quad (3.26a)$$

$$V_l^{S(*)} = \frac{V_l^{S(0)}}{1 + V_l^{S(0)} \int_0^\infty \tanh(\beta_0 e^{-t}) dt} . \quad (3.26b)$$

The system remains stable in the \hat{V} channel at any temperature if, for all harmonics, the following conditions are fulfilled:

$$V_l^{A(0)} < 0 , \quad (3.27a)$$

$$V_l^{S(0)} > 0 . \quad (3.27b)$$

At low temperature (i.e., when the low-energy action approach is supposed to work well), $\beta_0 \gg 1$ and the integrals of Eqs (3.26) can be evaluated exactly as in the theory of superconductivity (cf. section 33.3 of Ref. [5]):

$$V_l^{A(*)} = \frac{V_l^{A(0)}}{1 - V_l^{A(0)} \ln(2\tilde{\gamma}v_F\Lambda_0/\pi T)} , \quad (3.28a)$$

$$V_l^{S(*)} = \frac{V_l^{S(0)}}{1 + V_l^{S(0)} \ln(2\tilde{\gamma}v_F\Lambda_0/\pi T)} . \quad (3.28b)$$

wherein $\ln \tilde{\gamma} \equiv \gamma \approx 0.577$ is Euler's constant. If the conditions (3.27) are violated, the \hat{V} -interaction becomes marginally relevant, and a pole appears at temperature

$$T_{SC}^{(l)} = \frac{2\tilde{\gamma}}{\pi} v_F \Lambda_0 \exp\left(-\frac{1}{\Xi_l}\right) . \quad (3.29)$$

Here $\Xi_l \equiv \max\{V_l^{A(0)}, |V_l^{S(0)}|\}$, in which only the harmonics violating the condition (3.27) are included. If phonons provide the underlying mechanism of attraction, we may identify the characteristic energy scale $v_F\Lambda_0$ of the low-energy action with the Debye energy ω_D . In the present context, the BCS theory of superconductivity corresponds to the special case of a contact attractive interaction, for which only the zeroth harmonic $V_0^{S(0)} \propto U_0 < 0$ does not vanish.

3.3 RG equations in three dimensions

In order to obtain the tree-level marginal coupling functions for a spherical Fermi surface in 3D, we follow the same procedure as in Sec. 2.3. Eq. (2.46) enforcing the momenta of the coupling function both to satisfy the conservation law and to lie on the Fermi surface in the limit $\Lambda/K_F \rightarrow 0$, now allows more freedom for the vectors than the corresponding Case 1 in 2D [cf. Eq. (2.47)]. At $\Lambda/K_F = 0$ and for the pair of incoming momenta $(\mathbf{K}_F^1, \mathbf{K}_F^2)$, Eq. (2.46) allows the pair of outgoing momenta $(\mathbf{K}_F^3, \mathbf{K}_F^4)$ to lie on the cone swept by a rotation of $(\mathbf{K}_F^1, \mathbf{K}_F^2)$ around the vector $\mathbf{K}_F^1 + \mathbf{K}_F^2$. [We remind that in Eq. (2.46) \mathbf{n}_i denotes the unit vector in the direction of \mathbf{K}_F^i .] The 3D solution of Eq. (2.46) for Case 1 is illustrated in Fig. 2.

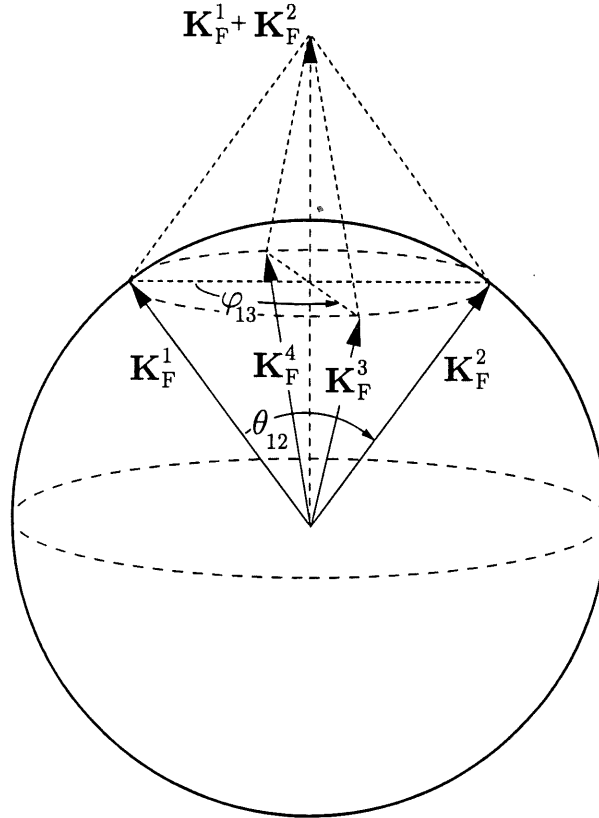


Figure 2: Conic configuration of the four momenta lying on the Fermi sphere and satisfying the momentum conservation law $\mathbf{K}_F^1 + \mathbf{K}_F^2 = \mathbf{K}_F^3 + \mathbf{K}_F^4$

Case 1. $\mathbf{n}_1 \neq -\mathbf{n}_2$. In order to find the tree-level marginal coupling function for this configuration, it is convenient to use the rotational symmetry of the problem. In general, in the action's effective interaction, we have three independent integrations over solid angles [cf. Eq. (2.39)]. Because of the symmetry of the problem, one of the axes of the spherical coordinate system can be chosen, e.g., along vector \mathbf{n}_1 , and then the other two angular integrations can be carried out relatively to the axial direction \mathbf{n}_1 , while the remaining integral over $d\Omega_3^{(1)}$ (or, in another conventional notation, over $d\mathbf{n}_1$) gives the overall solid angle, i.e., 4π . In order to treat the geometric constraint imposed by the delta function (2.49) in the simplest way, it is convenient to choose the polar axis along vector $\mathbf{n}_1 + \mathbf{n}_2$. Let us denote the unit vector in this direction as \mathbf{e}_{12} , i.e.,

$$\mathbf{e}_{12} = \frac{\mathbf{n}_1 + \mathbf{n}_2}{|\mathbf{n}_1 + \mathbf{n}_2|} . \quad (3.30)$$

Suppose that the angle between vectors \mathbf{n}_1 and \mathbf{n}_2 is θ_{12} . After choosing the polar axis along \mathbf{e}_{12} , we will specify one of the independent vectors left, say, \mathbf{n}_1 , by a pair of angular variables $(\theta' = \frac{1}{2}\theta_{12}, \varphi)$, wherein the polar angle $\theta' \in [0, \pi/2]$ and the azimuthal angle $\varphi \in [0, 2\pi]$. The condition (2.46) imposed by the delta function can be written, using Eq. (3.30) as

$$|2\mathbf{e}_{12} \cos \theta' - \mathbf{n}_3| = 1 . \quad (3.31)$$

The vector \mathbf{n}_3 can be decomposed as follows:

$$\mathbf{n}_3 = \mathbf{e}_{12} \cos \theta_3 + \mathbf{e}_\perp \sin \theta_3 , \quad (3.32)$$

wherein $\mathbf{e}_\perp = (\cos \varphi_3, \sin \varphi_3)$ is the unit vector lying in the azimuthal plane and specified by the angle φ_3 . As can easily be checked, Eq. (3.31) is satisfied for any vector \mathbf{n}_3 : $\theta_3 = \theta'$, $\forall \varphi_3 \in [0, 2\pi]$. Thus, the delta function (2.49) removes the integration over θ_3 . So, the angular integrations in the effective interaction (2.39) are done as follows:

$$\begin{aligned} & \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \delta(|\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3| - 1) \cdot \{...\} \\ &= \int d\mathbf{e}_{12} \int_0^{\frac{\pi}{2}} 2 \sin \theta' d\theta' \int_0^{2\pi} d\varphi_1 \int_0^\pi \sin \theta_3 d\theta_3 \int_0^{2\pi} d\varphi_3 \delta(|2 \cos \theta' \cdot \mathbf{e}_{12} - \mathbf{n}_3| - 1) \cdot \{...\} \\ &= 8\pi \int_0^{\frac{\pi}{2}} \sin^2 \theta' d\theta' \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_3 \cdot \{...\} . \end{aligned} \quad (3.33)$$

Accordingly, for the interaction of the low-energy effective action describing the processes of the quasi-particles scattering in the vicinity of the special (“conic”) momentum configuration shown in Fig. 2, we obtain [cf. Eqs (2.24,2.39)]:

$$S_{\text{int}}^{\text{eff}} \Big|_{\text{cone}} = - \frac{K_F^6}{(2\pi)^8 \beta^3} \sum_{\omega_1 \dots \omega_3} \left\{ \prod_{i=1}^3 \int_{-\Lambda_0}^{\Lambda_0} dk_i \right\} \int_0^{\frac{\pi}{2}} \sin^2 \theta' d\theta' \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_{13} \\ \times \hat{\Gamma}^{\Lambda_0}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^3, \mathbf{K}_F^4 | k_n, \omega_n) \cdot [\bar{\psi}\psi]^2 . \quad (3.34)$$

In the above formula we shifted the azimuthal angle $\varphi_3 \mapsto \varphi_{13}$ in the integration, in order to measure it relatively to angle φ_1 . Vectors $|\mathbf{K}_F^i| = K_F$ are parametrized by unit vectors \mathbf{n}_i , and the “conic” configuration of the latter is specified by their polar and azimuthal angles:

$$\mathbf{n}_1 \mapsto (\theta', \varphi_1) , \quad \mathbf{n}_3 \mapsto (\theta', \varphi_1 + \varphi_{13}) , \\ \mathbf{n}_2 \mapsto (\theta', \varphi_1 + \pi) , \quad \mathbf{n}_4 \mapsto (\theta', \varphi_1 + \varphi_{13} + \pi) . \quad (3.35)$$

By following the same steps as in the tree-level scaling analysis of Sec. 2.3, we find that the function $\Gamma^{\Lambda_0}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^3, \mathbf{K}_F^4 | 0, 0)$ is marginal for the momentum configuration (3.35), while the dependence of this function on k_n and ω_n is irrelevant. So, at tree-level, we can define the dimensionless coupling function

$$\hat{\Phi} \equiv \frac{1}{2} \nu_F \Gamma^{\Lambda_0}(\mathbf{K}_F^1, \mathbf{K}_F^2; \mathbf{K}_F^3, \mathbf{K}_F^4 | 0, 0) , \quad (3.36)$$

which, due to rotational symmetry, depends only on relative the angles $\theta' = \frac{1}{2}\theta_{12}$ and φ_{13} . It can be decomposed as

$$\hat{\Phi} = \Phi^A(\theta_{12}, \varphi_{13}) I_{\gamma\delta}^{\alpha\beta} + \Phi^S(\theta_{12}, \varphi_{13}) T_{\gamma\delta}^{\alpha\beta} . \quad (3.37)$$

The functions $\Phi^{A,S}$ have the symmetry properties:

$$\Phi^A(\theta_{12}, \varphi_{13} \pm \pi) = -\Phi^A(\theta_{12}, \varphi_{13}) , \quad (3.38a)$$

$$\Phi^S(\theta_{12}, \varphi_{13} \pm \pi) = \Phi^S(\theta_{12}, \varphi_{13}) , \quad (3.38b)$$

which follow from the Pauli principle. The minimal domain of definition allowing to recover all values of these functions is $\theta, \varphi \in [0, \pi]$.

