Studia Scientiarum Mathematicarum Hungarica 47 (2), 139–154 (2010)
DOI: 10.1556/SScMath.2009.1121
First published online 7 August, 2009

WEIGHTED APPROXIMATION VIA Θ-SUMMATIONS OF FOURIER–JACOBI SERIES

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Communicated by P. Vértesi

(Received November 23, 2007; accepted September 3, 2008)

Abstract

This paper is devoted to the study of Θ-summability of Fourier–Jacobi series. We shall construct such processes (using summations) that are uniformly convergent in a Banach space $\left(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}}\right)$ of continuous functions. Some special cases are also considered, such as the Fejér, de la Vallée Poussin, Cesàro, Riesz and Rogosinski summations. Our aim is to give such conditions with respect to Jacobi weights $w_{\gamma,\delta}$, $w_{\alpha,\beta}$ and to summation matrix Θ for which the uniform convergence holds for all $f \in C_{w_{\gamma,\delta}}$. Order of convergence will also be investigated. The results and the methods are analogues to the discrete case (see [16] and [17]).

1. Introduction

It is known that the sequence of partial sums of the trigonometric Fourier series is not uniformly convergent for all continuous functions. However, by using a suitable summation, one can get uniform convergence. We may refer to [15] and the references there. The algebraic case is more complicated.

The Fourier–Jacobi series has been studied extensively by many authors. It is known that there exists a continuous function on $[-1,1]$ such that its Fourier–Jacobi series is not uniformly convergent on $[-1,1]$ (see e.g. [12, Ch. IX], [10, p. 301], [1], [8, p. 160]). The *weighted* convergence of certain

2000 *Mathematics Subject Classification.* Primary 41A10, 42C10.

Key words and phrases. Weighted polynomial approximation, Fourier–Jacobi series, uniform convergence.

*Research supported by the Hungarian National Scientific Research Foundation (OTKA) under Grant No. T047132.

sums of Fourier–Jacobi series (obtaining best weighted polynomial approximation) was investigated by D. S. Lubinsky and V. Totik [6]. M. Felten [3] (see also [4], [5]) showed that their results can be largely extended. In a recent paper J. Szabados [11] proved a weighted error estimate for approximation by Cesàro means of Fourier–Jacobi series.

The present paper is devoted to the study of weighted uniform convergence of Θ-sums of Fourier–Jacobi series. The discrete version of this problem (when we use Lagrange interpolation polynomials instead of the partial sums of Fourier–Jacobi series) was studied by L. Szili and P. Vértesi [16], [17] (see also [13], [14]). Starting from these results we shall construct a wide class of linear processes (using Jacobi polynomials and summations) which are uniformly convergent in suitable weighted spaces of continuous functions.

Let $w_{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}$ be a Jacobi weight $(\alpha, \beta > -1,\ x \in [-1,1])$ and denote by $p_n(w_{\alpha,\beta})$ $(n \in \mathbb{N})$ the orthonormal polynomials with respect to the weight $w_{\alpha,\beta}$. For an infinite matrix Θ (see (2.2)) we shall consider suitable summations of Fourier–Jacobi series (see Section 2.2). These polynomials will be denoted by $S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot)$ (see (2.3)).

The aim of this paper is to give conditions for α, β, γ, δ and the summation matrix Θ satisfying

$$\lim_{n\to+\infty} \left\| \left(f - S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) \right) w_{\gamma,\delta} \right\| = 0$$

for all $f \in C_{w_{\gamma,\delta}}$ (see Section 2.1). Order of convergence will also be investigated.

2. Notation and preliminaries

2.1. Selection of the function space

Let $C(-1,1)$ be the linear space of real valued continuous functions defined on the interval $(-1,1)$. We define the weighted function space

$$C_{w_{\gamma,\delta}} := \left\{ f \in C(-1,1) : \lim_{|x|\to 1} \left(f w_{\gamma,\delta} \right)(x) = 0 \right\},$$

where

$$w_{\gamma,\delta}(x) := (1-x)^{\gamma}(1+x)^{\delta} \quad \left(x \in [-1,1],\ \gamma, \delta \geqq 0 \right)$$

is a Jacobi weight.

