On the Computation of the Nucleolus of Cooperative Transferable Utility Games

PhD Thesis Summary

by

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1. Introduction

The nucleolus was developed by Schmeidler (1969) and became one of the most applied solution concepts of cooperative game theory. Despite its good properties it lost some popularity in the last 20 or so years. The general opinion shifted partly because it was categorized to be a too complicated solution concept. It is not only difficult to compute, but it is also hard to interpret, since its axiomatization is less straightforward as for instance the Shapley value's. Not to mention that the latter one has many different axiomatizations while the nucleolus has basically just one. Even verifying whether an allocation is the nucleolus or not is conjectured to be $\mathcal{NP}$-hard in general (Faigle et al., 1998).

Ironically in this period there has been several breakthroughs regarding the computational aspect of the nucleolus that somehow escaped the attention of many theoreticians and researchers. The aim of this thesis is threefold.

- To aggregate the known results related to the computation of the nucleolus.
- To analyze the various existing methods in practice.
- To expand the theory of characterization sets and show their usefulness.

2. Methodology

Determining the nucleolus is a notoriously hard problem, even $\mathcal{NP}$-hard for various classes of games. While $\mathcal{NP}$-hardness was proven for minimum cost spanning tree games (Faigle et al., 1998), voting games (Elkind et al., 2009) and flow and linear production games (Deng et al., 2009), it is still unknown whether the corresponding decision problem – i.e. verifying whether an allocation is the nucleolus or not – belongs to $\mathcal{NP}$.

In recent years several polynomial time algorithms were proposed to find the nucleolus of important families of cooperative games, like standard tree (Maschler et al., 2010), assignment (Solymosi and Raghavan, 1994), matching (Kern and Paulusma, 2003) and bankruptcy games (Aumann and Maschler, 1985). In addition Kuipers (1996) and Arin and Inarra (1998) developed methods to compute the nucleolus for convex games.

The main breakthrough came from another direction. In their seminal paper Maschler et al. (1979) described the geometric properties of the nucleolus and devised a computational framework in the form of a sequence of linear programs. Although these LPs consist of exponentially many inequalities they can be solved efficiently if one knows which constraints are redundant. Huberman (1980); Granot et al. (1998); Reijnierse and Potters (1998) provided methods to identify coalitions that correspond to non-redundant constraints.

Granot et al. (1998) provided the most fruitful approach. They introduced the concept of characterization set which is a collection of coalitions that determines the nucleolus by itself. They proved that if the size of the characterization set is polynomially bounded in the number of players, then the nucleolus of the game can be computed in strongly polynomial time. A collection that characterizes the nucleolus in one game need not characterize it in another one. Thus we are interested in properties of coalitions that characterize the nucleolus independently of the realization of the coalitional function. Huberman (1980) was the first to show that such a property exists. He introduced the concept of essential coalitions which are coalitions that have no weakly minorizing partition (cf. Definition 3.4). Granot et al. (1998) provided another collection that characterize the nucleolus in cost games with non-empty cores. Saturated coalitions contain all the players that can join the coalition without imposing extra cost.

Using the concept of dual game we introduce two new characterization sets: dually essential and dually saturated coalitions. We show how these two sets are related to the existing characterization sets. Furthermore for two important classes of games we demonstrate how characterization sets can be used to determine the nucleolus in polynomial time.

3. Game theoretical framework

3.1. Cooperative games and solutions

A *cooperative game with transferable utility* is an ordered pair (N, v) consisting of the player set $N = \{1, 2, \ldots, n\}$ and a characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The value $v(S)$ represents the worth of coalition S. We denote by $\mathcal{P} = 2^N \setminus \{\emptyset, N\}$ the family of the non-trivial coalitions. A *solution* is a vector $x \in \mathbb{R}^N$ that represents the payoff of each player. We employ the following notations $x(S) = \sum_{i \in S} x_i$ for any $S \subseteq N$, and instead of $x(\{i\})$ we simply write $x(i)$. A cooperative game (N, v) is called *monotonic* if

$$S \subseteq T \subseteq N \;\Rightarrow\; v(S) \leq v(T),$$

superadditive if

$$(S, T \in \mathcal{P},\; S \cap T = \emptyset) \;\Rightarrow\; v(S) + v(T) \leq v(S \cup T),$$

and *convex* if

$$S, T \subseteq N \;\Rightarrow\; v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

