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ON THE WEIGHTED LEBESGUE FUNCTION OF FOURIER–JACOBI SERIES

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. S.A. Agahanov and G.I. Natanson [1] established lower and upper bounds for the Lebesgue functions $L_n^{(\alpha,\beta)}(x)$ of Fourier–Jacobi series on the interval $[-1,1]$. The bounds differ from each other only in a constant factor depending on Jacobi parameters α and β, so their result is of final character. The aim of this paper is to extend their estimation for the weighted Lebesgue functions $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$ using Jacobi weights with parameters γ and δ. We shall also give sufficient conditions with respect to α,β,γ and δ for which the order of the weighted Lebesgue functions is $\log(n+1)$ on the whole interval $[-1,1]$.

1. Introduction

It is known that the Lebesgue functions of an approximation process play an important role in the convergence of that process. The Lebesgue functions $L_n^{(\alpha,\beta)}(x)$ (see (2.1)) of Fourier–Jacobi series have been studied by many authors.

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G. Szegő [10, 9.3.] showed that for every fixed number $\varepsilon \in (0,1)$

$$\max_{x\in[-1+\varepsilon,1-\varepsilon]} L_n^{(\alpha,\beta)}(x) \sim \log(n+1)$$
$$(n \in \mathbb{N} := \{1,2,\ldots\}).$$

Here and in what follows for the positive functions $a_n, b_n : I \to \mathbb{R}$ (I is an interval of $\mathbb{R}$) the notation

$$a_n(x) \sim b_n(x) \qquad (x \in I,\ n \in \mathbb{N})$$

means that there exist positive constants c_1, c_2 independent of x and n such that

$$c_1 \le \frac{a_n(x)}{b_n(x)} \le c_2 \qquad (x \in I,\ n \in \mathbb{N}).$$

H. Rau [7] showed that the order of the Lebesgue functions at the points -1 and 1 is $n^{\sigma+\frac{1}{2}}$, where $\sigma = \max\{\alpha,\beta\}$.

S. A. Agahanov and G. I. Natanson [1] proved the following result: if $\alpha, \beta > > -\frac{1}{2}$ then

$$L_n^{(\alpha,\beta)}(x) \sim \log\left(n(1-x)^{\varepsilon(\alpha)}(1+x)^{\varepsilon(\beta)} + 1\right) + \sqrt{n}\left(|P_n^{(\alpha,\beta)}(x)| + |P_{n+1}^{(\alpha,\beta)}(x)|\right)$$
$$(x \in [-1,1],\quad n \in \mathbb{N}),$$

where

$$\varepsilon(t) = \begin{cases} \frac{1}{2}, & \text{if } t \in \mathbb{R} \setminus \{\frac{1}{2}\} \\ 0, & \text{if } t = \frac{1}{2} \end{cases}$$

and $P_n^{(\alpha,\beta)}(x)$ is the nth Jacobi polynomial.

The aim of this paper is to extend this estimation by using suitable Jacobi weights. We will give conditions for the weight parameters γ and δ such that the order of the weighted Lebesgue functions $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$ is $\log(n+1)$ on the whole interval $[-1,1]$.

2. Pointwise estimate of the weighted Lebesgue function

For parameters $\alpha, \beta > -1$ we shall denote by $P_n^{(\alpha,\beta)}$ the nth Jacobi polynomial with the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \qquad (n \in \mathbb{N}_0 := \{0,1,2,\ldots\}).$$

They are orthogonal with respect to the Jacobi weight function

$$w^{(\alpha,\beta)}(x) := (1-x)^\alpha(1+x)^\beta \qquad \big(x \in (-1,1)\big).$$

The nth *Lebesgue function of Fourier–Jacobi series* is defined by

$$L_n^{(\alpha,\beta)}(x) := \int_{-1}^{1} |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha,\beta)}(y)\,\mathrm{d}y \tag{2.1}$$
$$\big(n \in \mathbb{N},\ x \in [-1,1]\big),$$

where the kernel function $K_n^{(\alpha,\beta)}(x,y)$ can be expressed as

$$\begin{aligned} K_n^{(\alpha,\beta)}(x,y) &= \sum_{k=0}^{n} \left\{h_k^{(\alpha,\beta)}\right\}^{-1} P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y) = \\ &= \lambda_n^{(\alpha,\beta)} \frac{P_{n+1}^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x)P_{n+1}^{(\alpha,\beta)}(y)}{x-y}. \end{aligned} \tag{2.2}$$

Here

$$h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\,\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}\,, \tag{2.3}$$

and

$$\lambda_n^{(\alpha,\beta)} = \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2}\,\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \tag{2.4}$$

(see [10, (4.3.3) and (4.5.2)]), where $\Gamma(p)$ $(p > 0)$ is the Gamma function.

For $\gamma, \delta \geq 0$ we define the nth *weighted Lebesgue function of Fourier–Jacobi series* by

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) := w^{(\gamma,\delta)}(x) \int_{-1}^{1} |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y \tag{2.5}$$
$$\big(n \in \mathbb{N},\ x \in [-1,1]\big).$$

For the existence of this integral, we shall assume that the parameters γ, δ satisfy the inequalities

$$\gamma < \alpha + 1, \quad \delta < \beta + 1. \tag{2.6}$$

Theorem. *Suppose that $\alpha, \beta > -\frac{1}{2}$ and $\gamma, \delta \geq 0$ satisfy the inequalities*

$$\frac{\alpha}{2} + \frac{1}{4} < \gamma < \frac{\alpha}{2} + \frac{3}{4} \quad \textit{and} \quad \frac{\beta}{2} + \frac{1}{4} < \delta < \frac{\beta}{2} + \frac{3}{4}. \tag{2.7}$$

Then we have for all $n \in \mathbb{N}$ and $x \in [-1,1]$ that

$$c_1\, w^{(\gamma,\delta)}(x)\, \phi_n^{(\alpha,\beta)}(x) \le L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \le c_2\, \widetilde{w}_n^{(\gamma,\delta)}(x)\, \phi_n^{(\alpha,\beta)}(x) \tag{2.8}$$

with the constants $c_1, c_2 > 0$ independent of x and n, where

$$\phi_n^{(\alpha,\beta)}(x) := \log\left(n\sqrt{1-x^2}+1\right) +$$

$$+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(\sqrt{1+x}+\tfrac{1}{n}\right)^{\beta+\frac{1}{2}}\left(|P_n^{(\alpha,\beta)}(x)|+|P_{n+1}^{(\alpha,\beta)}(x)|\right),$$

and

$$\widetilde{w}_n^{(\gamma,\delta)}(x) := \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left(\frac{\sqrt{1+x}}{\sqrt{1+x}+\frac{1}{n}}\right)^{2\delta}.$$

We note that the conditions for the parameters $\alpha, \beta, \gamma, \delta$ in Theorem imply the inequalities in (2.6).

Corollary. *Suppose that $\alpha, \beta > -\frac{1}{2}$ and $\gamma, \delta \ge 0$ satisfy the inequalities* (2.7)*. Then we have*

$$\max_{x\in[-1,1]} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \;\sim\; \log(n+1) \qquad (n \in \mathbb{N}).$$

Remark. A result similar to this Corollary was proved by U. Luther and G. Mastroianni [5]. This paper does not contain a pointwise estimation (cf. (2.8)).

3. Preliminaries

In what follows for the functions $a_n, b_n : I \to \mathbb{R}$ (I is an interval of $\mathbb{R}$) the notation

$$a_n(x) = O(b_n(x)) \qquad (x \in I,\ n \in \mathbb{N})$$

means that there exists a positive constant c independent of x and n such that

$$|a_n(x)| \le c\, b_n(x) \qquad (x \in I,\ n \in \mathbb{N}).$$

3.1. Formulas for Jacobi polynomials. Here we list those well known formulas which we shall use throughout the paper.

If $\alpha, \beta > -1$ then for every $x \in [-1,1]$ and $n \in \mathbb{N}$ we have

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) \tag{3.1}$$

(see [10, (4.1.3)]) and

$$\frac{d}{dx}\left\{P_n^{(\alpha,\beta)}(x)\right\} = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{3.2}$$

(see [10, (4.21.7)]).

An important bound for Jacobi polynomials can be given in this form: if $\alpha, \beta > -1$ then

$$\left|P_n^{(\alpha,\beta)}(x)\right| = O\left(n^{-\frac{1}{2}}\right)\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} \qquad (0 \le x \le 1,\ n \in \mathbb{N}) \tag{3.3}$$

(see [6, 2.3.22]).

