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Discrete Nonlinear Planar Systems and Applications to Biological Population Models

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Discrete Nonlinear Planar Systems and Applications to Biological Population Models

by

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Dedicated to the memory of my father, Levon Lazaryan

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ABSTRACT

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We study planar systems of difference equations and their applications to biological models of species populations. Central to the analysis of this study is the idea of *folding* - the method of transforming systems of difference equations into higher order scalar difference equations. For example, a planar system is transformed into a core second order difference equation and a passive non-dynamic equation. Two classes of second order equations are studied in detail: quadratic fractional and exponential.

In the study of the quadratic fractional equation, we investigate the boundedness and persistence of solutions, the global stability of the positive fixed point and the occurrence of periodic solutions with non-negative parameters and initial values. These results are then applied to a class of linear/rational systems of difference equations that can be transformed into a quadratic fractional second order difference equation

via folding. These results apply to systems with negative parameters, instances not commonly considered in previous studies. Using the idea of folding, we also identify ranges of parameter values that provide sufficient conditions on existence of chaotic, as well as multiple stable orbits of different periods for the planar system.

We also study a second order exponential difference equation with time varying parameters. We obtain sufficient conditions for boundedness of solutions and global convergence to zero. For the special, autonomous case (with constant parameters), we show occurrence of multistable periodic and nonperiodic orbits. For the case where parameters are periodic, we show that the nature of the solutions differs significantly depending on whether the period of the parameters is even or odd.

The above results are applied to biological models of populations. We investigate a broad class of planar systems that arise in the study of so-called stage-structured (adult-juvenile) single species populations, with and without time-varying parameters. In some cases, these systems are of the rational sort (e.g. the Beverton-Holt type), while in other cases the systems involve the exponential (or Ricker) function. In biological contexts, these results include conditions that imply extinction or survival of the species in some balanced form, as well as possible occurrence of complex and chaotic behavior. Special rational and exponential cases of the model are considered where we explore the role of inter-stage competition, restocking strategies, as well as seasonal fluctuations in the vital rates.

CHAPTER I

Introduction and Preliminaries

Difference equations, in the form of recursions and finite differences have appeared in variety of contexts from early days of mathematics, one such instance being the Fibonacci numbers. The development of differential and integral calculus was made possible with the concept of limits of finite differences and sums. In numerical analysis, finite differences have been used for obtaining numerical solutions to differential equations. Difference equations often appear as discrete analogs of differential equations and have many applications in natural and social sciences.

In the last few decades, difference equations have gained increasing interest on their own and have been studied as an independent field. Current studies of difference equations concern not only topological properties and asymptotic behavior of the solutions, but also rigorous treatment of these equations as objects or constructs of their own merit (for example, see [83]). Advances in difference equations have led to the development of a variety methods and techniques concerning the analysis of scalar as well as higher dimensional systems of difference equations (see [33], [81]).

In this thesis, we study certain broad classes of planar systems with applications to biological models of species populations. Central to the analysis of these systems is the idea of *folding* - a procedure that relates the study of planar systems to corresponding second order equations. A standard technique for analyzing kth order difference (and differential) equations is to "unfold" the equation into k first order equations and study the resulting system. For certain systems, one may also apply the reverse process, by "folding" them into higher order scalar equations. While this method has not been widely applied to the study of difference equations, it has been used before, under different names, in applications. For example, folding linear systems in both continuous and discrete time is seen in control theory (see for example [8], [33], [55]). In this framework, the *controllability canonical form* is the folding: using standard algebraic methods, a completely controllable system is found to be equivalent to a linear equation whose order equals the rank of the controllability matrix. In addition, this method is used in [31] and [66] in a study of a variety of nonlinear differential systems displaying chaotic behavior. Here these systems are studied and classified by converting them to ordinary differential equations of order 3 that are defined by jerk functions. Among those are the well-known systems of Lorenz ([70]) and Rossler ([78]).

More recently, the ideas found in control theory and chaotic differential systems have been generalized in a systematic study of the folding of systems in [80], [87]. The work in [80] and [87] extends the method in a more general sense by developing an explicit algorithm that folds systems into equations. This is done by starting with a system and deriving a higher order equation through a sequence of *inversions* and *substitutions* together with *index shifts* for difference equations or *higher derivatives* for differential ones. The algorithmic approach allows one to apply the method to both difference and differential systems, whether they are autonomous or not. One possible advantage of the folding lies in the fact that in many instances, the folding can reduce the underlying system into a higher order equation that is more tractable or has been previously well explored. Hence the potential applicability of the method is manifold and can be employed in variety of mathematical, biological, physical systems, and other areas, such as probability, finance and economics.

We apply the method of folding to rational and exponential planar systems and study these systems through resulting scalar second order rational and exponential difference equations. The choice of these systems is twofold. First, rational and exponential systems and equations have gained great interest in the field of difference equations and have generated numerous studies (see [4], [5], [6], [9], [15], [17], [51], [79], [84] and references thereof). Second, these systems appear in applications to biological models of species populations (for example, see [3], [40], [45], [46], [75], [92]). Difference equations have been used in increasing frequency in biological models, since discrete systems may be more convenient in modeling biological phenomena, as they are computationally efficient (see, for example, [67], [89] and [91]). Systems of difference equations are used to model interactions of species, as seen in predator-prey, cooperative or competitive models, which are captured by systems of higher dimensions ([44], [88], [90]). Among many known discrete population models are Beverton-Holt ([12]), Pielou ([75]) and Ricker ([77]) equations. More recent examples of population models can be found in [20], [36] and [49].

The current work is organized as follows. In the rest of this chapter, we introduce the method of folding, together with preliminary concepts, definitions and results relevant to the study of difference equations. Since our study of planar systems relies, to a great extent, on an underlying second order difference equation, Chapters 2 and 3 investigate certain classes of rational and exponential second order equations. Besides their applicability to planar systems, the results of these chapters are self-contained and in the context of higher order scalar equations are of mathematical interest in their own right.

In Chapter 2, we study a second order quadratic fractional difference equation. We establish existence and boundedness of solutions, local and global stability of fixed points as well as occurrence of periodic orbits under relatively standard assumptions on parameter values. The results obtained in Chapter 2 are then applied to the study of a class of linear/rational planar systems considered in Chapter 4, many special cases of which can be found in a number of population models in biology. Folding also helps extend the results obtained in Chapter 2 to cases of the linear rational system where some of the parameters are allowed to be negative. Using the folding, we then identify a different set of parameter ranges and cases where the system exhibits chaotic behavior. The significance of these findings is twofold: Studies of planar systems are typically limited to assuming nonnegative parameter values, since in the presence of negative values for (some) parameters, issues such as existence of iterates become a nontrivial matter. Furthermore, prior studies of linear-fractional equations and systems have not been focused on demonstrating the occurrence of chaos or coexisting cycles.

In Chapter 3, a second order exponential equation in studied. Several sufficient conditions are obtained for boundedness and convergence of solutions to zero for the general nonautonomous equation. Occurrence of multistable periodic and chaotic solutions of the equations is explored for special cases with constant, as well as periodic parameters.

The results obtained in Chapters 2 and 3 are then applied to the study of biological models considered in Chapter 5. We begin with a general matrix model of stage-structured populations, where the members of a population are differentiated by age, between adult (reproducing) and juvenile (non-reproducing) members. Several results are derived that relate to the extinction of species both for autonomous and nonautonomous, as well as density dependent matrix models. Special cases of the model are then considered, to explore the role of intra-species competition, restocking strategies, as well as periodic or seasonal variations in vital rates.

1.1 Difference equations and maps

In the following sections, we introduce some preliminary concepts, definitions and results related to the study of difference equations. Unless otherwise indicated, these results are drawn from texts of [32], [33] and [81].

A k-th order difference equation on a metric space (X, d) is defined by

$$x_{n+1} = F(n, x_n, x_{n-1}, \cdots, x_{n-k+1}) \tag{1.1}$$

where $F : \mathbb{N} \times D \to X$ is a given function, $\mathbb{N}$ is the set of non-negative integers, X is a set and $D \subseteq X \times X \times \cdots \times X = X^k$. The solution of (1.1) obtained from initial point $(x_0, x_{-1}, \cdots, x_{-k+1})$ is a sequence $\{x_n\} \in X$ such that x_n satisfies (1.1) for all $n > 0$. An initial point $(x_0, x_{-1}, \cdots, x_{-k+1})$ generates a (forward) solution $\{x_n\}$ by iteration of the function

$$(n, x_n, x_{n-1}, \cdots x_{n-k+1}) \to F(n, x_n, x_{n-1}, \cdots x_{n-k+1}) : \mathbb{N} \times D \to X.$$

so long as each iterate x_n stays in D. When the function F does not depend on the index n, the difference equation in (1.1) is *autonomous*, i.e.

$$x_n = F(x_n, x_{n-1}, \cdots x_{n-k+1}) \tag{1.2}$$

Otherwise, it is *nonautonomous.* Solutions of (1.1) or (1.2) are also called *orbits* or *trajectories.*

1.2 First order autonomous difference equations

The equation

$$x_{n+1} = F(x_n) \tag{1.3}$$

is an example of a *first order difference equation*, where $F : D \to X$ is a map from a subset $D \subseteq X$ of a metric space X. The solutions of (1.3) from initial point $x_0 \in D$ are generated by

$$x_n = F^n(x_0) \text{ for } n > 0$$

where $F^n = F \circ F \circ \cdots \circ F$ is the composition of F with itself n times.

Definition 1.1. The set $S \subset D$ is called an *invariant set* if $F(S) \subseteq S$, i.e. for all initial values $x_0 \in S$, $x_n = F^n(x_0) \in S$ for all $n > 0$.

Definition 1.2. A point $\bar{x} \in D$ is an equilibrium point of (1.3) if it is a fixed point of F, i.e.

$$\bar{x} = F(\bar{x}) \tag{1.4}$$

In other words, $x_n = \bar{x}$ for all $n \geq 0$, or $\bar{x}$ is a constant solution of (1.3) .

Definition 1.3. A fixed point $\bar{x}$ of (1.3) is *stable*, if given $\epsilon > 0$, there exists a $\delta > 0$ such that for initial point $x_0 \in D$

$$d(F^n(x_0), \bar{x}) < \epsilon \text{ for all } n > 0 \text{ whenever } d(x_0, \bar{x}) < \delta$$

The fixed point $\bar{x}$ is *unstable* if it is not stable.

Intuitively, if a fixed point is stable, then the iterates obtained from initial points that are close enough to the fixed point will stay sufficiently close to it.

Definition 1.4. The fixed point $\bar{x}$ of (1.3) is *attracting* if there is a set $S \subseteq D$ such that for all initial points $x_0 \in S$

$$\lim_{n \to \infty} x_n = \bar{x}$$

If $S = D$, then $\bar{x}$ is globally attracting.

Definition 1.5. The difference equation in (1.3) has a *periodic solution of period* p, if there is a positive integer p such that

$$x_{n+p} = x_n \ \text{ for all } \ n \geq 0$$

A solution $\{x_n\} \in X$ of (1.3) is periodic with *prime* period p, if it is periodic with period p and p is the least integer for which $x_{n+p} = x_n$ for all $n \geq 0$. A point $s \in D$ is a p-periodic point of the map F if there is a positive integer p such that $F^p(s) = s$. The orbit of a p-periodic point s of F is the set $\{s, F(s), \cdots, F^{p-1}(s)\}$, also referred to as a p-cycle of F. A point $y \in D$ is eventually p-periodic, if there exists a positive integer k such that $F^k(y) = s$ and $F^{k+np}(y) = s$ for all $n \geq 0$.

The stability of a p-periodic solution, or a p-cycle, can then be determined via the composite map F^p. We say that the solutions of (1.3) converge to a p-cycle, if F^p has a fixed point that is attracting.

We next define several concepts, in order to characterize maps that are referred to as "chaotic." These are maps whose iterates behave in an unpredictable manner. Several definitions of chaos exist in literature. We discuss two of these definitions (see [28] and [65]). A more familiar definition of chaos in literature, in the sense of Li-Yorke ([65]), can be given as follows:

Definition 1.6. (Li-Yorke chaos) Let $F_n : (X, d) \to (X, d)$ be functions on a metric space and define $F_0^n = F_n \circ F_{n-1} \circ \cdots \circ F_0$, i.e. the composition of maps F_0 through F_n. The nonautonomous system (X, F_n) is chaotic if there is an uncountable set $S \subset X$ (the scrambled set) such that for every pair of points $x, y \in S$

$$\limsup_{n\to\infty} d(F_0^n(x), F_0^n(y)) > 0 \text{ and } \liminf_{n\to\infty} d(F_0^n(x), F_0^n(y)) = 0$$

Theorem 1.7. *(Li-Yorke)* Let I be an interval and let the map $F : I \to I$ be continuous. Assume that there is a point $a \in I$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$ satisfy

$$d \leq a < b < c \quad \text{or} \quad d \geq a > b > c$$

Then

1. for every integer $k \geq 0$, there is a periodic point in I having period k.

2. there is an uncountable set $S \subset I$, containing no periodic points, which satisfies the following conditions:

   (i) For every $x, y \in S$ with $x \neq y$

   $$\limsup_{n\to\infty} d(F_0^n(x), F_0^n(y)) > 0 \text{ and } \liminf_{n\to\infty} d(F_0^n(x), F_0^n(y)) = 0$$

   (ii) For every $x \in S$ and periodic point $p \in I$

   $$\limsup_{n\to\infty} d(F_0^n(x), F_0^n(p)) > 0$$

The above definition implies that if the interval map F has a periodic point with period 3, then the hypothesis of the theorem are satisfied and the map F is chaotic.[1]

[1] This has warranted the title of the paper "Period Three Implies Chaos" in [65].