A *cooperative cost game* is an ordered pair (N, c) consisting of the player set $N = \{1, 2, \ldots, n\}$ and a characteristic cost function $c : 2^N \to \mathbb{R}$ with $c(\emptyset) = 0$. The value $c(S)$ represent how much cost coalition S must bear if it chooses to act separately from the rest of the players. A cost game (N, c) is called *subadditive* if

$$(S, T \subset N,\ S \cap T = \emptyset)\ \Rightarrow\ c(S) + c(T) \geq c(S \cup T).$$

and *concave* if,

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T), \quad \forall\, S, T \subseteq N.$$

Given a game Γ and an allocation $x \in \mathbb{R}^N$, we define the *satisfaction* of a coalition S as

$$sat_\Gamma(S, x) := x(S) - v(S). \quad \text{if } \Gamma = (N, v) \tag{1}$$

$$sat_\Gamma(S, x) := c(S) - x(S). \quad \text{if } \Gamma = (N, c) \tag{2}$$

Note that the satisfaction of a coalition is defined differently for profit and cost games, but the meaning of the $sat_\Gamma(S, x)$ expression is the same: the contentment of coalition S with respect to the payoff vector x.

A solution x is called an *allocation* if it is *efficient*, i.e. $x(N) = v(N)$. Similarly we say that x is *individually rational* if $x(i) \geq v(i)$ for all $i \in N$. The *imputation set* of the game $I(N, v)$ consists of the efficient and individually rational solutions. The *core* of the game (N, v) is a set-valued solution where all the satisfaction values are non-negative. Formally,

$$\mathcal{C}(N, v) = \{x \in \mathbb{R}^N \mid x(N) = c(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}. \tag{3}$$

A game is called *balanced* if its core in non-empty. If $x \in \mathcal{C}(N, v)$ then no coalition can improve upon x, in the sense, that each coalition gets at least as much as it could gain on its own. Thus, each member of the core is a highly stable payoff distribution. Note that in the n-dimensional payoff space the core allocations form a convex polyhedron.

We say that a vector $x \in \mathbb{R}^m$ *lexicographically precedes* $y \in \mathbb{R}^m$ (denoted by $x \preceq y$) if, either $x = y$, or there exists a number $1 \leq j \leq m$ such that $x_i = y_i$ if $i < j$ and $x_j < y_j$.

Definition 3.1. *Let $\Gamma = (N, v)$ be a game, $X \subseteq \mathbb{R}^N$ a set of allocation, $x \in X$ an allocation, and let $\theta^{\mathcal{P}}(x) \in \mathbb{R}^{2^n}$ be the* satisfaction vector *that contains the 2^{n-2} satisfaction values of the non-trivial coalitions in a non-decreasing order. The* nucleolus of Γ with respect to X *is the subset of those payoff vectors $x \in X$ that lexicographically maximize $\theta^{\mathcal{P}}(x)$ over X. Formally,*

$$\mathcal{N}(\Gamma, X) = \{x \in X \mid \theta^{\mathcal{P}}(y) \preceq \theta^{\mathcal{P}}(x) \text{ for all } y \in X\}.$$

It is well known that if X is nonempty and compact then $\mathcal{N}(\Gamma, X) \neq \emptyset$ and if X is convex then $\mathcal{N}(\Gamma, X)$ consist of a single point (for proof see (Schmeidler, 1969)). Furthermore, the nucleolus is a continuous function of the characteristic function. If X is chosen to be the set of imputations then we speak of the *nucleolus of* Γ. Henceforward we will use the shorthand notation $\mathcal{N}(\Gamma)$ for $\mathcal{N}(\Gamma, I(N, v))$.

3.2. Characterization sets

We introduce two theoretical novelties, the dually essential and the dually saturated coalitions, by applying the ideas of Granot et al. (1998) and Huberman (1980) to the dual game. Among the two new characterization sets the former one proves to be especially useful in case of cost games.

Definition 3.2. *Let $\Gamma^{\mathcal{F}} = (N, \mathcal{F}, v)$ be a cooperative game with coalition formation restrictions, where $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}$ consists of all coalitions deemed permissible besides the grand coalition N. Let $\theta^{\mathcal{F}}(x) \in \mathbb{R}^{|\mathcal{F}|}$ be the restricted vector that contains the satisfaction values $sat_{\Gamma}(S, x)$, $S \in \mathcal{F}$ in a non-decreasingly order. Furthermore, let $\mathcal{N}(\Gamma^{\mathcal{F}})$ be defined as the set of imputations that lexicographically maximizes $\theta^{\mathcal{F}}(x)$. Then $\mathcal{F}$ is called a characterization set for the nucleolus of the game $\Gamma = (N, v)$, if $\mathcal{N}(\Gamma^{\mathcal{F}}) = \mathcal{N}(\Gamma)$.*

The next theorem provides a sufficient condition for a collection of coalitions to be a characterization set.