A more precise formula is the following. Let $\alpha, \beta > -1$. Then we have

$$P_n^{(\alpha,\beta)}(\cos s) = n^{-\frac{1}{2}}k(s)\Big(\cos(Ns+\nu) + \frac{O(1)}{n\sin s}\Big), \tag{3.4}$$

where

$$\frac{c}{n} \le s \le \pi - \frac{c}{n},\ k(s) = k^{(\alpha,\beta)}(s) = \pi^{-\frac{1}{2}}\left(\sin\frac{s}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{s}{2}\right)^{-\beta-\frac{1}{2}},$$
$$N = n + \tfrac{1}{2}(\alpha+\beta+1),\ \nu = -\left(\alpha+\tfrac{1}{2}\right)\tfrac{\pi}{2}.$$

Here c is a fixed positive number and the bound for the error term holds uniformly in the interval $\left[\frac{c}{n}, \pi - \frac{c}{n}\right]$ (see [10, (8.21.18)]).

If $\alpha, \beta, \mu > -1$ then we have uniformly in $n \in \mathbb{N}$ that

$$\int_0^1 |P_n^{(\alpha,\beta)}(y)|(1-y)^{\mu}\,\mathrm{d}y \sim \begin{cases} n^{\alpha-2\mu-2}, & \text{if } 2\mu < \alpha - \frac{3}{2} \\ n^{-\frac{1}{2}}\log n, & \text{if } 2\mu = \alpha - \frac{3}{2} \\ n^{-\frac{1}{2}}, & \text{if } 2\mu > \alpha - \frac{3}{2} \end{cases} \tag{3.5}$$

(see [10, (7.34.1)]).

Let $p > 0$ be a fixed real number. Then

$$\frac{\Gamma(n+p)}{\Gamma(n)} \sim n^p \qquad (n \in \mathbb{N})$$

(see [8, p. 166]). Thus for the numbers (2.3) and (2.4) we have

$$h_n^{(\alpha,\beta)} \sim \frac{1}{n} \qquad (n \in \mathbb{N}), \qquad \lambda_n^{(\alpha,\beta)} \sim n \qquad (n \in \mathbb{N}). \tag{3.6}$$

We introduce the notations

$$\overline{P}_n(x) := P_n^{(\alpha+1,\beta)}(x),$$
$$\widetilde{P}_n(x) := P_n^{(\alpha+1,\beta+1)}(x).$$

Using the formulas [10, (4.5.7)] we obtain that

$$\begin{aligned}\frac{1}{2}(1-x^2)\widetilde{P}_{n-1}(x) = \Big(x + \frac{\alpha-\beta}{2n+\alpha+\beta+2}\Big)P_n^{(\alpha,\beta)}(x)-\\ -\frac{2n+2}{2n+\alpha+\beta+2}P_{n+1}^{(\alpha,\beta)}(x).\end{aligned} \tag{3.7}$$

Moreover, by [10, (4.5.4)] we have

$$\Big(1+\frac{\alpha+\beta}{2n+2}\Big)(1-x)\overline{P}_n(x) = \frac{n+\alpha+1}{n+1}P_n^{(\alpha,\beta)}(x) - P_{n+1}^{(\alpha,\beta)}(x). \tag{3.8}$$

3.2. Auxiliary results.

Lemma 1. *Suppose that $R \geq 1$ and $A < 0$ are fixed real numbers. Then with a suitable index $N \in \mathbb{N}$ we have*

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \sim \Big(s+\frac{R}{n}\Big)^A \Big[\log\Big(\frac{ns}{R}+1\Big)+1\Big] \tag{3.9}$$

uniformly in $s \in [0, \frac{\pi}{2}]$ and $n \in \mathbb{N}$, $n > N$.

Proof. Let us introduce the following notation

$$I := I(n,s,A,R) := \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t$$
$$(n \in \mathbb{N},\ s \in [0,\tfrac{\pi}{2}],\ A<0,\ R \geq 1).$$

In order to prove the statement, we split the interval $[0, \frac{\pi}{2}]$ into three parts:

$$\big[0,\tfrac{\pi}{2}\big] = \big[0,\tfrac{R}{n}\big] \cup \big(\tfrac{R}{n},\tfrac{2\pi}{9}\big) \cup \big[\tfrac{2\pi}{9},\tfrac{\pi}{2}\big].$$

Case 1. Let $0 \leq s \leq \frac{R}{n}$ and $t \in [s+\frac{R}{n}, \frac{2\pi}{3}]$. From $2s \leq s+\frac{R}{n} \leq t$ it follows that

$$\tfrac{1}{2}\,t \leq t-s \leq t.$$

Therefore we have

$$(3.10)\qquad \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1}\,\mathrm{d}t \le \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \le 2\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1}\,\mathrm{d}t.$$

Since

$$(3.11)\qquad \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1}\,\mathrm{d}t = \frac{1}{|A|}\left[\left(s+\frac{R}{n}\right)^A - \left(\frac{2\pi}{3}\right)^A\right],$$

we obtain the following upper estimation of I:

$$(3.12)\qquad I \le \frac{2}{|A|}\left(s+\frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R}+1\right)+1\right].$$

Now, let us consider the lower estimation. If $n \ge \frac{6R}{\pi}$ and $A < 0$, then $\left(\frac{n\pi}{3R}\right)^A \le 2^A$. Therefore using (3.10) and (3.11) we get

$$\begin{aligned}
I &\ge \frac{1}{|A|}\left[\left(s+\frac{R}{n}\right)^A - \left(\frac{2\pi}{3}\right)^A\right] = \frac{1}{|A|}\left(s+\frac{R}{n}\right)^A \left[1-\left(\frac{\frac{2\pi}{3}}{s+\frac{R}{n}}\right)^A\right] \ge \\
&\ge \frac{1}{|A|}\left(s+\frac{R}{n}\right)^A \left[1-\left(\frac{\frac{2\pi}{3}}{\frac{2R}{n}}\right)^A\right] = \frac{1}{|A|}\left(s+\frac{R}{n}\right)^A \left[1-\left(\frac{n\pi}{3R}\right)^A\right] \ge \\
&\ge \frac{1-2^A}{|A|}\left(s+\frac{R}{n}\right)^A = \frac{1-2^A}{|A|}\left(s+\frac{R}{n}\right)^A \frac{1+\log 2}{1+\log 2} \ge \\
&\ge \frac{1-2^A}{|A|(1+\log 2)}\left(s+\frac{R}{n}\right)^A \left[1+\log\left(\frac{ns}{R}+1\right)\right],
\end{aligned}$$

where we used the fact that from $\frac{ns}{R} \le 1$ it follows that $\log 2 \ge \log\left(\frac{ns}{R}+1\right)$.

Consequently,

$$I \ge c\left(s+\frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R}+1\right)+1\right]$$
$$\left(s \in [0, \tfrac{R}{n}],\ A<0,\ R\ge 1,\ n \ge \tfrac{6R}{\pi}\right),$$

with a constant $c > 0$ independent of s and n.

This inequality together with (3.12) prove (3.9), if $0 \le s \le \frac{R}{n}$.

CASE 2. Let $\frac{R}{n} < s < \frac{2\pi}{9}$. Then $s + \frac{R}{n} < 2s < 3s < \frac{2\pi}{3}$. Now we split the integral I into two parts:

$$I = \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t = \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s}\,\mathrm{d}t + \int_{3s}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t =: I_1 + I_2.$$

For I_1 we have

$$I_1 = \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s}\,\mathrm{d}t \le \left(s + \frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{3s} \frac{1}{t-s}\,\mathrm{d}t =$$
$$= \left(s + \frac{R}{n}\right)^A \left[\log(2s) - \log\frac{R}{n}\right] = \left(s + \frac{R}{n}\right)^A \log\left(\frac{2ns}{R}\right) =$$
$$= \left(s + \frac{R}{n}\right)^A \left[\log 2 + \log\frac{ns}{R}\right] \le \left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R} + 1\right) + 1\right].$$

If $3s \le t$ then $s \le \frac{1}{3}\,t$, i.e. $s + \frac{2}{3}\,t \le t$. Thus

$$\tfrac{2}{3}\,t \le t - s \le t.$$

Therefore for I_2 we get

$$I_2 = \int_{3s}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \le \frac{3}{2} \int_{3s}^{\frac{2\pi}{3}} t^{A-1}\,\mathrm{d}t = \frac{3}{2|A|}\left[(3s)^A - \left(\frac{2\pi}{3}\right)^A\right] \le$$
$$\le \frac{3}{2|A|}(2s)^A \le \frac{3}{2|A|}\left(s + \frac{R}{n}\right)^A.$$