Theorem 1.7, however, pertains to interval maps (or first order scalar equations) only, and generally does not apply to systems of two or higher dimensions or to scalar equations of higher order. Sufficient conditions for existence of chaotic orbits in more general sense are discussed in the next section. Alternative definition of chaos in the sense of Devaney, can be given as follows:

Definition 1.8. (Devaney chaos) The map F on a metric space (X, d) is said to be chaotic if

(i) F is *transitive*, i.e for any pair of nonempty sets U and V of X, there exists a positive integer k such that

$$F^k(U) \cap V \neq \emptyset$$

(ii) the set of periodic points P of F is dense in X.

(iii) F has *sensitive dependence on initial conditions*, i.e. there exists an $\epsilon > 0$ such that for any $x_0 \in X$ and any open set U with $x_0 \in U$, there exists a $y_0 \in U$ and a positive integer k such that

$$d(F^k(x_0), F^k(y_0)) > \epsilon$$

1.3 Systems of difference equations

In many instances and applications, the set X is assumed to be a subset of $\mathbb{R}^m$, in which case the equations in (1.1) and (1.2) represent systems of m nonautonomous or autonomous difference equations of k-th order. In this case, the mapping F in component form can be given as $F = [f_1, f_2, \cdots f_m]$. A commonly used metric is defined in the usual way by the Euclidean norm

$$||x|| = \left[\sum_{i=1}^{m} r_i^2 \right]^{1/2}$$

where $x = [r_1, r_2, \cdots, r_m]$ in component form and the metric can be defined in the usual way as $d(x, y) = ||x - y||$. If $m = 2$, the systems defined by (1.1) and (1.2) with the usual topology are called *planar* systems.

In subsequent chapters we will study first order planar systems of type

$$x_{n+1} = F(n, x_n) \text{ or } x_{n+1} = F(x_n)$$

where in component form

$$x = [r_1, r_2], \quad F = [f_1, f_2], \quad F : \mathbb{N} \times \mathbb{R}^m \to \mathbb{R}^m$$

If the map $F(x) = Ax$ is linear where $x \in D$ and A is an $m \times m$ matrix with real entries, then (1.1) and (1.2) are called *linear* systems, otherwise they are *nonlinear* systems.

Theorem 1.9. *(Linear Maps)* Let A be an $m \times m$ matrix with real entries. For the linear map $L(x) = Ax$, the origin is an asymptotically stable fixed point if the modulus of the largest eigenvalue of A, or its spectral radius $\rho(A)$, is less than one. The origin is unstable if $\rho(A) > 1$.

Now let $F : D \to \mathbb{R}^m$, where $D \subseteq \mathbb{R}^m$ and assume $F \in C^1(D, \mathbb{R}^m)$. The derivative $DF(x)$ of F, commonly referred to as the *Jacobian*, is an $m \times m$ matrix with entries defined by

$$[\mu_{i,j}] = \frac{\partial f_i}{\partial r_j}(x) \quad i, j = 1, 2, \cdots m$$

where $x = [r_1, r_2, \cdots, r_m]$ and $F = [f_1, f_2, \cdots, f_m]$.

Definition 1.10. A fixed point $\bar{x}$ of a map $F \in C^1(D, \mathbb{R}^m)$ is *hyperbolic*, if no eigenvalue of $DF(\bar{x})$ has modulus equal to one. Otherwise $\bar{x}$ is *nonhyperbolic*.

Theorem 1.11. *(Linearlized Stability)* Let $\bar{x}$ be a fixed point of a map $F \in C^1(B_\epsilon(\bar{x}), \mathbb{R}^m)$ for some $\epsilon > 0$. Assume that $\bar{x}$ is hyperbolic and $DF(\bar{x})$ is invertible. If $\rho(DF(\bar{x})) < 1$ (or respectively, $\rho(DF(\bar{x})) > 1$), then $\bar{x}$ is asympotically stable (or respectively, unstable).

Definition 1.12. $F \in C^1(D, \mathbb{R})^m$ where $D \subseteq \mathbb{R}^m$ and $\overline{B}_\epsilon(\bar{x}) \subset D$ be the closed ball, where $\bar{x}$ is a fixed point of F and $\epsilon > 0$. If for every $x \in \overline{B}_\epsilon(\bar{x})$, all the eigenvalues of the Jacobian $DF(s)$ have magnitude greater than 1, then $\bar{x}$ is an *expanding* fixed point. If in addition there is an $x_0 \in \overline{B}_\epsilon(\bar{x})$ such that

(i) $x_0 \neq \bar{x}$

(ii) there is a positive integer k such that $F^k(x_0) = \bar{x}$

(iii) $\det[DF^k(x_0)] \neq 0$

then the expanding fixed point $\bar{x}$ is a *snap-back repeller*.

The next result establishes the connection between snap-back repellers and occurrence of chaotic behavior (see [72], [71]).

Theorem 1.13. *(Marotto)* Let $F \in C^1(D, \mathbb{R}^m)$ where $D \subseteq \mathbb{R}^m$. If F possesses a snap-back repeller, then the equation defined by

$$x_{n+1} = F(x_n)$$

is chaotic, i.e. there exists

1. a positive integer N such that F has a point of period p for every positive integer $p \geq N$.

2. a"scrambled set" of F, i.e. an uncountable set S containing no periodic points of F such that

   (i) $F(S) \subset S$ and there are no periodic points of F in S.

   (ii) for every $x, y \in S$, with $x \neq y$

   $$\limsup_{n\to\infty} d(F^n(x) - F^n(y)) > 0$$

   (iii) for every $x \in S$ and each periodic point y of F

   $$\limsup_{n\to\infty} d(F^n(x) - F^n(y)) > 0$$

3. an uncountable subset S_0 of S such that for every $x, y \in S_0$

   $$\liminf_{n\to\infty} d(F^n(x) - F^n(y)) = 0$$

Notice that unlike Theorem 1.7, Marotto's result is more general, as it applies to both scalar equations of any order, as well as to systems of higher dimensions.

1.4 Higher order scalar difference equations

A *scalar* difference equation of order k is defined as

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k+1}) \tag{1.5}$$

where $f : I^k \to I$ is a continuous function and $I \subset \mathbb{R}$ is an interval of the real line. Given the set of k initial values $x_0, x_{-1}, \cdots, x_{-k+1} \in I$, one may recursively generate the solution $\{x_n\}$, $n \geq 1$ of (1.5).

A standard technique for analyzing kth order scalar difference equations is to "unfold" the equation into k first order equations and study the resulting system. The equation in (1.5) may be converted to a system as follows:

Let $y_{1,n} = x_{n-k+1}$. $y_{2,n} = x_{n-k+2}, \cdots, y_{k,n} = x_n$. Then (1.5) can be written as

$$y_{n+1} = F(y_n) \tag{1.6}$$

where

$$y_n = [y_{1,n}, y_{2,n}, \cdots, y_{k,n}]^T$$

and

$$F(y_n) = [y_{2,n}, y_{3,n}, \cdots, y_{k,n}, f(y_{k,n}, y_{k-1,n}, \cdots, y_{1,n})]^T$$

We may also write

$$F = [F_1, F_2, \cdots F_k]^T \quad \text{where} \quad F_1(y_1) = y_2, F_2(y_2) = y_3, \cdots, F_k(y_k) = f(y_k, \cdots, y_1).$$

Then the properties, definitions and concepts pertaining to the solutions of (1.5) can be stated in terms of (1.6), as defined in Sections 1.2 and 1.3. Hence, the results for the higher order scalar equations can always be extended to an associated higher dimensional system. However, since systems of difference equations may not always be convertible to scalar difference equations, results obtained for systems may not always apply to scalar equations. In the next section we outline a general procedure for certain types of systems that may be converted into higher order scalar equations. This procedure also allows to extend the results obtained for higher order equations to much broader classes of systems besides the ones obtained by the standard unfolding discussed above.

1.5 Folding of planar systems into equations

Consider a second order difference equation

$$s_{n+2} = \phi(n, s_{n+1}, s_n) \tag{1.7}$$

where $\phi : \mathbb{N}_0 \times D' \to S$ is a function and $D' \subset S \times S$. As outlined in the previous section, a standard way of "unfolding" the second order equation in (1.7) to a system in (1.13) can be done as

$$\begin{cases} s_{n+1} = t_n \\ t_{n+1} = \phi(n, s_n, t_n) \end{cases} \tag{1.8}$$

Here the second order term (the temporal delay) in (1.7) is converted to a new variable in the state space. All solutions of (1.7) are reproduced from the solutions in (1.8) by $(s_n, s_{n+1}) = (s_n, t_n)$. However, (1.7) may be unfolded in different ways into systems of two equations, so (1.8) is not unique.

One may also apply the reverse process to systems, by "folding" them into higher order scalar equations. The method, in general form, is described in [80], [87] as an algorithm that folds systems into equations. This is done by starting with a system and deriving a higher order equation through a sequence of *inversions, substitutions* and *index shifts*

We demonstrate the idea of the folding on an example of a planar system as follows.

Example 1.14. Consider the following planar system:

$$x_{n+1} = ax_n + by_n \tag{1.9a}$$

$$y_{n+1} = \frac{x_n}{1 + cy_n + dx_n} \tag{1.9b}$$

Assuming that $b \neq 0$, from (1.9a) we can obtain an explicit expression for y_n given by

$$y_n = \frac{1}{b}(x_{n+1} - ax_n) \tag{1.10}$$

Next, we substitute (1.10) into (1.9b) to obtain

$$y_{n+1} = \frac{x_n}{1 + dx_n + \frac{c}{b}(x_{n+1} - ax_n)} \tag{1.11}$$

Finally, shifting the index of (1.9a) we get

$$x_{n+2} = ax_{n+1} + by_{n+1} = ax_{n+1} + \frac{bx_n}{1 + dx_n + \frac{c}{b}(x_{n+1} - ax_n)}$$

which can be further simplified to

$$x_{n+2} = ax_{n+1} + \frac{b^2 x_n}{cx_{n+1} + (db - ac)x_n + b} \tag{1.12}$$

The equation in (1.12) is a special case of a quadratic fractional second order difference equation which will be studied in the next chapter.

The idea shown in the above example can be formalized as follows. Consider a general, nonautonomous planar system given by

$$\begin{cases} x_{n+1} = f(n, x_n, y_n) \\ y_{n+1} = g(n, x_n, y_n) \end{cases} \tag{1.13}$$

where $n = 0, 1, 2, ...,$ $f, g : \mathbb{N}_0 \times D \to S$ are given functions, $\mathbb{N}_0$ is the set of non-negative integers, S is a non-empty set and $D \subset S \times S$.

Definition 1.15. Let S be a nonempty set and consider a function $f : \mathbb{N}_0 \times D \to S$ where $D \subset S \times S$. Then f is *semi-invertible* (or partially invertible) if there are sets

$M \subset D, M' \subset S \times S$ and a function $h : \mathbb{N}_0 \times M' \to S$ such that for all $(u, v) \in M$ if $w = f(n, u, v)$, then $(u, v) \in M'$ and $v = h(n, u, w)$ for all $n \in \mathbb{N}_0$.

Semi-inversion refers to the *solvability* of the equation $w - f(n, u, v) = 0$ for v via the implicit function theorem. (see [80]). On the other hand, the function f is semi-invertible, if it is separable, which we define as follows:

Definition 1.16. Let $(G, *)$ be a nontrivial group and let $f : \mathbb{N}_0 \times G \times G \to G$. If there are functions $f_1, f_2 : \mathbb{N}_0 \times G \to G$ such that

$$f(n, u, v) = f_1(n, u) * f_2(n, v)$$

for all $u, v \in G$ and $n \geq 1$, then f is said to be *separable* on G and is given by $f = f_1 * f_2$.

For example, an affine function $f(n, u, v) = a_n u + b_n v + c_n$ where a_n, b_n, c_n are real parameters, is separable on $\mathbb{R}$ with addition for all n with $f_1(n, v) = a_n u$, $f_2(n, v) = b_n v + c_n$. Similarly, $f(n, u, v) = a_n \frac{u}{v}$ is separable on $\mathbb{R} \setminus \{0\}$ relative to multiplication.

Now, suppose that $f_2(n, .)$ is a bijection for every n and $f_2^{-1}(n, .)$ is its inverse, i.e. $f_2(n, f_2^{-1}(nv)) = v$ and $f_2^{-1}(n, f_2(n, v)) = v$ for all v. Then a separable function f is semi-invertible if $f_2(n, .)$ is a bijection for each fixed n, since for every $u, v, w \in G$

$$w = f_1(n, u) * f_2(n, v) \Rightarrow v = f_2^{-1}(n, [f_1(n, v)]^{-1} * w)$$

where the map inversion and group inversion (denoted by -1) are distinguished from the context. In this case, one may obtain an explicit expression for the semi-inversion h by

$$h(n, u, w) = f_2^{-1}(n, [f_1(n, u)]^{-1} * w) \tag{1.14}$$

with $M = M' = G \times G$. This observation is summarized as follows:

Theorem 1.17. Let $(G, *)$ be a nontrivial group and $f = f_1 * f_2$ be separable. If $f_2(n, .)$ is a bijection for each n, then f is semi-invertible on $G \times G$ with a semi-inversion uniquely defined by (1.14).