Theorem 3.3. *(Granot et al., 1998) Let Γ be a cooperative (cost or profit) game, $\mathcal{F} \subseteq \mathcal{P}$ a non-empty collection, and x an element of the nucleolus of $\Gamma^{\mathcal{F}}$. Collection $\mathcal{F}$ is a characterisation set for the nucleolus of Γ if for every $S \in \mathcal{P} \setminus \mathcal{F}$ there exists a non-empty subcollection $\mathcal{F}_S$ of $\mathcal{F}$, such that*

i. *$sat_{\Gamma}(T, x) \leq sat_{\Gamma}(S, x)$ for every $T \in \mathcal{F}_S$,*

ii. *e_S can be expressed as a linear combination of the vectors in $\{e_T : T \in \mathcal{F}_S \cup \{N\}\}$.*

The first property that defines a characterization set was suggested by Huberman (1980).

Definition 3.4 (Essential coalitions)**.** *Let N be a set of players, (N, v) a profit, (N, c) a cost game. Coalition $S \neq \emptyset$ is called* essential *in profit game $\Gamma = (N, v)$ if it cannot be partitioned as $S = S_1 \dot{\cup} \cdots \dot{\cup} S_k$ with $k \geq 2$ and $S_j \neq \emptyset$ for all $1 \leq j \leq k$ such that*

$$v(S) \leq v(S_1) + \cdots + v(S_k).$$

Similarly, coalition $S \neq \emptyset$ is called essential in cost game $\Gamma = (N, c)$ if it cannot be partitioned as $S = S_1 \dot{\cup} \cdots \dot{\cup} S_k$ with $k \geq 2$ and $S_j \neq \emptyset$ for all $1 \leq j \leq k$ such that

$$c(S) \geq c(S_1) + \cdots + c(S_k).$$

The set of essential coalitions is denoted by $\mathcal{E}(\Gamma)$, where Γ is either (N, v) or (N, c). A coalition that is not essential is called inessential*.*

The *dual game* $\Gamma^* = (N, g^*)$ of game $\Gamma = (N, g)$ is defined by the coalitional function $g^*(S) := g(N) - g(N \setminus S)$ for all $S \subseteq N$, where Γ is either (N, v) or (N, c). Notice that $g^*(\emptyset) = 0$, $g^*(N) = g(N)$, and $(g^*)^*(S) = g(S)$ for all $S \subseteq N$. It will be useful to think of the dual game of a profit game as a cost game and vice versa.

We can identify small redundant coalitions, if we apply Huberman's argument to the dual game.

Definition 3.5 (Dually essential coalitions). *Let N be a set of players, (N,v) a profit, (N,c) a cost game. Coalition $S \neq \emptyset, N$ is called* dually essential *in game (N,v) if its complement cannot be partitioned as $N \setminus S = (N \setminus T_1) \dot{\cup} \cdots \dot{\cup} (N \setminus T_k)$ with $k \geq 2$ and $T_j \neq N$ for all $1 \leq j \leq k$ such that*

$$v^*(N \setminus S) \geq v^*(N \setminus T_1) + \cdots + v^*(N \setminus T_k),$$

or equivalently,

$$v(S) \leq v(T_1) + \cdots + v(T_k) - (k-1)v(N).$$

Similarly, $S \neq \emptyset, N$ is called dually essential *in cost game (N,c) if its complement cannot be partitioned as $N \setminus S = (N \setminus T_1) \dot{\cup} \cdots \dot{\cup} (N \setminus T_k)$ with $k \geq 2$ and $T_j \neq N$ for all $1 \leq j \leq k$ such that*

$$c^*(N \setminus S) \leq c^*(N \setminus T_1) + \cdots + c^*(N \setminus T_k),$$

or equivalently,

$$c(S) \geq c(T_1) + \cdots + c(T_k) - (k-1)c(N).$$

The set of dually essential coalitions is denoted by $\mathcal{DE}(\Gamma)$, where Γ is either (N,v) or (N,c). A coalition that is not dually essential is called dually inessential.