Case 2. $\mathbf{n}_1 = -\mathbf{n}_2$. This case of opposite ingoing (outcoming) momenta can be treated exactly as in Sec. 2.3 for the 2D case, providing us with the

marginal coupling function defined by Eq. (2.70), with the symmetry properties (2.71).

The one-loop analysis of the RG flow for the four-point vertex with momenta in the configuration (3.35) shows that contributions from one-loop diagrams are severely cut off by phase space restrictions, rendering the one-loop corrections irrelevant, and so preserving the tree-level marginality of the coupling function (3.37). The only exception is a special (“degenerate”) case of the conic configuration (3.35) when $\varphi_{13} \rightarrow 0$. From a more physical standpoint the case $\varphi_{13} \rightarrow 0$ corresponds to a vanishing momentum transfer $\mathbf{Q} \rightarrow 0$ [cf. definition (2.59)].[†] For small transfer \mathcal{Q} , all the phase space of the ZS diagram in 3D is available for integration, as in 2D, and the Green function’s contribution is sensitive to the order of taking the limit $\mathcal{Q} \rightarrow 0$ [cf. Eqs (3.1,3.2)]. In order to treat this scattering geometry, i.e., the Landau channel of (nearly) forward scattering quasiparticles, we will use the action’s effective interaction in the Landau channel as is defined by Eqs (2.65), $d = 3$. Such physical quantities of the Fermi Liquid as the forward scattering vertex and the Landau interaction function are identified with the fixed points of the running couplings $\hat{\Gamma}^{\mathcal{Q}}$ and $\hat{\Gamma}^{\Omega}$, respectively [cf. Eqs (2.66)]. The functions $\hat{\Gamma}^{\mathcal{Q}}$, $\hat{\Gamma}^{\Omega}$ depend on the relative angle θ_{12} . The bare coupling in the Landau channel defined by Eq. (2.61) is just a value of the coupling $\hat{\Phi}$ [cf. Eq. (3.37)] at $\varphi_{13} = 0$, i.e., $\hat{U}(\theta_{12}) \equiv \hat{\Phi}(\theta_{12}, 0)$.

The technical particularity of dimension three is that, for a spherical Fermi surface, we expand the coupling functions into Legendre polynomials.

[†] Notice that the four-point vertex (2.59) with all four momenta taken on the Fermi surface in the conic configuration (3.35) has the momentum transfer $\mathbf{Q} = \mathbf{K}_F^3 - \mathbf{K}_F^1$ lying in the azimuthal plane. It is easy to find from Fig. 2 that $|\mathbf{Q}| = 2K_F \cdot \sin(\frac{1}{2}\theta_{12}) \cdot \sin(\frac{1}{2}\varphi_{13})$, so the momentum transfer in this configuration can be as large as $2K_F$.

Using again the short-hand notation \mathbf{X} for the set of coupling functions [cf Eq. (2.67)], this expansion is

$$\mathbf{X}(\theta) = \sum_{l=0}^{\infty} (2l+1) \mathbf{X}_l P_l(\cos\theta) , \quad (3.39a)$$

$$\mathbf{X}_l = \frac{1}{2} \int_{-1}^1 d(\cos\theta) \mathbf{X}(\theta) P_l(\cos\theta) . \quad (3.39b)$$

The rest of the calculations for the Landau channel is conducted in the same way as in Sec. 3.1, resulting in the Fermi Liquid results (3.22) with the Pomeranchuk stability conditions (1.11). The treatment of the BCS interacting channel made for the 2D case in Sec. 3.2 may be repeated step by step in the 3D case, resulting in the same formulas and conclusions.

3.4 Response functions

In this section we apply our RG technique to the study of response functions. (For the definitions of these functions and their relationship with physical observables, see, for instance, Ref. [58].) All the results presented below are valid for spatial dimensions $d = 2, 3$. In order to calculate the compressibility, we consider the density response function $\kappa(\mathcal{Q})$, defined as follows:

$$\delta^{(d+1)}(0)\kappa(\mathcal{Q}) \equiv \langle \bar{\rho}(\mathcal{Q})\bar{\rho}^\dagger(\mathcal{Q}) \rangle , \quad (3.40)$$

wherein the density operator $\rho(\mathcal{Q})$ is

$$\rho(\mathcal{Q}) = \int_{(1)} \bar{\psi}_\alpha(\mathbf{1})\psi_\alpha(\mathbf{1} + \mathcal{Q}) \quad (3.41)$$

and $\bar{\rho}(\mathcal{Q}) = \rho(\mathcal{Q}) - \langle \rho(\mathcal{Q}) \rangle$ stands for the density fluctuation. Here $\mathcal{Q} \equiv (\mathbf{q}, \Omega_n)$, and Ω_n is the bosonic Matsubara frequency. The zeroth component of this function $\kappa(\mathcal{Q})$, taken in the physical limit, gives us the derivative of the particle concentration n with respect to the chemical potential, i.e.

$$\kappa \equiv \lim_{\mathbf{q} \rightarrow 0} \left[\kappa(\mathcal{Q}) \Big|_{\Omega_n=0} \right] = \frac{\partial n}{\partial \mu} . \quad (3.42)$$

The RG flow equations are simplest when obtained in a scheme analogous to the standard field-theory renormalization technique, with a finite cutoff,^{41,42,43} i.e., momentum integrations are taken over the interval $k \in [-\Lambda, \Lambda]$, instead of the (infinitesimal) Wilson-Kadanoff (WK) integration scheme. In this “field-theoretic” scheme the RG parameter τ [cf. Eq. (3.18)] runs from $\tau = 0$ to $\tau = \tau_0 \approx 1$ (the fixed point), and the r.h.s. of Eqs (3.20) changes its sign. The first two terms of the perturbative expansion for κ give us [cf. notations (3.17)]

$$\kappa \simeq \kappa_0 - \kappa_0 \tau A_0^{(0)} , \quad (3.43)$$

wherein $\kappa_0 = \frac{N}{2} \nu_F \tau$ is the contribution from the free part of the effective action. Introducing the auxiliary function $\bar{\kappa}(\tau) \equiv \kappa/\kappa_0$, and using Eqs (3.20), we get the RG equation:

$$\frac{\partial \ln \bar{\kappa}(\tau)}{\partial \tau} = -A_0(\tau) , \quad (3.44)$$

which yields $\partial n/\partial \mu$ as its solution at $\tau = 0$ ($t = \infty$):

$$\kappa^* = \frac{N}{2} \frac{\nu_F}{1 + F_0^*} . \quad (3.45)$$

From the thermodynamic formula for the compressibility

$$K \equiv -\frac{1}{V} \frac{\partial V}{\partial P} = \frac{1}{n^2} \frac{\partial n}{\partial \mu}$$

(see, for instance, Ref. [9]), we easily recover the result for the electron Fermi liquid:

$$K = \frac{1}{n^2} \frac{\nu_F}{1 + F_0^*} . \quad (3.46)$$

To find *the spin susceptibility* in our $SU(N)$ formalism, we consider the flavor response function $\chi(\mathcal{Q})$, defined as follows

$$\delta_{ab} \delta^{(d+1)}(0) \chi(\mathcal{Q}) \equiv \frac{1}{N^2 - 1} \langle S_a(\mathcal{Q}) S_b^\dagger(\mathcal{Q}) \rangle , \quad (3.47)$$

wherein the flavor density operator is

$$S_a(\mathcal{Q}) = \frac{g}{2} \int_{(1)} \lambda_{\alpha\beta}^a \bar{\psi}_\alpha(\mathbf{1}) \psi_\beta(\mathbf{1} + \mathcal{Q}) , \quad (3.48)$$

and g is the gyromagnetic ratio. The uniform susceptibility χ is given by the physical limit $\chi(\mathcal{Q} \rightarrow 0)$ [cf. Eq. (3.42)]. Defining $\chi(\mathcal{Q})$ in the fashion (3.47), we used the fact that in the paramagnetic state the response is the same along all of the $(N^2 - 1)$ directions a . The rest of the calculations is carried out in the same way as above. For the auxiliary function $\bar{\chi}(\tau) \equiv \chi/\chi_0$ ($\chi_0 = \frac{1}{4}g^2\nu_F\tau$ is the contribution in the absence of interaction) we obtain the RG equation:

$$\frac{\partial \ln \bar{\chi}(\tau)}{\partial \tau} = -B_0(\tau) . \quad (3.49)$$

Again, we recover the FL theory result:⁹

$$\chi^* = \frac{1}{4}g^2 \frac{\nu_F}{1 + G_0^*} . \quad (3.50)$$

Notice, that the stability conditions for the solutions (3.45) and (3.50) are just a special case ($l = 0$) of the Pomeranchuk conditions (1.11).

It is interesting to show how the results (3.45) and (3.50) can be obtained in the Wilson-Kadanoff scheme with source fields.[†] Below we specialize to the electron ($N = 2$) spin susceptibility, but the compressibility may be computed in the same fashion. We add to the effective action a source field $h(\mathcal{Q})$, conjugated to the z -component of the spin density (3.48). For details on this approach, see, for instance, Ref. [39]. The successive integration over momentum modes generates a vertex correction $z(t)$ to the source term along with higher order terms in the external field h . The effective action, at some intermediate value of t (the flow parameter), takes the form

$$\begin{aligned} S[\psi, h, t] = & S[\psi, 0, t] + \int_{(\mathcal{Q})} z(t, \mathcal{Q}) [h(\mathcal{Q})S_3(\mathcal{Q}, t) + h.c.] + \\ & + \int_{(\mathcal{Q})} \chi(t, \mathcal{Q}) h^*(\mathcal{Q}) h(-\mathcal{Q}) + \mathcal{O}(h^3) . \end{aligned} \quad (3.51)$$

In linear response theory it is sufficient to keep track of terms up to second order in h . The recursion relations for the vertex $z(t)$ and the susceptibility $\chi(t)$

[†] I am indebted in great part to N. Dupuis for the calculation of response functions using this technique.