Now, suppose that $\{(x_n, y_n\}$ is an orbit of (1.13) in D. If one of the functions in (1.13), say f, is semi-invertible, then by Definition 1.15 there is a set $M \subset D$, a set $M' \subset S \times S$ and a function $h : \mathbb{N}_0 \times M' \to S$ such that if $(x_n, y_n) \in M$, then $(x_n, x_{n+1}) = (f_n, f(n, x_n, y_n)) \in M'$ and $y_n = h(n, x_n, x_{n+1})$. Therefore

$$\begin{aligned} x_{n+2} = f(n+1, x_{n+1}, y_{n+1}) &= f(n+1, x_{n+1}, g(n, x_n, y_n)) \\ &= f(n+1, x_{n+1}, g(n, x_n, h(n, x_n, x_{n+1}))) \end{aligned} \tag{1.15}$$

and the function

$$\phi(n, u, w) = f(n+1, w, g(n, u, h(n, u, w))) \tag{1.16}$$

is defined on $\mathbb{N}_0 \times M'$. If $\{s_n\}$ is a solution to (1.7) with initial conditions $s_0 = x_0, s_1 = x_1 = f(0, x_0, y_0)$, and ϕ is defined by (1.16) then

$$\begin{aligned} s_2 &= f(1, s_1, g(0, s_0, h(0, s_0, s_1))) \\ &= f(1, x_1, g(0, x_0, h(0, x_0, x_1))) = f(1, x_1, g(0, x_0, y_0)) = x_2 \end{aligned}$$

Continuing this way inductively, we obtain $s_n = x_n$ and thus

$$h(n, s_n, s_{n+1}) = h(n, x_n, x_{n+1}) = y_n$$

and therefore

$$(x_n, y_n) = (s_n, h(n, s_n, s_{n+1})) \tag{1.17}$$

which implies that the solution $\{(x_n, y_n)\}$ may be obtained from a solution $\{s_n\}$ via (1.17). The above can be summarized in the following theorem.

Theorem 1.18. Suppose that f in (1.13) is semi-invertible with M, M' and h given by Definition 1.15. Then each orbit of (1.13) in M may be derived from a solution of (1.7) via (1.17) with ϕ given by (1.16).

Thus, we define the folding as follows:

Definition 1.19. (Folding) The pair of equations

$$s_{n+2} = \phi(n, s_n, s_{n+1}) \quad \text{(core)} \tag{1.18}$$

$$y_n = h(n, x_n, x_{n+1}) \quad \text{(passive)} \tag{1.19}$$

where ϕ is defined by (1.16) is a folding of the system in (1.13). The initial values of the core equation are determined from the initial point (x_0, y_0) as $s_0 = x_0, s_1 = f(0, x_0, y_0)$.

The equation in (1.19) is called *passive* since it simply evaluates the function h on a solution of the core equation (1.18) without any iterations involved, i.e. it is nondynamic. On the other hand, (1.13) can be thought of as a nonstandard the unfolding of the second order equation (1.18) that is generally not equivalent to the standard unfolding (1.8).

In many instances, the folding can reduce the underlying system into a higher order equation that is more tractable or has been previously well explored. However, the folding method does not always guarantee that the resulting equation will have the aforementioned properties. Therefore, for practical reasons, it is important to identify systems that do fold into known and tractable equations, which is done by what [80] and [87] describe as the *inverse problem*. The idea behind this for planar systems is as follows: We start with one of the two equations of the system, say the one given by f and a known function ϕ that defines a second-order equation with desired properties. Then a function g is determined so that the system with components f and g folds into a second order equation defined by ϕ. More formally, the inverse problem can be described as follows:

Suppose that a function f satisfies Definition 1.15. Then by (1.16)

$$f(n+1, w, g(n, u, h(n, u, w))) = \phi(n, u, w)$$

is a function of n, u, w. Since f is semi-invertible, then by Definition 1.15 we obtain

$$g(n, u, h(n, u, w)) = h(n+1, w, \phi(n, u, w)) \tag{1.20}$$

Now, suppose that $\phi(n, u, w)$ is prescribed on a set $\mathbb{N}_0 \times M'$ where $M' \subset S \times S$ and we need to find g that satisfies (1.20). Assume that a subset $M \subset D$ exists with the property that $f(\mathbb{N}_0 \times M) \times \phi(\mathbb{N}_0 \times M') \subset M'$. For $(n, u, v) \in \mathbb{N}_0 \times M$ define

$$g(n, u, v) = h(n+1, f(n, u, v), \phi(n, u, f(n, u, v))) \tag{1.21}$$

In particular, if $v \in h(\mathbb{N}_0 \times M')$, then g above satisfies (1.21). Then

Theorem 1.20. Let f be a semi-invertible function with h given by Definition 1.15. Further, let ϕ be a given function on $\mathbb{N}_0 \times M'$. If g is given by (1.21) then (1.13) folds to the difference equation

$$s_{n+2} = \phi(n, s_n, s_{n+1})$$

together with a passive equation.

We demonstrate the usefulness of the method on the following example:

Example 1.21. Consider the second-order rational difference equation

$$x_{n+2} = \frac{\alpha x_{n+1}}{Bx_n + C}$$

Under the change of variable $x_n = \frac{C}{B} z_n$, the above equation can be written as

$$z_{n+2} = \frac{p z_{n+1}}{1 + z_n} \tag{1.22}$$

where $p = \frac{\alpha}{B}$. The equation in (1.22) is known as Pielou's difference equation (see [75], [76]), which is a discrete analogue of the delay logistic equation used as a prototype of modelling single-species dynamics. The study of dynamical properties of (1.22) can be found in [51] and [54].

The system

$$x_{n+1} = 2y_n + 1 \tag{1.23a}$$

$$y_{n+1} = \frac{-0.25x_n + 0.8y_n + 0.2}{0.5x_n + 0.4} \tag{1.23b}$$

has the folding

$$x_{n+2} = \frac{0.8x_{n+1}}{0.5x_n + 0.4} \tag{1.24}$$

which by the change of variables described above can be converted to (1.22).

On the other hand, the system

$$x_{n+1} = 2y_n + 1 \tag{1.25a}$$

$$y_{n+1} = \frac{-0.26x_n + 0.8y_n + 0.2}{0.5x_n + 0.4} \tag{1.25b}$$

has the folding

$$x_{n+2} = \frac{0.8x_{n+1} - 0.02x_n}{0.5x_n + 0.4} \tag{1.26}$$

is of different functional form than that of (1.24), even though the systems in (1.23) and (1.25) are nearly identical. Moreover, the solutions of (1.24) from positive initial values will always be defined, whereas it may not be the case for those of (1.26). These distinctions would not be obvious by simply looking at the systems alone.

1.6 Second order difference equations

In previous sections, we showed the connection between systems of difference equations and higher order scalar equations, both via folding and unfolding. The method of folding allows one to study planar systems by means of the core second order difference equation obtained from the folding, which, in some instances, may be more tractable to rigorous analysis, especially in light of the fact that a number of results on local and global behavior of the solutions of second order difference equations are known in the literature. In the final section of this chapter, we state several of these results (see [51] and references thereof).

Consider a second order autonomous difference equation

$$x_{n+1} = f(x_n, x_{n-1}) \;\; n = 0, 1, \cdots \tag{1.27}$$

where I is an interval of the real line and $f : I \times I \to I$ is a map. Define

$$p = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) \;\text{ and }\; q = \frac{\partial f}{\partial u}(\bar{x}, \bar{x})$$

as partial derivatives of $f(u, v)$ evaluated at the fixed point $\bar{x}$. Then the equation

$$x_{n+1} = px_n + qx_{n-1} \tag{1.28}$$

is called the *linearization equation* associated with (1.27) around the fixed point $\bar{x}$ and the quadratic equation

$$\lambda^2 - p\lambda - q = 0 \tag{1.29}$$

is the *characteristic equation* associated with the linearization of (1.27) around the fixed point.

Theorem 1.22. *(Linearized Stability)* Let (1.28) be the linearization of (1.27) around a fixed point $\bar{x}$.

(i) If both roots of the characteristic equation (1.29) lie in an open disk $|\lambda| < 1$, then the fixed point of (1.27) is *locally asymptotically stable*.

(ii) If at least one of the roots of (1.29) has modulus greater than one, then the fixed point $\bar{x}$ of (1.27) is *unstable*.

(iii) If one of the roots of (1.29) has modulus greater than one and the other root has modulus smaller than one, then the fixed point $\bar{x}$ is a *saddle*.

(iv) If both roots of (1.29) have moduli greater than one, then the fixed point $\bar{x}$ is a *repeller*.

The next results pertain to global attractivity of the fixed point.

Theorem 1.23. *(Stability Trichotomy)* Assume

$$f \in C^1[[0,\infty) \times [0,\infty), [0,\infty)]$$

is such that

$$u\left|\frac{\partial f}{\partial u}\right| + v\left|\frac{\partial f}{\partial v}\right| < f(u,v) \text{ for all } u, v \in (0,\infty)$$

Then the difference equation in (1.27) has stability trichotomy, that is exactly one of the following three cases holds for all solutions of (1.27):

(i) $\lim_{n\to\infty} x_n = \infty$ for all $(x_{-1}, x_0) \neq (0,0)$.

(ii) $\lim_{n\to\infty} x_n = 0$ for all initial points and 0 is the only equilibrium of (1.27).

(iii) $\lim_{n\to\infty} x_n = \bar{x}$ for all $(x_{-1}, x_0) \neq (0,0)$ and $\bar{x}$ is the only positive equilibrium of (1.27).

The final set of results are known in literature as M & m theorems (see [51] and [52]), and rely on the assumption that the function $f(u,v)$ defining the second order difference equation is monotone in its arguments.

Theorem 1.24. Let $[a,b]$ be an interval of real numbers and assume that

$$f : [a,b] \times [a,b] \to [a,b]$$

is a continuous function with the following properties:

(i) $f(x,y)$ is non-decreasing in $x \in [a,b]$ for each $y \in [a,b]$, and $f(x,y)$ is non-increasing in $y \in [a,b]$ for each $x \in [a,b]$;

(ii) If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system

$$f(m,M) = m \quad f(M,m) = M$$

then $m = M$.

Then (1.27) has a unique fixed point $\bar{x} \in [a,b]$ and every solution of (1.27) converges to $\bar{x}$.

Theorem 1.25. Let $[a,b]$ be an interval of real numbers and assume that

$$f : [a,b] \times [a,b] \to [a,b]$$

is a continuous function with the following properties:

(i) $f(x, y)$ is non-increasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is non-decreasing in $y \in [a, b]$ for each $x \in [a, b]$;

(ii) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(M, m) = m \quad f(m, M) = M$$

then $m = M$, i.e. the difference equation (1.27) has no solution of prime period two in $[a, b]$.

Then (1.27) has a unique fixed point $\bar{x} \in [a, b]$ and every solution of (1.27) converges to $\bar{x}$.

Theorem 1.26. Let $[a, b]$ be an interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function with the following properties:

(i) $f(x, y)$ is non-increasing in each of its arguments.

(ii) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, m) = M \quad f(M, M) = m$$

them $m = M$

Then (1.27) has a unique fixed point $\bar{x} \in [a, b]$ and every solution of (1.27) converges to $\bar{x}$.

CHAPTER II

Dynamics of a Second Order Rational Difference Equation

In this chapter, we study the dynamics of the second-order equation

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{2.1}$$

where

$$0 \leq a < 1, \quad \alpha, \beta, \gamma, A, B \geq 0, \quad \alpha + \beta + \gamma, A + B, C > 0 \tag{2.2}$$

We investigate the boundedness and persistence of solutions, the global stability of the positive fixed point and the occurrence of periodic solutions.[1]

Equation (2.1) is a quadratic-fractional equation since it can be written as

$$x_{n+1} = \frac{aAx_n^2 + aBx_nx_{n-1} + (aC + \alpha)x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{2.3}$$

and (2.3) is a special case of the equation

$$x_{n+1} = \frac{px_n^2 + qx_nx_{n-1} + \delta x_{n-1}^2 + c_1x_n + c_2x_{n-1} + c_3}{Ax_n + Bx_{n-1} + C} \tag{2.4}$$

[1] The content of this chapter, unless otherwise indicated, is from [58].

which includes rational equations that are the sum of linear equation and a linear/linear rational equation mentioned in [25]:

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C}$$

When $a = 0$, the equation in (2.1) reduces to linear/linear case that has been studied extensively together with its sub-cases in [51], [4], [5], [10], as well as [16], [38], [53] and references thereof. More recently, second order linear/linear rational equations have appeared in [7]- [10].

The study of rational equations with quadratic terms has been less systematic, although the equation in (2.4) has been studied in [25], [26] and in more general cases in [47] and [48].

In particular, in [25] it has been shown that depending on the values of parameters and initial conditions, the equation in (2.4) can exhibit a wide variety of dynamic behaviors, including coexisting periodic solutions and chaotic trajectories. In contrast, we show that when (2.2) holds, the trajectories of (2.1) are relatively well behaved: when the function

$$f(u, v) = au + \frac{\alpha u + \beta v + \gamma}{Au + Bv + C}$$

is monotone in its arguments, (2.1) cannot have periodic solution of period greater than two. Moreover, we show that if (2.1) has no prime or minimal period two solutions then the trajectories of (2.1) converge to the unique positive fixed point. We further demonstrate how these results can be applied to the study of linear-rational planar systems.

2.1 Existence and boundedness of solutions

When (2.2) holds we may assume that $C = 1$ in (2.1) without loss of generality by dividing the numerator and denominator of the fractional part by C and relabeling the parameters. Thus we consider

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + 1} \tag{2.5}$$

Note that the underlying function

$$f(u,v) = au + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}$$

is continuous on $\mathbb{R}^+ = [0,\infty)$. The next result gives sufficient conditions for the positive solutions of (2.5) to be uniformly bounded from above and below by positive bounds.

Theorem 2.1. *Let (2.2) hold and assume further that*

$$\alpha = 0 \text{ if } A = 0 \quad \text{and} \quad \beta = 0 \text{ if } B = 0. \tag{2.6}$$

Then the following are true:

(a) Every solution $\{x_n\}$ of (2.5) with non-negative intial values is uniformly bounded from above, i.e. there is a number $M > 0$ such that $x_n \le M$ for all n sufficiently large.