Notice that, by definition, the grand coalition is not dually essential. On the other hand, all $(n-1)$-player coalitions are dually essential in any game. If $S \in \mathcal{P}$ is dually inessential then it is contained in each of the coalitions $T_1, \ldots, T_k \in \mathcal{P}$ in the above expression, but every player in $N \setminus S$ appears exactly $k-1$ times in this family. We call such a system of coalitions an *overlapping decomposition* of S.

The next characterization set was proposed by Granot et al. (1998) for balanced monotonic cost games.

Definition 3.6 (Saturated coalitions). *Let (N,c) be a monotonic cost game. A coalition $S \subseteq N$ is said to be* saturated *if $c(S) = c(S \cup i)$ implies $i \in S$.*

In other words if S is a saturated coalition then every new member will impose extra cost on the coalition. We now convert the concept of saturatedness to monotonic profit games based on the dualization correspondence between profit and cost games.

Definition 3.7 (Dually saturated coalitions). *Let (N,v) be a monotonic profit game and $S \subseteq N$ be an arbitrary coalition. We say that S is* dually saturated *if $v(S \setminus i) < v(S)$ for any $i \in S$.*

In other words every member contributes to the worth of coalition S. Notice that S is saturated in the monotonic cost game c if and only if $N \setminus S$ is dually saturated in the monotonic profit game c^*.

4. Results

4.1. Theoretical contributions

The larger a characterization set is the easier to uncover it in a particular game class. However, with smaller characterization set it comes a faster LP. Hence there is a tradeoff between the difficulty in identifying the members of a characterization set and its efficiency. In order to exploit this technique we analyzed the relationship of the four known characterization sets. We proved that

- dually essential and dually saturated coalitions indeed form characterization sets for the nucleolus,
- essential coalitions are a subset of dually saturated coalitions in monotonic profit games,
- dually essential coalitions are a subset of saturated coalition in case of monotonic cost games,
- in general essential and dually essential coalitions do not contain each other (for additive games their intersection is empty).

A collection of coalitions $\mathcal{B}_S \subseteq \mathcal{P}$ is said to be *S-balanced* if there exist positive weights λ_T for $T \in \mathcal{B}_S$, such that $\sum_{T\in\mathcal{B}_S} \lambda_T e_T = e_S$. An N-balanced collection is simply called *balanced.* A coalition S is called *vital* if for any S-balanced collection $\mathcal{B}_S$ and any system $(\lambda_T)_{T\in\mathcal{B}_S}$ of balancing weights for $\mathcal{B}_S$, $\sum_{T\in\mathcal{B}_S} \lambda_T v(T) < v(S)$.

The main theoretical contribution is gathered by the following theorem.

Theorem 4.1. *Let $\Gamma = (N, v)$ be a game with a non-empty core. If the grand coalition is vital, the collection $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ forms a characterization set of $\mathcal{N}(\Gamma)$.*

We demonstrated how these novelties can be applied in case of bankruptcy and directed acyclic graph games.

4.2. Bankruptcy problems and hydraulic rationing

Bankruptcy situations, the fair division of the estate is one of the oldest problems in economy. In the ancient times bankruptcy situations arose mostly with connection to inheritance. In the following we present the bankruptcy game and the related results.

Let $N = \{1, 2, \ldots, n\}$ be the set of creditors. The *bankruptcy problem* is defined as a pair (d, E) where $E \in \mathbb{R}^+$ represents the firm's liquidation value (or *estate/endowment*) and $d \in (\mathbb{R}^+)^n$ is the collection of claims with $\sum_{i=1}^n d_i > E$. Again we employ the shorthand notation $d(S) = \sum_{i\in S} d_i$. Let $\mathbb{B}$ denote the class of such problems. A *solution of a bankruptcy problem* is a vector $x \in (\mathbb{R}_0^+)^n$ such that $\sum_i^n x_i = E$. A *rule* $r : \mathbb{B} \to \mathbb{R}^n$ is a mapping that assigns a unique solution to each bankruptcy problem. The characteristic function corresponding to the bankruptcy problem (d, E) is

$$v_{(d,E)}(S) = \max\{E - d(N \setminus S), 0\}$$

The *dual of a rule* r (denoted by r^*) assigns awards in the same way as r assigns losses, namely

$$r^*(d, E) = d - r(d, d(N) - E).$$

A *self-dual rule* is one with $r^* = r$, such rule treats losses and awards the same way. We say that a hydraulic $\mathcal{H}$ *corresponds* to a bankruptcy problem (d, E) if the following conditions hold

- $\mathcal{H}$ has n vessels
- the volume of the ith vessel is equal to the size of the claim of agent i
- there is E amount of water distributed among the vessels.