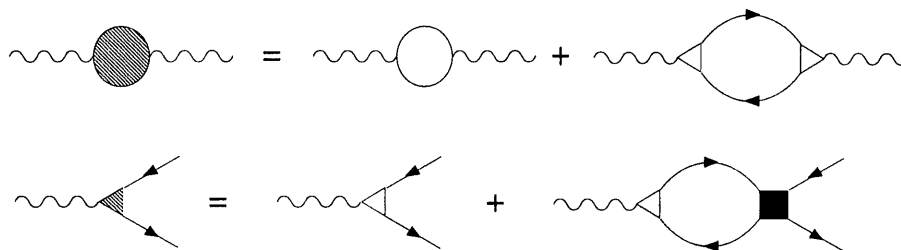


Figure 3: Diagrammatic form of the recursion relations for the susceptibility $\chi(t)$ (circle) and the vertex $z(t)$ (triangle). The hatched symbols stand for the functions at $t + dt$, whereas light ones stand for functions at t . The black square stands for the vertex $\hat{\Gamma}(t)$. Integration over loop momenta is carried out in an infinitesimal shell $d\Lambda(t)$.

are illustrated in Fig. 3. In the physical limit $\mathcal{Q} \rightarrow 0$, we obtain the following pair of RG equations:

$$\frac{\partial \chi}{\partial \tau} = -\frac{1}{4} g^2 \nu_F z^2(\tau), \quad \frac{\partial z}{\partial \tau} = z(\tau) B_0(\tau) \quad (3.52)$$

with the initial conditions $z(\tau_0) = 1$ and $\chi(\tau_0) = 0$. Using the solution (3.21b) for $B_0(\tau)$, we can easily solve this system. The uniform susceptibility is again given by Eq. (3.50).

CHAPTER IV

Role of the interference in the Landau channel

In this chapter we present the quantitative RG theory which provides the constructive calculation of the parameters of the Fermi Liquid (the Landau function and the forward scattering vertex) in terms of the parameters of the effective action. The derived RG equations explicitly preserve the (anti)symmetry of the scattering vertex (i.e., they preserve the Pauli principle) on the whole trajectory of the RG flow. This is achieved by accounting for the interference between the direct and exchange processes of the particle-hole scattering. This chapter is organized as follows. In Section 4.1, which is rather technical, coupled RG equations that take into account both the direct (ZS) and exchange (ZS') particle-hole one-loop diagrams are derived for the two-dimensional case. Section 4.2 explains some of the weak points of the standard FLT results and argues for their partial revision. In Section 4.3 we give a numerical and approximate analytical solution of the coupled RG equations for spinless fermions. In Section 4.4 we present and discuss our results for the Landau function and the scattering vertex calculated at different temperatures. In Section 4.5 we relate this study to the standard treatment of Fermi Liquid Theory. The consequences of the RG corrections on FLT results are discussed.

4.1 Coupled RG equations in the Landau channel

As already discussed in Sec. 3.1, in order to respect the Pauli principle it is necessary to take into account in the equations both the ZS and ZS' contributions to the RG flow, since only their combined contribution ($ZS + ZS'$) is antisymmetric under exchange of incoming (or outgoing) particles. We discard the symmetry-preserving contribution of the BCS graph to the RG flow of the vertices in the Landau channel. Thus, we leave out the interference near $\theta = \pi$ of the Landau channel with the BCS channel [cf. Eq. (2.73)], which leads to the Kohn-Luttinger effect.²⁸

The implicit form of the RG flow equations in the Landau channel wherein both (ZS and ZS') contributions are taken into account, is given by Eqs (3.11). Let us now explicitly derive the analytic form of the contribution coming from the exchange graph ZS' . We recall that the analytic expression of that graph is given by Eq. (3.6), and the Green's function's contribution at zero transfer Q by Eq. (3.8). The calculation of the ZS' contribution to the RG equations is complicated by the fact that, at arbitrary angle between vectors \mathbf{K}_1 and \mathbf{K}_2 , not all the phase space is available for integration. [Cf. discussion in the beginning of Sec. 3.1, especially after Eq. (3.10).] Thus, we have to keep track of the phase space constraints (2.40) for the momenta entering the r.h.s. of Eq. (3.6).

Let us consider the ZS' graph (see Fig. 1) when all external momenta satisfy momentum conservation and lie in C_Λ^2 . It suffices then to check whether the internal momenta \mathbf{K}_5 and $\mathbf{K}_5 - \mathbf{Q}'$ [\mathbf{Q}' is defined by Eq. (3.7)] lie in C_Λ^2 when \mathbf{K}_F^5 runs around the Fermi surface during the integration. From Fig. 4A we see that if $|\mathbf{Q}'| > 2\Lambda$, the loop momenta lie both in C_Λ^2 only at special values of \mathbf{K}_F^5 (the shaded regions), i.e., only small fragments of phase space are available for integration. At smaller \mathbf{Q}' (cf. Fig. 4B) these intersections form a connected region and \mathbf{K}_F^5 is free to run around the Fermi surface. If we completely neglect the ZS' graph when the intersection is disconnected (in Fig. 4A), the contribution of this graph to the RG flow at $|\mathbf{Q}'| < 2\Lambda$ is calculated in the same way as that of the ZS graph. Since $|\mathbf{K}_1| = |\mathbf{K}_2| = K_F$ and $\mathbf{Q}'|_{Q=0} = \mathbf{K}_2 - \mathbf{K}_1$, the condition $|\mathbf{Q}'| < 2\Lambda$ is equivalent to the condition $|\sin((\theta_1 - \theta_2)/2)| < \Lambda/K_F$ for the angle between \mathbf{K}_1 and \mathbf{K}_2 .

After taking care of the phase space constraints, the rest of the calculations of the term coming from the ZS' contribution is straightforward. The explicit form of the ZS' terms (3.10) is found to be:

$$\begin{aligned} \left. \frac{\partial \hat{\Gamma}^\Omega(2\phi)}{\partial t} \right|_{ZS'} &= \left. \frac{\partial \hat{\Gamma}^Q(2\phi)}{\partial t} \right|_{ZS'} = \\ &= -\beta_R \Theta(\theta_c - |\phi|) \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \Gamma_{46}^{15(Q)}(\phi - \theta) \Gamma_{53}^{62(Q)}(\theta + \phi) Y(\phi, \theta; \beta_R) . \end{aligned} \quad (4.1)$$

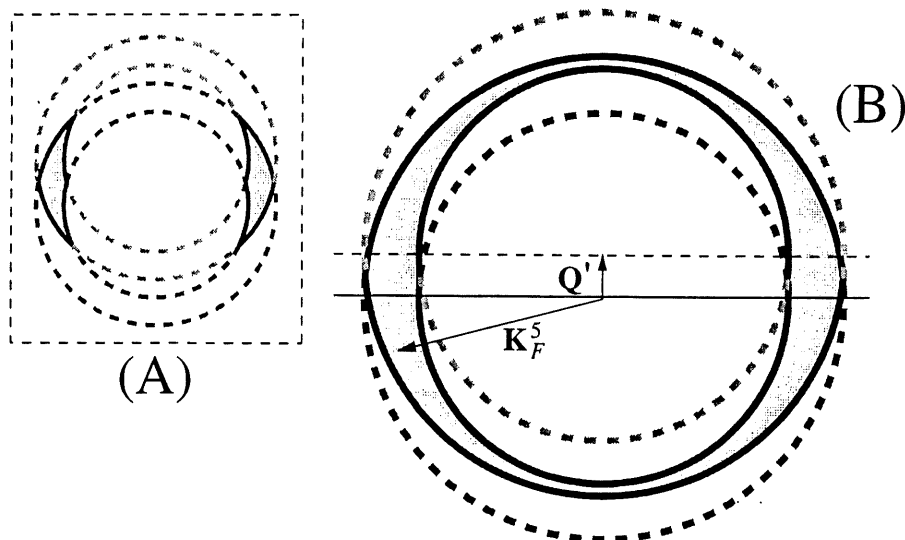


Figure 4: If $|\mathbf{Q}'| > 2\Lambda$, the intersection (shaded) of the supports of \mathbf{K}_5 and $\mathbf{K}_5 - \mathbf{Q}'$ are disconnected (A). If $|\mathbf{Q}'| < 2\Lambda$, this intersection forms a connected area (B). Note that the RG flow is governed by the boundaries of this intersection, not by their interior directly.

To simplify the formulas we parametrize the angular dependence of the vertices by the angle ϕ between \mathbf{K}_F^1 and $(\mathbf{K}_F^1 + \mathbf{K}_F^2)$, $|\phi| \in [0, \pi/2]$. The small ZS' contribution coming from $|\sin \phi| > \Lambda(t)/K_F$ (Fig. 2A) was neglected, and this is accounted for by the Heaviside step function Θ , wherein

$$\theta_c \equiv \arcsin(\Lambda(t)/K_F) . \quad (4.2)$$

We also defined the function

$$Y(\phi, \theta; \beta_R) \equiv \frac{1}{\beta_{Q'}} \frac{\sinh(2\beta_{Q'})}{\cosh(2\beta_R) + \cosh(2\beta_{Q'})} , \quad (4.3)$$

$$\beta_{Q'} \equiv \beta_F \sin \theta \sin \phi, \quad \beta_F \equiv \beta_{v_F} K_F .$$

which arises in the calculation of the ZS' contribution (3.8).

Now we have the analytic expressions (3.4,4.1) for the contributions of both graphs (ZS and ZS') entering the implicit form (3.11) of the flow

equations. Summing up all formulas, we obtain the following system of RG equations:

$$\frac{\partial \Gamma_{34}^{12(Q)}(2\phi)}{\partial t} = \frac{\beta_R}{\cosh^2 \beta_R} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \Gamma_{36}^{15(Q)}(\phi - \theta) \Gamma_{54}^{62(Q)}(\theta + \phi) + \frac{\partial \Gamma_{34}^{12(\Omega)}(2\phi)}{\partial t} , \quad (4.4a)$$

$$\frac{\partial \Gamma_{34}^{12(\Omega)}(2\phi)}{\partial t} = -\beta_R \Theta(\theta_c - |\phi|) \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \Gamma_{46}^{15(Q)}(\phi - \theta) \Gamma_{53}^{62(Q)}(\theta + \phi) Y(\phi, \theta; \beta_R) \quad (4.4b)$$

wherein we explicitly indicated the flavor indices in order to avoid any confusion. Notice that

$$\lim_{\beta_{Q'} \rightarrow 0} Y(\phi, \theta; \beta_R) = \frac{1}{\cosh^2 \beta_R} . \quad (4.5)$$

From Eqs (4.4a,4.3,4.5) one can see that at small angles ($|\phi| \lesssim T/v_F K_F$) there is a strong interference between the ZS and ZS' contributions. This interference depletes the RG flow of $\Gamma^{A(Q)}(\phi)$ at small angles. Moreover, using the convolution relationships (2.31) one can check that at $\phi = 0$ the flow is exactly zero, for the two contributions have the same thermal factor $\beta_R \cosh^{-2}(\beta_R)$:[†]

$$\frac{\partial \Gamma^{A(Q)}(\phi = 0, t)}{\partial t} = 0 , \quad \forall t . \quad (4.6)$$

The initial conditions for the flow equations (4.4) are:

$$\hat{\Gamma}^Q(\phi, t = 0) = \hat{\Gamma}^\Omega(\phi, t = 0) = \hat{U}(\phi) . \quad (4.7)$$

Recall that the fixed points $\hat{\Gamma}^{Q*}$ and $\hat{\Gamma}^{\Omega*}$ of the vertices $\hat{\Gamma}^Q$ and $\hat{\Gamma}^\Omega$ are the forward scattering vertex and the Landau interaction function, respectively. From Eqs (4.6,4.7,2.64) we conclude that the RG equations for the forward scattering vertex preserve the Pauli principle at any point of the RG flow trajectory

$$\Gamma^{A(Q)}(\phi = 0, t) = 0 , \quad \forall t , \quad (4.8)$$

while the ‘‘uncompensated’’ RG flow generated by the ZS' graph drives the vertex $\hat{\Gamma}^\Omega$ to a fixed point value (the Landau function), which does not satisfy the Pauli principle, i.e., $\hat{\Gamma}^{\Omega*}(\phi = 0) \neq 0$.

