The application of the theory of dynamical systems in conceptual models of environmental physics

The thesis points of the PhD thesis

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Introduction, motivation

Environmental physics is an important area of modern physics which applies the tools of classical physics. Environmental fluid dynamics (Pedlosky, 1979; Vallis, 2006) grew out from general hydrodynamics in the first part of the 20^{th} century. Compared to the latter it is special in that its phenomena are considerably influenced, what is more, typically determined by the following factors: the Earth's rotation and spherical geometry, and the investigated medium's shallowness and stratification. Taking into account these effects environmental fluid dynamics has proven to be considerably successful in understanding the atmospheric and oceanic processes. This knowledge is, on the one hand, necessary for understanding how real weather and, consequently, climate works, and, on the other hand, for interpreting appropriately the solutions obtained numerically from the basic equations of the atmosphere and the ocean. The full exploration of this range of phenomena is, however, a duty for the future.

Within environmental fluid dynamics, phenomena on a rotating sphere still need a more precise understanding. My research, aimed at investigating the motion of vortex pairs and the chaotic advection in their velocity fields, and providing a testbed for a generally used approximation, the so-called β-plane approximation, contributes to this issue. The description of climate dynamics, in turn, is not at all matured, this poses in fact one of the most important tasks of our age. Its probabilistic approach is based on the so-called snapshot attractor picture. My aim was to work out in detail the method of applying snapshot attractors, and to quantify their deviation from the attractors of time-independent systems.

Model setups, applied methods

In an ideal two-dimensional incompressible flow the simplest vortices are the so-called point vortices. Their motion is described by ordinary differential equations in which the constant circulation of the point vortices appear as a parameter. This construction is applicable also for spherical geometry (Newton, 2001). The motion of point vortices on a rotating sphere can be taken into account by making the circulations of the vortices depend on the position, in particular, on the latitudinal coordinate. This is the so-called the modulation which ensures the conservation of the angular momentum of the vortices. The idea (Zabusky and McWilliams, 1982) is more than three decades old, its application, however, has been restricted to the β-plane approximation (Pedlosky, 1979) so far when one investigates a small neighborhood of a given latitude. The extension of the method to the full sphere was done in [1]. In the first major topic of our work we considered pairs composed of two modulated vortices. We defined a pair such that its members' circulations on the reference latitude φ_{r} are identical in absolute value and are opposite in sign.

In the β-plane approximation of environmental fluid dynamics (Pedlosky, 1979) the geometry is approximated by a plane near a particular latitude considered to be the origin, and the latitudinal dependence of quantities are taken into account in the linear order of the Taylor expansion around the origin. The model of the modulated point vortices is especially suitable for investigating the appropriateness of this approximation, since, unlike in usual hydrodynamical systems, the motion of vortices is determined by ordinary differential equations.

In the velocity field of modulated vortex pairs on the β-plane passive tracers are advected chaotically (Velasco Fuentes et al. 1995; about chaotic advection in general: Aref, 1984, 2002). If the tracers leave the close neighborhood of the vortex pair, the advection is open and is governed by a so-called chaotic saddle which is a fractal set consisting of unstable periodic orbits. if the tracers always move together with the vortex pair, the advection is closed, and is determined by a space-filling chaotic set. In the model of modulated vortex pairs on the β-plane both behavior types are observable. What is more, the type of the behavior with a given set of parameters depends on the initial conditions of the vortex pair: between the two types of advection, a kind of "phase transition" occurs at a particular latitude (Benczik et al., 2007).

In the other major topic of our work, the studying of climate dynamics, we applied the model referred to as the Lorenz 84 model (Lorenz, 1984). This describes (on a conceptual level) the atmospheric dynamics of midlatitudes by three variables: the speed of the Westerlies and the amplitudes of two cyclonic modes. We incorporated a climate change into this model (in what follows: the modified Lorenz 84 model) by the linear time-dependence of a parameter which corresponds to the decrease of the temperature contrast between the Equator and the polar region and thus mimics the increase of the CO_2 level. This way we obtained a nonautonomous dynamical system.

The usual attractors (Ott, 1993) appearing in autonomous dynamical systems (or in those forced periodically in time) cannot form if the forcing is not periodic in time. The long term behavior is described by the so-called snapshot attractor (Romeiras et al., 1993; Chekroun et al., 2011) in such cases which changes in time perpetually and without any periodicity. This object is obtained by initializing an ensemble of trajectories in a given time instant t_0, being in the remote past, applying an identical forcing on them, and registering the endpoints of these trajectories at the time instant t of the observation.