Summarizing the above formulas we obtain that there exists a constant $c > 0$ independent of n and s such that

$$I \le c\left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R} + 1\right) + 1\right]$$
$$\left(s \in \left(\tfrac{R}{n}, \tfrac{2\pi}{9}\right),\ A < 0,\ R \ge 1,\ n \ge \tfrac{6R}{\pi}\right). \tag{3.13}$$

For the lower estimation of I it is enough to consider the integral I_1. Since

$s + \frac{R}{n} \le t \le 3s \le 3\left(s + \frac{R}{n}\right)$, thus by $A < 0$ we get that

$$
\begin{aligned}
I_1 = \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s}\,\mathrm{d}t \ge 3^A \left(s + \frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{3s} \frac{1}{t-s}\,\mathrm{d}t = \\
= 3^A \left(s + \frac{R}{n}\right)^A \left(\log(2s) - \log\frac{R}{n}\right) = \\
= 3^A \left(s + \frac{R}{n}\right)^A \log\left(2\frac{ns}{R}\right).
\end{aligned} \tag{3.14}
$$

The following inequality holds:

$$
\frac{\log(2x)}{\log(x+1)+1} > \frac{\log 2}{1+\log 2} \qquad (x \ge 1). \tag{3.15}
$$

Indeed, if $x \ge 1$ then

$$
\begin{aligned}
\frac{\log(2x)}{\log(x+1)+1} \ge \frac{\log(2x)}{\log(2x)+1} = 1 - \frac{1}{\log(2x)+1} \ge \\
\ge 1 - \frac{1}{1+\log 2} = \frac{\log 2}{1+\log 2}.
\end{aligned}
$$

Since $\frac{ns}{R} \ge 1$ we obtain from (3.14) and (3.15) that

$$
I \ge I_1 \ge \frac{3^A \log 2}{1+\log 2}\left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R}+1\right)+1\right],
$$

which together with (3.13) prove (3.9), if $\frac{R}{n} < s < \frac{2\pi}{9}$.

Case 3. Let $\frac{2\pi}{9} \le s \le \frac{\pi}{2}$ and $t \in \left[s + \frac{R}{n}, \frac{2\pi}{3}\right]$. Then

$$
s + \frac{R}{n} \le t \le \frac{2\pi}{3} \le 3s \le 3\left(s + \frac{R}{n}\right), \tag{3.16}
$$

so we have the following upper estimation of I:

$$
\begin{aligned}
I = \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \le \left(s + \frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{1}{t-s}\,\mathrm{d}t = \\
= \left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{2\pi}{3} - s\right) - \log\frac{R}{n}\right] = \\
= \left(s + \frac{R}{n}\right)^A \log\left[\frac{n}{R}\left(\frac{2\pi}{3} - s\right)\right] \le \left(s + \frac{R}{n}\right)^A \log\left(\frac{2ns}{R}\right) = \\
= \left(s + \frac{R}{n}\right)^A \left[\log\frac{ns}{R} + \log 2\right] \le \left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R}+1\right)+1\right].
\end{aligned} \tag{3.17}
$$

For the lower estimation of I we use the condition $A<0$ and (3.16). Then we have

$$I=\int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}}\frac{t^A}{t-s}\,\mathrm{d}t\geq 3^A\left(s+\frac{R}{n}\right)^A\int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}}\frac{1}{t-s}\,\mathrm{d}t=$$

$$(3.18)\qquad =3^A\left(s+\frac{R}{n}\right)^A\log\left[\frac{n}{R}\left(\frac{2\pi}{3}-s\right)\right]=$$

$$=3^A\left(s+\frac{R}{n}\right)^A\log\left(\frac{\pi}{2}\cdot\frac{4}{3}\frac{n}{R}-\frac{ns}{R}\right)\geq$$

$$\geq 3^A\left(s+\frac{R}{n}\right)^A\log\left(\frac{1}{3}\frac{ns}{R}\right).$$

The following inequality is true:

$$(3.19)\qquad \frac{\log\left(\frac{1}{3}\,x\right)}{\log(x+1)+1}>\frac{\log\frac{4}{3}}{\log(8e)}\qquad (x\geq 4).$$

Indeed, if $x\geq 4$, then

$$\frac{\log\left(\frac{1}{3}\,x\right)}{\log(x+1)+1}>\frac{\log\left(\frac{1}{3}\,x\right)}{\log(2x)+1}=\frac{\log\left(\frac{1}{3}\,x\right)}{\log\left(\frac{1}{3}\,x\right)+\log 6+1}=$$

$$=1-\frac{\log(6e)}{\log\left(\frac{1}{3}\,x\right)+\log(6e)}\geq 1-\frac{\log(6e)}{\log\frac{4}{3}+\log(6e)}=\frac{\log\frac{4}{3}}{\log(8e)}.$$

Let $n\geq\frac{18R}{\pi}$. Then $\frac{ns}{R}\geq\frac{n}{R}\frac{2\pi}{9}\geq 4$. Thus using (3.18) and (3.19) we obtain

$$I\geq 3^A\frac{\log\frac{4}{3}}{\log(8e)}\left(s+\frac{R}{n}\right)^A\left[\log\left(\frac{ns}{R}+1\right)+1\right],$$

which together with (3.17) prove (3.9), if $\frac{2\pi}{9}\leq s\leq\frac{\pi}{2}$.

Lemma 1 is proved. ■

Lemma 2. *If $A>-1$, $n\in\mathbb{N}$ and $s\in\left(\frac{1}{n},\frac{\pi}{2}\right]$, then there exists a constant $c>0$ independent from s and n such that*

$$\int\limits_0^{s-\frac{1}{n}}\frac{t^A}{s-t}\,\mathrm{d}t\leq c\left(s+\frac{1}{n}\right)^A\log\left(ns+1\right).$$

Proof. Consider the following identity:

$$\int_0^{s-\frac{1}{n}} \frac{t^A}{s-t}\,\mathrm{d}t = \frac{1}{s}\int_0^{s-\frac{1}{n}} \frac{t^A[(s-t)+t]}{s-t}\,\mathrm{d}t =$$

$$= \frac{1}{s}\int_0^{s-\frac{1}{n}} t^A\,\mathrm{d}t + \frac{1}{s}\int_0^{s-\frac{1}{n}} \frac{t^{A+1}}{s-t}\,\mathrm{d}t =: I_1 + I_2.$$

For I_1 we have

$$I_1 = \frac{1}{s}\int_0^{s-\frac{1}{n}} t^A\,\mathrm{d}t = \frac{1}{s}\,\frac{\left(s-\frac{1}{n}\right)^{A+1}}{A+1} \le c\,s^A,$$

where $c>0$ is independent of s and n. From $A+1>0$ it follows that

$$I_2 = \frac{1}{s}\int_0^{s-\frac{1}{n}} \frac{t^{A+1}}{s-t}\,\mathrm{d}t \le s^A \int_0^{s-\frac{1}{n}} \frac{1}{s-t}\,\mathrm{d}t = s^A \log(ns),$$

therefore

$$I_1 + I_2 \le c\,s^A\big(1+\log(ns)\big) \le c\,s^A \log(ns+1).$$

Since

$$\frac{1}{2} \le \frac{s}{s+\frac{1}{n}} = 1 - \frac{1}{ns+1} \le 1,$$

we have that there exists a $c>0$ independent of s and n such that

$$s^A \le c\left(s+\frac{1}{n}\right)^A,$$

which proves our statement. ∎

4. Proof of Theorem

In this section we shall use the following notations:

$$P_n(x) := P_n^{(\alpha,\beta)}(x), \quad \lambda_n := \lambda_n^{(\alpha,\beta)}.$$

By (3.1) we have the following symmetry property of the kernel function (2.2)

$$K_n^{(\alpha,\beta)}(x,y) = K_n^{(\beta,\alpha)}(-x,-y)$$
$$\big(x,y \in [-1,1], \quad n \in \mathbb{N}, \quad \alpha,\beta > -1\big).$$

Using this we obtain the symmetry property of the weighted Lebesgue function:

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(-x) = L_n^{(\beta,\alpha),(\delta,\gamma)}(x) \tag{4.1}$$
$$\big(x,y \in [-1,1], \quad n \in \mathbb{N}, \quad \alpha,\beta > -1, \quad \gamma,\delta \geq 0\big),$$

which means that it is enough to prove (2.8) for $x \in [0,1]$ only.

From now on we will assume that $x \in [0,1]$.

