## Institute of Mathematics

Results of the Ph.D. thesis

## Some Ramsey- and

 anti-Ramsey-type results in combinatorial number theory and geometryJózsef Borbély

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## Summary of the thesis

In this thesis my goal is to present some applications of combinatorial methods in number theory, group theory, and geometric graph theory. Combinatorial methods are considered as profound and current topics of mathematics nowadays. In particular, I will study Ramsey and anti-Ramsey (or rainbow type) problems. The great number of journals publishing papers in the concerned areas attests to the important role of these branches. The following journals publish papers of this type (without making any claims of completeness): Combinatorica, Discrete and Computational Geometry, Integers, Journal of Combinatorics and Number Theory, Moscow Journal of Combinatorics and Number Theory, The Electronic Journal of Combinatorics. We always will concentrate us on Ramsey-type problems. It is important to clarify what we mean by Ramsey-type results. In this thesis we speak about a Ramsey-type theorem, if the result can be formulated in the following way: if one colors every element of a "large" structure with exactly one of $k$ fixed colors, then one can always find a monochromatic partial structure of a desired size (independent of the chosen coloring). The first known theorem of this sort was proved by Frank P. Ramsey (see [13]). Since his death in 1930 many new Ramsey-type problems have been posed and solved. This branch of mathematics has a widespread literature. The book of Graham, Rotschild and Spencer ([13]) is one of the most profound summaries of the most important results. In addition, the book of Landman ([27]) contains many important results concerning Ramsey-type number theoretical problems. We have to distuingish between Ramsey-type theorems and density theorems. We speak about a density theorem if we can formulate our results in the following way: if one chooses a "large" subset of a given set, then one can find another subset of the chosen subset with the desired properties. It is clear that density theorems are stronger than Ramseytype theorems. This means that it is possible, that one can find a Ramsey-type result without even when no density result exists, but from the existence of a density theorem one can conclude that there is also a Ramsey-type theorem. We have to explain what we understand by rainbow Ramsey-type theorems.

In these cases one colors every element of a given set by exactly one of the $k$ given colors (in the same manner as in the case of Ramsey-type theorems), but now we look for multicolored substructures of given type. In this thesis we focus primarily on Ramsey-type theorems in number theory and its related areas.

The first chapter is based on [2]. In this chapter we proceed from a problem of Roth (see [17]) that was solved by Erdös, Sárközy and T. Sós in an elementary way (see [11]). We will extend their methods in order to achieve more general results. For any $k$-coloring of a special subset of $r$-dimensional vectors we investigate asymptotically the minimal number of $r$-dimensional vectors that are representable as a sum of distinct $r$-dimensional vectors of the same color. Throughout the first chapter we use the following notations: for any integers $k$ and $l$, where $k \leq l,(k, l)$ denotes the set $\{k, k+1, \cdots, l\}$. Moreover, for any positive integer $r$ and for any sets $H_{1}, H_{2}, \cdots, H_{r}$, we denote the Cartesian product of these $r$ sets by $H_{1} \times H_{2} \times \cdots \times H_{r}$. If $H_{1}=H_{2}=\cdots=H_{r}=H$, then we write $H^{r}=H_{1} \times H_{2} \times \cdots \times H_{r}$. In Chapter 1 first we prove the following additive theorem:

Theorem 1. For fixed positive integers $r, s, k$ there is a positive integer $m_{0}$ with the following property: For any positive integer $m>m_{0}$ and for any $k$-coloring of the elements of the set $(1, m)^{r}$ there are at least

$$
\left[\frac{m}{s}\right]^{r}-3 \cdot\left(2^{r-1} m^{r}\right)^{1-2^{-k s+k-1}}
$$

vectors $\vec{x}$ in $(1, m)^{r}$, such that there are pairwise distinct vectors $\overrightarrow{x_{1}}, \overrightarrow{x_{2}}, \cdots, \overrightarrow{x_{s}}$ of the same color in $(1, m)^{r}$, whose sum is $\vec{x}$.