Ergodicity cannot hold in such systems, i.e., the ensemble average (taken with respect to the natural measure) and a temporal average taken along a single trajectory (on an infinitely long time window) do not coincide. Chekroun et al. (2011) introduced an alternative definition — which is, however, not applicable for single time series, only for ensembles — for temporal averaging by means of which the mentioned equality can be made valid.

Thesis points related to the modulated point vortex pairs

1. Time scale separation in the advection: a crossover from locally open advection to global mixing [1]

The advection of passive tracers in the velocity field of a vortex pair on a rotating sphere, described by modulating the circulations, may locally be closed and open. We showed that whether the advection is closed or open depends on the initial conditions of the vortex pair when the set of parameters is given: between the two types of advection, a kind of "phase transition" occurs at a particular latitude. The locally closed and open advection types are observed on short time scales. On asymptotically long times, however, we found even the open advection to become closed since the vortex pair periodically reenters its own wake due to the spherical geometry. The chaotic saddle governing the open dynamics was shown to be less and less repelling, to gradually become space filling and to spread to the full sphere in a zonal band. Such a time scale separation is expected to occur in any chaotic advection problem where the flow is locally open but globally closed.

2. The inconsistency of the β-plane approximation in the dipole limit [2]

In the modulated point vortex pair model on the sphere we considered the equations of motion in the dipole limit. We derived the β-plane approximation for these equations around the reference latitude φ_{r} of the model, and we also carried out a consistent linearization of the original (full spherical) dipole equations in the latitudinal coordinate φ around the reference latitude φ_{r}. We showed that two terms of the consistently linearized equations of motion are missing in the β-plane equations. Even in the geophysically relevant context of small velocities one of the two additional terms turns out to be of equal order as the only term present in the β-plane equations. The missing of the additional terms is due to the inconsistent treatment of the spherical geometry in the β-plane approximation. These results strongly suggest that conventional β-plane approximations might not be suitable under all circumstances for analyzing phenomena with a small meridional extension in the realm of environmental fluid dynamics.

3. The different trajectories of the β-plane approximation and the spherical treatment [2]

In the β-plane approximation it is the reference latitude φ_r on which a uniform eastward and a uniform westward propagation of the dipole occurs (depending on the initial conditions). The uniform eastward propagation serves as a center for the trajectories initiated closely to φ_r and exhibiting a meandering motion to the east. The uniform westward propagation separates the large-sized, circular trajectories that are restricted to the northern and to the southern side of φ_r. In the full spherical dipole equations and also in the consistently linearized ones we analytically showed that the uniform eastward and westward propagations take place on *different* latitudes from φ_r and also from each other (although the differences are typically not very large). We call them the special latitudes, denoted by $\varphi_\pm$, with φ_+ corresponding to the eastward propagation. In particular, we found that $\varphi_+ < \varphi_0$ and $\varphi_- > \varphi_0$ if $\varphi_r > 0$. At the same time, we numerically pointed out that the special latitudes exhibit the same roles of center and separator as φ_r in the β-plane approximation. The difference between φ_+ and φ_- is interpreted as a symmetry breaking, observable the best in a phase space representation, and which is hidden by the conventional β-plane approximation. By solving numerically the dipole equations of the β-plane approximation and the spherical treatment we concluded that the β-plane approximation makes the largest relative errors in the vicinity of its origin where it would be expected to perform the best. We found that it is inapplicable approximately in the middle one fourth of its validity range. Similar results were obtained for finite-sized vortex pairs as well. These numerical findings confirm also from a practical point of view that the conventional β-plane approximation may end up in misleading results.

4. Advectional aspects of the inconsistency of the β-plane approximation [2]

We compared the chaotic advection of passive tracers in the field of a finite-sized vortex pair under β-plane and full spherical treatments. We found considerably different advection patterns on a quantitative level: the clouds of tracers left behind the vortex pair are rather different under the two treatments. What is more, we showed transport properties (e.g. escape rates) to be systematically enhanced or attenuated by the β-plane approximation, depending on the initial conditions of the vortex pair. We managed to explain this qualitatively by comparing the vortex pair trajectories under the different treatments. It is nevertheless true that the advection patterns and the level of their error in the β-plane approximation cannot be deduced purely from the vortex pair trajectories.