[†] Both the ZS contribution and the zero-angular part of the ZS' contribution become singular in the limit $T \rightarrow 0$ since $\lim_{\beta \rightarrow \infty} (\beta/4) \cosh^{-2}(\beta x/2) = \delta(x)$.

In order to make the discussion as clear as possible, we concentrate from now on in this chapter on **2D spinless** ($N = 1$) fermions. This simple model has nevertheless all the necessary qualities to illustrate our key points and to demonstrate the impact on the FLT results, caused by the the interference in the Landau channel. In the spinless case the RG equations (4.4) take their simplest form, since only the antisymmetric over momentum-frequency exchange components of the interaction and vertices are present [they are labeled by A in our notations (2.62,2.41)]. Then, according to the representation (3.16,3.17) for the components of the “charge” and “flavor” components, we introduce for $N = 1$ the running vertex $F \equiv -2\Gamma^{\Omega(A)}$, whose fixed point gives the Landau function (for the case $N = 1$ the Landau function has only one component), and the running vertex $\Gamma \equiv -2\Gamma^Q(A)$, whose fixed point gives the single component of the forward scattering vertex.[†]

Let us explicitly write the RG equations (4.4) for the special case $N = 1$, using the new notations:

$$\frac{\partial \Gamma(2\phi)}{\partial t} = -\frac{\beta_R}{\cosh^2 \beta_R} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \Gamma(\phi - \theta) \Gamma(\theta + \phi) + \frac{\partial F(2\phi)}{\partial t} \quad (4.9a)$$

$$\frac{\partial F(2\phi)}{\partial t} = \beta_R \Theta(\theta_c - |\phi|) \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \Gamma(\phi - \theta) \Gamma(\theta + \phi) Y(\phi, \theta; \beta_R) . \quad (4.9b)$$

These equations have the initial conditions [cf. Eqs (4.7)]:

$$\Gamma(\phi, t = 0) = F(\phi, t = 0) = U(\phi) . \quad (4.10)$$

and the vertex satisfies the Pauli principle [cf. Eqs (4.6,4.8)]:

$$\frac{\partial \Gamma(\phi = 0, t)}{\partial t} = 0 , \quad \Gamma(\phi = 0, t) = 0 , \quad \forall t , \quad (4.11)$$

while the Landau function does not, and $F^*(\phi = 0) \neq 0$.

[†] We appologize for some abuse of notation. Normally, according to (3.17), we should have called this running vertex A , but we will reserve A for the “charge” component of the vertex in the general $SU(N)$ case, $N > 1$, and we will keep calling Γ the single component of the vertex in the spinless case.

4.2 Deficiencies of the decoupled approximations in the Landau channel

Before finding a solution (exact or approximate) to the flow equations (4.9) which fully takes into account the coupling of Γ and F , we will comment on approximate solutions in which this coupling is neglected. The Landau channel, as we defined it, includes, at one-loop RG, both the direct (ZS) and exchange (ZS') quasiparticle-quasihole loops with a small transfer \mathcal{Q} . We will call *decoupled* any treatment of the Landau channel which does not explicitly take into account both the direct and exchange contributions. It is shown below that solutions for the forward scattering vertex provided by decoupled methods fail to meet the requirements of the Fermi statistics. Tackling the Pauli principle by imposing additional constraints on the solutions (sum rules) leads to conceptual difficulties discussed below.

Let us first solve the RG equations in the decoupled approximation. If we neglect completely the ZS' contribution in Eqs (4.9) and perform a Fourier transformation, we recover a familiar system of equations [i.e., simplified for the $N = 1$ version of Eqs (3.20)], with its RPA-like solution in which all harmonics are decoupled:

$$\frac{\partial \Gamma_l}{\partial \tau} = \Gamma_l^2 \quad \Longrightarrow \quad \Gamma_l^{\text{RPA}}(\tau) = \frac{\Gamma_l(\tau_0)}{1 + (\tau_0 - \tau)\Gamma_l(\tau_0)} , \quad (4.12a)$$

$$\frac{\partial F_l}{\partial \tau} = 0 \quad \Longrightarrow \quad F_l^{\text{RPA}}(\tau) = \text{const} . \quad (4.12b)$$

We recall that since the temperature in the effective action is restricted by the condition (2.43), we can set $\tau_0 = 1$ [cf. notations (3.19)].

With the initial conditions $\Gamma_l(\tau_0) = F_l(\tau_0) = U_l$ [cf. Eq. (4.10)], the solutions of the flow equations (4.12) at $\tau = 0$ ($t = \infty$) are

$$(a) \quad \Gamma_l^* = \frac{U_l}{1 + U_l} \quad (b) \quad F_l^* = U_l , \quad (4.13)$$

with the following stability conditions:

$$U_l > -1 , \quad \forall l , \quad (4.14)$$

which are the Stoner criteria, well known from the RPA approach. The bare interaction satisfies the Pauli principle [cf. Eq.(2.64)]

$$\sum_{l=-\infty}^{\infty} U_l = 0 . \quad (4.15)$$

If the vertex Γ is to satisfy the Pauli principle, the condition

$$\sum_{l=-\infty}^{\infty} \frac{U_l}{1+U_l} = 0 \quad (4.16)$$

must be imposed on the r.h.s. of (4.13a). However, it has been known for a long time that conditions (4.16) and (4.15) are incompatible, unless the stability conditions (4.14) are broken.⁴⁷ Indeed, subtracting (4.15) from (4.16), we find

$$\sum_{l=-\infty}^{\infty} \frac{U_l^2}{1+U_l} = 0 , \quad (4.17)$$

which cannot be satisfied without violation of (4.14).

This proves that the antisymmetric bare interaction U cannot be at the same time a fixed point of the RG flow and the Landau function, unless the classic FLT formulas are unapplicable. The accepted cure to this paradox is to give up the Pauli principle on the Landau function, because of the neglected ZS' contribution.⁴⁷ In the RG approach this is accomplished (in the decoupled approximation) by letting the ZS' contribution drive the bare interaction U towards the Landau function F^* [e.g., during an earlier stage of mode elimination], and then by solving the RG equations (4.12) with F^* as a new renormalized “bare” interaction. This leads to the standard relationship between the scattering vertex and the Landau function

$$\Gamma_l^* = \frac{F_l^*}{1+F_l^*} . \quad (4.18)$$

The above equation is just a special case ($N = 1$) of Eqs (3.22). Because of the ZS' contribution, the Pauli principle does not apply to F^* (a), while it is enforced on the vertex Γ^* through a sum rule (b):[†]

$$(a) \quad \sum_{l=-\infty}^{\infty} F_l^* \neq 0 \quad (b) \quad \sum_{l=-\infty}^{\infty} \frac{F_l^*}{1+F_l^*} = 0 . \quad (4.19)$$

[†] The l.h.s. of Eq. (4.19b) is the sum (3.23) in case $N = 1$.

In doing so, the stability conditions (4.14) are modified as follows

$$F_l^* > -1, \quad \forall l, \quad (4.20)$$

i.e., they become Pomeranchuk's stability conditions for the Fermi liquid, originally obtained on thermodynamic grounds.² Such a decoupled RG treatment of the direct and exchange loops makes Eqs (4.19) compatible with the conditions (4.20).

However, the sum rule (4.19b) is “unnatural”, in the following sense. The bare interaction can in principle be traced from a microscopic Hamiltonian. For instance, let us consider the spinless extended Hubbard Hamiltonian on a square lattice (with lattice spacing a) at low filling, with nearest-neighbor repulsive interaction (U^{nn}). The coupling function of such microscopic Hamiltonian is given by Eq. (3.25a). Let us choose this interaction (3.25a) as a trial bare dimensionless coupling function of the effective action:

$$U(\theta_1 - \theta_2) = \mathcal{U} \sin^2 \left(\frac{\theta_1 - \theta_2}{2} \right), \quad (4.21)$$

wherein all parameters are hidden within a single coefficient \mathcal{U} . The only nonzero Fourier components U_l of the interaction are:

$$U_0 = \frac{1}{2}\mathcal{U}, \quad U_{\pm 1} = -\frac{1}{4}\mathcal{U}. \quad (4.22)$$

The interaction (4.21) satisfies the Pauli principle (2.64,4.15). The RPA sum rule (4.16) imposes an additional constraint, which the interaction (4.21) does not satisfy. If we suppose that the “improved” results (4.18,4.19b) are always true, then, starting from any kind of microscopic interaction [e.g., the bare interaction (4.21)] and integrating “fast modes” outside the immediate vicinity of the Fermi surface, we have to end up with a “fine tuned” interaction, for any interaction has to be “fine tuned” in order to satisfy (4.19b). The integral of the flow (4.11) (or, equivalently the sum rule (4.32) below) is not a fine tuning, since firstly, the bare interaction at the initial point can be always antisymmetrized, and, secondly, we have an exact cancellation of the RG flow for the vertex Γ at zero angle due to direct and exchange contributions, thus

preserving (4.11). On the contrary, there is no reason for any bare interaction to satisfy (4.16) at the beginning, nor is there a mechanism to provide the fine tuning (4.19b) on other parts of the RG trajectory.