In what follows, C or c (or $C_1, C_2, \dots, c_1, c_2, \dots$) will always denote a positive constant (not necessarily the same at different occurrences) independent of n and x. Also, N will always denote a fixed natural number, not necessarily the same at different occurrences.

4.1. Upper estimation of $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$. In order to estimate (2.5) we split the integral into two parts:

$$\int_{-1}^{1} |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y = \int_{-1}^{-\frac{1}{2}} \dots\,\mathrm{d}y \;+ \int_{-\frac{1}{2}}^{1} \dots\,\mathrm{d}y.$$

In the second integral we use the substitutions

$$y = \cos t \;\; \left(0 \leq t \leq \tfrac{2\pi}{3}\right) \quad \text{and} \quad x = \cos s \;\; \left(0 \leq s \leq \tfrac{\pi}{2}\right),$$

and consider the following two cases:

$$\text{(i)} \;\; \tfrac{1}{n} \leq s \leq \tfrac{\pi}{2} \quad \text{and} \quad \text{(ii)} \;\; 0 \leq s \leq \tfrac{1}{n}.$$

In the first case we split the second integral into three parts:

$$\int_{-\frac{1}{2}}^{1} \dots\,\mathrm{d}y = \int_{0}^{\frac{2\pi}{3}} \dots\,\mathrm{d}t = \int_{0}^{s-\frac{1}{n}} \dots\,\mathrm{d}t + \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} \dots\,\mathrm{d}t + \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \dots\,\mathrm{d}t.$$

Thus we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) =: \sum_{k=1}^{4} J_k,$$

where

$$J_1 = w^{(\gamma,\delta)}(x) \int_{-1}^{-\frac{1}{2}} |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y,$$

$$J_2 = w^{(\gamma,\delta)}(x) \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t,$$

$$J_3 = w^{(\gamma,\delta)}(x) \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t,$$

$$J_4 = w^{(\gamma,\delta)}(x) \int_{0}^{s-\frac{1}{n}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t.$$

In the second case the lower bound in J_3 is 0 and $J_4 := 0$.

4.1.1. Estimation of J_1. Here we use the formula (2.2). Since $x \geq 0$ we have $|x-y| \geq \frac{1}{2}$ $(-1 \leq y \leq -\frac{1}{2})$. Consequently,

$$J_1 = w^{(\gamma,\delta)}(x) \int_{-1}^{-\frac{1}{2}} \lambda_n \frac{|P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)|}{|x-y|} w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y \leq$$

$$\leq 2\lambda_n w^{(\gamma,\delta)}(x)|P_n(x)| \int_{-1}^{-\frac{1}{2}} |P_{n+1}(y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y +$$

$$+2\lambda_n w^{(\gamma,\delta)}(x)|P_{n+1}(x)| \int_{-1}^{-\frac{1}{2}} |P_n(y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y.$$

By (3.1) we have

$$\int_{-1}^{-\frac{1}{2}} |P_n(y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y = \int_{-1}^{-\frac{1}{2}} |P_n^{(\alpha,\beta)}(y)|(1-y)^{\alpha-\gamma}(1+y)^{\beta-\delta}\,\mathrm{d}y \leq$$

$$\leq c \int_{-1}^{-\frac{1}{2}} |P_n^{(\alpha,\beta)}(y)|(1+y)^{\beta-\delta}\,\mathrm{d}y = c \int_{\frac{1}{2}}^{1} |P_n^{(\beta,\alpha)}(y)|(1-y)^{\beta-\delta}\,\mathrm{d}y \leq$$

$$\leq c \int_0^1 |P_n^{(\beta,\alpha)}(y)|(1-y)^{\beta-\delta}\,\mathrm{d}y.$$

Since $\delta < \frac{\beta}{2} + \frac{3}{4}$, i.e. $2(\beta-\delta) > \beta - \frac{3}{2}$ it follows by (3.5) that the last integral has the upper bound $cn^{-\frac{1}{2}}$. Consequently,

$$\int_{-1}^{-\frac{1}{2}} |P_n(y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y = O\big(n^{-\frac{1}{2}}\big) \qquad (n \in \mathbb{N}).$$

Collecting the above formulas and using (3.6) we obtain

$$J_1 = O(\sqrt{n}) w^{(\gamma,\delta)}(x) \Big(|P_n^{(\alpha,\beta)}(x)| + |P_{n+1}^{(\alpha,\beta)}(x)| \Big) \qquad \big(x \in [0,1],\ n \in \mathbb{N}\big). \tag{4.2}$$

4.1.2. Estimation of J_2. The expression

$$J_2 = w^{(\gamma,\delta)}(x) \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t) \sin t\,\mathrm{d}t$$

may be simplified by using the following formulas:

$$w^{(\gamma,\delta)}(x) = (1-x)^{\gamma}(1+x)^{\delta} \sim (1-x)^{\gamma} \qquad \big(x \in [0,1]\big),$$

$$w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t = (1-\cos t)^{\alpha-\gamma}(1+\cos t)^{\beta-\delta}\sin t \sim t^{2(\alpha-\gamma)+1} \qquad \big(t \in [0, \tfrac{2\pi}{3}]\big),$$

$$x - y = \cos s - \cos t = 2\sin\frac{t+s}{2}\sin\frac{t-s}{2} \sim t^2 - s^2 \sim t(t-s) \qquad \big(s \in [0, \tfrac{\pi}{2}],\quad t \in [s, \tfrac{2\pi}{3}]\big).$$

Thus by (2.2) and (3.6) we have uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$J_2 \sim (1-x)^{\gamma} \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)|\, t^{2(\alpha-\gamma)+1}\,\mathrm{d}t \sim$$

$$\sim n(1-x)^{\gamma} \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \Big| P_{n+1}(x)P_n(\cos t) - P_n(x)P_{n+1}(\cos t) \Big| \frac{t^{2(\alpha-\gamma)}}{t-s}\,\mathrm{d}t.$$

Following the idea of [1, p. 15] we use the identity

$$P_{n+1}(y)P_n(x) - P_n(y)P_{n+1}(x) =$$
$$= \left(1 + \frac{\alpha+\beta}{2n+2}\right)\left[(1-x)\overline{P}_n(x)P_n(y) - (1-y)\overline{P}_n(y)P_n(x)\right], \tag{4.3}$$

which may be verified by using (3.8).

Thus we have uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$J_2 = O(n)(1-x)^{\gamma+1}|\overline{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |P_n(\cos t)| \frac{t^{2(\alpha-\gamma)}}{t-s}\,\mathrm{d}t +$$
$$+ O(n)(1-x)^{\gamma}|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |\overline{P}_n(\cos t)| \frac{t^{2(\alpha-\gamma)+2}}{t-s}\,\mathrm{d}t =$$
$$= O(\sqrt{n})(1-x)^{\gamma+1}|\overline{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t +$$
$$+ O(\sqrt{n})(1-x)^{\gamma}|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s}\,\mathrm{d}t =: J_{21} + J_{22},$$

where we used (3.3) and $\sqrt{1-\cos t} \sim t$ $(t \in [0, \frac{2\pi}{3}])$.

From the condition $\frac{\alpha}{2} + \frac{1}{4} < \gamma$ it follows that $\alpha - 2\gamma - \frac{1}{2} < -1$, so by Lemma 1, $s \sim \sqrt{1-x}$ $(\cos s = x \in [0,1])$ and (3.3) we obtain

$$J_{21} = O(\sqrt{n})(1-x)^{\gamma+1}|\overline{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t =$$
$$= O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma+2} (\log(n\sqrt{1-x}+1)+1).$$

Similarly, for J_{22} we have (since $\alpha - 2\gamma + \frac{1}{2} \in (-1, 0)$)

$$J_{22} = O(\sqrt{n})(1-x)^{\gamma}|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s}\,\mathrm{d}t =$$

$$= O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\Big(\log\left(n\sqrt{1-x}+1\right)+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|\Big).$$

Finally we obtain the estimate

$$\begin{aligned}J_2 = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\Big(\log\left(n\sqrt{1-x}+1\right)+\\ +\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(|P_n(x)|+|P_{n+1}(x)|\right)+1\Big),\end{aligned} \tag{4.4}$$

which holds uniformly in $x\in[0,1]$ and $n\in\mathbb{N}$, $n>N$.