Thesis points related to snapshot attractors and climate dynamics

5. The relevance of snapshot attractors in changing climates [3]

The relevance of smooth deterministic forcings in climate change science appears as a novelty in the literature of snapshot attractors. In this context we argued that the internal variability of the climate is described by the natural probability measure of the snapshot attractor. We note furthermore that climate changes should be interpreted as the time-dependence of the natural measure. As an illustration, we considered the time-dependent snapshot attractor (which is different in every time instant) of the modified Lorenz 84 model, and we evaluated time-dependent averages and variances over its natural measure. In contrast to how snapshot attractors are treated in the mathematical literature, we demonstrated, with the help of the statistics (through the investigation of their independence on the set of initial conditions), that an initialization of the numerical ensemble of the trajectories, representing the snapshot attractor and its natural measure, may take place even only a short time before the time instant of observation, because the convergence to the snapshot attractor (and to its natural measure) is exponentially fast. This property enables one to numerically generate the snapshot attractor of any particular problem without any additional effort compared to generating usual attractors.

6. A deviation from ergodicity [4]

In nonautonomous dynamical systems with arbitrary time-dependence ergodicity, the equality of temporal averages along single trajectories and ensemble averages, may not be fulfilled, because the natural probability measure depends on time. To characterize the mismatch between the two kinds of averages we introduced what we call the signed ergodicity deficit $\delta_\tau(t)$: this is the difference of two terms. One of them is the temporal average $\mathcal{A}_\tau$ which is taken along a single realization over a finite-length time window τ, and the other is the ensemble average $\mathcal{A}_\mu$ which is taken over the natural measure of the snapshot attractor at the time t of observation. $\delta_\tau(t)$ depends on the time instant t, on the choice of the window length τ, and on the particular realization. As such, it has its on probability distribution. We investigated this probability density function (pdf) in detail in the modified Lorenz 84 model. We considered separately a case with a periodic forcing (i.e., an ergodic case) and that when a parameter of the model is varied linearly in time (nonergodic case). In a climatic context these cases describe a stationary and a changing climate, respectively. We found that the pdf of $\delta_\tau(t)$ approximates well a rather extended Gaussian both in the ergodic and in the nonergodic case. Single realization in finite-length time windows thus typically exhibit a nonzero ergodicity deficit, they are "not ergodic".

7. Measures of nonergodicity [4]

We also investigated the quantitative characteristics of the pdf introduced in Thesis Point 6. The standard deviation $\sigma_\mu(\delta_\tau)$ of the pdf, describing the *spread* among different realizations, was numerically found with increasing τ to decrease according to a law of $1/\sqrt{\tau}$, which implies a scale-free convergence to fulfilling the ergodic relation (when this relation holds at all). In the ergodic case we found the average $\mathcal{A}_\mu(\delta_\tau)$ of the pdf to be zero for any window length τ, which is a generalization of the ergodic theorem for finite τ. In the nonergodic case, however, we observed this average to be typically different from zero (i.e., it represents a *bias*), and its modulus exhibits an increasing tendency with increasing τ. We proposed this latter, i.e., the modulus $|\mathcal{A}_\mu(\delta_\tau)|$ of the average signed ergodicity deficit, to regard as a measure of how much the dynamics is nonergodic. The mechanisms underlying our findings are the central limit theorem and the considerable time-dependence of the natural measure. The average ergodicity deficit modulus, $\mathcal{A}_\mu(|\delta_\tau|)$, incorporates both the spread and the bias, and stands for the expected distance of a single-realization temporal average from the ensemble average. In the modified Lorenz 84 model it is never small, according to a trade-off situation, pointed out by us, between the spread and the bias, decreasing and increasing with increasing τ, respectively. We conclude that single-realization temporal averages are never expected to approximate well ensemble averages in general nonautonomous dynamical systems. In particular, we do not expect 30-year averages along single realizations — applied widely — to give useful hints on internal variability and on climate change in climate science. Instead, the measures themselves introduced for the degree of nonergodicity measure well the degree of climate change.