Then $\big(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}}\big)$ is a Banach space where the norm $\|\cdot\|_{w_{\gamma,\delta}}$ is defined by

$$\|f\|_{w_{\gamma,\delta}} := \|fw_{\gamma,\delta}\| := \max_{x\in[-1,1]} \big|(fw_{\gamma,\delta})(x)\big| \quad (f \in C_{w_{\gamma,\delta}}).$$

2.2. Fourier–Jacobi series

For $\alpha, \beta > -1$ we can uniquely define the sequence of orthonormal Jacobi polynomials

$$p_n(x) := p_n(w_{\alpha,\beta}, x) := \gamma_n x^n + \dots + \gamma_0 \quad \big(\gamma_n > 0,\ n \in \mathbb{N}_0 := \{0,1,2,\dots\}\big)$$

for the weight $w_{\alpha,\beta}$ satisfying

$$\int_{-1}^{1} p_n(w_{\alpha,\beta}, x)p_m(w_{\alpha,\beta}, x)w_{\alpha,\beta}(x)\,dx = \delta_{m,n} \quad (n, m \in \mathbb{N}_0). \tag{2.1}$$

If $f \in C_{w_{\gamma,\delta}}$ is a given function then we can construct the Fourier–Jacobi series of f by

$$S(f, w_{\alpha,\beta}, x) := \sum_{k\in\mathbb{N}_0} c_k(f)p_k(w_{\alpha,\beta}, x) \quad \big(x \in [-1,1]\big)$$

where

$$c_k(f) := c_k^{(\alpha,\beta)}(f) := \int_{-1}^{1} f(x)p_k(w_{\alpha,\beta}, x)w_{\alpha,\beta}(x)\,dx \quad (k \in \mathbb{N}_0)$$

are the Fourier–Jacobi coefficients. Denote the nth partial sum of the Fourier–Jacobi series by

$$S_n(f, w_{\alpha,\beta}, x) := \sum_{k=0}^{n} c_k(f)p_k(w_{\alpha,\beta}, x) \quad \big(x \in [-1,1],\ n \in \mathbb{N}_0\big).$$

2.3. Θ-summation of the Fourier–Jacobi series

Let us fix a summation matrix

$$\Theta := \begin{pmatrix} \theta_{0,1} & & \\ \theta_{0,2} & \theta_{1,2} & \\ \theta_{0,3} & \theta_{1,3} & \theta_{2,3} \\ \vdots & \vdots & \vdots \end{pmatrix}, \tag{2.2}$$

where $\theta_{k,n}$'s are real numbers.

The Θ-*sum of the Fourier–Jacobi series* is defined by

$$S_n^{\Theta}(f, w_{\alpha,\beta}, x) := \sum_{k=0}^{n-1} \theta_{k,n} c_k(f) p_k(w_{\alpha,\beta}, x) \tag{2.3}$$

$$\big(x \in [-1,1],\ n \in \mathbb{N},\ f \in C_{w_{\gamma,\delta}}\big).$$

Before starting the analysis of the problem of convergence, we discuss some possible choices of the summation matrix Θ.

Example 1. *Partial sums of Fourier–Jacobi series.* Let

$$\theta_{k,n} := 1 \quad (k = 0, 1, \dots, n-1,\ n \in \mathbb{N}).$$

Then $S_n^{\Theta}(f, w_{\alpha,\beta}, x)$ is the usual nth partial sum of the Fourier–Jacobi series of f.

Example 2. *Summation functions.* Let $\varphi : [0,1] \to \mathbb{R}$ be a given function with $\varphi(0) = 1$ and $\varphi(1) = 0$. Now we can give the summation matrix Θ by $\theta_{k,n} := \varphi\big(\frac{k}{n}\big)$ $(k = 0, 1, \dots, n-1,\ n \in \mathbb{N})$. These cases will be called φ-summation of the Fourier–Jacobi series.

Example 3. *Fejér summation of the Fourier–Jacobi series.* Let

$$\theta_{k,n} := 1 - \frac{k}{n} \quad (k = 0, 1, \dots, n-1,\ n \in \mathbb{N})$$

$$\big(\varphi_F(t) := 1 - t,\ t \in [0,1]\big).$$

Then $\Theta_F := (\theta_{k,n})$ is the Fejér summation matrix and

$$\sigma_n(f, w_{\alpha,\beta}, x) := S_n^{\Theta_F}(f, w_{\alpha,\beta}, x) := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) c_k(f) p_k(w_{\alpha,\beta}, x)$$

$$\big(x \in [-1,1],\ n \in \mathbb{N}\big)$$

are the Fejér means of the Fourier–Jacobi series.