(b) If $\gamma > 0$ then there is $L \in (0,M)$ such that $L \le x_n \le M$ for all large n. Moreover, $[L,M]$ is an invariant interval for (2.5).

Proof. (a) Let

$$\rho_1 = \begin{cases} \alpha/A \text{ if } A > 0 \\ 0 \qquad \text{if } A = 0 \end{cases} \qquad \rho_2 = \begin{cases} \beta/B \text{ if } B > 0 \\ 0 \qquad \text{if } B = 0 \end{cases}$$

By (2.2), $\delta = \rho_1 + \rho_2 + \gamma > 0$ and for all $n \geq 0$

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + 1} \leq ax_n + \rho_1 + \rho_2 + \gamma = ax_n + \delta$$

Let N be an integer. Then

$$x_{N+1} \leq ax_N + \delta$$

$$x_{N+2} \leq ax_{N+1} + \delta \leq a^2 x_N + \delta(1 + a)$$

Proceeding this way inductively, we obtain for all $n > N$

$$x_n \leq a^{n-N-1} x_{N+1} + \delta(1 + a + \ldots + a^{n-N-2}) \leq \frac{\delta}{1-a} + a^{n-N-1}\left[x_0 - \frac{\delta}{1-a}\right]$$

As $n \to \infty$, the second term on the right hand side of the above equation approaches zero. In particular, for all n sufficiently large

$$a^{n-N-1}\left[x_0 - \frac{\delta}{1-a}\right] \leq \frac{a}{1-a}$$

Therefore, for all n sufficiently large

$$x_n \leq \frac{\delta}{1-a} + \frac{a}{1-a} = \frac{\delta + a}{1-a} := M$$

(b) Suppose that $\gamma > 0$. Then for all $n \geq 1$

$$x_n \geq \frac{\gamma}{(A+B)M + 1} := L$$

To verify that $L < M$ we observe that

$$M \geq (1-a)M = a + \delta \geq a + \gamma > a + L \geq L.$$

Finally, we establish that $f(u,v) \in [L,M]$ for all $u,v \in [L,M]$. If $u,v \in [L,M]$ then

$$f(u,v) \leq aM + \delta = \frac{a\delta + a^2}{1-a} + \delta = \frac{\delta + a^2}{1-a} \leq \frac{\delta + a}{1-a} = M$$

Further,

$$f(u,v) \geq \frac{\gamma}{(A+B)M+1} = L \text{ for all } 0 \leq u,v \leq M$$

and the proof is complete. $\square$

We emphasize that conditions (2.6) allow $A > 0$ with $\alpha = 0$ and $B > 0$ with $\beta = 0$. More instances of invariant intervals for the special case $a = 0$ can be found in [51].

Remark 2.2. If $a \geq 1$ then the solutions of (2.5) may not be uniformly bounded. In fact, all non-trivial solutions of (2.5) are unbounded since $x_{n+1} \geq ax_n$ for all n if $a > 1$. When $a = 1$ solutions may still be unbounded as is readily seen in the following, first-order special case:

$$x_{n+1} = x_n + \frac{\alpha x_n}{Ax_n + 1}$$

2.2 Existence and local stability of a unique positive fixed point

The fixed point of (2.5) must satisfy the following equation:

$$x = ax + \frac{\alpha x + \beta x + \gamma}{Ax + Bx + 1}$$

Combining and rearranging terms yields

$$(1-a)(A+B)x^2 - [\alpha + \beta - (1-a)]\, x - \gamma = 0$$

i.e. the fixed points must be the roots of the quadratic equation

$$S(t) = d_1 t^2 - d_2 t - d_3 \tag{2.7}$$

where

$$d_1 = (1-a)(A+B), \quad d_2 = \alpha + \beta - (1-a), \quad d_3 = \gamma$$

If (2.2) holds then $d_1 > 0$ and $d_3 \geq 0$. There are two more cases to consider.

Case 1: If $d_2 = 0$ then (2.7) has two roots given by

$$t_{\pm} = \pm\sqrt{\frac{d_3}{d_1}}$$

Thus if $\gamma > 0$ then the unique positive fixed point of (2.5) is

$$\bar{x} = \sqrt{\frac{\gamma}{(1-a)(A+B)}}$$

Case 2: When $d_2 \neq 0$ then the roots of (2.7) are given by

$$t_{\pm} = \frac{\alpha + \beta - (1-a) \pm \sqrt{[\alpha + \beta - (1-a)]^2 + 4(1-a)(A+B)\gamma}}{2(1-a)(A+B)}$$

In particular, if $\gamma > 0$ then the unique positive fixed point of (2.5) is

$$\bar{x} = \frac{\alpha + \beta - (1-a) + \sqrt{[\alpha + \beta - (1-a)]^2 + 4(1-a)(A+B)\gamma}}{2(1-a)(A+B)} \tag{2.8}$$

The above discussions imply the following.

Lemma 2.3. *If (2.2) holds and $\gamma > 0$ then (2.5) has a positive fixed point $\bar{x}$ that is uniquely given by (2.8).*

We now consider the local stability of $\bar{x}$ under the hypotheses of the above lemma. The characteristic equation associated with the linearization of (2.5) at the point $\bar{x}$

is given by

$$\lambda^2 - f_u(\bar{x},\bar{x})\lambda - f_v(\bar{x},\bar{x}) = 0 \tag{2.9}$$

where

$$f(u,v) = au + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}.$$

Now,

$$f_u = a + \frac{\alpha(Au + Bv + 1) - A(\alpha u + \beta v + \gamma)}{(Au + Bv + 1)^2} = a + \frac{(B\alpha - A\beta)v + \alpha - A\gamma}{(Au + Bv + 1)^2}$$

Similarly

$$f_v = \frac{\beta(Au + Bv + 1) - B(\alpha u + \beta v + \gamma)}{(Au + Bv + 1)^2} = \frac{(A\beta - B\alpha)u + \beta - B\gamma}{(Au + Bv + 1)^2}$$

Alternatively we can express f_u in terms of f as

$$f_u = a + \frac{\alpha - A(f(u,v) - au)}{Au + Bv + 1} \tag{2.10}$$

and likewise,

$$f_v = \frac{\beta - B(f(u,v) - au)}{Au + Bv + 1} \tag{2.11}$$

Define

$$\begin{aligned} f_u(\bar{x},\bar{x}) &= a + \frac{\alpha - (1-a)A\bar{x}}{(A+B)\bar{x} + 1} := p \\ f_v(\bar{x},\bar{x}) &= \frac{\beta - (1-a)B\bar{x}}{(A+B)\bar{x} + 1} := q \end{aligned}$$

and note that the fixed point $\bar{x}$ is locally asymptotically stable if both roots of (2.9), namely,

$$\lambda_1 = \frac{p - \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad \lambda_2 = \frac{p + \sqrt{p^2 + 4q}}{2}$$

are inside the unit disk of the complex plain. Both roots are complex if and only if $p^2+4q<0$ or $q<-(p/2)^2$. In this case, $|\lambda_1|=|\lambda_2|=-q$ so both roots have modulus less than 1 if and only if $q>-1$ or equivalently, $q+1>0$, i.e.,

$$\begin{aligned}\beta-(1-a)B\bar{x}+(A+B)\bar{x}+1&>0\\(A+aB)\bar{x}+\beta+1&>0\end{aligned}$$

This is clearly true if (2.2) holds. So if (2.2) holds and $\gamma>0$ and if $-1<q<-p^2/4$ then $\bar{x}$ is locally asymptotically stable with complex roots or eigenvalues.

Now suppose that $q\geq -p^2/4$ and the eigenvalues are real. A routine calculation shows that $\lambda_2<1$ if and only if $q<1-p$ or equivalently,

$$\begin{aligned}[(2a-1)A+aB]\bar{x}+\alpha-(A+B)\bar{x}-(1-a)+\beta-(1-a)B\bar{x}&<0\\2(1-a)(A+B)\bar{x}&>\alpha+\beta-(1-a)\end{aligned}\tag{2.12}$$

which is true if (2.2) holds and $\gamma>0$; see (2.8). Note that (2.12) is equivalent to

$$p+q<1.$$

Next, $\lambda_2>-1$ if and only if

$$p+\sqrt{p^2+4q}>-2\tag{2.13}$$

If $p>-2$ then (2.13) holds trivially. On the other hand, if $p\leq -2$ or $p+2\leq 0$ then

$$\begin{aligned}(2+a)[(A+B)\bar{x}+1]+\alpha-(1-a)A\bar{x}&\leq 0\\(1+2a)A\bar{x}+(2+a)(B\bar{x}+1)+\alpha&\leq 0\end{aligned}$$

which is not possible if (2.2) holds. It follows that $|\lambda_2| < 1$ if (2.2) holds and $\gamma > 0$.

Next, consider λ_1 and note that $\lambda_1 < 1$ if and only if $p - \sqrt{p^2 + 4q} < 2$. This is clearly true if $p < 2$ which is in fact the case. To see why, note that $p - 2 < 0$ if and only if

$$\alpha - (1-a)A\bar{x} - (2-a)[(A+B)\bar{x} + 1] < 0 \tag{2.14}$$

Since by (2.12)

$$\begin{aligned}(2-a)(A+B)\bar{x} &= 2(1-a)(A+B)\bar{x} + a(A+B)\bar{x} \\ &> \alpha + \beta - (1-a) + a(A+B)\bar{x}\end{aligned}$$

it follows that

$$\begin{aligned}\alpha - (1-a)A\bar{x} - (2-a)[(A+B)\bar{x} + 1] &= -(1-a)A\bar{x} - (2-a) + \alpha - (2-a)(A+B)\bar{x} \\ &< -(1-a)A\bar{x} - (2-a) - \beta + (1-a) - a(A+B)\bar{x} \\ &= -(1-a)A\bar{x} - 1 - \beta - a(A+B)\bar{x} \\ &< 0\end{aligned}$$

This proves that (2.14) is true and we conclude that $\lambda_1 < 1$ if (2.2) holds and $\gamma > 0$. Next, $\lambda_1 > -1$ if and only if

$$p - \sqrt{p^2 + 4q} > -2.$$

This requires that $p > -2$ which is true if (2.2) holds and $\gamma > 0$. Now the above inequality reduces to $p + 1 > q$ or

$$\begin{aligned}\beta - (1-a)B\bar{x} - a[(A+B)\bar{x} + 1] - \alpha + (1-a)A\bar{x} &< (A+B)\bar{x} + 1 \\ \beta - \alpha - (1+a) &< 2(aA+B)\bar{x} \qquad (2.15)\end{aligned}$$

We also note that if the reverse of the above inequality holds, i.e.,

$$2(Aa+B)\bar{x} < \beta - \alpha - (1+a). \tag{2.16}$$

then the above calculation show that $\lambda_1 < -1$ while $|\lambda_2| < 1$. Therefore in this case $\bar{x}$ is a saddle point. If $\beta - \alpha - (1+a) \leq 0$ then (2.16) does not hold and $\bar{x}$ is locally asymptotically stable.

The preceding calculations in particular prove the following.

Lemma 2.4. *Let (2.2) hold and $\gamma > 0$. Then the positive fixed point $\bar{x}$ of (2.5) is locally asymptotically stable if and only if (2.15) holds and a saddle point if and only if (2.16) holds.*

Since $\bar{x}$ is non-hyperbolic if neither (2.15) nor (2.16) holds, Lemma 2.4 gives a complete picture of the local stability of $\bar{x}$ under its stated hypotheses.

Remark 2.5. For $\bar{x}$ to be a saddle point it is necessary that $\beta - \alpha - (1+a) > 0$, i.e.

$$\beta > 1 + a + \alpha \geq 1. \tag{2.17}$$

To get a more detailed picture, we insert the value of $\bar{x}$ in (2.16) and obtain, after some routine calculations, the following equivalent version of (2.16)

$$(\mu^2-1)(\beta-1)^2 - 2(1+\theta\mu)(\alpha+a)(\beta-1) + (\theta^2-1)(\alpha+a) > 4(1-a)(A+B)\gamma \tag{2.18}$$

where, assuming that $a > 0$ or $B > 0$,

$$\mu = \frac{1-\rho}{aA+B}, \quad \theta = \frac{1+\rho}{aA+B}, \quad \rho = \frac{aA+B}{(1-a)(A+B)}.$$

In light of (2.17) the inequality in (2.18) is more likely to hold if $\mu > 1$, i.e. if

$$1 - \frac{aA + B}{(1-a)(A+B)} > aA + B$$

or equivalently,

$$aA + B < \frac{A + B}{A + B + 1/(1-a)}.$$

It is clear that this is not possible if a is sufficiently close to 1, indicating that increasing the value of a (other parameters being fixed) is likely to stabilize the fixed point $\bar{x}$.

2.3 Global stability and convergence of solutions

We next discuss global convergence results, one of which needs the following familiar result from [41].

Lemma 2.6. *Let I be an open interval of real numbers and suppose that $f \in C(I^m, \mathbb{R})$ is nondecreasing in each coordinate. Let $\bar{x} \in I$ be a fixed point of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-m+1}) \tag{2.19}$$

and assume that the function $h(t) = f(t, \ldots, t)$ satisfies the conditions

$$h(t) > t \text{ if } t < \bar{x} \quad \text{and} \quad h(t) < t \text{ if } t > \bar{x}, \quad t \in I. \tag{2.20}$$

Then I is an invariant interval of (2.19) and $\bar{x}$ attracts all solutions with initial values in I.

We now use the preceding result to obtain sufficient conditions for the global attractivity of the positive fixed point.