8. The application of a different construction for ergodicity [4]

We also investigated a different, purely mathematically motivated construction for ergodicity. This transforms the ensemble average taken over the endpoints, corresponding to the time t of observation, of different trajectories to the temporal average taken over the initial time instants of these trajectories. We found this artificial averaging to always tend to the traditional ensemble average with the increasing length of the time window for averaging (i.e., with taking into account more and more trajectories initialized in different time instants), both in the autonomous and nonautonomous case, it thus makes no distinction between these cases. As follows from what is said in Thesis Point 7, distinguishing autonomous and nonautonomous dynamics is important from a climatic point of view. For this reason, and also because the artificial averaging implies the use of an ensemble just as averaging over the numerically represented natural measure itself, we do not really see a reason for considering this different construction for ergodicity to be very useful in climate science.

Conclusions

In view of the results detailed in the thesis points, we may draw two conclusions. On the one hand, numerous uncovered phenomena may be found even in the better-studied areas of environmental fluid dynamics (such as in the dynamics of vorticity). On the other hand, finding the appropriate framework (such as that of snapshot attractors) may be essential for describing environmental systems that are not yet understood (such as the climate system). Still, we have been applying the theory of dynamical systems for all our purposes, and the success of this approach demonstrates the power of this theory. (This is also illustrated well by the research made parallel to the PhD work in the topic of chaotic scattering [E1-E4].)

Let us consider briefly the details, too. In the model of vortex pairs we have managed to describe a new phenomenon with a general perspective: how the chaotic saddle of a locally open dynamics transforms, when investigating long times, to the space-filling chaotic set of a globally closed dynamics. Within the same model it has also become clear that results originating in the traditional β-plane approximation have to be treated with caution.

We have seen that the probabilistic characterization of a continuously changing climate is provided by the natural measure of the snapshot attractor, and that its numerical generation does not require special efforts due to the exponentially fast convergence. We note that in a different work [E5] we have already applied this approach for a high-degrees-of-freedom, intermediate-complexity climate model as well. From the results of the present thesis points it becomes clear that climate change and nonergodicity are closely related. As an important consequence, the widely used temporal averages taken along single realizations are typically irrelevant in a changing climate.

The publications providing the basis for the thesis points

[1] G. Drótos, T. Tél, and G. Kovács. Modulated point vortex pairs on a rotating sphere: Dynamics and chaotic advection. *Physical Review E*, 87:063017, 2013.

[2] G. Drótos and T. Tél. On the validity of the β-plane approximation in the dynamics and the chaotic advection of a point vortex pair model on a rotating sphere. *Journal of the Atmospheric Sciences*, 72:415–429, 2015.

[3] G. Drótos, T. Bódai, and T. Tél. Probabilistic concepts in a changing climate: A snapshot attractor picture. *Journal of Climate*, 28:3275–3288, 2015.

[4] G. Drótos, T. Bódai, and T. Tél. Quantifying nonergodicity in nonautonomous dissipative dynamical systems: An application to climate change. *Submitted to Phys. Rev. E*, 2015.

Further, parallel publications

[E1] G. Drótos, C. Jung and T. Tél. When is high-dimensional scattering chaos essentially two dimensional? Measuring the product structure of singularities. *Phys. Rev. E*, 86:056210, 2012.

[E2] F. Gonzalez, G. Drotos and C. Jung. The decay of a normally hyperbolic invariant manifold to dust in a three degrees of freedom scattering system. *J. Phys. A: Math. Theor.*, 47:045101, 2014.

[E3] G. Drótos, F. González Montoya, C. Jung and T. Tél. Asymptotic observability of low-dimensional powder chaos in a three-degrees-of-freedom scattering system. *Phys. Rev. E*, 90:022906, 2014.

[E4] G. Drótos and C. Jung. The chaotic saddle of a three degrees of freedom scattering system reconstructed from cross section data. *Submitted to Proc. Roy. Soc. A*, 2015.

[E5] M. Herein, J. Márfy, G. Drótos and T. Tél. Probabilistic concepts in intermediate-complexity climate models: A snapshot attractor picture. *Accepted for publication by J. Climate*, `http://dx.doi.org/10.1175/JCLI-D-15-0353.1`, 2015.

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- O. U. Velasco Fuentes, G. J. F. van Heijst and B. E. Cremers, 1995, "Chaotic transport by dipolar vortices on a β-plane". *J. Fluid Mech.* **291**, 139–161.
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