4.1.3. Estimation of J_3. The expression J_3 may be simplified (see the estimate of J_2):

$$J_3 \sim (1-x)^{\gamma}\int\limits_{s-\frac{1}{n}}^{s+\frac{1}{n}}|K_n^{(\alpha,\beta)}(x,\cos t)|\,t^{2(\alpha-\gamma)+1}\,\mathrm{d}t$$
$$(x\in[0,1],\ \ s\in[0,\tfrac{\pi}{2}]),$$

if $s\ge\frac{1}{n}$ (the lower bound of the integral is 0 if $0\le s\le\frac{1}{n}$). For the kernel function we shall use the following estimates (see (3.3) and (3.6))

$$\begin{aligned}\left|K_n^{(\alpha,\beta)}(x,\cos t)\right| &= \Big|\sum_{k=0}^{n}\frac{1}{h_k}P_k(x)P_k(\cos t)\Big| = \Big|\frac{1}{h_0}+\sum_{k=1}^{n}\frac{1}{h_k}P_k(x)P_k(\cos t)\Big| =\\ &= O(1)\Big(1+\sum_{k=1}^{n}k|P_k(x)|\,|P_k(\cos t)|\Big) =\\ &= O(1)\Big(1+\sum_{k=1}^{n}k\,k^{-\frac{1}{2}}\left(\sqrt{1-x}+\tfrac{1}{k}\right)^{-\alpha-\frac{1}{2}}k^{-\frac{1}{2}}\left(t+\tfrac{1}{k}\right)^{-\alpha-\frac{1}{2}}\Big) =\\ &= O(1)\Big(1+n\big(\sqrt{1-x}+\tfrac{1}{n}\big)^{-\alpha-\frac{1}{2}}t^{-\alpha-\frac{1}{2}}\Big)\end{aligned}$$
$$(x\in[0,1],\, t\in[0,\tfrac{2\pi}{3}]).$$

If $\frac{1}{n}<s\le\frac{\pi}{2}$ then we have uniformly in $x=\cos s$ that

$$J_3 = O(1)\,(1-x)^{\gamma}\left\{\int\limits_{s-\frac{1}{n}}^{s+\frac{1}{n}}t^{2(\alpha-\gamma)+1}\,\mathrm{d}t+\frac{n}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{\alpha+\frac{1}{2}}}\int\limits_{s-\frac{1}{n}}^{s+\frac{1}{n}}t^{\alpha-2\gamma+\frac{1}{2}}\,\mathrm{d}t\right\}.$$

Since

$$\int_{s-\frac{1}{n}}^{s+\frac{1}{n}} t^A \sim \frac{s^A}{n} \qquad \big(\tfrac{1}{n} \le s \le \pi,\ n \in \mathbb{N},\ A > -1\big),$$

we obtain by $s \sim \sqrt{1-x}$ that

$$\begin{aligned}
J_3 &= O(1)(1-x)^\gamma \left\{ \frac{s^{2(\alpha-\gamma)+1}}{n} + \frac{s^{\alpha-2\gamma+\frac{1}{2}}}{\big(\sqrt{1-x}+\frac{1}{n}\big)^{\alpha+\frac{1}{2}}} \right\} = \\
&= O(1)(1-x)^\gamma \left\{ s^{2(\alpha-\gamma+1)} + \frac{1}{\big(\sqrt{1-x}+\frac{1}{n}\big)^{2\gamma}} \right\} = \\
&= O(1)(1-x)^\gamma \frac{1}{\big(\sqrt{1-x}+\frac{1}{n}\big)^{2\gamma}} = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}} \right)^{2\gamma}.
\end{aligned}$$

If $0 \le s \le \frac{1}{n}$ then (see the definition of J_3 in Section 4.1) we get

$$J_3 = O(1)\,(1-x)^\gamma \left\{ \int_0^{s+\frac{1}{n}} t^{2(\alpha-\gamma)+1}\,\mathrm{d}t + \frac{n}{\big(\sqrt{1-x}+\frac{1}{n}\big)^{\alpha+\frac{1}{2}}} \int_0^{s+\frac{1}{n}} t^{\alpha-2\gamma+\frac{1}{2}}\,\mathrm{d}t \right\}.$$

Since $\gamma < \alpha+1$ and $\gamma < \frac{\alpha}{2}+\frac{3}{4}$ we have $2(\alpha-\gamma)+1 > -1$ and $\alpha-2\gamma+\frac{1}{2} > -1$. So by

$$\int_0^{s+\frac{1}{n}} t^A\,\mathrm{d}t \sim \left(s+\frac{1}{n}\right)^{A+1} \qquad (s \ge 0,\ A > -1)$$

we obtain

$$\begin{aligned}
J_3 &= O(1)\,(1-x)^\gamma \left\{ \big(s+\tfrac{1}{n}\big)^{2(\alpha-\gamma)+2} + \frac{n\,\big(s+\frac{1}{n}\big)^{\alpha-2\gamma+\frac{3}{2}}}{\big(\sqrt{1-x}+\frac{1}{n}\big)^{\alpha+\frac{1}{2}}} \right\} = \\
&= O(1)\,(1-x)^\gamma \left\{ \frac{1}{n^{2(\alpha+1-\gamma)}} + n\big(\sqrt{1-x}+\tfrac{1}{n}\big)\,\frac{1}{\big(\sqrt{1-x}+\frac{1}{n}\big)^{2\gamma}} \right\} = \\
&= O(1)\,(1-x)^\gamma \left\{ 1 + \frac{1}{\big(\sqrt{1-x}+\frac{1}{n}\big)^{2\gamma}} \right\} = \\
&= O(1)\,(1-x)^\gamma\, \frac{1}{\big(\sqrt{1-x}+\frac{1}{n}\big)^{2\gamma}} = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}} \right)^{2\gamma}.
\end{aligned}$$

Finally we get the estimate

$$J_3 = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}, \tag{4.5}$$

which holds uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$.

4.1.4. Estimation of J_4. First we remark that $J_4 = 0$ if $0 \le s \le \frac{1}{n}$, so we suppose that $s \in \left[\frac{1}{n}, \frac{\pi}{2}\right]$, i.e. $x = \cos s \in \left[0, 1 - \frac{c}{n^2}\right] =: I_n$. The expression J_4 may be simplified (see the estimation of J_2) by using the relation

$$|x-y| \sim |t^2 - s^2| \sim s|t-s| \sim \sqrt{1-x}|t-s|$$
$$\left(\tfrac{1}{n} \le s \le \tfrac{\pi}{2},\quad t \in \left[0, s - \tfrac{1}{n}\right]\right).$$

Namely, we have (uniformly in $x \in I_n$ and $n \in \mathbb{N}$)

$$J_4 = w^{(\gamma,\delta)}(x)\int_0^{s-\frac{1}{n}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t \sim$$
$$\sim n(1-x)^{\gamma-\frac{1}{2}}\int_0^{s-\frac{1}{n}} |P_{n+1}(x)P_n(\cos t) - P_n(x)P_{n+1}(\cos t)|\,\frac{t^{2(\alpha-\gamma)+1}}{s-t}\,\mathrm{d}t.$$

Using the identity (4.3) and the estimate (3.3) we obtain

$$J_4 = O(n)(1-x)^{\gamma-\frac{1}{2}}\Bigg\{(1-x)|\overline{P}_n(x)|\int_0^{s-\frac{1}{n}} |P_n(\cos t)|\frac{t^{2(\alpha-\gamma)+1}}{s-t}\,\mathrm{d}t +$$
$$+|P_n(x)|\int_0^{s-\frac{1}{n}} t^2\,|\overline{P}_n(\cos t)|\frac{t^{2(\alpha-\gamma)+1}}{s-t}\,\mathrm{d}t\Bigg\} =$$
$$= O(\sqrt{n})(1-x)^{\gamma+\frac{1}{2}}|\overline{P}_n(x)|\int_0^{s-\frac{1}{n}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{s-t}\,\mathrm{d}t +$$
$$+O(\sqrt{n})(1-x)^{\gamma-\frac{1}{2}}|P_n(x)|\int_0^{s-\frac{1}{n}} \frac{t^{\alpha-2\gamma+\frac{3}{2}}}{s-t}\,\mathrm{d}t =: J_{41} + J_{42}$$
$$\left(\tfrac{1}{n} \le s = \arccos x \le \tfrac{\pi}{2},\quad n \in \mathbb{N}\right).$$