These difficulties are not specific to the decoupled RG approximation, since the latter is strictly equivalent to the diagrammatic microscopic derivation of FLT^{3,5,7} leading to the same results (4.18,4.19,4.20). The decoupled RG treatment is equivalent to applying the Bethe-Salpeter equation with the particle-hole ZS loop singled out, F being the vertex irreducible in this loop. There are no *a priori* reasons in that approach to demand this vertex to satisfy the Pauli principle. The rearrangement of diagram summations in the Bethe-Salpeter equation leading to (4.18) is based on the assumption that the vertex irreducible in the direct particle-hole loop (ZS) is a regular function of its variables, neglecting the zero-angle singularity (at $T = 0$; see footnote on p. 61) in the ZS' loop. As a consequence, the Pauli principle for the scattering vertex Γ^* is not guaranteed in the final result and “the amplitude sum rule” (4.19b) must be imposed by hand. The solution (4.18) of the Bethe-Salpeter equation is tantamount to the summation of the ladder diagrams built up from the ZS loops, wherein the Landau function stands as the bare interaction. For this reason, the solution (4.18) we will call the “*the ZS-ladder approximation*” in the following. We refer the reader to a paper of A. Hewson³¹ wherein a “generalized” Bethe-Salpeter equation for Fermi liquids, which explicitly takes into account both the ZS and ZS' loops, is derived. For further discussion on this issue, see also Ref. [47].

4.3 Solution of the coupled RG equations

4.3.1 Exact numerical solution

The coupled integro-differential flow equations (4.9) may be solved numerically. The functions $\Gamma(\theta)$ and $F(\theta)$ are then defined on a discrete grid of angles, and simple linear interpolation is used to represent them between the

grid points. The grid spacing is not uniform: it has to be very small near $\theta = 0$, where the flow is singular, but may be larger elsewhere. The RG equations then reduce to a large number of coupled nonlinear differential equations, which are solved by a fourth-order Runge-Kutta method with adaptive step-size. Typically, a grid of a few hundred points is sufficient (we take advantage of the symmetry of the functions). Of course, the numerical solution was checked to be indistinguishable from the (exact) RPA solution when the ZS' contribution is discarded.

An example of solution for the spinless case with the interaction function (4.21) is shown on Fig. 5(A), at various temperatures. The interaction function $U(\theta)$ and the RPA solution $\Gamma^{\text{RPA}}(\theta)$ are also shown. This solution will be discussed in Sec. 4.4.

4.3.2 Approximate analytical solution

The flow equations (4.9) may also be solved analytically, albeit only approximately. In this section we give the approximate solution for the fixed points Γ^* and F^* both in terms of Fourier components and in terms of angular variables. (See Eqs (4.31,4.33) below.)

The Fourier transform of Eqs (4.9) is

$$\frac{\partial \Gamma_n}{\partial \beta_R} = \frac{1}{\cosh^2 \beta_R} \Gamma_n^2 + \frac{\partial F_n}{\partial \beta_R} , \quad (4.23a)$$

$$\frac{\partial F_n}{\partial \beta_R} = - \sum_{l,m=-\infty}^{\infty} \mathcal{Y}_{n-m,2l-2m}(\beta_R) \Gamma_l \Gamma_{l-2m} , \quad (4.23b)$$

$$\mathcal{Y}_{n',m'}(\beta_R) \equiv \frac{2}{\pi^2} \int_0^{\frac{\pi}{2}} d\phi \int_0^{\pi} d\theta \cos(2\phi n') \cos(\theta m') \Theta(\theta_c - |\phi|) Y(\phi, \theta; \beta_R) \quad (4.23c)$$

On the plane (ϕ, θ) , the function $Y(\phi, \theta; \beta_R)$ has a maximum on the line $\theta = \pi/2$, which moves from the position $(\pi/2, \pi/2)$ at the beginning of renormalization procedure (when $\beta_R \sim \beta_F$) towards the position $(0, \pi/2)$ when approaching the fixed point ($\beta_R \rightarrow 0$). Elsewhere, $Y(\phi, \theta; \beta_R)$ is either quite

flat, or its contribution is eliminated by the cutoff factor $\Theta(\theta_c - |\phi|)$ during the renormalization flow. Therefore, we approximated the function $Y(\phi, \theta; \beta_R)$ on the plane (ϕ, θ) by its value on the line $(\phi, \pi/2)$. This approximation, simplifying considerably our equations, allows an analytical treatment and a qualitative insight harder to find in purely numerical results. The approximate analytical solution of the RG equations given below justifies that simplification *a posteriori*, when compared with the direct numerical solution of Eqs (4.9).

The approximate RG equations are:

$$\frac{\partial \Gamma_n}{\partial \beta_R} = \sum_{m=-\infty}^{\infty} \left[\frac{1}{\cosh^2 \beta_R} \delta_{nm} - Y_{n-m}(\beta_R) \right] \Gamma_m^2, \quad (4.24a)$$

$$\frac{\partial F_n}{\partial \beta_R} = - \sum_{m=-\infty}^{\infty} Y_{n-m}(\beta_R) \Gamma_m^2, \quad (4.24b)$$

wherein

$$Y_n(\beta_R) = \frac{2}{\pi} \int_0^{\arcsin(2\beta_R/\beta_F)} d\phi Y\left(\phi, \frac{\pi}{2}; \beta_R\right) \cos(2n\phi). \quad (4.25)$$

The key difference between Eqs (4.12) and (4.24) is that the former do not generate new harmonics since all harmonics are decoupled, whereas the latter couple all harmonics (because of the ZS' contribution) in such a way that an infinite number of new harmonics are generated by the RG flow, even if only a finite number of harmonics are nonzero at the start. For instance, the trial interaction (4.22) has only three nonzero components, but according to Eqs (4.24) the fixed points Γ^* and F^* will possess an infinite number of them. The generation of new harmonics is not an artefact of the approximation which was used to go from Eqs (4.9) to Eqs (4.24), but is a generic consequence of the interference in the Landau channel (cf. Eqs (4.23)).

Let us start the analysis of Eqs (4.24) with a heuristic observation. Whereas the component $Y_0(\beta_R)$ is a nonnegative function of β_R , the others ($Y_n(\beta_R)$, $n \geq 1$) are increasingly oscillating functions of β_R when n increases. These oscillations along the whole RG trajectory $[0, \beta_0]$ will effectively decrease the contributions from the harmonics Γ_m ($m \neq n$) to the flow of Γ_n . Because

of this, we expect the diagonal terms ($m = n$) of Eqs (4.24) to be more important, and this justifies a perturbative approach, in which the nondiagonal terms are ignored at zeroth order. Let $\gamma_n(\beta_R)$ be the zeroth order solution:

$$\frac{\partial \gamma_n}{\partial \beta_R} = \left[\frac{1}{\cosh^2 \beta_R} - Y_0(\beta_R) \right] \gamma_n^2 . \quad (4.26)$$

The solution is

$$\gamma_n(\beta_R) = \frac{U_n}{1 + \left[\tanh \beta_0 - \tanh \beta_R - I_0(\beta_0) + I_0(\beta_R) \right] U_n} , \quad (4.27)$$

with

$$\begin{aligned} I_n(\beta_R) &\equiv \int_0^{\beta_R} d\beta'_R Y_n(\beta'_R) \\ &= \frac{1}{\pi \beta_F} \int_0^{\arcsin(2\beta_R/\beta_F)} d\phi \left[\ln \frac{\cosh(\beta_R + \beta_F \sin \phi)}{\cosh(\beta_R - \beta_F \sin \phi)} - \ln \frac{\cosh(\frac{3}{2}\beta_F \sin \phi)}{\cosh(\frac{1}{2}\beta_F \sin \phi)} \right] \frac{\cos(2n\phi)}{\sin \phi} . \end{aligned} \quad (4.28)$$

The fixed point γ_n^* is

$$\gamma_n^* = \frac{U_n}{1 + \left[1 - I_0(\beta_0) \right] U_n} . \quad (4.29)$$

The integrals $I_n(\beta_0)$ can be evaluated analytically, since $(\beta_F, \beta_0) \gg 1$ according to condition (2.43). In the following we shall need the first two components only:

$$\begin{aligned} I_0(\beta_0) &\approx \frac{\Lambda_0}{K_F} \frac{1}{\pi} \left[\ln 2 + \ln \frac{1 + \sqrt{1 - \Lambda_0^2/(2K_F)^2}}{1 + \sqrt{1 - \Lambda_0^2/K_F^2}} \right] \\ &\quad + \frac{1}{\pi} \left(2 \arcsin \frac{\Lambda_0}{2K_F} - \arcsin \frac{\Lambda_0}{K_F} \right) + \frac{T}{v_F K_F} \frac{(\ln 2)(\ln 3)}{\pi} , \end{aligned} \quad (4.30a)$$

$$\begin{aligned} I_1(\beta_0) &\approx \frac{\Lambda_0}{K_F} \frac{1}{\pi} \left[\ln 2 + \ln \frac{1 + \sqrt{1 - \Lambda_0^2/(2K_F)^2}}{1 + \sqrt{1 - \Lambda_0^2/K_F^2}} \right] \\ &\quad + \sqrt{1 - \Lambda_0^2/K_F^2} - \sqrt{1 - \Lambda_0^2/(2K_F)^2} + \frac{T}{v_F K_F} \frac{(\ln 2)(\ln 3)}{\pi} . \end{aligned} \quad (4.30b)$$

[The next term in the temperature dependence, omitted in Eqs (4.30), is of the order $(T/v_F K_F)^2$]. Treating the off-diagonal terms ($n \neq m$) on the r.h.s. of

(4.24a) as perturbations, we obtain the following approximate solution at first order:

$$\Gamma_n(\beta_R) \approx \gamma_n(\beta_R) + \sum_{m \neq n} \int_{\beta_R}^{\beta_0} d\beta'_R Y_{n-m}(\beta'_R) \gamma_m^2(\beta'_R) , \quad (4.31a)$$

$$F_n(\beta_R) = \Gamma_n(\beta_R) + \int_{\beta_R}^{\beta_0} d\beta'_R \frac{\Gamma_n^2(\beta'_R)}{\cosh^2 \beta'_R} . \quad (4.31b)$$

It is straightforward to check that the solution (4.31a) satisfies the sum rule (i.e., the Pauli principle (4.11)):

$$\sum_n \Gamma_n(\beta_R) = 0 , \quad \forall \beta_R . \quad (4.32)$$

The solution (4.31) can be converted back in terms of the relative angle $\theta \in [-\pi, \pi]$ with a little help from Eq. (4.26):

$$\begin{aligned} \Gamma^*(\theta) = & U(\theta) - \int_0^{\beta_0} \frac{d\beta_R}{\cosh^2 \beta_R} \sum_{n=-\infty}^{\infty} \cos(n\theta) \gamma_n^2(\beta_R) \\ & + \Theta(\theta_0 - |\theta|) \int_{\frac{1}{2}\beta_F |\sin(\theta/2)|}^{\beta_0} d\beta_R Y\left(\frac{\theta}{2}, \frac{\pi}{2}; \beta_R\right) \sum_{n=-\infty}^{\infty} \cos(n\theta) \gamma_n^2(\beta_R) , \end{aligned} \quad (4.33a)$$

$$F^*(\theta) = \Gamma^*(\theta) + \int_0^{\beta_0} \frac{d\beta_R}{\cosh^2 \beta_R} \sum_{n=-\infty}^{\infty} \cos(n\theta) \Gamma_n^2(\beta_R) . \quad (4.33b)$$

wherein $\theta_0 \equiv 2 \arcsin(2\beta_0/\beta_F)$. A comparison of Eqs (4.33a) and (4.9a) shows that – with the aforementioned approximation of the angular dependence of the function Y – the approximate solution (4.31a) may be obtained by replacing the vertex components Γ_n on the r.h.s. of Eq. (4.9a) by the “renormalized” RPA ansatz (4.27). It would be a mistake, however, to conclude that the ZS' diagram contributes only to the third term on the r.h.s. of Eq. (4.33a) since the γ_n -s partially include its contribution. It is worth noting that Eqs (4.31b,4.33b) are not approximations in the sense of Eqs (4.31a) or (4.33a), but they are exact relations for F , derived from the basic RG equations (4.24).