A hydraulic that corresponds to a bankruptcy problem always implicitly defines an allocation rule. The nature of the rule depends on the shape of the vessels. Figure 1 shows some examples.

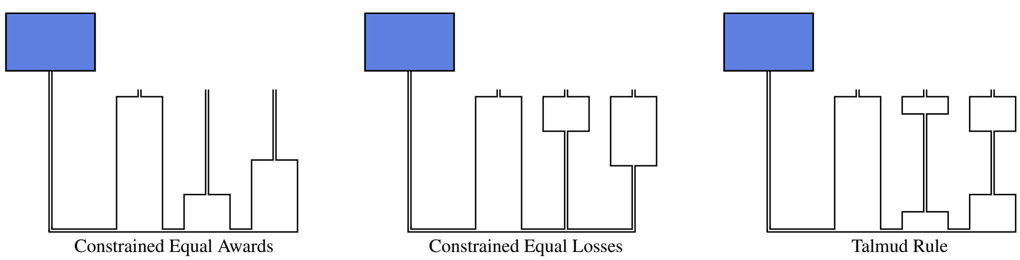

Figure 1: The representation of CEA, CEL and Talmud rules

The hydraulic representation was propagated by Kaminski (2000). Using the hydraulic approach we showed that

- a rule is consistent if and only if it corresponds to a connected hydraulic in which the shape of a vessel depends only on the respective claim size,
- a rule is self-dual if and only if it corresponds to some horizontally symmetric connected hydraulic,
- the satisfaction of coalition S is the minimum of the following two amounts: the water contained in S and the air contained in $N \setminus S$
- the nucleolus and the proportional rule are the two extremes of a family of solutions that correspond to hydraulics where the vessels have k cylinders of equal size.

With the help of the hydraulic representation we also gave a direct and elementary proof of the famous theorem of Aumann and Maschler.

Theorem 4.2. *(Aumann and Maschler, 1985) The solution generated by the Talmud rule is the nucleolus of the corresponding coalitional game.*

Furthermore using the concept of characterization sets we gave an alternative way to compute the nucleolus. The next two lemmata together with Observation 4.5 mark the cornerstones of the proof.

Lemma 4.3. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then $\mathcal{E}(\Gamma)$ contains the singleton coalitions, the grand coalition and coalitions with non-zero characteristic function value.*

Lemma 4.4. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then $\mathcal{DE}(\Gamma)$ contains the $n-1$ player coalitions, the grand coalition and coalitions with characteristic function value of zero.*

Observation 4.5. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then $\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma)$ contains the grand coalition, $n-1$ person coalitions with non-zero characteristic function value and singleton coalitions with characteristic function value of zero.*

By showing that the grand coalition is vital we immediately obtained the following theorem.

Theorem 4.6. *Let (d, E) be a bankruptcy problem and $\Gamma = (N, v_{(d,E)})$ be the corresponding bankruptcy game. Then*

$$\mathcal{E}(\Gamma) \cap \mathcal{DE}(\Gamma) = \{i \in N \mid v_{(d,E)}(i) = 0\} \cup \{N \setminus i \mid i \in N,\ v_{(d,E)}(N \setminus i) > 0\} \cup \{N\}$$

is a characterization set for $\mathcal{N}(\Gamma)$.

We obtained a characterization set which size is linear in the number of players. Thus, there is a fast LP to compute the nucleolus. Note, that an even faster algorithm can be derived from the hydraulic framework. The point here is that we do not need to check axioms or apply various tricks to find the nucleolus. The problem is reduced to a simple algebraic computation.

4.3. Directed acyclic graph games

Network-based cost games have many applications: with them we can model the cost sharing of infrastructural projects, carpooling or the cost sharing of irrigation systems. Here we introduce a new network-based game and the related results.

A directed acyclic graph network $\mathcal{D}$ or shortly a *DAG-network* is given by the following:

- $G(V, A)$ is a directed acyclic graph, with a special node - the so called *root* of G, denoted by $\mathbf{r}$ - such that from each other node of G there leads at least one directed path to the root. G is considered to be a simple graph, i.e. it has no loops or parallel arcs.