Since $\gamma < \frac{\alpha}{2} + \frac{3}{4}$, thus $\alpha - 2\gamma + \frac{1}{2} > -1$ we have by using Lemma 2 and $s \sim \sqrt{1-x}$ that

$$J_{41} = O(\sqrt{n})(1-x)^{\gamma+\frac{1}{2}} \left|\overline{P}_n(x)\right| \left(s + \tfrac{1}{n}\right)^{\alpha-2\gamma+\frac{1}{2}} \log(ns+1) =$$

$$= O(\sqrt{n}) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \left|\overline{P}_n(x)\right| \left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{3}{2}} \log(ns+1) =$$

$$= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \log\left(n\sqrt{1-x}+1\right)$$

$$\big(x \in I_n, \quad n \in \mathbb{N}\big).$$

Similarly,

$$J_{42} = O(\sqrt{n})(1-x)^{\gamma-\frac{1}{2}} |P_n(x)| \left(s + \tfrac{1}{n}\right)^{\alpha-2\gamma+\frac{3}{2}} \log(ns+1) =$$

$$= O(\sqrt{n}) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \left|P_n(x)\right| \frac{\left(\sqrt{1-x}+\frac{1}{n}\right)^{\alpha+\frac{3}{2}}}{\sqrt{1-x}} \log(ns+1) =$$

$$= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \log\left(n\sqrt{1-x}+1\right)$$

$$\big(x \in I_n, \quad n \in \mathbb{N}\big).$$

Summarizing the above formulas we obtain

$$J_4 = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \log\left(n\sqrt{1-x}+1\right) \qquad (4.6)$$

$$\big(x \in I_n, \quad n \in \mathbb{N}\big).$$

4.1.5. The final upper estimate. Using (4.2), (4.4), (4.5) and (4.6) we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \Big(\log\left(n\sqrt{1-x}+1\right) +$$

$$\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} \left(|P_n(x)| + |P_{n+1}(x)|\right) + 1\Big)$$

$$(x \in [0,1],\ n \in \mathbb{N},\ n > N).$$

Let $\bar{x} \in (0,1)$ be the closest number to 1 for which

$$P_n(\bar{x}) = \frac{1}{2} P_n(1) \sim n^{\alpha}$$

holds. If $x \in [0, \bar{x}]$ then

$$1 - x \ge 1 - \bar{x} = \frac{P_n(1) - P_n(\bar{x})}{P_n'(\xi)} \sim \frac{1}{n^2} \quad \big(\xi \in (\bar{x}, 1)\big) \tag{4.7}$$

(see (3.2)). Thus

$$\log\left(n\sqrt{1-x}+1\right) \ge c.$$

If $x \in (\bar{x}, 1]$ then $P_n(x) \sim n^\alpha$, so

$$\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} \left(|P_n(x)| + |P_{n+1}(x)|\right) \ge c.$$

This means that also

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \Bigg(\log\left(n\sqrt{1-x}+1\right) +$$

$$\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} \left(|P_n(x)| + |P_{n+1}(x)|\right) \Bigg)$$

$$(x \in [0,1],\ n \in \mathbb{N},\ n > N)$$

is true.

From this we have uniformly in $x \in [-1, 1]$ and $n \in \mathbb{N}$, $n > N$ that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \left(\frac{\sqrt{1+x}}{\sqrt{1+x}+\frac{1}{n}}\right)^{2\delta} \phi_n^{(\alpha,\beta)}(x),$$

where

$$\phi_n^{(\alpha,\beta)}(x) = \log\left(n\sqrt{1-x^2}+1\right) +$$

$$+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} \left(\sqrt{1+x}+\tfrac{1}{n}\right)^{\beta+\frac{1}{2}} \left(|P_n^{(\alpha,\beta)}(x)| + |P_{n+1}^{(\alpha,\beta)}(x)|\right).$$

Thus the upper estimation in (2.8) is proved.

4.2. Lower estimation of $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$. Because of symmetry, it is enough to consider $x \in [0, 1]$. We shall give three different lower estimations for the weighted Lebesgue function.

4.2.1. The first lower estimation. If $\alpha, \beta > -1$ and $\gamma, \delta \ge 0$, then there exists a constant $c > 0$ independent of x and n such that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \ge c\, w^{(\gamma,\delta)}(x) \qquad \big(x \in [0,1],\ n \in \mathbb{N}\big). \tag{4.8}$$

Indeed, using the orthogonality of Jacobi polynomials we have

$$\int_{-1}^{1} K_n^{(\alpha,\beta)}(x,y)w^{(\alpha,\beta)}(y)\,\mathrm{d}y = 1 \quad \big(x \in [0,1],\ n \in \mathbb{N}\big).$$

Therefore

$$\begin{aligned} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &= w^{(\gamma,\delta)}(x) \int_{-1}^{1} \big|K_n^{(\alpha,\beta)}(x,y)\big| \frac{w^{(\alpha,\beta)}(y)}{(1-y)^\gamma(1+y)^\delta}\,\mathrm{d}y \geq \\ &\geq c\, w^{(\gamma,\delta)}(x) \int_{-1}^{1} \big|K_n^{(\alpha,\beta)}(x,y)\big| w^{(\alpha,\beta)}(y)\,\mathrm{d}y \geq \\ &\geq c\, w^{(\gamma,\delta)}(x) \int_{-1}^{1} K_n^{(\alpha,\beta)}(x,y) w^{(\alpha,\beta)}(y)\,\mathrm{d}y = c\, w^{(\gamma,\delta)}(x). \end{aligned}$$

4.2.2. The second lower estimation. If $\alpha, \beta > -1$ and $\gamma, \delta \geq 0$, then there exists a constant $c > 0$ independent of x and n such that

$$\begin{gathered} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c\, w^{(\gamma,\delta)}(x)\sqrt{n}\,\big(|P_n(x)| + |P_{n+1}(x)|\big) \\ (x \in [0,1],\ n \in \mathbb{N}). \end{gathered} \tag{4.9}$$

In [1, p. 18] it was proven that

$$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} |K_n^{(\alpha,\beta)}(x,\cos t)|\,\mathrm{d}t \geq c\sqrt{n}\,\big(|P_n(x)| + |P_{n+1}(x)|\big),$$

$$(x \in [0,1],\ n \in \mathbb{N}),$$

from which (4.9) follows immediately.

4.2.3. The third lower estimation. It is clear that

$$\begin{gathered} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq \\ \geq w^{(\gamma,\delta)}(x) \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t \end{gathered} \tag{4.10}$$

for all $x = \cos s \in [0,1]$ and $R > 0$. Using the ideas of [1], we shall give a lower estimation for the right hand side of (4.10) with a suitable number $R > 1$.

Since

$$w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t \sim t^{2\alpha-2\gamma+1}$$

$$\big(s\in[0,\tfrac{\pi}{2}],\quad t\in[s,\tfrac{2\pi}{3}]\,\big),$$

we obtain from (4.10) that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\geq c\,(1-x)^{\gamma}\int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}}|K_n^{(\alpha,\beta)}(x,\cos t)|\cdot t^{2\alpha-2\gamma+1}\,\mathrm{d}t. \tag{4.11}$$

The estimation the above integral is performed in several steps.

Step 1. From (3.7) it follows that

$$F_n(x,y):=P_{n+1}(y)P_n(x)-P_n(y)P_{n+1}(x)=\frac{2n+\alpha+\beta+2}{4(n+1)}\times$$
$$\times\Big\{(1-x^2)\widetilde{P}_{n-1}(x)P_n(y)-(1-y^2)\widetilde{P}_{n-1}(y)P_n(x)+(y-x)P_n(x)P_n(y)\Big\},$$

so by (3.6) we have uniformly for all $x\in[0,1]$ and $n\in\mathbb{N}$ that

$$\big|K_n^{(\alpha,\beta)}(x,y)\big|=\lambda_n^{(\alpha,\beta)}\left|\frac{F_n(x,y)}{x-y}\right|\geq$$

$$\geq c\,n\left|\frac{(1-x^2)\widetilde{P}_{n-1}(x)P_n(y)-(1-y^2)\widetilde{P}_{n-1}(y)P_n(x)}{x-y}-P_n(x)P_n(y)\right|\geq$$

$$\geq c_1\,n\left|\frac{(1-x^2)\widetilde{P}_{n-1}(x)P_n(y)-(1-y^2)\widetilde{P}_{n-1}(y)P_n(x)}{x-y}\right|-c_2\,n\,|P_n(x)|\,|P_n(y)|.$$

Since $|x-y|=|\cos s-\cos t|\sim t(t-s)$ we have

$$\int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}}|K_n^{(\alpha,\beta)}(x,\cos t)|\cdot t^{2\alpha-2\gamma+1}\,\mathrm{d}t\geq$$

$$\geq c_1\int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}}\Big|(1-x^2)\widetilde{P}_{n-1}(x)P_n(y)-(1-y^2)\widetilde{P}_{n-1}(y)P_n(x)\Big|\frac{t^{2\alpha-2\gamma}}{t-s}\,\mathrm{d}t-$$

$$-c_2\,n|P_n(x)|\int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}}|P_n(\cos t)|t^{2\alpha-2\gamma+1}\,\mathrm{d}t.$$