Moreover, in the first chapter we achieve a result concerning the number of representations:

Theorem 2. For every positive real number $\alpha$ and $\beta$ with the property

$$
\alpha^{r}+\beta^{r} \leq \frac{1}{2^{2 r+1} k}
$$

there is a positive integer $m_{\alpha \beta}$, such that for every $m>m_{\alpha \beta}$ and for every $k$-coloring of $\mathbb{N}^{r}$ the number of elements in $(1, m)^{r}$ having representations as
a sum of two monochromatic distinct vectors in more than $\frac{\beta^{r}}{2} m^{r}$ ways is more than $\alpha^{r} m^{r}$.

The second chapter is based on [3]. In the second chapter we approach the problem of Roth from another point of view. This means that we look for analogous results in some abelian groups, but this time we focus on rainbowtype Ramsey problems. More precisely, for any $k$-coloring of some special abelian groups $G$ we investigate the minimal number of elements that are representable as a sum of $r$ elements of distinct colors. First we prove the following theorem in an elementary way:

Theorem 3. For every $k$-coloring of the positive integers $1,2, \ldots, n$, where color $i$ occurs exactly $n_{i}$ times $(i=1,2, \ldots, k)$, where $n_{1} \geq n_{2} \geq \cdots \geq n_{k}$, there are at most $r-1+\sum_{i=1}^{r-1}(r-i) n_{i}$ numbers in the set $\{1,2, \cdots, n\}$ having no representation as a sum of $r$ integers of distinct colors ( $r \geq 1$ ).

Then we investigate analogous questions in finite abelian groups. We prove the following theorem by using Kneser's theorem:

Theorem 4. For every $k$-coloring of the elements of the additive abelian group $G$ of finite order $m$, where every color class is used, there are at least

$$
\min \left\{m-\frac{(k-1) \cdot m}{D_{k}(G)}, \frac{m}{d_{k}(G)}\right\}
$$

elements of the group having a representation as the sum of $k$ elements of $G$ of different colors (where $d_{k}(G)$ is the largest positive integer not greater than $k$ dividing the number $|G|$, and $D_{k}(G)$ is the smallest positive integer greater than $k$ dividing the number $|G|$ ).

Then, by using Vosper's theorem, we achieve the following result in $\mathbb{Z}_{p}$ :
Theorem 5. If the elements of $\mathbb{Z}_{p}$ are colored by $k \geq 2$ colors such that every color class is used, then at least $p-1$ elements of $\mathbb{Z}_{p}$ have a representation as a sum of $k-1$ elements of pairwise distinct colors.

Moreover, we have the conjecture, that even the following stronger result holds:

Conjecture 1. If the elements of $\mathbb{Z}_{p}$ are colored by $k \geq 4$ colors such that every color class is used, then each element of $\mathbb{Z}_{p}$ has a representation as a sum of $k-1$ elements of pairwise distinct colors.

It is a natural question how to extend the result of Theorem 5 in other finite abelian groups. We could not solve the problem, but we have the following a conjecture:

Conjecture 2. Let $G$ be a finite abelian group having $m$ elements, and let $r$ and $k$ be positive integers such that $2 \leq r<k$. If $m=p_{1} \cdot p_{2} \cdots p_{s}$, where $p_{1} \leq p_{2} \leq \cdots \leq p_{s}$ are prime numbers, then the following statement is true:
for every $k$-coloring of the given abelian group $G$, where every color class is used, the number of elements of $G$ having a representation as a sum of $r$ elements of distinct colors is
(i) exactly $m$, if $r=2$ and $m<2 k-1$
(ii) at least $m-\frac{m}{p_{1}}$, if $r=2$ and $2 k-1 \leq m$
(iii) at least $m-\frac{m}{p_{1} \cdot p_{2} \cdots p_{k-r+1}}$, if $2<r$ and $k-r+1 \leq s$
(iv) is eactly $m$, if $2<r$ and $k-r+1>s$.

The bounds expressed above are tight.
It is trivial that Conjecture 1 is a consequence of Conjecture 2.
In the third chapter we generalize analogous Ramsey-type results concerning products. For any $k$-coloring of the positive integers up to $M$ will estimate the minimal number of positive integers that are representable as a product of $r$ integers of the same color. The results of this chapter are written in [4]. In the third chapter we first prove the following theorem:

Theorem 6. There is a positive absolute constant $c$ such that if $r$ and $k$ are positive integers, where $k$ is greater than $r$, and $M$ is a positive integer large enough depending on $k$, then every $k$-coloring of the elements of the set $\{1,2, \cdots, M\}$ satisfies the inequality

$$
\sum_{b \in B} \frac{1}{b}>c \cdot \frac{2}{r} \cdot \frac{1}{k^{r-1}} \cdot \log M,
$$

where $B$ denotes the set of positive integers that are representable as a product of $r$ elements of the set $\{1,2, \cdots, M\}$ of the same color.