- There is a cost function $\delta : A \to \mathbb{R}_0^+$ that assigns a non-negative real number to each arc. This value is regarded as the construction cost of the arc.

For a subgraph T, $V(T)$ denotes the node set of T. Similarly $A(T)$ denotes its arc set, while $A_{\mathbf{p}}$ is used for the set of arcs that leave node $\mathbf{p}$.

Let N be a set of players and let $\mathcal{R} : N \to V$ be the residency function that maps N to the node set of G. If player i is assigned to node $\mathbf{p}$ we say that player i *resides* at $\mathbf{p}$. A node is *occupied* if at least one player resides in it. Note that unoccupied leafs are redundant and can be omitted from the network. The residency function is not assumed to be injective and/or surjective, but it is a proper function. It means that any one player resides at exactly one node, but there can be unoccupied nodes or nodes having more than one residents. The set of residents of a subgraph T is denoted by $N(T)$, formally, $N(T) = \mathcal{R}^{-1}(V(T))$. A network $\mathcal{D}$ together with a residency mapping $\mathcal{R}$ defined on $\mathcal{D}$ is called a *player network*.

For a subgraph T, we define its construction cost $C(T)$ as the total cost of the arcs in T, i.e. $C(T) = \sum_{a \in A(T)} \delta(a)$. A path whose end point is the root is called a *rooted path*. A connected subgraph of G that is a union of rooted paths is called a *trunk*. For each coalition S, let T_S denote the set of trunks that have maximum number of arcs among the cheapest trunks that connect all players in S to the root. The maximality requirement may seem odd at first, but it is needed to ensure the uniqueness of T_N.

We say that a trunk T corresponds to a node set B if $V(T) = B$. Similarly we say that a coalition S corresponds to the trunk T if $T \in T_S$. Note that more than one coalition can correspond to the same trunk.

The characteristic function of the cost allocation game that is associated with the player network $(\mathcal{D}, \mathcal{R})$ is defined as follows.

$$c_{(\mathcal{D},\mathcal{R})}(S) \stackrel{def}{=} C(T) \qquad T \in T_S.$$

The pair $(N, c_{(\mathcal{D},\mathcal{R})})$ is called a *DAG-game*. We show that the condition

(*) there must be a resident at each node with more than one entering arc and with leaving arc(s) all of positive cost

is sufficient for a DAG-game to have a non-empty core. The next theorem underlines the significance of DAG-games.

Theorem 4.7. *Let N be a player set and $\hat{c} : 2^N \to \mathbb{R}$ a monotonic, subadditive cost function. There exists a DAG-network $\mathcal{D} = (G(V, A), \delta)$ and a residency mapping $\mathcal{R} : N \to V$ such that*

$$c_{(\mathcal{D},\mathcal{R})}(S) = \hat{c}(S) \quad \forall S \subseteq N.$$

In other words every monotonic, subadditive cost game can be modeled as a DAG-game. Furthermore, we obtained the following results.

- We introduced a graph canonization process that does not change the characteristic function of the game and showed that all networks that satisfy *(*)* can be canonized.
- We proved that canonized DAG-games are balanced.
- A node is called *free* if its residents do not have to pay anything in any core allocation. We characterized the set of free nodes, and gave a simple algorithm to identify them.

We proved many other structural results with respect to the core. To present them we need some further notation. The subgraph that is constructed by the grand coalition[1] T_N specifies a partial order on the node set. We say that node $\mathbf{p}$ is an *ancestor* of node $\mathbf{q} \neq \mathbf{p}$ if $\mathbf{p}$ can be reached from $\mathbf{q}$ via a path in T_N. The node set that contains $\mathbf{p}$ together with its descendants is called a *full branch* and denoted by $B_{\mathbf{p}}$. A specific *branch*, denoted by $B_{\mathbf{p}}^Q$ is a subset of $B_{\mathbf{p}}$ that collects nodes that still can reach $\mathbf{p}$ using only arcs from T_N after removing the node set Q from $B_{\mathbf{p}}$. A branch B is *proper* if deleting B along with the entering and leaving arcs of B, the root can still be reached on a directed path from any of the remaining nodes.

The main contribution is that we uncovered the graph structure of dually essential coalitions. It turned out it is simple and easy to deal with.