4.3.3 Extension of the effective action

In the numerical and analytical results presented in the following sections the initial cutoff Λ_0 of the effective action is extended to K_F , i.e., $\beta_0 = \beta_F/2$. [Cf. notations (3.19,4.3)]. This point should be clarified. Notice first that the ZS contribution is not sensitive to the bandwidth cutoff Λ_0 – provided condition (2.43) is satisfied – since $\tanh\beta_0$ is unity with exponential accuracy. On the other hand, the angular cutoff of the ZS' contribution [cf. Eqs (4.9,4.23,4.25,4.28)] comes from a cutoff imposed on the momentum transfer in this graph [cf. Eq. (3.6)]. It is $\theta_c = \arcsin(2\beta_R/\beta_F)$ (with $2\beta_R/\beta_F \equiv \Lambda/K_F$) if $\Lambda_0 \leq K_F$, and $\theta_c = \pi/2$ otherwise. The specific choice $\beta_0 = \beta_F/2$ ($\Lambda_0 = K_F$) means that at the initial point of the RG flow the angle ϕ is allowed to take all values (i.e., the momentum transfer \mathbf{Q}' is not cut off), while the bandwidth is extended to the full depth of the Fermi sea. It can be checked that the results are not sensitive to the choice of a bigger cutoff $\Lambda_0 \gtrsim K_F$, since then not only is the ZS contribution to the flow exponentially small, but that of ZS' as well, until the cutoff decreases to $\Lambda \sim K_F$ (this was also confirmed by direct numerical tests). The formulas for the approximate analytic solution are derived for $\Lambda_0 \leq K_F$.

Such an extension of the low-energy cutoff to large values is analogous to what is routinely done in 1D models (e.g., the Tomonaga-Luttinger model¹²). In that context, deviations of the real excitation spectrum from linearity and the approximated integration measure are expected to affect only the numerical values of the renormalized physical parameters.

Choosing $\Lambda_0 \sim K_F$ renders the RG fixed points (observables) sensitive only to the two independent physical scales present in the model: T and $v_F K_F = 2E_F$, and not to the arbitrary scale Λ_0 , which divides fast and slow modes. Lowering the running cutoff until it reaches some intermediate scale Λ_X (such that $\Lambda_X \ll K_F$ and $v_F \Lambda_X \gg T$) provides us with Λ_X -dependent parameters for the action. We regard Λ_X as the scale of the low-energy effective

action. However, the observable quantities (the fixed points) do not depend on a particular choice of Λ_X .

4.4 Analysis and discussion of the RG results

We will now discuss the main novelties brought by quantum interference in the Landau channel and compare with the results of decoupled approximations. The solutions $\Gamma^*(\theta)$ and $F^*(\theta)$ at different temperatures and for the interaction (4.22) are shown on Fig.5 [A: direct numerical solution of Eqs (4.9); B: solution (4.33)]. For this interaction the sum in the second and third terms on the r.h.s. of Eq. (4.33a) is $\gamma_0^2(\beta_R) + 2\gamma_1^2(\beta_R) \cos\theta$. The curves were calculated for $\mathcal{U} = 1$ [cf. Eq. (4.21)], which is four times smaller than the critical value $\mathcal{U}_{cr}^{RPA} = 4$ at which the instability appears in the RPA solution (4.12a) for Γ_1^* . Comparison of the approximate solutions (4.31,4.33) with the direct numerical solution shows good agreement.

In Fig. 5 the differences between the RG solution and the RPA solution (4.12a) are minor at large angles, but they become especially striking at small angles θ , where the interference between the ZS and ZS' contributions is very strong. The RG solution gives $\Gamma^*(\theta = 0) = 0$ (the Pauli principle), while $\Gamma^{RPA}(\theta = 0) = -1/3$ for this interaction strength. The Landau interaction function $F^*(\theta)$ differs from the bare interaction $U(\theta)$, and $F^*(\theta = 0) \neq 0$. If the ZS' contribution is neglected (the RPA solution (4.12b)), these two quantities coincide.

An interesting feature of the RG result is the temperature dependence of the vertices $\Gamma^*(\theta)$ and $F^*(\theta)$. As T decreases, the “beak” of $\Gamma^*(\theta)$ in the region of strong interference becomes narrower. The characteristic angular width of this “beak” is $|\theta| \sim T/v_F K_F$. A similar narrowing is noticeable in the temperature dependence of $F^*(\theta)$. One can also see from the figures a weakening of the interference effect at lower temperatures, for then the RG solutions lie closer to

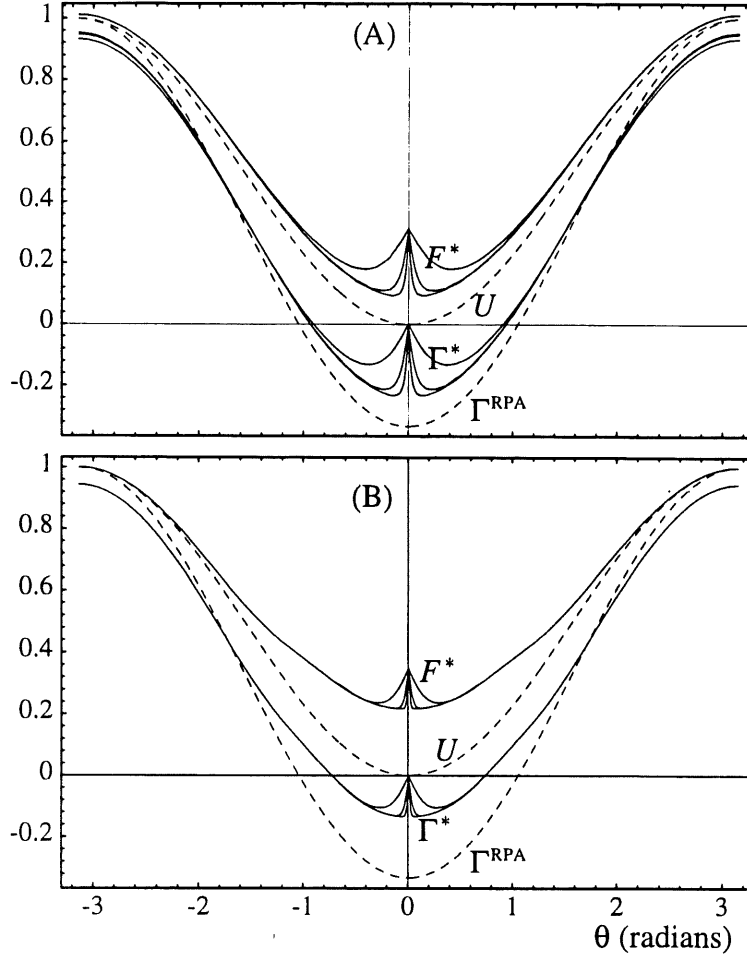


Figure 5: (A) Results of the numerical solution of the coupled RG equations. The curves labeled Γ^* and F^* are the forward scattering vertex and the Landau function, respectively, at temperatures $T/v_F K_F = 0.1, 0.025,$ and 0.01 . The narrowest central peak corresponds to the smallest temperature, and vice versa. (B) Approximate analytical solution of the coupled RG equations, for the same parameters as in (A), calculated numerically from Eq. (4.33). In both cases the initial cutoff was $\Lambda_0 = K_F$.

the RPA curves, but the distinctions between them do not disappear as $T \rightarrow 0$, and the RG never reproduces the RPA result.[†]

[†] At exactly zero temperature the angular features of the vertices reduce to finite discontinuities at zero angle, accompanied by finite-angle deviations from the RPA

In terms of Fourier components this behavior manifests itself in a linear temperature dependence of Γ_n^* and F_n^* . This linearity is found both in the direct numerical solution of Eqs (4.9), and from the solution of Eqs (4.25,4.27,4.28,4.31). This temperature dependence can be revealed analytically. Integrating by parts and using Eq. (4.26), we can rewrite Eq. (4.31a) at the fixed point as

$$\Gamma_n^* = \gamma_n^* + \sum_{m \neq n} I_{n-m}(\beta_0) U_m^2 - 2 \int_0^{\beta_0} d\beta_R \sum_{m \neq n} I_{n-m}(\beta_R) \left[\frac{1}{\cosh^2 \beta_R} - Y_0(\beta_R) \right] \gamma_m^3(\beta_R) . \quad (4.34)$$

The leading term on the r.h.s. of Eq. (4.34) is γ_n^* . Using then Eqs (4.29,4.30a), we obtain for $n = 0, 1$:

$$\Gamma_n^*(T) \approx \gamma_n^*(T) \approx \gamma_n^*(0) + \frac{T}{v_F K_F} \frac{(\ln 2)(\ln 3)}{\pi} [\gamma_n^*(0)]^2 , \quad (n = 0, 1) , \quad (4.35)$$

wherein

$$\gamma_n^*(0) = \frac{U_n}{1 + \left[1 - I_0(\beta_0) \right]_{T=0} U_n} . \quad (4.36)$$

For the interaction (4.22) $U_n = 0$ and so $\gamma_n^* = 0$ for $n > 1$. Thus, the higher harmonics $\Gamma_{n>1}^*$, are entirely generated by the RG flow. To leading order, we obtain from Eq. (4.34):

$$\Gamma_2^* \approx I_1(\beta_0) U_1^2 . \quad (4.37)$$

This component also has a linear temperature dependence, according to Eqs (4.30). To estimate the components of the Landau function, we first rewrite Eq. (4.31b) in another, equivalent form (cf. Eqs (4.24)):

$$F_n(\beta_R) = U_n + \sum_{m=-\infty}^{\infty} \int_{\beta_R}^{\beta_0} d\beta'_R Y_{n-m}(\beta'_R) \Gamma_m^2(\beta'_R) . \quad (4.38)$$

curve depending on the parameters of the effective action, e.g., the radius of Fermi surface, the strength of the interaction, etc. The zero-temperature limit is, however, mostly an academic question since the effect of interference between the Landau and the BCS channels, neglected in this study, would result in a Kohn-Luttinger instability in the V -interaction function²⁸ and destroy the regime of Fermi liquid before the system attains $T = 0$. Due to this second interference our results are not reliable near $\theta = \pi$, since $\Gamma(\pi) = V(0)$. Cf. Eq. (2.73).