Theorem 2.7. *Assume that (2.2) holds with $\gamma > 0$ and suppose that $f(u,v)$ is non-decreasing in both arguments. Then (2.5) has a unique fixed point $\bar{x} > 0$ that is asymptotically stable and attracts all positive solutions of (2.5).*

Proof. The existence and uniqueness of $\bar{x} > 0$ follows from Lemma 2.3. Next, the function h in (2.20) takes the form

$$h(t) = at + \frac{(\alpha+\beta)t + \gamma}{(A+B)t+1}.$$

Note that the fixed point $\bar{x}$ of (2.5) is a solution of the equation $h(t) = t$ so we verify that conditions (2.20) hold. For $t > 0$ the function h may be written as

$$h(t) = \phi(t)t, \qquad \text{where } \phi(t) = a + \frac{\alpha+\beta+\gamma/t}{(A+B)t+1}.$$

Note that $\phi(\bar{x}) = h(\bar{x})/\bar{x} = 1$. Further,

$$\phi'(t) = \frac{-[(A+B)t+1]\gamma/t^2 - (A+B)[\alpha+\beta+\gamma/t]}{[(A+B)t+1]^2}$$

so ϕ is decreasing (strictly) for all $t > 0$. Therefore,

$$t < \bar{x} \text{ implies } h(t) = \phi(t)t > \phi(\bar{x})t = t,$$

$$t > \bar{x} \text{ implies } h(t) = \phi(t)t < \phi(\bar{x})t = t.$$

Now by Lemma 2.6 $\bar{x}$ attracts all positive solutions of (2.5). In particular, $\bar{x}$ is not a saddle point so by Lemma 2.4 it is asymptotically stable. □

The following is a corollary of the above result.

Corollary 2.8. *Assume that (2.2) holds with $\gamma > 0$ and the following inequalities are satisfied:*

$$B\alpha \leq A\beta \leq B\alpha + 2aB, \quad A\gamma \leq a + \alpha, \quad B\gamma \leq \beta. \tag{2.21}$$

Then (2.5) has a unique fixed point $\bar{x} > 0$ that is asymptotically stable and attracts all positive solutions of (2.5).

Proof. We show that if the inequalities (2.21) hold then the function

$$f(u, v) = au + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}$$

is nondecreasing in each of its two coordinates u, v. This is demonstrated by computing the partial derivatives f_u and f_v to show that $f_u \geq 0$ and $f_v \geq 0$. By direct calculation $f_u \geq 0$ iff

$$a(Au + Bv)^2 + 2aAu + (2aB + B\alpha - A\beta)v + a + \alpha - A\gamma \geq 0.$$

The above inequality holds for all $u, v > 0$ if

$$2aB + B\alpha - A\beta \geq 0, \quad A\gamma \leq a + \alpha. \tag{2.22}$$

Similarly, $f_v \geq 0$ iff

$$(A\beta - B\alpha)u + \beta - B\gamma \geq 0$$

which is true for all $u, v > 0$ if

$$A\beta - B\alpha \geq 0, \quad B\gamma \leq \beta \tag{2.23}$$

By the inequalities (2.22) and (2.23), conditions (2.21) are sufficient for f to be nondecreasing in each of its coordinates. The rest of the result follows from Theorem 2.7. $\square$

The next result pertains to the case when the function $f(u, v)$ is nonincreasing in both of its arguments. We use the Stability Trichotomy Theorem 1.23:

Theorem 2.9. *Assume that (2.2) holds with $\gamma > 0$ and $f(u,v)$ is nonincreasing in both arguments. Then (2.5) has a unique fixed point $\bar{x} > 0$ that is asymptotically stable and attracts all positive solutions of (2.5).*

Proof. Since $f(u,v)$ is nonincreasing in both arguments, then $f_u, f_v \leq 0$ for all $u, v \geq 0$. By (2.10) and (2.11) we have

$$\begin{aligned} u|f_u| + v|f_v| &= u\left[-a - \frac{\alpha - A(f(u,v) - au)}{Au + Bv + 1}\right] + v\left[\frac{B(f(u,v) - au) - \beta}{Au + Bv + 1}\right] \\ &= -au + \frac{(f(u,v) - au)(Au + Bv)}{Au + Bv + 1} - \frac{\alpha u + \beta v}{Au + Bv + 1} \\ &< f(u,v) - au < f(u,v) \;\text{ for all }\; u, v \in (0, \infty). \end{aligned} \tag{2.24}$$

The rest follows from Theorem 1.23, in light of the fact that the solutions to (2.5) are bounded by Theorem 2.1 and that $\bar{x}$ is the unique positive fixed point in $[0, \infty)$. □

Remark 2.10. When (2.2) holds, finding parameter values that ensure $f_u \leq 0$ is a nontrivial task. It is possible for the special case $a = 0$ and we refer the readers to [51] for more details. For the case when $a > 0$, in lieu of a corollary we pose the following open problems.

Problem 2.11. *Assume that parameters of (2.5) satisfy (2.2). Find parameter values so that $f_u \leq 0$ for all $u, v \geq 0$. In addition, find parameter values that ensure that (i) $f_u \leq 0, f_v \leq 0$, and (ii) $f_u \leq 0, f_v \geq 0$ for all $u, v \geq 0$.*

Problem 2.12. *Assume that parameters of (2.5) satisfy (2.2). Find invariant intervals where (i) $f(u,v)$ is nonincreasing in both arguments; (ii) $f(u,v)$ is nonincreasing in u and nondecreasing in v, for all $u, v \geq 0$.*

Next, we consider the case where $f(u,v)$ is nondecreasing in u and nonincreasing in v which involves application of Theorem (1.24.

Theorem 2.13. *Let (2.2) hold with $\gamma > 0$ and further assume that*

$$\alpha = 0 \ \text{ if } \ A = 0.$$

If $f(u, v)$ is nondecreasing in u and nonincreasing in v, then (2.5) has a positive fixed point $\bar{x}$ that attracts every solutions with non-negative initial values.

Proof. Note that by hypothesis $f_v \leq 0$ and this implies that $\beta = 0$ if $B = 0$. Now Theorem 2.1 implies that for arbitrary positive initial values there are real numbers $L_0, M_0 > 0$ and a positive integer N such that $x_n \in [L_0, M_0]$ for $n \geq N$. Therefore, to prove the global attractivity of $\bar{x}$ we need only show that the hypotheses of Lemma 1.24 are satisfied with $[a, b] = [L_0, M_0]$.

Next, consider the system

$$f(m, M) = m \ \text{ and } \ f(M, m) = M.$$

Clearly, $m = M = \bar{x}$ is a solution to the above system. If we assume that $m \neq M$, then the above system will have a positive solution if $m, M > 0$ and satisfy the following equations:

$$m = am + \frac{\alpha m + \beta M + \gamma}{Am + BM + 1} \tag{2.25a}$$

$$M = aM + \frac{\alpha M + \beta m + \gamma}{AM + Bm + 1}. \tag{2.25b}$$

From (2.25a) we get

$$(1 - a)(Am^2 + BMm + m) = \alpha m + \beta M + \gamma. \tag{2.26}$$

Similarly, from (2.25b) we get

$$(1-a)(AM^2 + BMm + M) = \alpha M + \beta m + \gamma. \tag{2.27}$$

Taking the difference of both sides of the above two equations in (2.26) and (2.27) yields

$$\begin{aligned}(1-a)[A(M^2-m^2)+(M-m)] &= \alpha(M-m)+\beta(m-M)\\ (1-a)(M-m)(A(m+M)+1) &= (M-m)(\alpha-\beta).\end{aligned}$$

When $A = \alpha = 0$, then the last expression implies that the system $f(m, M) = m$ $f(M, m) = M$ has no positive solution besides $M = m = \bar{x}$ and we are done. We next assume that $A > 0$. Since $M \neq m$ we get

$$(1-a)A(m+M) = \alpha - \beta - (1-a). \tag{2.28}$$

From (2.28) we infer that $\alpha - \beta - (1-a)C > 0$, or stated differently, when $\alpha - \beta - (1-a) \leq 0$, then the above system has no positive solution besides $m = M = \bar{x}$. Next, we sum the equations in (2.26) and (2.27) to get

$$(1-a)A(m^2+M^2) + 2(1-a)BMm = (\alpha+\beta-(1-a))(M+m) + 2\gamma.$$

Adding and subtracting $2A(1-a)Mm$ from the right hand side of the above yields

$$(1-a)A(m+M)^2 + 2(1-a)(B-A)Mm = (\alpha+\beta-(1-a))(M+m) + 2\gamma.$$

Thus

$$\begin{aligned}
2(1-a)(B-A)Mm &= (M+m)\left[(\alpha+\beta-(1-a)-(1-a)A(M+m)\right]+2\gamma \\
&= (M+m)\left[(\alpha+\beta-(1-a)-\alpha+\beta+(1-a)\right]+2\gamma \\
&= \frac{2\beta(\alpha+\beta-(1-a))}{(1-a)A}+2\gamma
\end{aligned}$$

i.e.

$$(1-a)(B-A)Mm = \frac{\beta[\alpha+\beta-(1-a)]}{(1-a)A}+\gamma$$

from which we infer that $B-A>0$, since the right hand side of (2.29) is positive. Stated differently, this implies that when $B<A$, the above system has no positive solution besides $m=M=\bar{x}$.

Now, let

$$m+M = \frac{\alpha-\beta-(1-a)}{(1-a)A} := P$$

and

$$Mm = \frac{\beta(\alpha+\beta-(1-a))}{(1-a)^2A(B-A)}+\frac{\gamma}{(1-a)(B-A)} := Q.$$

Then $m=P-M$ and $M(P-M)=Q$. Similarly, $M=P-M$ and $m(P-m)=Q$. Thus M and m must be the roots of the quadratic equation

$$S(t) = t^2-Pt+Q$$

therefore, for the roots of $S(t)$ to be real, we require that $P^2-4Q>0$, i.e.

$$\frac{[\alpha-\beta-(1-a)]^2}{(1-a)^2A^2}-\frac{4\beta[\alpha-\beta-(1-a)]}{(1-a)^2A(B-A)}-\frac{4\gamma}{(1-a)(B-A)}>0$$

which is equivalent to

$$\frac{4\gamma(1-a)}{B-A} < \frac{\alpha-\beta-(1-a)}{A}\left[\frac{\alpha-\beta-(1-a)}{A} - \frac{4\beta}{B-A}\right]. \tag{2.29}$$

Now

$$\frac{\alpha-\beta-(1-a)}{A} - \frac{4\beta}{B-A} = \frac{(B-A)(\alpha-\beta-(1-a)-4A\beta}{A(B-A)}$$

and

$$\begin{aligned}(B-A)[\alpha-\beta-(1-a)]-4A\beta &= (B-A)(\alpha-\beta-(1-a)]-4A\beta \\ &\quad + A[\alpha-\beta-(1-a)] - A[\alpha-\beta-(1-a)] \\ &= (A+B)[\alpha-\beta-(1-a)] - 2A[\alpha+\beta-(1-a)].\end{aligned}$$

Thus the inequality in (2.29) becomes

$$\begin{aligned}\frac{4\gamma(1-a)}{B-A} &< \frac{\alpha-\beta-(1-a)}{A}\left[\frac{\alpha-\beta-(1-a)}{A} - \frac{4\beta}{B-A}\right] \\ &= \frac{\alpha-\beta-(1-a)}{A^2(B-A)}\left[(A+B)[\alpha-\beta-(1-a)] - 2A[\alpha+\beta-(1-a)]\right].\end{aligned}$$

Multiplying both sides by $(B-A)(A+B)$ yields

$$4\gamma(1-a)(A+B) < \frac{(A+B)^2}{A^2}[\alpha-\beta-(1-a)]^2 - \frac{2(A+B)}{A}[\alpha+\beta-(1-a)][\alpha-\beta-(1-a)].$$

Adding $[\alpha+\beta-(1-a)]^2$ to both sides we get

$$\begin{aligned}
&[\alpha+\beta-(1-a)]^2+4\gamma(1-a)(A+B)\\
&\qquad< [\alpha+\beta-(1-a)]^2-\frac{2(A+B)}{A}[\alpha+\beta-(1-a)][\alpha-\beta-(1-a)]\\
&\qquad+\frac{(A+B)^2}{A^2}[\alpha-\beta-(1-a)]^2\\
&\qquad= [\alpha+\beta-(1-a)-\frac{A+B}{A}(\alpha-\beta-(1-a))]^2\\
&\qquad< [\alpha+\beta-(1-a)-(\alpha-\beta-(1-a)]^2=4\beta^2
\end{aligned}$$

which implies that

$$[\alpha+\beta-(1-a)]^2+4\gamma(1-a)(A+B)-4\beta^2<0. \tag{2.30}$$

But since for the above system to have a solution, $\alpha-\beta-(1-a)>0$, then $\alpha-(1-a)>\beta$. This implies that the inequality in (2.30) is false (i.e. the roots of $S(t)$ cannot be real), as

$$\begin{aligned}
[\alpha+\beta-(1-a)]^2+4\gamma(1-a)(A+B)-4\beta^2 &> (2\beta)^2+4\gamma(1-a)(A+B)-4\beta^2\\
&= 4\gamma(1-a)(A+B)>0.
\end{aligned}$$

Thus the system $f(m,M)=m$, $f(M,m)=M$ has no positive solution where $m\neq M$. Theorem 2.1 implies that for arbitrary positive initial values, there is an integer N such that $x_n\in[L,M]$ for $n>N$, so with $[a,b]=[L,M]$ and x_N and x_{N+1} as initial values, x_n must converge to $\bar{x}$ by Lemma 1.24. □

Corollary 2.14. *Assume that (2.2) holds with $\gamma, A, B>0$ and the following conditions hold:*

$$\frac{\alpha}{A}\geq\gamma\geq\frac{\beta}{B}. \tag{2.31}$$

Then (2.5) has a unique positive fixed point $\bar{x}$ that is asymptotically stable and attracts all solutions of (2.5).