EXAMPLE 4. *The* (C,μ) *Cesàro means of the Fourier–Jacobi series* are defined by

$$\theta_{k,n} := \frac{A^{(\mu)}_{n-k-1}}{A^{(\mu)}_{n-1}} \quad (\mu \geqq 0,\ k = 0,1,\dots,n-1,\ n \in \mathbb{N})$$

where

$$A^{(\mu)}_0 := 1, \quad A^{(\mu)}_m := \binom{m+\mu}{m} = \frac{(\mu+1)\cdots(\mu+m)}{m!} \quad (m \in \mathbb{N}).$$

If $\mu = 1$ or $\mu = 0$ then we obtain the Fejér summation or the partial sums of the Fourier–Jacobi series, respectively.

EXAMPLE 5. *The* (R,ν,μ) *Riesz summation of the Fourier–Jacobi series* is defined by the summation function

$$\varphi_{\nu,\mu}(t) := (1-t^{\nu})^{\mu} \quad \big(t \in [0,1]\big),$$

where $\nu, \mu \geqq 0$ are fixed real numbers.

EXAMPLE 6. *The de la Vallée Poussin summation* is defined by the summation function

$$\varphi_s(t) := \begin{cases} 1, & \text{if } 0 \leqq t \leqq s \\ (t-1)/(s-1), & \text{if } s < t \leqq 1 \end{cases}$$

where $s \in (0,1)$.

EXAMPLE 7. *The Rogosinski summation* is defined by the summation function

$$\varphi_R(t) := \cos\frac{\pi t}{2} \quad \big(t \in [0,1]\big).$$

3. Results

First we give a necessary and sufficient condition for the uniform convergence (cf. [17, Statement 3.1]).

THEOREM 3.1. *If* $\alpha, \beta > -1$ *and* $\gamma, \delta \geqq 0$ *then for the summation matrix* $\Theta = (\theta_{k,n})$ *we have*

$$\lim_{n\to+\infty} \left\| \left(f - S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) \right) w_{\gamma,\delta} \right\| = 0 \tag{3.1}$$

for all $f \in C_{w_{\gamma,\delta}}$ *if and only if*

$$\lim_{n\to+\infty} (1 - \theta_{k,n}) = 0 \quad \textit{for all fixed } k = 0, 1, 2, \dots \tag{T1}$$

and

$$(B) \qquad \begin{cases} \textit{there exists } C > 0 \textit{ independent of } n \textit{ such that for all } n \in \mathbb{N} \\ \sup\limits_{x\in[-1,1]} \displaystyle\int_{-1}^{1} \left| \sum_{k=0}^{n-1} \theta_{k,n} p_k(x) p_k(y) \right| \frac{w_{\gamma,\delta}(x) w_{\alpha,\beta}(y)}{w_{\gamma,\delta}(y)} \, dy \leqq C. \end{cases}$$

However, to verify (B) generally is not easy, so we are going to give sufficient conditions for the uniform convergence.

First we define some further conditions ($(T1)$–$(T5)$) corresponding to the summation matrix Θ:

$$\theta_{n-1,n} = O\left(\frac{1}{n}\right) \qquad (n \in \mathbb{N}), \tag{T2}$$

$$\triangle^2 \theta_{k-1,n} = O\left(\frac{1}{n^2}\right) \qquad (k = 1, 2, \dots, n-1,\ n \in \mathbb{N}), \tag{T3}$$

$$\triangle^2 \theta_{k-1,n} \ (k = 1, 2, \dots, n-1,\ n \in \mathbb{N}) \text{ is of constant sign}, \tag{T4}$$

$$\operatorname{sgn} \triangle^2 \theta_{k-1,n} = \operatorname{sgn} \theta_{n-1,n} \quad (k = 1, 2, \dots, n-1,\ n \in \mathbb{N}), \tag{T5}$$

where

$$\triangle^2 \theta_{k,n} := \triangle \theta_{k+1,n} - \triangle \theta_{k,n}, \quad \triangle \theta_{k,n} := \theta_{k+1,n} - \theta_{k,n} \quad (\theta_{n,n} := 0).$$

THEOREM 3.2 (cf. [17, Theorem 3.3]). *Suppose that* $\alpha, \beta \geqq -1/2$ *and* $\gamma, \delta \geqq 0$ *satisfy the inequalities*

$$\frac{\alpha}{2} - \frac{1}{4} < \gamma < \frac{\alpha}{2} + \frac{3}{4} \quad \textit{and} \quad \frac{\beta}{2} - \frac{1}{4} < \delta < \frac{\beta}{2} + \frac{3}{4}. \tag{3.2}$$

Then

(a)$(T1)$, $(T2)$ *and* $(T3)$

or

(b)$(T1)$, $(T2)$ *and* $(T4)$

or

(c)$(T1)$ *and* $(T5)$

imply

$$\lim_{n\to+\infty} \left\| \left(f - S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) \right) w_{\gamma,\delta} \right\| = 0 \tag{3.3}$$

for all $f \in C_{w_{\gamma,\delta}}$.