Proceeding in the same fashion as above, we obtain the linear temperature-dependent components F_n^* :

$$F_n^* \approx U_n + I_0(\beta_0)U_n^2 + (|n-1|+1)I_1(\beta_0)U_{|n-1|}^2, \quad (n=0,1) \quad (4.39a)$$

$$F_2^* \approx I_1(\beta_0)U_1^2. \quad (4.39b)$$

We should emphasize that simple formulas like (4.35,4.37,4.39) serve only to illustrate how the temperature dependence comes about, and give only the order of magnitude of the higher harmonics ($n > 2$). The latter should rather be calculated numerically. The temperature dependence of the lowest harmonics (e.g., F_0^* and F_1^*) does not seem to be a relevant issue in the calculation of quantities such as the compressibility, effective mass and heat capacity, since, in the total ZS' contribution, the temperature corrections, of the order of $T/v_F K_F$, are very small in comparison with the main corrections of order Λ_0/K_F . As a consequence, the actual values of the lowest harmonics vary within a few percent at most, even in the entire temperature interval $0 \leq T/v_F K_F \leq 0.1$ (the maximum temperature studied is really high: $T = 0.2E_F$).

The temperature dependence is more pertinent as a “collective” effect of the higher harmonics generated by the RG flow. Let us explain this point with the example of the interaction (4.22). The “improved” RPA ansatz (4.27) renormalizes the bare components U_n into γ_n ($n = 0, \pm 1$). The latter form almost perfectly the function $\Gamma^*(\theta)$, except at small angles. For those three components γ_n the sum rule (4.32) is less violated than for the “pure” RPA components (4.13a). The generation of the new harmonics by the second term on the r.h.s. of Eq. (4.31a) gives “a final touch” to the curve $\Gamma^*(\theta)$, resulting mostly in the formation of a temperature-dependent feature near $\theta = 0$. The actual calculation of the components Γ_n^* showed that, in order to obtain with acceptable accuracy the right form of $\Gamma^*(\theta)$ provided by Eq. (4.33a) via the Fourier transformation of Eq. (4.31a), at least $N_{\max} \sim v_F K_F/T$ components are necessary. So, the lower the temperature is, the more harmonics are needed for the formation of the vertex $\Gamma^*(\theta)$. The same conclusion can be drawn from a

numerical solution of the equations, but since it is carried out in terms of angles on a discrete grid, a reliable calculation of higher harmonics is difficult.

Another physical consequence of the quantum interference in the Landau channel is the increased robustness of the system against instabilities induced by strong interactions. Even from the approximate solution (4.31), we see that the maximum interaction strength allowed is now larger than the one provided by the RPA solution [cf. (4.13,4.14)]. From Eq. (4.29) we obtain the stability conditions for the approximate solution (4.31): $U_l > -[1 - I_0(\beta_0)]^{-1}$, $\forall l$ with $0 < I_0(\beta_0) < 1$ according to (4.30a). Since $I_0(\beta_0)$ grows with temperature, larger values of $|U_l|$ are allowed as T increases: the higher the temperature, the more stable the system is, as it should be from physical grounds. At the optimal choice of the initial cutoff ($\Lambda_0 = K_F$), $I_0(\beta_0)$ grows from 0.255 at $T = 0$ to 0.27 at $T/v_F K_F = 0.1$. This value of temperature is the largest we can try without violating the condition of applicability of our model (2.43). Thus, within this approximate solution, the effect of interference increases the critical coupling by 40% compared to the RPA critical value (4.14). Since we are retaining only two one-loop diagrams, linearized excitation spectrum and integration measure, we cannot be more conclusive on the role of the modes deep into the Fermi sea in screening a microscopic interaction of arbitrary strength, and in stabilizing the Fermi liquid phase.

4.5 Contact with the Landau FLT and discussion

In this section we explain how the present RG theory, which takes into account the interference in the Landau channel, is related to the standard results of the Landau FLT.^{1,3} This will also allow us to relate the results of this chapter to the decoupled RG approach to the Fermi liquid presented in Chapter 3.

It is important to notice that the two contributions to the RG flow, coming from the ZS and ZS' graphs, behave quite differently as the flow

parameter β_R runs from $\beta_0 \gg 1$ towards $\beta_R = 0$. At large β_R the ZS contribution to the flow, which gives the term proportional to $\cosh^{-2} \beta_R$ on the r.h.s. of Eq. (4.23), is virtually negligible, up to $\beta_R \sim 1$. On this part of the RG trajectory, the main contribution to the renormalization of Γ and F comes from the ZS' graph. On the other hand, closer to the fixed point ($\beta_R \lesssim 1$), the ZS contribution grows since $\cosh^{-2} \beta_R \sim 1$ for all harmonics, while $Y_n(\beta_R)$ decreases for the lower-order harmonics. At $\beta_R \ll 1$:

$$Y_n(\beta_R) \approx \frac{1}{\pi n} \sin \frac{4n\beta_R}{\beta_F} . \quad (4.40)$$

Using the approximated form (4.25) is justified here, since at $\beta_R \ll 1$ there is no difference between the exact form of the RG equations (4.23) and Eqs (4.24). Indeed, when $\beta_R \ll 1$, the largest allowed ϕ is roughly $2\beta_R/\beta_F$, so in Eq.(4.3) $\max|\beta_{Q'}| \approx 2\beta_R \ll 1$ and the limit (4.5) of the function Y can be taken. The Kronecker delta appearing after the integration over θ removes one summation, and we recover exactly Eqs (4.24) with $Y_n(\beta_R)$ given by (4.40). It should be also kept in mind that the ZS' flow is localized within the angle $|\phi| \sim 2\beta_R/\beta_F$.

Such different behavior of the two contributions (ZS and ZS') to the total RG flow explains why approximations based on the decoupling of these two contributions (RPA, ZS -ladder^{5,7,33,34}) are reasonable. To clarify to what extent the standard results of FLT (4.18,4.19) can be corroborated by RG, we will make a two-step approximation of our RG equations. In doing so we will follow exactly the “recipe” of the ZS -ladder approximation discussed in Sec. 4.2, but now we can check each step by direct comparison with the RG solution of Eqs (4.9).

In the first step we neglect the contribution of the ZS graph above an intermediate flow parameter β_X . As one can see from the RG equations (4.23), this removes the exponentially small difference between $\Gamma_n(\beta_R)$ and $F_n(\beta_R)$ at $\beta_R > \beta_X$. This approximation is asymptotically exact as $T \rightarrow 0$ (see footnote on p. 61). Neglecting, in the second stage of this approximation, the ZS' flow for $\beta_R < \beta_X$, localized by that time within the angle $\theta_X = 2\arcsin(2\beta_X/\beta_F)$,

we recover the exactly solvable equations (4.12) with the new initial point $\beta_R = \beta_X$, instead of $\beta_R = \beta_0$. Then according to Eqs (4.12), $F_n^X \equiv F_n(\beta_X)$ is the (approximate) fixed point value of the Landau function, while $\Gamma_n(\beta_R)$ flows towards the (approximate) fixed point Γ_n^{ph} from the new bare value $\Gamma_n^X \equiv \Gamma_n(\beta_X) = F_n^X$. This second step of approximation violates the Pauli principle, no matter how close we are to the Fermi surface [cf. Eq.(4.11) and footnote on p. 61]. Afterwards the theory says nothing about the values of the functions $\Gamma(\theta)$ and $F(\theta)$ inside the interval $2\theta_X$ and, of course, there are no more correlations between these functions.

To preserve the correct zero-temperature limit and to minimize the angle within which the approximation gives completely wrong results for Γ^* and F^* , the intermediate cutoff Λ_X corresponding to $\beta_X = v_F \Lambda_X / 2T$ should be chosen such that $\tanh \beta_X \approx 1$ [cf. Eqs (4.12,4.13)] and $2\beta_X / \beta_F = \Lambda_X / K_F \ll 1$. Summing up what is said above, we obtain:

$$\Gamma_n^{\text{ph}} = \frac{\Gamma_n^X}{1 + \tanh(\beta_X) \Gamma_n^X} = \frac{F_n^X}{1 + F_n^X}, \quad (4.41a)$$

$$\Gamma_n^X = F_n^X = U_n + \sum_{l,m=-\infty}^{\infty} \int_{\beta_X}^{\beta_0} d\beta'_R \mathcal{Y}_{n-m,2l-2m}(\beta'_R) \Gamma_l(\beta'_R) \Gamma_{l-2m}(\beta'_R). \quad (4.41b)$$

In Fig. 6 we illustrated all this by the direct numerical calculation of F^X , Γ_n^X from Eqs (4.9) for the interaction (4.21), followed by a calculation of Γ^{ph} from Eqs (4.41). The RG solutions for Γ^* and F^* are also presented. The function $F^X(\theta)$ follows almost perfectly the Landau function (the real fixed point $F^*(\theta)$), except within $2\theta_X$ of $\theta = 0$. In the part of the RG trajectory $\beta_X \leq \beta_R \leq \beta_0$ ($\beta_0 = 100$, $\beta_X = 5$, $T/v_F K_F = 0.005$), not only is the ZS flow exponentially weak, but the central part of the ZS' flow as well [cf. Eq. (4.5)]. So, the evolution of both vertices is due mostly to the “tail” $\theta > \theta_X$ of the function Y at $\beta_R \gtrsim 1$. That is why $\Gamma^X(\theta)$ and $F^X(\theta)$ are virtually identical. Only the slowing down of the ZS' flow almost everywhere at $\beta_R \lesssim 1$ – except on the central part (cf. Eq. (4.40)) wherein it is always as strong as the other one (ZS) – results in the drastic differences between the two limits of the four-point vertex at the fixed point. The function $\Gamma^{\text{ph}}(\theta)$ is featureless and looks