Therefore by (3.5) we get uniformly for all $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$
\begin{aligned}
(4.12) \qquad & L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq \\
& \geq c_1\, n\,(1-x)^{\gamma} \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left|(1-x^2)\widetilde{P}_{n-1}(x)P_n(y) - (1-y^2)\widetilde{P}_{n-1}(y)P_n(x)\right| \frac{t^{2\alpha-2\gamma}}{t-s}\,\mathrm{d}t - \\
& -c_2\sqrt{n}(1-x)^{\gamma}|P_n(x)|.
\end{aligned}
$$

Step 2. For the estimation of the integral

$$
I := \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left|(1-x^2)\widetilde{P}_{n-1}(x)P_n(y) - (1-y^2)\widetilde{P}_{n-1}(y)P_n(x)\right| \frac{t^{2\alpha-2\gamma}}{t-s}\,\mathrm{d}t
$$

we use the asymptotic formula (3.4) for the Jacobi polynomials

$$
P_n(y) = P_n^{(\alpha,\beta)}(y) \quad \text{and} \quad \widetilde{P}_{n-1}(y) = P_{n-1}^{(\alpha+1,\beta+1)}(y),
$$

which gives

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(\cos t) &= \frac{k^{(\alpha,\beta)}(t)}{\sqrt{n}}\left(\cos(Nt+\nu) + \frac{O(1)}{n\sin t}\right), \\
P_{n-1}^{(\alpha+1,\beta+1)}(\cos t) &= \frac{k^{(\alpha+1,\beta+1)}(t)}{\sqrt{n-1}}\left(\cos(\overline{N}t+\overline{\nu}) + \frac{O(1)}{n\sin t}\right) = \\
&= \frac{2\,k^{(\alpha,\beta)}(t)}{\sqrt{n-1}\,\sin t}\left(\cos(\overline{N}t+\overline{\nu}) + \frac{O(1)}{(n-1)\sin t}\right),
\end{aligned}
$$

where

$$
\overline{N} = n-1+\frac{(\alpha+1)+(\beta+1)+1}{2} = N
$$

and

$$
\overline{\nu} = -\frac{2(\alpha+1)+1}{4}\pi = \nu - \frac{\pi}{2}.
$$

We have

$$
\begin{aligned}
&(1-x^2)\widetilde{P}_{n-1}(x)P_n(y) - (1-y^2)\widetilde{P}_{n-1}(y)P_n(x) = \\
&= \frac{k^{(\alpha,\beta)}(t)}{\sqrt{n}}\left\{(1-x^2)\widetilde{P}_{n-1}(x)\cos(Nt+\nu) - 2\sqrt{\tfrac{n}{n-1}}P_n(x)\sin t\cdot\sin(Nt+\nu)\right\} + \\
&+O\left(\frac{1}{n^{3/2}}\right)(1-x^2)\widetilde{P}_{n-1}(x)\cdot\frac{k^{(\alpha,\beta)}(t)}{\sin t} + O\left(\frac{1}{(n-1)^{3/2}}\right)P_n(x)\cdot k^{(\alpha,\beta)}(t).
\end{aligned}
$$

If $0 < s + \frac{R}{n} \le t \le \frac{2\pi}{3}$, then

$$k^{(\alpha,\beta)}(t) = \frac{1}{\sqrt{\pi}}\Big(\sin\tfrac{t}{2}\Big)^{-\alpha-\frac{1}{2}}\Big(\cos\tfrac{t}{2}\Big)^{-\beta-\frac{1}{2}} \sim t^{-\alpha-\frac{1}{2}}.$$

Therefore

$$I \ge \frac{c_1}{\sqrt{n}} \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \Big|(1-x^2)\widetilde{P}_{n-1}(x)\cos(Nt+\nu) -$$

$$-2\sqrt{\tfrac{n}{n-1}}P_n(x)\sin t \cdot \sin(Nt+\nu)\Big| \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t -$$

$$-\frac{c_2}{n^{3/2}}\left\{(1-x^2)|\widetilde{P}_{n-1}(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{3}{2}}}{t-s}\,\mathrm{d}t + |P_n(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t\right\}.$$

Step 3. Using the above inequality and (4.12) we have

(4.13)
$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \ge c_1\sqrt{n}\,(1-x)^\gamma \times$$

$$\times \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \Big|(1-x^2)\widetilde{P}_{n-1}(x)\cos(Nt+\nu) - 2\sqrt{\tfrac{n}{n-1}}P_n(x)\sin t \cdot \sin(Nt+\nu)\Big| \times$$

$$\times \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t - c_2\sqrt{n}\,(1-x)^\gamma|P_n(x)| - c_3\,\varrho_1(n,x),$$

where

$$\varrho_1(n,x) = \frac{(1-x)^\gamma}{\sqrt{n}} \times$$

$$\times\left\{(1-x^2)|\widetilde{P}_{n-1}(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{3}{2}}}{t-s}\,\mathrm{d}t + |P_n(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t\right\}.$$

Since $t \ge \frac{R}{n}$ we have

$$\varrho_1(n,x) \le c\,\frac{\sqrt{n}}{R}(1-x)^\gamma \times$$

$$\times\left\{(1-x^2)|\widetilde{P}_{n-1}(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t + |P_n(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s}\,\mathrm{d}t\right\}.$$

Using Lemma 1, $s \sim \sqrt{1-x}$ and (3.3) we get uniformly for all $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$\varrho_1(n,x) \le c \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}} \right)^{2\gamma} \times$$
$$\times \left\{ \frac{1}{R} \Big[\log\left(n\sqrt{1-x}+1\right)+1 \Big] + \sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} |P_n(x)| \right\}.$$

Step 4. Now, we consider the integral in (4.13) and write $\sin s = \sqrt{1-x^2}$ instead of $\sin t$. Then by the Lagrange mean value theorem we have

$$\sin t = \sin s + \tau = \sqrt{1-x^2} + \tau$$

with $|\tau| \le t - s$. Thus we obtain an error term in the integral, which we shall denote by $\varrho_2(n,x)$. Therefore we have uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \ge c_1 \sqrt{n}\,(1-x)^{\gamma} \sqrt{1-x^2} \times$$
$$\times \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left| \sqrt{1-x^2}\,\widetilde{P}_{n-1}(x) \cos(Nt+\nu) - 2\sqrt{\frac{n}{n-1}}P_n(x)\sin(Nt+\nu) \right| \times$$
$$\times \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t - c_2\,\varrho_2(n,x) - c_3\,\varrho_1(n,x) - c_4\sqrt{n}(1-x)^{\gamma}\,|P_n(x)|,$$

where

$$\varrho_2(n,x) = 2\sqrt{n}\,(1-x)^{\gamma}\frac{n}{n-1}|P_n(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |\sin(Nt+\nu)|t^{\alpha-2\gamma-\frac{1}{2}}\,\mathrm{d}t \le$$

$$\le c\sqrt{n}\,(1-x)^{\gamma}|P_n(x)| \left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha-2\gamma+\frac{1}{2}} \le c \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}} \right)^{2\gamma}$$

(using $s \sim \sqrt{1-x}$ and (3.3)).

Let

$$\psi := \arg\left(\sqrt{1-x^2}\,\widetilde{P}_{n-1}(x) + i2\sqrt{\frac{n}{n-1}}P_n(x) \right).$$

Then we have uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \ge c_1\,(1-x)^{\gamma} \times$$
$$\times \left(n(1-x^2)\left((1-x^2)\widetilde{P}_{n-1}^2(x) + \frac{4n}{n-1}P_n^2(x) \right) \right)^{\frac{1}{2}} \times$$

$$\times \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |\cos(Nt+\nu+\psi)| \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t-$$

$$-c_2\,\varrho_2(n,x)-c_3\,\varrho_1(n,x)-c_4\sqrt{n}(1-x)^{\gamma}\,|P_n(x)|.$$

Step 5. Now we will estimate the integral

$$B:=\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left|\cos(Nt+\nu+\psi)\right| \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t.$$

Since $|\cos t| \geq \cos^2 t = \frac{1+\cos(2t)}{2}$ it follows that

$$B \geq \frac{1}{2}\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \Big(1+\cos 2(Nt+\nu+\psi)\Big) \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t.$$