Proof. The condition in (2.31) are sufficient to ensure that $f_u \geq 0$ and $f_v \leq 0$ for all $u, v \geq 0$, and the result follows from Theorem 2.13. $\square$

The case when $f(u, v)$ is nonincreasing in the first argument and nondecreasing in the second argument is considered in one of the following sections, where we discuss periodic solutions.

2.4 Periodic solutions

We consider some conditions that lead to the occurrence of periodic solutions of (2.5). In this section, we explicitly assume that $Aa + B > 0$. By assumption in (2.2), $Aa + B = 0$ implies that $a = B = 0$, which reduces (2.5) to the second-order rational equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma}{A x_n + 1} \tag{2.32}$$

which has been studied in [51], p. 167. In particular, it was shown that when

$$\beta = \alpha + 1$$

then every solution of (2.32) converges to a period-two solution. For $Aa + B > 0$, we show that when the function f is monotone in its arguments, then (2.5) does not have periodic solutions of prime period greater than two.

2.4.1 Prime period two solutions

The equation

$$x_{n+1} = a x_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{A x_n + B x_{n-1} + 1}$$

has a positive prime period two solution if there exist real numbers $m, M > 0$, with $m \neq M$, such that

$$m = aM + \frac{\alpha M + \beta m + \gamma}{AM + Bm + 1} \ \text{ and } \ M = am + \frac{\alpha m + \beta M + \gamma}{Am + BM + 1}. \tag{2.33}$$

From (2.33) we obtain

$$(m - aM)(AM + Bm + 1) = \alpha M + \beta m + \gamma \tag{2.34}$$

$$(M - am)(Am + BM + 1) = \alpha m + \beta M + \gamma. \tag{2.35}$$

Taking the difference of right and left hand sides of (2.34) and (2.35) and rearranging the terms yields

$$(Aa + B)(m - M)(m + M) = (m - M)(\beta - \alpha - (1 + a))$$

or

$$m + M = \frac{\beta - \alpha - (1 + a)}{Aa + B}. \tag{2.36}$$

Since $Aa + B > 0$, we infer from (2.36) that $\beta - \alpha - C(1 + a) > 0$ is a necessary condition for existence of positive period two solutions. Similarly, adding the right and left hand sides of (2.34) and (2.35) and rearranging the terms yields

$$2(A - aB)Mm = (\alpha + \beta - (1 - a))(M + m) + (Aa - B)(m^2 + M^2) + 2\gamma.$$

Adding an subtracting $2(Aa - B)$ yields

$$2(1 + a)(A - B)Mm = (m + M)\left[(\alpha + \beta - (1 - a)) + (Aa - B)(m + M)\right] + 2\gamma.$$

Inserting from (2.36) the expression for $m+M$ inside the square bracket yields

$$\begin{aligned}2(1+a)(A-B)Mm &= (M+m)\left[(\alpha+\beta-(1-a))+\frac{Aa-B}{Aa+B}(\beta-\alpha-C(1+a))\right]+2\gamma\\ &= \frac{2(M+m)}{Aa+B}\left[Aa(\beta-1)+B(\alpha+a)\right]+2\gamma.\end{aligned}$$

Thus

$$(1+a)(A-B)Mm=\left[\frac{\beta-\alpha-(1+a)}{(Aa+B)^2}\right]\left[Aa(\beta-1)+B(\alpha+a)\right]+\gamma. \qquad (2.37)$$

Since from (2.36) we have $\beta-\alpha-(1+a)>0$, then $\beta-1>0$. Thus the right hand side of (2.37) is positive and therefore, $A-B>0$ is another necessary condition for existence of positive period two solution and

$$Mm=\frac{1}{(1+a)(A-B)}\left[\left[\frac{\beta-\alpha-(1+a)}{(Aa+B)^2}\right]\left[Aa(\beta-1)+B(\alpha+a)\right]+\gamma\right]. \qquad (2.38)$$

Let

$$P=\frac{\beta-\alpha-(1+a)}{Aa+B}$$

and

$$K=\frac{1}{(1+a)(A-B)}\left[\left[\frac{\beta-\alpha-(1+a)}{(Aa+B)^2}\right]\left[Aa(\beta-1)+B(\alpha+a)\right]+\gamma\right]$$

with $P,K>0$. From (2.36) we obtain

$$m=\frac{\beta-\alpha-(1+a)}{Aa+B}-M=P-M.$$

Inserting the above into (2.38) yields

$$M(P-M)=K \ \text{ or } \ M^2-PM+K=0. \qquad (2.39)$$

Similarly, an identical expression can be obtained for m, i.e.

$$m^2 - Pm + K = 0. \tag{2.40}$$

Thus M and m must be the real and positive roots of the quadratic equation

$$Q(t) = t^2 - Pt + K$$

with

$$t = \frac{P \pm \sqrt{P^2 - 4K}}{2}$$

which will be the case if and only if

$$P^2 - 4K > 0$$

or equivalently,

$$\frac{4\gamma}{(1+a)(A-B)} < \left[\frac{\beta - \alpha - (1+a)}{Aa+B}\right]\left[\frac{\beta - \alpha - (1+a)}{Aa+B} - \frac{4\left[Aa(\beta-1) + B(\alpha+a)\right]}{(1+a)(A-B)(Aa+B)}\right]. \tag{2.41}$$

We summarize the above results as follows:

Theorem 2.15. *Assume that (2.2) holds with $\gamma, Aa + B > 0$. Then (2.5) has a positive prime period two solution if and only if the following conditions are satisfied:*

1. $\beta - \alpha - (1+a) > 0$

2. $A - B > 0$

3. $\frac{4\gamma}{(1+a)(A-B)} < \left[\frac{\beta-\alpha-(1+a)}{Aa+B}\right]\left[\frac{\beta-\alpha-(1+a)}{Aa+B} - \frac{4[Aa(\beta-1)+B(\alpha+a)]}{(1+a)(A-B)(Aa+B)}\right].$

The next results pertain to the case when $f(u, v)$ is monotone in its arguments and this holds for any difference equation of second-order.

Theorem 2.16. *Let D be a subset of real numbers and assume that*

$$f : D \times D \to D$$

is non-decreasing in $x \in D$ for each $y \in D$ and non-increasing in $y \in D$ for each $x \in D$. Then the difference equation

$$x_{n+1} = f(x_n, x_{n-1})$$

has no prime period two solution.

Proof. Assume that the above difference equation has prime period two solution. Then there exist real numbers m and M, such that

$$f(m, M) = M \text{ and } f(M, m) = m.$$

When $m = M$, we're done. So assume that $m \neq M$.

If $m < M$, then by the hypothesis

$$f(m, M) \leq f(M, M) \leq f(M, m)$$

which implies that $M \leq m$, which is a contradiction. Similarly, if $m > M$, then by the hypothesis

$$f(M, m) \leq f(M, M) \leq f(m, M)$$

which implies that $m \geq M$, which is also a contradiction. □

Remark 2.17. If (2.2) holds and $\gamma, Aa + B > 0$, we observe the following:

(a) When $f(u, v)$ is nondecreasing in both arguments, then by Theorem 2.7, the fixed point $\bar{x}$ is globally asymptotically stable, so no periodic solutions exist.

(b) When $f(u,v)$ is nonincreasing in both arguments, then by Theorem 2.9, the fixed point $\bar{x}$ is globally asymptotically stable, so no periodic solutions exist.

(c) When $f(u,v)$ is nondecreasing in u and nonincreasing in v, then by Theorem 2.16 no period two solution exists. Moreover, by Theorem 2.13 $\bar{x}$ is globally asymptotically stable, so no periodic solutions of other periods exist.

(d) From (a)-(c) we conclude that the only case that periodic solutions may exist is when $f(u,v)$ is nonincreasing in u and nondecreasing in v (or f is non-monotone). In addition, all the conditions in Theorem 2.15 must also be satisfied.

Theorem 2.18. *Let (2.2) hold with $\gamma, Aa+B>0$ and assume that $f(u,v)$ is nonincreasing in u and nondecreasing in v. Then (2.5) has a positive fixed point $\bar{x}$ that attracts every solution of (2.5) if either of the conditions below fails:*

1. $\beta-\alpha-(1+a)>0$

2. $A-B>0$

3. $\frac{4\gamma}{(1+a)(A-B)} < \left[\frac{\beta-\alpha-(1+a)}{Aa+B}\right]\left[\frac{\beta-\alpha-(1+a)}{Aa+B} - \frac{4[Aa(\beta-1)+B(\alpha+a)]}{(1+a)(A-B)(Aa+B}\right].$

Proof. The failure of either of the above conditions implies that (2.5) has no positive prime period two solution. Theorem 1 implies that for arbitrary positive initial values, there is an integer N such that $x_n \in [L,M]$ for $n>N$ so with $[a,b]=[L,M]$ and x_N and x_{N+1} as initial values, x_n must converge to $\bar{x}$ by Lemma 1.25. □

Our final result of this section establishes the connection between existence of prime period two solution and the stability of the fixed point.

Theorem 2.19. *Let(2.2) holds with $\gamma, Aa+B>0$. Then (2.5) has a positive prime period two solution if and only it $\bar{x}$ is a saddle.*

Proof. First, when $\alpha+\beta-(1-a)=0$, then the fixed point $\bar{x}$ is given by $\bar{x}=\sqrt{\frac{\gamma}{(1-a)(A+B)}}$. This implies that

$$\beta-\alpha-(1+a)<0$$

and on the one hand, $\bar{x}$ must be stable and more importantly, (2.5) has no prime period two solution and there is nothing further to consider for this case.

Now assume that $\alpha+\beta-(1-a)\neq 0$. Then the fixed point is given by

$$\bar{x}=\frac{\alpha+\beta-(1-a)+\sqrt{(\alpha+\beta-(1-a))^2+4(1-a)(A+B)\gamma}}{2(1-a)(A+B)}.$$

By Lemma 2.4, $\bar{x}$ is a saddle if and only if

$$\bar{x}<\frac{\beta-\alpha-(1+a)}{2(Aa+B)}$$

which implies that $\beta-\alpha-(1+a)>0$.

Now

$$\bar{x}<\frac{\beta-\alpha-(1+a)}{2(Aa+B)}$$

iff

$$\frac{\alpha+\beta-(1-a)+\sqrt{(\alpha+\beta-(1-a))^2+4(1-a)(A+B)\gamma}}{2(1-a)(A+B)}<\frac{\beta-\alpha-(1+a)}{2(Aa+B)}$$

iff

$$\frac{\sqrt{(\alpha+\beta-(1-a))^2+4(1-a)(A+B)\gamma}}{(1-a)(A+B)}<\frac{\beta-\alpha-(1+a)}{(Aa+B)}-\frac{\alpha+\beta-(1-a)}{(1-a)(A+B)}$$

iff

$$\sqrt{(\alpha+\beta-(1-a))^2+4(1-a)(A+B)\gamma} < \frac{(1-a)(A+B)(\beta-\alpha-(1-a)}{Aa+B} - [\alpha+\beta-(1-a)]$$

iff

$$(\alpha+\beta-(1-a))^2+4(1-a)(A+B)\gamma < \frac{(1-a)^2(A+B)^2(\beta-\beta-(1+a)^2}{(Aa+B)^2} - \frac{2(1-a)(A+B)(\alpha+\beta-(1-a))(\beta-\alpha-(1+a)}{Aa+B} + (\alpha+\beta-(1-a))^2$$

iff

$$4(1-a)(A+B)\gamma < \frac{(1-a)^2(A+B)^2(\beta-\beta-(1+a)^2}{(Aa+B)^2} - \frac{2(1-a)(A+B)(\alpha+\beta-(1-a))(\beta-\alpha-(1+a))}{Aa+B}$$

iff

$$\begin{aligned}
4\gamma &< \frac{(1-a)(A+B)(\beta-\alpha-(1+a))}{(Aa+B)^2} - \frac{2(\alpha+\beta-(1-a))(\beta-\alpha-(1+a))}{Aa+B} \\
&= \frac{(\beta-\alpha-(1+a))}{Aa+B}\left[\frac{(1-a)(A+B)(\beta-\alpha-(1+a))}{Aa+B} - 2(\alpha+\beta-(1-a))\right] \\
&= \frac{(\beta-\alpha-(1+a))}{Aa+B}\left[\frac{(1-a)(A+B)(\beta-\alpha-(1+a))-2(Aa+B)(\alpha+\beta-(1-a))}{Aa+B}\right].
\end{aligned}$$

Adding and subtracting $(1+a)(A-B)[\beta-\alpha-(1+a)]$ to the numerator of the second fraction in previous equation yields

$$
\begin{aligned}
(1-a)(A+B)(\beta-\alpha-(1+a)) &- 2(Aa+B)(\alpha+\beta-(1-a)) \\
&= (1+a)(A-B)[\beta-\alpha-(1+a)] - 4Aa(\beta-1) - 4B(\alpha+a).
\end{aligned}
$$

Thus we have

$$
4\gamma < \frac{\beta-\alpha-(1+a)}{Aa+B}\left[\frac{(1+a)(A-B)(\beta-\alpha-(1+a)) - 4\left[Aa(\beta-1)+B(\alpha+a)\right]}{Aa+B}\right].
$$

Note that since $\gamma > 0$, it must be the case that the right hand side of the last expression is positive, which implies that $A-B>0$. Dividing both sides of the above expression by $(1+a)(A-B)$ then yields:

$$
\frac{4\gamma}{(1+a)(Aa+B)} < \frac{\beta-\alpha-(1+a)}{Aa+B}\left[\frac{(\beta-\alpha-(1+a))}{Aa+B} - \frac{4\left[Aa(\beta-1)+B(\alpha+a)\right]}{(Aa+B)(1+a)(A-B)}\right]
$$

and the proof is complete, since the conditions Theorem 2.15 are satisfied. $\square$

We end our discussion with the following corollaries that immediately follow from the results discussed in previous sections.