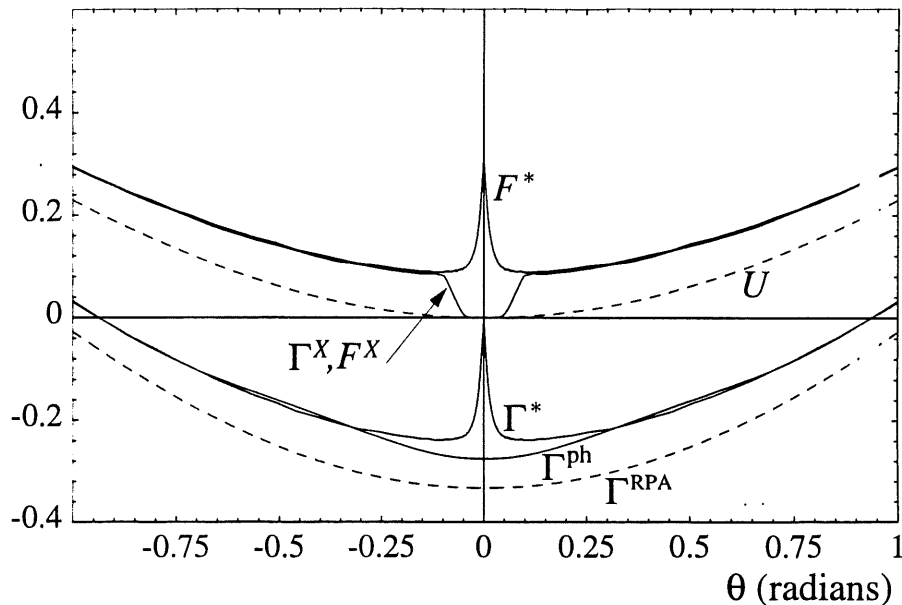


Figure 6: Comparison between the exact numerical solution of the coupled RG equations for $T/v_F K_F = 0.005$ (Γ^* and F^*), the intermediate values of Γ^X, F^X obtained from the initial value U by stopping the flow at $\beta_R = 5$, and the phenomenological vertex Γ^{ph} (the result of the standard FLT derivations) obtained by applying the RPA solution to Γ^X (F^X) considered as a new initial point of the flow. Γ^{ph} practically coincides with Γ^* , except in the central region.

like a corrected RPA solution. The differences between Γ_n^* and Γ_n^X (F_n^* and F_n^X) are negligible, i.e. less than 1%, only for the components $n = 0, 1$.

As it should be clear by now, there is no real incompatibility of the stability conditions with the Pauli principle, since this is a mere artefact of the ZS-ladder approximation. It is pointless to impose the sum rule either to Γ_n^{ph} in the form (4.32), or to F_n^X in the form (4.19b). Both sums would give the value of the “uncorrelated” function $\Gamma^{\text{ph}}(\theta)$ at $\theta = 0$. This function goes smoothly from the right patch $[\theta_X, \pi]$ towards $\theta = 0$ (cf. Fig. 6) – or, equivalently, from the left, because of parity. Actually, it can be proved exactly, turning the arguments of Sec. 4.2 around, that in a stable Fermi liquid, it is impossible to obtain $\Gamma^{\text{ph}}(\theta = 0) = 0$, even by chance. Thus, there is no need for the Landau

function F^* to be “fine tuned” in the sense of the sum rule (4.19b), since only the relation (4.41) – between the approximate vertex Γ^{ph} and F^X – is an exact relationship (more precisely, asymptotically exact when $T \rightarrow 0$), not (4.18), which relates the physical quantities F^* and Γ^* .

In the context of our discussion at the end of Sec. 4.3, notice that the cutoff Λ_X ($v_F \Lambda_X / T \gg 1$, $\Lambda_X / K_F \ll 1$) corresponds to the initial cutoff of the *low-energy* effective action wherein Γ^X is the bare interaction function (coupling) of that action. The equality of the functions Γ^X and F^X illustrates the point of Sec. 2.4 that, at the beginning, the action’s coupling function can be defined independently of the order in which the zero-transfer limit is taken.

When the RG flow reaches the scale Λ_X , the contribution of the ZS' graph to the flow of Γ_n and F_n is strictly irrelevant in the RG sense, and could have been neglected in a model with a finite number of couplings (e.g., the φ^4 -theory, 1D g-ology models, and so on), keeping only marginal terms [cf. Eqs (4.12)]. But, as pointed out by Shankar,²⁸ in the vicinity of the Fermi surface we are dealing with coupling *functions*, i.e., with an *infinite* set of couplings. Our RG solution provides a curious example of a finite deviation of the RG trajectory at the fixed point due to an infinite number of irrelevant terms. The *right* fixed point ($\Gamma^*(\theta = 0) = 0$) cannot be reached if those terms are neglected, since $\Gamma^{\text{ph}}(\theta_X \rightarrow 0) \neq \Gamma^*(\theta = 0)$ (even at $T = 0$; see footnote on p. 73) and we would return to the problems caused by the solution Γ^{ph} (the ZS -ladder approximation) discussed in Sec. 4.2. To put it differently, neglecting those irrelevant terms at some part of the flow (solution (4.41)) violates the invariance of the RG trajectory at the point $\theta = 0$, expressed by Eqs (4.10,4.11).

The ZS -ladder approximation seems acceptable in the normal Fermi liquid regime with moderate interaction ($F_n \lesssim 10$), when the narrow-angle features of vertices revealed by the RG theory are not too large (see footnote of p. 73) because the forward ($\theta = 0$) singularity has little effect on the first components ($\Gamma_n^* \approx \Gamma_n^{\text{ph}}$, $F_n^* \approx F_n^X$ for $n = 0, 1$ and, in the case of a weak interaction, for $n = 2$). This singularity affects mostly the higher Fourier components. So, the

relationship (4.18) is valid only for small n . It should not be used for F_n^* ($n \geq 2$) neither directly, nor via the sum rule from the scattering vertex provided experimentally. For the physical vertex Γ_n^* the sum rule (4.32) is always valid, but our results indicate that its angular shape may require a large number of harmonics to adequately represent it. The existence of a finite solution for $\Gamma^{\text{ph}}(\theta)$ under conditions

$$\Gamma_n^X > -1, \quad \forall n \quad (4.42)$$

guarantees not only finite RG solutions for Γ^* and F^* , but also the fulfillment of the thermodynamic Pomeranchuk conditions (4.20) by F^* .

The major consequence of the interference in the Landau channel on the standard results of the FLT is reducing the relationship (4.18) between the components of the scattering vertex and the Landau function to the rank of approximation and invalidating of the sum rule (4.19). The rest of results for normal Fermi liquids would not be affected seriously by the RG corrections. For example, the temperature dependence of the vertices would give a weak correction to the leading terms. These conclusions are neither related to the specific choice of the model considered, nor to the spatial dimension. Including spin doubles the number of vertices involved, changing nothing essentially. [Coming back to the general RG equations (4.4) for the $SU(N)$ case and introducing the components $\{F, G\}$ and $\{A, B\}$ according to (3.16, 3.17), one can easily obtain the RG equations in terms of those variables. Contrary to the case (3.20), the variables are coupled. The equations preserve the Pauli principle for the scattering vertex [cf. Eq. (4.8)] in the form

$$A(\phi = 0, t) + (N - 1)B(\phi = 0, t) = \sum_{l=-\infty}^{\infty} [A_l(t) + (N - 1)B_l(t)] = 0, \quad \forall t. \quad (4.43)$$

However, the components of the scattering vertex and of the Landau function are not related by the simple RPA-type relationships (3.22)]. The differences for the case $d = 3$ are only quantitative (e.g., the type of temperature dependence) because of different angular functions and solid angle integrations.

Conclusion

The finite-temperature renormalization-group (RG) method was developed to study interacting fermions in spatial dimensions $d = 2, 3$. A model with a ψ^4 -Grassmann effective action with $SU(N)$ -invariant short-range interaction and rotationally invariant Fermi surface is adopted as a starting point of the RG analysis. Applying a decoupled (RPA-type) approximation at the one-loop RG level, we showed how the key results of the Landau Fermi liquid theory (FLT) can be recovered by the finite-temperature RG technique. In particular, this RG approximation allows us to obtain the relationship between the components of the forward scattering vertex and of the Landau interaction function of quasiparticles. In the standard diagrammatic FLT this relationship is provided by the solution of the Bethe-Salpeter equation. Using the decoupled RG approximation for calculation of response functions, we find the FLT results for the compressibility and the spin susceptibility as solutions of the RG flow equations.

We discussed subtleties stemming from the crossing-symmetry properties of the four-point vertex (the implications of the Pauli principle). We pointed out symmetry-related distinctions between three quantities: the bare interaction of the low-energy effective action, the Landau function and the forward scattering vertex.

Our results show that the bare interaction of the effective action is not a RG fixed point, as was concluded in earlier RG studies of the Fermi Liquid, but a common starting point of the flow trajectories of two limiting forms of the four-point vertex. In order to explicitly preserve the (anti)symmetry of the four-point vertex (crossing symmetry) in RG calculations for the Landau channel of (nearly) forward scattering quasiparticles, we have derived RG equations that take into account both the contributions of the direct (ZS) and exchange (ZS') particle-hole graphs at the one-loop level. From those RG flow equations, the basic quantities of Fermi Liquid theory, the Landau interaction

function and the forward scattering vertex, are calculated in terms of the effective action's interaction function.

The classic derivations of Fermi Liquid theory applying the Bethe-Salpeter equation for the four-point vertex at $T = 0$ is based on the approximation that the vertex, irreducible in the direct particle-hole loop (ZS) is a regular function of its variables, neglecting the zero-angle singularity in the exchange loop (ZS'). This approach, and other analogous (decoupled) ones, including the decoupled RG approximation, are tantamount to a summation of the direct particle-hole ladder diagrams, wherein the Landau function stands as the bare interaction.

One of the major deficiencies of the decoupled approximation is that the antisymmetry of the forward scattering vertex related by the RPA-type formula to the Landau interaction function, is not guaranteed in the final result, and the amplitude sum rule must be imposed by hand on the components of the Landau function to satisfy the Pauli principle. This sum rule, which is not indispensable in the original phenomenological formulation of the Landau FLT, is equivalent, from the RG point of view, to a fine tuning of the effective interaction.

We demonstrated that the strong interference of the direct and exchange processes in particle-hole scattering near zero angle invalidates the RPA (decoupled) approximation in this region, resulting in temperature-dependent narrow-angle anomalies in the Landau function and forward scattering vertex, revealed by the RG analysis. In the RG approach, which explicitly takes into account the interference in the Landau channel, the Pauli principle is automatically satisfied. As follows from the RG solution, the amplitude sum rule, being an artefact of the RPA approximation, is not needed to respect the Pauli principle and, moreover, is not valid.

In perspective, the RG approach developed here, which takes into account the interference of different processes at one-loop level, opens the possibility to generalize this technique in order to calculate vertices and

response functions at non-zero energy and momentum transfer, starting from different kinds of effective actions. In particular, a straightforward extension of the present technique for the case of nonzero momentum-energy transfers with the same effective action as considered in this study, would allow to find (presumably) subleading RG corrections to the standard results for normal Fermi liquids, e.g., for susceptibilities and collective modes.

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