Moreover, we investigate analogous problem in some finite abelian groups. We prove the following theorem:

Theorem 7. For any positive integers $k$ and $r$ there exists a positive integer $T_{k, r}$, such that for every positive integer $M$ greater than $T_{k, r}$ the following statement holds: for any $k$-coloring of the elements of any cyclic multiplicative group $G$ of order $M$ there are at least

$$
\frac{M}{(M, r)}-3 \cdot\left(\frac{M}{(M, r)}\right)^{1-2^{-(k \cdot(r-1)+1)}}
$$

elements of $G$ that have a representation as a product of $r$ distinct elements of the same color (here ( $M, r$ ) denotes the greatest common divisor of the numbers $M$ and $r$ ).

In the fourth chapter we study the corresponding rainbow problems concerning products. We achieve an exact asymptotic result. More precisely, if $k$ is a fixed positive integer and we color the positive integers up to $n$ by $k$ colors so that every color is used, then by denoting the minimal number of integers having a representation as a product of $r$ numbers of distinct colors by $R_{k}^{(r)}(n)$, we prove the following:

Theorem 8. $\lim _{n \rightarrow \infty} \frac{R_{k}^{(r)}(n)}{n}=2-\frac{1}{2^{(r-1)(k-1)-(r-1)^{2}}}$.
In the last chapter we present and solve a Ramsey-type problem in geometric graph theory concerning multiplicity of special subgraphs. The problem was posed by Gyula Károlyi (see [22]). A geometric graph is a graph drawn in the plane, where every vertex is represented by a point and the edges are straight lines connecting some of the vertices (we assume, that an edge joining two vertices does not pass through a third vertex). A convex geometric graph is a special geometric graph, where the vertices of the graph lie in a convex position.
The study of the Ramsey-multiplicity for abstract graphs was introduced in
the paper of Harary and Prins (see [19]). One can similarly introduce this definition for geometric graphs.

For a graph $G$ the geometric Ramsey number $R_{g}(G)$ denotes the smallest integer $r$ with the property that for every 2 -coloring of the edges of a complete geometric graph $H$ on at least $r$ vertices, there is a subgraph of $H$ with noncrossing edges, that is isomorphic to $G$ and has all of its edges of the same color.

One can similarly define the number $R_{c}(G)$. In this case we restrict the definition to convex geometric graphs.

The above numbers exist if and only if the graph $G$ is outerplanar (see [14]). The concept of Ramsey multiplicity is the same as in the case of abstract graphs. $R M_{g}(G)$ denotes the minimum number of monochromatic copies of a given geometric graph $G$ as subgraphs of the complete geometric graph $H$ on $R_{g}(G)$ vertices, which is colored by 2 colors. One can similarly interprete the number $R M_{c}(G)$ in the case of convex geometric graphs.

Let us denote the non-crossing matching with $2 n$ vertices by $M_{2 n}$ for geometric graphs. In [24] the authors proved the following theorem for geometric graphs:

Theorem 9. $R_{g}\left(M_{2 n}\right)=R_{c}\left(M_{2 n}\right)=3 n-1$
In [22] Károlyi asks for a sharp estimate in the case of the numbers $R M_{g}\left(M_{2 n}\right)$ and $R M_{c}\left(M_{2 n}\right)$. He means that these numbers are exponentially large in terms of the number of the vertices. This conjecture seems reasonable, because in the case of abstract graphs the Ramsey-multiplicity tends to be large. According to a conjecture of Harary and Prins, in the case of abstract graphs the Ramsey-multiplicity can only be in some special cases non-exponential in terms of the number of vertices (see [19]). That is why the following can seem unexpected:

Theorem 10. $R M_{g}\left(M_{2 n}\right)=R M_{c}\left(M_{2 n}\right)=1$.

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