Corollary 2.20. *Let (2.2) hold with $\gamma, Aa+B>0$. If $f(u,v)$ is monotone in its arguments, then (2.5) has no periodic solution of period greater than two.*

Corollary 2.21. *Let (2.2) hold with $\gamma, Aa+B>0$. If $f(u,v)$ is monotone in its arguments and if (2.5) has no period two solution, then all solutions of (2.5) converge to $\bar{x}>0$.*

The above results give partial answers to two conjectures posed by [51] in their monograph for the special case $a=0$.

2.5 Concluding remarks and further considerations

We studied the dynamics of a second order quadratic fractional difference equation with non-negative parameters and initial values. We showed that under the above assumptions, the equation typically does not have periodic solutions of period greater than two. Further, we showed that if period two cycles do not occur then the solutions converge to the unique positive fixed point. When $aA + B, \gamma > 0$ we obtained necessary and sufficient conditions for the occurrence of periodic solutions and in particular proved that such solutions may appear if and only if the positive fixed point is a saddle.

The results establishing convergence to the positive fixed point essentially require the function defining the second order equation to be monotone. Instances when this hypothesis fails were not addressed and could be investigated next.

A natural extension for future research involves addition of a linear delay term to the above equation, i.e. the study of

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{2.42}$$

as well as the more general case given by

$$x_{n+1} = \frac{px_n^2 + qx_n x_{n-1} + \delta x_{n-1}^2 + c_1 x_n + c_2 x_{n-1} + c_3}{Ax_n + Bx_{n-1} + C} \tag{2.43}$$

Possible generalizations could include instances where quadratic terms also appear in the denominator.

CHAPTER III

Dynamics of a Second Order Exponential Difference Equation

In this chapter, we study the nonautonomous second order difference equation given by

$$r_{n+1} = \mu_n r_{n-1} e^{-r_{n-1}-r_n} \tag{3.1}$$

where coefficients $\{\mu_n\}$ are assumed to be a sequence of positive real numbers.[1] The equation in (3.1) can be written as

$$r_{n+1} = r_{n-1} e^{a_n - r_{n-1} - r_n} \tag{3.2}$$

where $a_n = \ln \mu_n$ may not always be positive. We prove general results on boundedness and convergence of solutions of (3.2) to zero for arbitrary sequence of coefficients $\{a_n\}$. We then examine the equation in (3.2) where the sequence $\{a_n\}$ is assumed to be periodic with period $p \geq 1$. When $p = 1$, a_n are constant, in which case (3.2) reduces to an autonomous second order difference equation. For this case, we establish that the solutions of (3.2) can exhibit complex and multistable behavior. We then examine cases where $p > 1$ and show that the nature of the solutions of (3.2) is

[1] The content of this chapter, unless otherwise noted, is from [57] and [59].

qualitatively different depending on whether p is even or odd. In both cases, we show that the solutions of (3.2) can exhibit periodic and non-periodic multistable behavior.

3.1 General results

We first establish several general results that apply to (3.2) where $\{a_n\}$ is assumed to be an arbitrary sequence. We look at boundedness of solutions, as well as conditions under which solutions converge to zero. We then consider the reduction of order of (3.2) by semiconjugate factorization that significantly facilitates further analysis (see [83] for further details).

3.1.1 Boundedness and global convergence to zero

Theorem 3.1. *Let $\{a_n\}$ be a sequence of real numbers satisfying*

$$\sup_{n\geq 0} a_n = a < \infty \tag{3.3}$$

Then the solutions of (3.2) from initial values $r_0, r_{-1} \geq 0$ are uniformly bounded by e^{a-1}.

Proof. Clearly, for $r_0, r_{-1} \geq 0$, $r_n \geq 0$ for all $n > 0$. Next

$$\begin{aligned} r_{n+1} &= r_{n-1}e^{a_n - r_{n-1} - r_n} \\ &\leq e^{a_n} r_{n-1} e^{-r_{n-1}} \\ &\leq e^{a_n - 1} \leq e^{a-1} < \infty \end{aligned}$$

as the function xe^{-x} attains a maximum at e^{-1}. Thus $0 \leq r_n \leq L$ for all $n > 0$ and the proof is complete. $\square$

Theorem 3.2. *Assume $\{a_n\}$ is a sequence of real numbers satisfying*

$$\limsup_{n\to\infty} a_n < 0 \tag{3.4}$$

Then the solutions of (3.2) from initial values $r_0, r_{-1} > 0$ *converge to* 0.

Proof. Given initial values $r_0, r_{-1} \geq 0$, we have

$$0 \leq r_1 = r_{-1}e^{a_0 - r_{-1} - r_0} \leq e^{a_0} r_{-1}$$

$$0 \leq r_2 = r_0 e^{a_1 - r_0 - r_1} \leq e^{a_1} r_0$$

$$0 \leq r_3 = r_1 e^{a_2 - r_1 - r_2} \leq e^{a_2} r_1 \leq e^{a_0} e^{a_2} r_{-1}$$

$$0 \leq r_4 = r_2 e^{a_3 - r_2 - r_3} \leq e^{a_3} r_2 \leq e^{a_3} e^{a_1} r_0$$

Continuing this way inductively, one may show that for all $n \geq 0$

$$0 \leq r_{2n+1} \leq r_{-1} \prod_{j=0}^{n} e^{a_{2j}}$$

$$0 \leq r_{2n+2} \leq r_0 \prod_{j=0}^{n} e^{a_{2j+1}}$$

Since by hypothesis $e^{a_n} < 1$ for infinitely many n, then $r_n \to 0$ as $n \to \infty$, which completes the proof. □

3.1.2 Reduction of order

The study of (3.2) is facilitated by the fact that it admits a semiconjugate factorization that splits it into two equations of order one. Following [83], we define

$$t_n = \frac{r_n}{r_{n-1}e^{-r_{n-1}}}$$

for each $n \geq 1$ and note that

$$t_{n+1}t_n = \frac{r_{n+1}}{r_n e^{-r_n}} \frac{r_n}{r_{n-1}e^{-r_{n-1}}} = \frac{r_{n+1}}{r_{n-1}e^{-r_{n-1}-r_n}} = e^{a_n}$$

or equivalently,

$$t_{n+1} = \frac{e^{a_n}}{t_n}. \tag{3.5}$$

Now

$$r_{n+1} = e^{a_n} r_{n-1} e^{-r_{n-1}} e^{-r_n} = e^{a_n} \frac{r_n}{t_n} e^{-r_n} = \frac{e^{a_n}}{t_n} r_n e^{-r_n} = t_{n+1} r_n e^{-r_n} \tag{3.6}$$

The pair of equations (3.5) and (3.6) constitute the semiconjugate factorization of (3.2):

$$t_{n+1} = \frac{e^{a_n}}{t_n}, \quad t_0 = \frac{r_0}{r_{-1}e^{-r_{-1}}} \tag{3.7}$$

$$r_{n+1} = t_{n+1} r_n e^{-r_n} \tag{3.8}$$

Every solution $\{r_n\}$ of (3.2) is generated by a solution of the system (3.7)-(3.8). Using the initial values r_{-1}, r_0 we obtain a solution $\{t_n\}$ of the first-order equation (3.7), called the factor equation. This solution is then used to obtain a solution of the cofactor equation (3.8) and thus also of (3.2).

For an arbitrary sequence $\{a_n\}$ and a given $t_0 \neq 0$ by iterating (3.7) we obtain

$$t_1 = \frac{e^{a_0}}{t_0}, \quad t_2 = \frac{e^{a_1}}{t_1} = t_0 e^{-a_0+a_1}, \quad t_3 = \frac{e^{a_2}}{t_2} = \frac{1}{t_0} e^{a_0-a_1+a_2}, \quad t_4 = \frac{e^{a_3}}{t_3} = t_0 e^{-a_0+a_1-a_2+a_3}, \cdots$$

This pattern of development implies the following result.

Lemma 3.3. *Let $\{a_n\}$ be an arbitrary sequence of real numbers and $t_0 \neq 0$. The* *general solution of (3.7) is given by*

$$t_n = t_0^{(-1)^n} e^{(-1)^n s_n}, \quad n = 1, 2, ... \tag{3.9}$$

where

$$s_n = \sum_{j=1}^{n} (-1)^j a_{j-1} \tag{3.10}$$

Proof. For $n = 1$, (3.9) yields

$$t_1 = t_0^{-1} e^{-s_1} = \frac{1}{t_0} e^{-(-a_0)} = \frac{e^{a_0}}{t_0}$$

which is true. Suppose that (3.9) is true for $n \le k$. Then by (3.9) and (3.10)

$$t_{k+1} = t_0^{(-1)^{k+1}} e^{(-1)^{k+1} s_{k+1}} = \frac{1}{t_0^{(-1)^k} e^{(-1)^k s_k}} e^{(-1)^{2k+2} a_k} = \frac{e^{a_k}}{t_k}$$

which is again true and the proof is now complete by induction. □

Note that the solution $\{t_n\}$ of (3.7) in the preceding lemma need not be bounded even if $\{a_n\}$ is a bounded sequence.

From the cofactor equation (3.8) we obtain

$$r_{2n+2} = t_{2n+2} r_{2n+1} e^{-r_{2n+1}} = t_{2n+2} t_{2n+1} r_{2n} \exp(-r_{2n} - t_{2n+1} r_{2n} e^{-r_{2n}})$$
$$r_{2n+1} = t_{2n+1} r_{2n} e^{-r_{2n}} = t_{2n+1} t_{2n} r_{2n-1} \exp\left(-r_{2n-1} - t_{2n} r_{2n-1} e^{-r_{2n-1}}\right)$$

For every solution $\{t_n\}$ of (3.7), $t_{n+1} t_n = e^{a_n}$ for all n, so the even terms of the sequence $\{r_n\}$ are

$$r_{2n+2} = r_{2n} \exp\left(a_{2n+1} - r_{2n} - t_{2n+1} r_{2n} e^{-r_{2n}}\right) \tag{3.11}$$

and the odd terms are

$$r_{2n+1} = r_{2n-1} \exp\left(a_{2n} - r_{2n-1} - t_{2n} r_{2n-1} e^{-r_{2n-1}}\right) \tag{3.12}$$

In the next sections, we explore the behavior of $\{r_n\}$ when the sequence $\{a_n\}$ is periodic with minimal period $p \geq 1$. We start with the case when $p = 1$, which reduces the equation in (3.2) into an autonomous equation. We then consider the case when the period p is odd, followed by the case when the period p is even.

3.2 Autonomous equation: the case when $p = 1$

When the sequence $\{a_n\}$ is constant, i.e. $a_n = a$ for $n \geq 0$, the equation in (3.2) reduces to the autonomous case given by

$$r_{n+1} = r_{n-1} e^{a - r_{n-1} - r_n} \tag{3.13}$$

The boundedness of the solutions of (3.13) follows as a consequence of Theorem 3.1, which we state as a corollary below:

Corollary 3.4. *Let a be a real number. Then the solutions of (3.13) from initial values of $r_0, r_{-1} > 0$ are uniformly bounded.*

All solutions of the factor equation in (3.7) with constant $a_n = a$ and $t_0 \neq e^{a/2}$ are periodic with period 2:

$$\left\{t_0, \frac{e^a}{t_0}\right\} = \left\{\frac{r_0}{r_{-1} e^{-r_{-1}}}, \frac{r_{-1} e^{a - r_{-1}}}{r_0}\right\}.$$

3.2.1 Fixed points, global stability

It is useful to begin the study of (3.13) by examining the fixed points, which must satisfy the equation

$$r = re^{a-2r}$$

Theorem 3.5. *Assume that $a < 0$. Then*

(a) The equation in (3.13) has a unique fixed point at 0 that is locally asymptotically stable.

(b) All solutions from nonnegative initial values r_0, r_{-1} converge to 0.

Proof. Clearly, 0 is the fixed point of (3.13) and when $a < 0$, 0 is the only fixed point. It is straightforward to check that the eigenvalues of the linearization of (3.13) at 0 are given by $\pm e^{a/2}$ which proves part (a). Part(b) follows from Theorem 3.2. □

If $a > 0$, then (3.13) has two fixed points: 0 and a positive fixed point $\bar{r} = a/2$. In this case, the eigenvalues of the linearization at 0 are greater than 1 in modulus, hence 0 is an unstable fixed point. On the other hand, the eigenvalues of the linearization of (3.13) are -1 and $a/2$, showing that $\bar{r}$ is nonhyperbolic.

The next result is proved in [37].

Theorem 3.6. *If $a \in (0, 1]$ then every non-constant, positive solution of (3.13) converges to a 2-cycle whose consecutive points satisfy $r_n + r_{n+1} = a$, i.e. the mean value of the limit cycle is the fixed point $\bar{r} = a/2$.*

The two-cycle in Theorem 3.6 is not unique–it is determined by the initial values. In the next section, we derive the precise mechanism that explains this, and much more complex behavior below. In particular, we extend Theorem 3.6 by showing that it holds for $a \in (0, 2]$.

3.2.2 Complex multistable behavior

The behavior of solutions of (3.13) is sufficiently unusual that we use the numerical simulation depicted in Figure 3.1 to motivate the subsequent discussion.