If the summation matrix is given by a summation function (see Section 2.3, Example 2), then we can give an even simpler sufficient condition for the uniform convergence.

THEOREM 3.3 (cf. [16, Theorem 4.3]). *Suppose that* $\alpha, \beta \geqq -1/2$ *and* $\gamma, \delta \geqq 0$ *fulfil requirements* (3.2). *Let* $\varphi : [0,1] \to \mathbb{R}$ *be a continuous summation function, moreover assume that*

(a)φ *is nonnegative and convex from below on* $[0,1]$

or

(b)φ *is convex (concave) from below on* $[0,1]$ *and there exist* $\varepsilon > 0$ *and* $c > 0$ *such that*

$$\left|\varphi(x)\right| \leqq c(1-x) \quad \left(x \in [1-\varepsilon, 1]\right). \tag{3.4}$$

Then (3.3) *holds for all* $f \in C_{w_{\gamma,\delta}}$.

COROLLARY 3.4 (cf. [17, Corollary 3.4]). *Suppose that* $\alpha, \beta \geqq -1/2$ *and* $\gamma, \delta \geqq 0$ *fulfil requirements* (3.2). *Then*

(a)*for* $\mu \geqq 1$ *the* (C, μ) *Cesàro,*

(b)*for* $\nu, \mu \geqq 1$ *the* (R, ν, μ) *Riesz,*

(c)*for every* $s \in (0,1)$ *the de la Vallée Poussin,*

(d)*the Rogosinski*

summations of Fourier–Jacobi series are uniformly convergent in the space $\left(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}}\right)$.

By choosing different summation matrices, different *orders of convergence* can be attained. Recall that the number

$$E_n(f, w_{\gamma,\delta}) := \inf_{p\in\mathcal{P}_n} \left\| (f-p) w_{\gamma,\delta} \right\|$$

is called the best nth degree weighted polynomial approximation of $f \in C_{w_{\gamma,\delta}}$. (By [16, p. 329, Example 1], $E_n(f, w_{\gamma,\delta}) \to 0$.)

The following result states that the best possible order of approximation by summations of Fourier–Jacobi series can be attained using the de la Vallée Poussin summation.

THEOREM 3.5 (cf. [17, Theorem 3.5]). *Suppose that* $\alpha, \beta \geqq -1/2$ *and* $\gamma, \delta \geqq 0$ *satisfy the requirements* (3.2). *Then for every* $s \in (0,1)$ *we have*

$$\big\| \big(f - S_n^{\varphi_s}(f, w_{\alpha,\beta}, \cdot) \big) w_{\gamma,\delta} \big\| \leqq C E_{q_n}(f, w_{\gamma,\delta}) \quad (n \in \mathbb{N}) \tag{3.5}$$

for all $f \in C_{w_{\gamma,\delta}}$ *with some constant* $C > 0$ *independent of* f *and* n, *where* φ_s *is a de la Vallée Poussin summation function and* $q_n := [sn]$.

The next result asserts that for many summation matrices the order of convergence is at least of Stechkin-type.

THEOREM 3.6 (cf. [17, Theorem 3.6]). *Suppose that* $\alpha, \beta \geqq -1/2$ *and* $\gamma, \delta \geqq 0$ *satisfy the requirements* (3.2), *moreover* $\theta_{0,n} = 1 + O\big(\frac{1}{n}\big)$ $(n \in \mathbb{N})$. *Then*

(a) $(T1)$, $(T2)$ *and* $(T3)$

or

(b) $(T1)$, $(T2)$, $(T4)$ *and* $1 - \theta_{1,n} = O\big(\frac{1}{n}\big)$ $(n \in \mathbb{N})$ *imply*

$$\big\| \big(f - S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) \big) w_{\gamma,\delta} \big\| \leqq \frac{C}{n} \sum_{k=0}^{n-1} E_k(f, w_{\gamma,\delta}) \tag{3.6}$$

for all $f \in C_{w_{\gamma,\delta}}$ *and* $n \in \mathbb{N}$ *with some constant* $C > 0$ *independent of* f *and* n.