Theorem 4.8. *The dually essential coalitions of the cost game $(N, c_{(\mathcal{D},\mathcal{R})})$ are the coalitions with $n-1$ player and saturated coalitions whose trunks correspond to node sets of the form $V \setminus B_{\mathbf{q}}^U$, where $B_{\mathbf{q}}^U$ is a proper branch and $\mathbf{q}$ has only one leaving arc.*

In other words, dually essential coalitions correspond to subgraphs that have one missing branch. This is a very specific structure that can be used to derive further results. In particular with the help of the dually essential coalitions we prove that contracting the free nodes with the root or rerouting some arcs to the root does not alter the core or the nucleolus of the game.

Finally we provided a large family of DAG-games where there are polynomial many proper branches. We define the *width of a DAG-network* as the maximum number of nodes that can be chosen from the node set, such that any two chosen node is incomparable, i.e. neither of them is an ancestor or descendant of the other[2].

[1] This is the cheapest subgraph that connects every player to the root.

[2] If we think about V as a partially ordered set then width is equivalent to the cardinality of the *maximum antichain* the poset has.

Lemma 4.9. *Let $\mathcal{D}$ be a canonized DAG-network of width k and let m denote the number of nodes in the graph. Let k be fixed, independent of m. There are at most $\mathcal{O}(m^{k+1})$ number of proper branches in $\mathcal{D}$.*

From which it immediately follows the main result of this chapter.

Theorem 4.10. *There exists a polynomial time algorithm in the number of players to compute the core and the nucleolus of any DAG-game that is induced by a fixed width canonized DAG-network.*

5. Conclusions

This thesis focused on the computational aspects of the nucleolus. We established several methods on the computation of the nucleolus as well as the corresponding theory. Furthermore, for two important classes of games we presented efficient algorithms that calculate the nucleolus in polynomial time.

One of the primary contributions of this thesis is the expansion of the theory of characterization sets. When the game in question is well-structured, characterization sets can simplify the proof substantially. Even when the structure is more complicated characterization sets can make the proof significantly simpler or at least possible. The main advantage of characterization sets is their algebraic formalism. The proofs need less bag-tricks and more mechanical computation which is perhaps aesthetically less pleasing but much more effective in terms of results.

Naturally characterization sets do not make other approaches obsolete. In case of bankruptcy games the hydraulic representation is much more instructive. From didactical point of view it is much easier to explain ideas using figures of vessels and water than handling abstract algebraic equations. With the help of the hydraulic framework we also uncovered the relationship of the nucleolus and the proportional rule which in other case would have remained hidden.

There are a variety of unsolved problems related to nucleolus. One straightforward open question how to extend the nucleolus to partition function form games (Csercsik and Sziklai, 2013). Another theoretical question is the verification of the nucleolus and its variants, like the per capita nucleolus and other lexicographic solution concepts (Sziklai et al., 2014). There are some possible research directions on the application side as well. We expect that both of the cooperative games that we presented here and their applications, which include multilateral bankruptcy situations and various cost sharing methods on networks, will gain more popularity in the future.

This thesis is based on the following manuscripts

Fleiner, T. and Sziklai, B. (2012). The Nucleolus Of The Bankruptcy Problem By Hydraulic Rationing. *International Game Theory Review*, 14(01):1250007–1–1

Solymosi, T. and Sziklai, B. (2015). Universal characterization sets for the nucleolus in balanced games. IEHAS Discussion Papers MT-DP 2015/12, Centre for Economic and Regional Studies, Hungarian Academy of Sciences. submitted to Games and Economic Behavior May 8. 2015

Sziklai, B., Fleiner, T., and Solymosi, T. (2014). On the core of directed acyclic graph games. IEHAS Discussion Papers MT-DP 2014/18, Centre for Economic and Regional Studies, Hungarian Academy of Sciences. submitted to EJOR August 6. 2014, revised version submitted April 8. 2015

Further papers on the topic

Biró, P., Kóczy, A. L., and Sziklai, B. (2013). Fair apportionment in the view of the venice commission's recommendation. IEHAS Discussion Papers MT-DP 2013/38, Centre for Economic and Regional Studies, Hungarian Academy of Sciences. under minor revision at Mathematical Social Sciences

Csercsik, D. and Sziklai, B. (2013). Traffic routing oligopoly. *Central European Journal of Operations Research*. forthcoming

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