Using Lemma 1 we have

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t \geq c\left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}}\left[\log\left(\frac{ns}{R}+1\right)+1\right] \geq$$

$$\geq c\left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}}\Big[\log(ns+1)+1-\log R\Big],$$

and by the second mean value theorem

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \cos 2(Nt+\nu+\psi)\frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t = \frac{(s+\frac{R}{n})^{\alpha-2\gamma-\frac{1}{2}}}{R/n}\times$$

$$\times\int_{s+\frac{R}{n}}^{\xi} \cos 2(Nt+\nu+\psi)\,\mathrm{d}t \leq c\,\left(s+\tfrac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \quad \Big(\xi\in\left(s+\tfrac{R}{n},\tfrac{2\pi}{3}\right)\Big).$$

Then we get

$$B \geq c_1\left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}}\Big[\log(ns+1)+1-c_2\Big].$$

STEP 6. From this we obtain

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c_1\,(1-x)^{\gamma} \left(n(1-x^2)\left((1-x^2)\widetilde{P}_{n-1}^2(x) + \frac{4n}{n-1}P_n^2(x)\right)\right)^{\frac{1}{2}} \times$$

$$\times \left(s + \frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \Big[\log(ns+1) + 1 - c_2\Big] -$$

$$-c_3\,\varrho_2(n,x) - c_4\,\varrho_1(n,x) - c_5\,\sqrt{n}(1-x)^{\gamma}\,|P_n(x)|.$$

$$\big(x \in [0,1],\ n \in \mathbb{N},\ n > N\big).$$

By (3.3) and $s \sim \sqrt{1-x}$ we have

$$C(x) := (1-x)^{\gamma}\left(s + \frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \times$$

$$\times\Big\{n(1-x^2)\Big((1-x^2)\widetilde{P}_{n-1}^2(x) + \frac{4n}{n-1}P_n^2(x)\Big)\Big\}^{\frac{1}{2}} \leq$$

$$\leq c_1\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \leq c_2,$$

which means that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c_1\,C(x)\Big[\log(n\sqrt{1-x}+1) + 1\Big] -$$

$$-c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\Bigg[\frac{1}{R}\big(\log(n\sqrt{1-x}+1)+1\big) +$$

$$+\sqrt{n}\,\big(\sqrt{1-x}+\tfrac{1}{n}\big)^{\alpha+\frac{1}{2}}\,|P_n(x)| + 1\Bigg] - c_3\,\sqrt{n}\,(1-x)^{\gamma}\,|P_n(x)|$$

$$(x \in [0,1],\ n \in \mathbb{N},\ n > N).$$

Let $\bar{x} \in (0,1)$ be the closest number to 1 for which

$$P_n(\bar{x}) = \frac{1}{2}P_n(1) \sim n^{\alpha}$$

holds. If $x \in [0,\bar{x}]$ then by (4.7) we have

$$s \sim \sqrt{1-x} \geq \sqrt{1-\bar{x}} \geq \frac{c}{n},$$

thus

$$\left(s + \frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \geq c\,s^{\alpha-2\gamma-\frac{1}{2}},$$

which means that

$$C(x) \geq c\, s^{\alpha-\frac{1}{2}} \Big\{ n(1-x^2)\Big((1-x^2)\widetilde{P}_{n-1}^2(x) + \frac{4n}{n-1} P_n^2(x)\Big)\Big\}^{\frac{1}{2}}.$$

It is proved in [1, p. 21] that

$$s^{\alpha-\frac{1}{2}} \Big\{ n(1-x^2)\Big((1-x^2)\widetilde{P}_{n-1}^2(x) + \frac{4n}{n-1} P_n^2(x)\Big)\Big\}^{\frac{1}{2}} > c \quad (x \in [0,\bar{x}]),$$

so for every $x \in [0,\bar{x}]$ and $n \in \mathbb{N}$, $n > N$ we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c_1 \Big[\log\big(n\sqrt{1-x}+1\big)+1\Big] - c_2 \Bigg\{ \sqrt{n}(1-x)^{\gamma}(x)|P_n(x)| + \\ + \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \Big(\frac{1}{R}\Big[\log\big(n\sqrt{1-x}+1\big)+1\Big] + \\ + \sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} |P_n(x)| + 1\Big)\Bigg\}.$$

Here

$$c_1 - \frac{c_2}{R}\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \geq c_1 - \frac{c_2}{R} =: c_3 \geq c_3 \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}.$$

The number R can be chosen such that $c_3 > 0$. Then we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c_3 \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \Big[\log\big(n\sqrt{1-x}+1\big)+1\Big] - \\ - c_2\sqrt{n}(1-x)^{\gamma}(x)|P_n(x)| - c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} - \\ - c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} |P_n(x)|$$

for all $x \in [0,\bar{x}]$ and $n \in \mathbb{N}$, $n > N$. If $x \in [\bar{x}, 1]$ then

$$1 - x \leq 1 - \bar{x} \sim \frac{1}{n^2}$$

(see (4.7)), and so

$$\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \Big[\log\big(n\sqrt{1-x}+1\big)+1\Big] \leq c \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \leq$$

$$\leq c\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|$$

(since $P_n(x)\sim n^{\alpha}$ on this interval), which means that with a suitable $c_4>0$ we have

$$\begin{aligned}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\geq c_3\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left[\log\left(n\sqrt{1-x}+1\right)+1\right]-\\ -c_2\sqrt{n}(1-x)^{\gamma}(x)|P_n(x)|-c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}-\\ -c_4\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|\end{aligned}\tag{4.14}$$

for all $x\in[0,1]$ and $n\in\mathbb{N}$, $n>N$.

4.2.4. The final lower estimation. From (4.8) we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\geq c_6\,(1-x)^{\gamma}\quad(x\in[0,1],\,n\in\mathbb{N}).\tag{4.15}$$

(4.9), (4.14) and (4.15) imply

$$\begin{aligned}c_3\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left[\log\left(n\sqrt{1-x}+1\right)+1\right]\leq L_n^{(\alpha,\beta),(\gamma,\delta)}(x)+\\ +c_2\sqrt{n}(1-x)^{\gamma}(|P_n(x)|+|P_{n+1}(x)|)+c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}+\\ c_4\sqrt{n}\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}(|P_n(x)|+|P_{n+1}(x)|)\leq\\ \leq L_n^{(\alpha,\beta),(\gamma,\delta)}(x)+\frac{c_2}{c}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)+\frac{c_2}{c_6}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{-2\gamma}+\\ +\frac{c_4}{c}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha-2\gamma+\frac{1}{2}}.\end{aligned}$$

Hence we obtain

$$c_3(1-x)^{\gamma}\left[\log\left(n\sqrt{1-x}+1\right)+1\right]\leq c_7\,L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$$

$$(x\in[0,1],\,n\in\mathbb{N},\,n>N).$$

Since (by (3.3))

$$\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(|P_n(x)|+|P_{n+1}(x)|\right)\le c$$

$$(x\in[0,1],\ n\in\mathbb{N}),$$

we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\ge c\,(1-x)^{\gamma}\Big(\log\left(n\sqrt{1-x}+1\right)+$$

$$+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(|P_n(x)|+|P_{n+1}(x)|\right)\Big)\ge$$

$$\ge c\,w^{(\gamma,\delta)}(x)\,\phi_n^{(\alpha,\beta)}(x),$$

where

$$\phi_n^{(\alpha,\beta)}(x)=\log\left(n\sqrt{1-x^2}+1\right)+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\times$$

$$\times\left(\sqrt{1+x}+\tfrac{1}{n}\right)^{\beta+\frac{1}{2}}\left(|P_n^{(\alpha,\beta)}(x)|+|P_{n+1}^{(\alpha,\beta)}(x)|\right).$$

The above estimate holds uniformly in $x\in[0,1]$ and $n\in\mathbb{N}$.

Theorem is proved. ■

5. Proof of Corollary

Since $L_n^{(\alpha,\beta),(\gamma,\delta)}(\pm1)=0$ we have

$$\max_{x\in[-1,1]} L_n^{(\alpha,\beta),(\gamma,\delta)}(x)=L_n^{(\alpha,\beta),(\gamma,\delta)}(x_0)$$

with $x_0\in(-1,1)$.

From Theorem and (3.3) it follows that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x_0)\le c_1\cdot 1\cdot\left(\log(n+1)+c_2\right)\le c_3\log(n+1)$$

and

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x_0)\ge c_4\,w^{(\gamma,\delta)}(x_0)\,\log\left(n\sqrt{1-x_0^2}+1\right)\ge$$

$$\ge c_5\log(c_6n+1)\ge c_7\log(n+1),$$

where the c_i $(i=1\dots7)$ constants are positive and independent of n. This proves the statement. ■

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