4. Proofs

The proofs follow the methods of [16] and [17].

4.1. PROOF OF THEOREM 3.1. The statement is a consequence of the Banach–Steinhaus theorem. It is clear that for all fixed $n \in \mathbb{N}$

$$\mathcal{S}_n^{\Theta} : \big(C_{w_{\gamma,\delta}}, \| \cdot \|_{w_{\gamma,\delta}} \big) \to \mathcal{P}_{n-1} \subset \big(C_{w_{\gamma,\delta}}, \| \cdot \|_{w_{\gamma,\delta}} \big),$$

$$\mathcal{S}_n^{\Theta} f := S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot)$$

is a bounded linear operator, where the norm of $\mathcal{S}_n^{\Theta}$ is

$$\big\| \mathcal{S}_n^{\Theta} \big\| := \sup_{0 \not\equiv f \in C_{w_{\gamma,\delta}}} \frac{\big\| \mathcal{S}_n^{\Theta} f \big\|_{w_{\gamma,\delta}}}{\|f\|_{w_{\gamma,\delta}}} = \sup_{0 \not\equiv f \in C_{w_{\gamma,\delta}}} \frac{\big\| S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) w_{\gamma,\delta} \big\|}{\|f w_{\gamma,\delta}\|}.$$

Since

$$S_n^{\Theta}(f, w_{\alpha,\beta}, x) = \sum_{k=0}^{n-1} \theta_{k,n} c_k(f) p_k(x)$$

$$= \int_{-1}^{1} (f w_{\gamma,\delta})(y) \left(\sum_{k=0}^{n-1} \theta_{k,n} p_k(x) p_k(y) \right) \frac{w_{\alpha,\beta}(y)}{w_{\gamma,\delta}(y)} \, dy$$

thus we have

$$\| \mathcal{S}_n^{\Theta} \| = \sup_{x\in[-1,1]} \int_{-1}^{1} \left| \sum_{k=0}^{n-1} \theta_{k,n} p_k(x) p_k(y) \right| \frac{w_{\gamma,\delta}(x) w_{\alpha,\beta}(y)}{w_{\gamma,\delta}(y)} \, dy$$

i.e. by (B) we get the boundedness of $\left(\|\mathcal{S}_n^{\Theta}\|, \, n \in \mathbb{N} \right)$.

From (2.1) we have

$$c_k(p_j) = \int_{-1}^{1} p_j(x) p_k(x) w_{\alpha,\beta}(x) \, dx = \delta_{j,k}$$

which means that

$$p_j - S_n^{\Theta}(p_j, w_{\alpha,\beta}, \cdot) = p_j - \theta_{j,n} p_j = (1 - \theta_{j,n}) p_j.$$

This by $(T1)$ ensures that for all fixed $j = 0, 1, \dots$.

$$\lim_{n\to+\infty} \left\| \left(p_j - S_n^{\Theta}(p_j, w_{\alpha,\beta}, \cdot) \right) w_{\gamma,\delta} \right\| = 0.$$

Since the polynomials are dense in $\left(C_{w_{\gamma,\delta}}, \| \cdot \|_{w_{\gamma,\delta}} \right)$, thus the conditions of the Banach–Steinhaus theorem hold. □

4.2. Proof of Theorem 3.2. We showed in the proof of Theorem 3.1. that if $(T1)$ holds then we have

$$\lim_{n\to+\infty} \left\| \left(p_j - S_n^{\Theta}(p_j, w_{\alpha,\beta}, \cdot) \right) w_{\gamma,\delta} \right\| = 0$$

for all fixed $j = 0, 1, \dots$.

Thus by the Banach–Steinhaus theorem it is enough to prove

LEMMA 4.1. *Suppose that for* $\alpha, \beta \geqq -1/2$ *and* $\gamma, \delta \geqq 0$ *the conditions* (3.2) *hold. Then* (a) *or* (b) *or* (c) *imply that there exists* $C > 0$ *independent of* n *such that*

$$\left\| S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) w_{\gamma,\delta} \right\| \leqq C \left\| f w_{\gamma,\delta} \right\| \quad (n \in \mathbb{N},\ f \in C_{w_{\gamma,\delta}}). \tag{4.1}$$

PROOF OF LEMMA 4.1. In [3] M. Felten showed that if α, β, γ and δ satisfy the conditions (3.2), then there exists a $K > 0$ independent of n such that

$$\left\| \sigma_n(f, w_{\alpha,\beta}, \cdot) w_{\gamma,\delta} \right\| \leqq K \left\| f w_{\gamma,\delta} \right\| \quad (n \in \mathbb{N},\ f \in C_{w_{\gamma,\delta}}) \tag{4.2}$$

where $\sigma_n(f, w_{\alpha,\beta}, \cdot)$ $(n \in \mathbb{N})$ are the Fejér summations.

Let $q_k(x) := c_k(f) p_k(x)$ $\big(x \in [-1,1],\ k = 0, 1, \ldots, n-1,\ n \in \mathbb{N}\big)$. Then

$$q_k = \sum_{l=0}^{k} q_l - \sum_{l=0}^{k-1} q_l = \left(\sum_{j=0}^{k} \sum_{l=0}^{j} q_l - \sum_{j=0}^{k-1} \sum_{l=0}^{j} q_l \right)$$

$$- \left(\sum_{j=0}^{k-1} \sum_{l=0}^{j} q_l - \sum_{j=0}^{k-2} \sum_{l=0}^{j} q_l \right) = (k+1)\sigma_{k+1} f - 2k\sigma_k f + (k-1)\sigma_{k-1} f$$

$$\big(\sigma_k f := \sigma_k(f, w_{\alpha,\beta}, \cdot),\ \sigma_0 f := \sigma_{-1} f := 0\big),$$

therefore we have

$$S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) = \sum_{k=0}^{n-1} \theta_{k,n} q_k = \sum_{k=1}^{n} \theta_{k-1,n} k \sigma_k f - 2 \sum_{k=1}^{n-1} \theta_{k,n} k \sigma_k f$$

$$+ \sum_{k=1}^{n-2} \theta_{k+1,n} k \sigma_k f = \sum_{k=1}^{n-1} \big((\theta_{k+1,n} - 2\theta_{k,n} + \theta_{k-1,n}) k \sigma_k f \big) + n\theta_{n-1,n} \sigma_n f$$

$$= \sum_{k=1}^{n-1} \triangle^2 \theta_{k-1,n} k \sigma_k f + n \theta_{n-1,n} \sigma_n f.$$

Using (4.2) we get

$$\left\| S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) w_{\gamma,\delta} \right\| \leqq K \left\| f w_{\gamma,\delta} \right\| \left(\sum_{k=1}^{n-1} \left| \triangle^2 \theta_{k-1,n} \right| k + n |\theta_{n-1,n}| \right).$$

From $(T2)$ and $(T3)$ it follows that

$$\sum_{k=1}^{n-1} \left| \triangle^2 \theta_{k-1,n} \right| k + n|\theta_{n-1,n}| \leqq c_1 + c_2 \quad (c_1, c_2 \geqq 0)$$

which proves the statement in the case (a).

Since

$$\sum_{k=1}^{n-1} \triangle^2 \theta_{k-1,n} k = -\sum_{k=0}^{n-2} \triangle \theta_{k,n} + (n-1)\triangle \theta_{n-1,n} = \theta_{0,n} - n\theta_{n-1,n}, \tag{4.3}$$

thus when (b) holds we have

$$\sum_{k=1}^{n-1} \left| \triangle^2 \theta_{k-1,n} \right| k + n|\theta_{n-1,n}| \leqq |n\theta_{n-1,n} - \theta_{0,n}| + n|\theta_{n-1,n}|$$

$$\leqq 2c_1 + |\theta_{0,n}| \leqq c_2 \quad (c_1, c_2 \geqq 0).$$

If (c) holds then

$$\sum_{k=1}^{n-1} \left| \triangle^2 \theta_{k-1,n} \right| k + n|\theta_{n-1,n}| \leqq |\theta_{0,n}| \leqq c \quad (c \geqq 0),$$

thus our statement is proved. □

4.3. Proof of Theorem 3.3. By the continuity of φ we have $(T1)$.

If φ is convex on $[0,1]$ then for every $0 < x_1 < x_2 < 1$ and $y_1, y_2 \in (x_1, x_2)$ we have

$$\frac{\varphi(y_1) - \varphi(x_1)}{y_1 - x_1} \leqq \frac{\varphi(y_2) - \varphi(x_2)}{y_2 - x_2}.$$

Therefore for every $k = 1, 2, \ldots, n-1$ $(n \in \mathbb{N})$ we get

$$\triangle^2 \theta_{k-1,n} = \frac{1}{n} \left(\frac{\theta_{k+1,n} - \theta_{k,n}}{1/n} - \frac{\theta_{k,n} - \theta_{k-1,n}}{1/n} \right)$$

$$= \frac{1}{n} \left(\frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{(k+1)/n - k/n} - \frac{\varphi\left(\frac{k}{n}\right) - \varphi\left(\frac{k-1}{n}\right)}{k/n - (k-1)/n} \right) \geqq 0.$$

Similarly, if φ is concave on $[0,1]$ then

$$\triangle^2\theta_{k-1,n} \leqq 0 \quad (k=1,2,\dots,n-1,\ n\in\mathbb{N}),$$

which means that $(T4)$ holds in both case.

Now if φ is nonnegative and convex we get

$$\operatorname{sgn}\triangle^2\theta_{k-1,n} = 1 = \operatorname{sgn}\varphi\left(\frac{n-1}{n}\right) = \operatorname{sgn}\theta_{n-1,n}$$
$$(k=1,2,\dots,n-1,\ n\in\mathbb{N}),$$

thus we have $(T5)$ which proves the statement in the case (a).

From (3.4) it follows that

$$\theta_{n-1,n} = \varphi\left(\frac{n-1}{n}\right) = O\left(\frac{1}{n}\right),$$

thus we have $(T2)$ which proves our statement in the case (b). □

4.4. Proof of Corollary 3.4. See Theorem 3.3 and Section 2.3, Examples. □

4.5. Proof of Theorem 3.5. Let $S_n^{\varphi_s}f := S_n^{\varphi_s}(f,w_{\alpha,\beta},\cdot)$. First we show that

$$p(x) = (S_n^{\varphi_s}p)(x) \quad \big(p\in\mathcal{P}_{q_n},\ x\in(-1,1),\ n\in\mathbb{N}\big). \tag{4.4}$$

Consider an arbitrary polynomial $p := \sum_{j=0}^{q_n} a_j p_j$. Then

$$c_k(p) = \int_{-1}^{1}\Big(\sum_{j=0}^{q_n} a_j p_j(x)\Big)p_k(x)w_{\alpha,\beta}(x)\,dx$$

$$= \sum_{j=0}^{q_n} a_j \int_{-1}^{1} p_j(x)p_k(x)w_{\alpha,\beta}(x)\,dx = \begin{cases} a_k, & \text{if } 0\leqq k\leqq q_n\\ 0, & \text{if } q_n<k\leqq n-1.\end{cases}$$

From the condition $\varphi_s(t)=1$ $\big(t\in[0,s]\big)$ it follows that $\varphi_s\big(\frac{k}{n}\big)=1$ $(k=0,1,\dots,q_n)$ therefore

$$(S_n^{\varphi_s}p)(x) = \sum_{j=0}^{q_n} a_j p_j(x) = p(x)$$

which proves (4.4).

Let Q be the best weighted approximating polynomial of f of order at most q_n, that is $E_{q_n}(f, w_{\gamma,\delta}) = \left\| (f - Q) w_{\gamma,\delta} \right\|$. Then by (4.4) we have

$$\left| f(x) - (S_n^{\varphi_s} f)(x) \right| w_{\gamma,\delta}(x) \leqq \left| f(x) - Q(x) \right| w_{\gamma,\delta}(x)$$

$$+ \left| Q(x) - (S_n^{\varphi_s} Q)(x) \right| w_{\gamma,\delta}(x) + \left| (S_n^{\varphi_s} Q)(x) - (S_n^{\varphi_s} f)(x) \right| w_{\gamma,\delta}(x)$$

$$\leqq E_{q_n}(f, w_{\gamma,\delta}) + 0 + \left| \int_{-1}^{1} \big((Q - f) w_{\gamma,\delta} \big)(y) \right.$$

$$\left. \times \left(\sum_{k=0}^{n-1} \varphi_s \left(\frac{k}{n} \right) p_k(y) p_k(x) \right) \frac{w_{\alpha,\beta}(y)}{w_{\gamma,\delta}(y)} \, dy \right| w_{\gamma,\delta}(x)$$

$$\leqq \big(1 + \left\| \mathcal{S}_n^{\varphi_s} \right\| \big) E_{q_n}(f, w_{\gamma,\delta}).$$

By Theorem 3.3 (b) we have (3.3), thus by Theorem 3.1 (B) we have

$$\sup_{n \in \mathbb{N}} \left\| \mathcal{S}_n^{\varphi_s} \right\| < +\infty. \qquad \square$$

4.6. Proof of Theorem 3.6. Using a standard argument similar to the one used by Stechkin [9] it may be proved that the order of convergence of the Fejér sums of the Fourier–Jacobi series is at least of Stechkin-type. Namely, if $\alpha, \beta \geqq -1/2$ and $\gamma, \delta \geqq 0$ satisfy the requirements (3.2), then we have

$$\left\| \big(f - \sigma_n(f, w_{\alpha,\beta}, \cdot) \big) w_{\gamma,\delta} \right\| \leqq \frac{C}{n} \sum_{k=0}^{n-1} E_k(f, w_{\gamma,\delta}) \tag{4.5}$$

for all $f \in C_{w_{\gamma,\delta}}$ and $n \in \mathbb{N}$ with some constant $C > 0$ independent of f and n. For the proof of this statement we refer to [11, (3.1)].

We showed in the proof of Lemma 4.1 that

$$S_n^{\Theta} f := S_n^{\Theta}(f, w_{\alpha,\beta}, \cdot) = \sum_{k=1}^{n-1} \triangle^2 \theta_{k-1,n} k \sigma_k f + n \theta_{n-1,n} \sigma_n f$$

where $\sigma_n f$ are the Fejér sums ($n \in \mathbb{N}$).

Using the identity (see (4.3) and $\theta_{0,n} = 1 + O\big(\frac{1}{n}\big)$)

$$\sum_{k=1}^{n-1} \triangle^2 \theta_{k-1,n} k + n\theta_{n-1,n} = 1 + O\left(\frac{1}{n}\right)$$

we get

$$f - S_n^\Theta f = \sum_{k=1}^{n-1} \triangle^2 \theta_{k-1,n} k (f - \sigma_k f) + n\theta_{n-1,n}(f - \sigma_n f) + fO\left(\frac{1}{n}\right).$$

From $(T2)$ and (4.5) it follows that

$$\big\| \big(n\theta_{n-1,n}(f - \sigma_n f)\big) w_{\gamma,\delta} \big\| \leqq \frac{c}{n} \sum_{k=0}^{n-1} E_k(f, w_{\gamma,\delta})$$

for all $f \in C_{w_{\gamma,\delta}}$ and $n \in \mathbb{N}$ with some constant $c > 0$ independent of f and n.

If $(T3)$ holds then using (4.5) we have

$$A := \left\| \left(\sum_{k=1}^{n-1} \triangle^2 \theta_{k-1,n} k (f - \sigma_k f) \right) w_{\gamma,\delta} \right\|$$

$$\leqq c \sum_{k=1}^{n-1} \big| \triangle^2 \theta_{k-1,n} \big| \left(\sum_{j=0}^{k-1} E_j(f, w_{\gamma,\delta}) \right) \leqq \frac{c_1}{n^2} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} E_j(f, w_{\gamma,\delta})$$

$$= \frac{c_1}{n} \sum_{k=1}^{n-1} \left(1 - \frac{k}{n} \right) E_{k-1}(f, w_{\gamma,\delta}) \leqq \frac{c_1}{n} \sum_{k=0}^{n-1} E_k(f, w_{\gamma,\delta}).$$

Since there exist $c_1, c_2 > 0$ such that

$$\left\| \left(fO\left(\frac{1}{n}\right) \right) w_{\gamma,\delta} \right\| \leqq \frac{c_1}{n} \leqq \frac{c_2}{n} \sum_{k=0}^{n-1} E_k(f, w_{\gamma,\delta}),$$

thus combining the above relations we obtain the statement in the case (a).

Now if (b) holds and $\triangle^2\theta_{k-1,n} \geqq 0$ (say) for $k = 1, 2, \dots, n-1$ and $n \in \mathbb{N}$ then

$$A \leqq c\left\{\sum_{j=0}^{n-1} E_j(f, w_{\gamma,\delta})\right\}\sum_{k=1}^{n-1}\triangle^2\theta_{k-1,n}$$

$$= c(\theta_{0,n} - \theta_{1,n} - \theta_{n-1,n})\sum_{k=0}^{n-1} E_k(f, w_{\gamma,\delta}) \leqq \frac{c_1}{n}\sum_{k=0}^{n-1} E_k(f, w_{\gamma,\delta}),$$

thus our statement is proved in this case, too. □

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