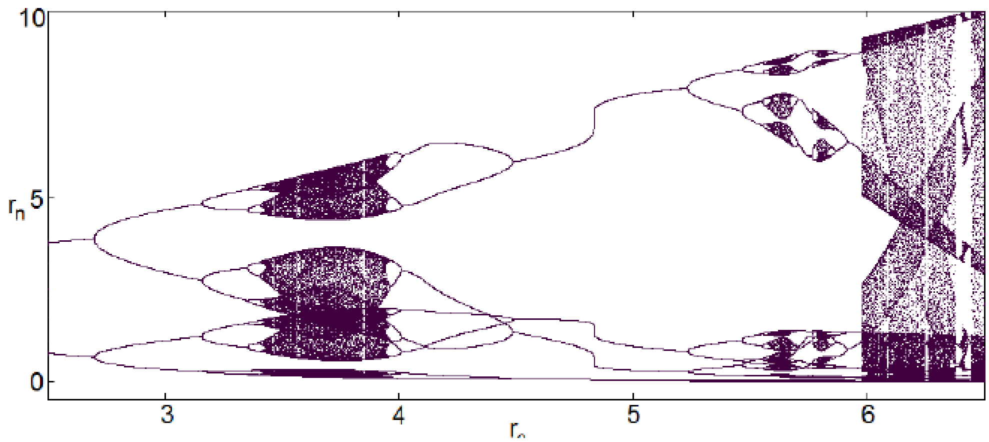

Figure 3.1: Bifurcation of multiple stable solutions in the state-space

In Figure 3.1, $a = 4.5$, $r_{-1} = a/2 = 2.25$ is fixed and $r_0 \in (0, \infty)$ acts as a bifurcation parameter. The changing values of r_0 are shown on the horizontal axis in the range 2.5 to 6.5. For every grid value of r_0 in the indicated range, 300 points of the corresponding solution $\{r_n\}$ are plotted vertically. In this figure, coexisting solutions with periods 2, 4, 8 and 16 are easily identified. The solutions shown in Figure 3.1 are stable since they are generated by numerical simulation, so that qualitatively different, stable solutions exist for (3.13) *for different initial values*. In the remainder of this section we explain this abundance of multistable solutions for (3.13) using the reduction (3.7)-(3.8).

Since the solutions of (3.7) with constant $a_n = a$ and $t_0 \neq e^{a/2}$ are periodic with period 2, the orbit of each nontrivial solution $\{r_n\}$ of (3.13) in its state-space, namely, the (r_n, r_{n+1})-plane, is restricted to the class of curve-pairs

$$g_0(r, t_0) = t_0 r e^{-r} \quad \text{and} \quad g_1(r, t_0) = t_1 r e^{-r}, \quad t_1 = \frac{e^a}{t_0} \tag{3.14}$$

These one-dimensional mappings form the building blocks of the two-dimensional, standard state-space map F of (3.13), i.e.

$$F(u,r) = (r, ue^{a-u-r}).$$

There are, of course, an infinite number of initial value-dependent curve-pairs for the map F.

The next result indicates the specific mechanism for generating the solutions of (3.13) from its semiconjugate factorization.

Lemma 3.7. *Let $a > 0$ and let $\{r_n\}$ be a solution of (3.13) with initial values $r_{-1}, r_0 > 0$.*

(a) For $k = 0, 1, 2, \ldots$ and t_0 as defined in (3.7)

$$r_{2k+1} = g_1 \circ g_0(r_{2k-1}, t_0), \quad r_{2k+2} = g_0 \circ g_1(r_{2k}, t_0)$$

Thus, the odd terms of every solution of (3.13) are generated by the class of one-dimensional maps $g_1 \circ g_0$ and the even terms by $g_0 \circ g_1$;

(b) If the initial values r_{-1}, r_0 satisfy

$$r_0 = r_{-1}e^{a/2-r_{-1}} \tag{3.15}$$

then $g_0(r, t_0) = g_1(r, t_0) = re^{a/2-r}$; i.e. the two curves g_0 and g_1 coincide with the curve

$$g(r) \doteq re^{a/2-r}$$

The trace of g contains the fixed point $(\bar{r}, \bar{r})$ in the state-space and is invariant under F.

Proof. (a) For $k = 0, 1, 2, \dots$ (3.8) implies that

$$r_{2k+1} = t_{2k+1} r_{2k} e^{-r_{2k}} = t_1 r_{2k} e^{-r_{2k}} = g_1(r_{2k}, t_0)$$
$$r_{2k} = t_{2k} r_{2k-1} e^{-r_{2k-1}} = t_0 r_{2k-1} e^{-r_{2k-1}} = g_0(r_{2k-1}, t_0)$$

Therefore,

$$r_{2k+1} = g_1(g_0(r_{2k-1}, t_0), t_0) = g_1 \circ g_0(r_{2k-1}, t_0)$$

A similar calculation shows that

$$r_{2k+2} = g_0(g_1(r_{2k}, t_0), t_0) = g_0 \circ g_1(r_{2k}, t_0)$$

and the proof of (a) is complete.

(b) Note that $g(\bar{r}) = \bar{r}e^{a/2-\bar{r}} = \bar{r}$ so the trace of g contains $(\bar{r}, \bar{r})$. The curves g_0, g_1 coincide if $t_0 = e^a/t_0$, i.e. $t_0 = e^{a/2}$. This happens if the initial values r_{-1}, r_0 satisfy (3.15). In this case, (r_{-1}, r_0) is clearly on the trace of g and by (3.8)

$$r_1 = t_1 r_0 e^{-r_0} = \frac{e^a}{t_0} r_0 e^{-r_0} = t_0 r_0 e^{-r_0} = g(r_0)$$

Therefore, the point (r_0, r_1) is also on the trace of g. Since $t_n = t_0$ for all n if $t_0 = e^{a/2}$ the same argument applies to (r_n, r_{n+1}) for all n and completes the proof by induction. □

Note that the invariant curve g does not depend on initial values. There is also the following useful fact about g.

Lemma 3.8. *The mapping g has a period-three point for $a \geq 6.26$.*

Proof. Let $d = a/2$. The third iterate of g is

$$g^3(r) = r \exp(3d - r - 2re^{d-r} + e^{d-re^{d-r}})$$

In particular,

$$g^3(1) < \exp(3d - 1 - e^{d-1}) \doteq h(d)$$

Solving $h(d) = 1$ numerically yields the estimate $d \approx 3.12$. Since $h(d)$ is decreasing if $d > 2.1$ it follows that $h(d) < 1$ if $d \geq 3.13$. Therefore, $g^3(1) < 1$ for $a \geq 6.26$. Further, for $\varepsilon \in (0, d)$

$$\begin{aligned} g^3(d-\varepsilon) &> (d-\varepsilon)\exp\left[2d + \varepsilon - 2(d-\varepsilon)e^{\varepsilon} + e^{d(1-e^{\varepsilon})}\right] \\ &> (d-\varepsilon)\exp[e^{-d(e^{\varepsilon}-1)} - 2d(e^{\varepsilon} - 1)] \end{aligned}$$

For sufficiently small ε the exponent is positive so we may assert that

$$g^3(1) < 1 < d - \varepsilon < g^3(d-\varepsilon)$$

Hence, there is a root of $g^3(r)$, or a period-three point of g in the interval $(1, a)$ if $d \geq 3.13$, i.e. $a \geq 6.26$. $\square$

The function compositions in Lemma 3.7 are specifically the following mappings:

$$\begin{aligned} g_1 \circ g_0(r, t_0) &= re^{a-r-t_0re^{-r}}, \\ g_0 \circ g_1(r, t_0) &= re^{a-r-t_1re^{-r}}, \quad t_1 = \frac{e^a}{t_0}. \end{aligned}$$

To simplify our notation, for each $t \in (0, \infty)$ define the class of functions $f_t : (0, \infty) \to (0, \infty)$ as

$$f_t(r) = re^{a-r-tre^{-r}}.$$

We also abbreviate f_{t_0} as f_0, f_{t_1} as f_1, $g_0(\cdot, t_0)$ as g_0 and $g_1(\cdot, t_0)$ as g_1. Then we see from the preceding discussion that

$$g_1 \circ g_0 = f_0, \quad g_0 \circ g_1 = f_1. \tag{3.16}$$

According to Lemma 3.7, iterations of f_0 generate the odd-indexed terms of a solution of (3.13) and iterations of f_1 generate the even-indexed terms.

The next result furnishes a relationship between f_i and g_i for $i = 0, 1$.

Lemma 3.9. *Let $t_0 \in (0, \infty)$ be fixed and $t_1 = e^a/t_0$. Then*

$$f_1 \circ g_0 = g_0 \circ f_0 \quad \text{and} \quad f_0 \circ g_1 = g_1 \circ f_1. \tag{3.17}$$

Proof. This may be established by straightforward calculation using the definitions of the various functions, or alternatively, using (3.16) to obtain

$$f_1 \circ g_0 = (g_0 \circ g_1) \circ g_0 = g_0 \circ (g_1 \circ g_0) = g_0 \circ f_0$$

This proves the first equality in (3.17) and the second equality is proved similarly. □

The equalities in (3.17) are not conjugacies since g_0 and g_1 are not one-to-one. However, the following is implied.

Lemma 3.10. *(a) If $\{s_1, s_2, \ldots, s_q\}$ is a q-cycle of f_0, i.e. a solution (listed in the order of iteration) of*

$$s_{n+1} = f_0(s_n) = s_n e^{a - s_n - t_0 s_n e^{-s_n}} \tag{3.18}$$

with minimal (or prime) period $q \geq 1$ then $\{g_0(s_1), g_0(s_2), \ldots, g_0(s_q)\}$ is a q-cycle of f_1, i.e. a solution of

$$u_{n+1} = f_1(u_n) = u_n e^{a - u_n - t_1 u_n e^{-u_n}} \tag{3.19}$$

with period q (listed in the order of iteration). Similarly, if $\{u_1, u_2, \ldots, u_q\}$ is a q-cycle of f_1, i.e. a solution of (3.19) with minimal period $q \geq 1$ then $\{g_1(u_1), g_1(u_2), \ldots, g_1(u_q)\}$ is a q-cycle of f_0, i.e. solution of (3.18) with period q.

(b) If $\{s_n\}$ is a non-periodic solution of (3.18) then $\{g_0(s_n)\}$ is a non-periodic solution of (3.19). Similarly, if $\{u_n\}$ is a non-periodic solution of (3.19) then $\{g_1(u_n)\}$

is a non-periodic solution of (3.18).

Proof. (a) By the hypothesis, $f_0(s_{n+q}) = s_n$ for all n and in the order of iteration

$$f_0(s_k) = s_{k+1} \quad \text{for } k = 1, \dots, q-1 \quad \text{and} \quad f_0(s_q) = s_1.$$

By Lemma 3.9,

$$f_1(g_0(s_{n+q})) = g_0(f_0(s_{n+q})) = g_0(s_n)$$

and also

$$f_1(g_0(s_k)) = g_0(f_0(s_k)) = g_0(s_{k+1}) \quad \text{for } k = 1, \dots, q-1,$$
$$f_1(g_0(s_q)) = g_0(f_0(s_q)) = g_0(s_1)$$

It follows that $\{g_0(s_1), g_0(s_2), \dots, g_0(s_q)\}$ is a periodic solution of (3.19) with period q, listed in the order of iteration. The rest of (a) is proved similarly.

(b) Let $\{s_n\}$ be a solution of (3.18) such that $\{g_0(s_n)\}$ is a periodic solution of (3.19). Then $\{g_1(g_0(s_n))\}$ is a periodic solution of (3.18) by (a). Since $g_1(g_0(s_n)) = f_0(s_n)$ by (3.16) we may conclude that there is a positive integer q such that $f_0^q(s_n) = f_0(s_n) = s_{n+1}$ for all n. Thus $s_{n+1} = f_0^{q-1}(s_{n+1})$ for all n and it follows that $\{s_n\}$ is a periodic solution of (3.18). This proves the first assertion in (b); the second assertion is proved similarly. □

The next result concerns the local stability of the periodic solutions of (3.18) and (3.19).

Lemma 3.11. *If $\{s_1, s_2, \dots, s_q\}$ is a periodic solution of (3.18) with minimal period q such that $s_k \neq 1$ for $k = 1, 2, \dots, q$ and*

$$\prod_{k=1}^{q} f_0'(s_k) < 1 \tag{3.20}$$

then $\{g_0(s_1), \dots, g_0(s_q)\}$ *is a solution of (3.19) of period* q *with* $\prod_{k=1}^{q} f_1'(g_0(s_k)) < 1$. *Similarly, if* $\{u_1, u_2, \dots, u_q\}$ *is a periodic solution of (3.19) with* $u_k \neq 1$ *for* $k = 1, 2, \dots, q$ *and*

$$\prod_{k=1}^{q} f_1'(u_k) < 1$$

then $\{g_1(u_1), g_1(u_2), \dots, g_1(u_q)\}$ *is a solution of (3.18) of period* q *with* $\prod_{k=1}^{q} f_0'(g_1(u_k)) < 1$.

Proof. By Lemma 3.9 and the chain rule

$$f_1'(g_0(r))g_0'(r) = g_0'(f_0(r))f_0'(r)$$

Now $g_0'(r) = (1-r)t_0e^{-r} \neq 0$ if $r \neq 1$. Thus if $s_k \neq 1$ for $k = 1, 2, \dots, q$ then

$$\begin{aligned}\prod_{k=1}^{q} f_1'(g_0(s_k)) &= \frac{g_0'(f_0(s_1))f_0'(s_1)}{g_0'(s_1)} \frac{g_0'(f_0(s_2))f_0'(s_2)}{g_0'(s_2)} \cdots \frac{g_0'(f_0(s_q))f_0'(s_q)}{g_0'(s_q)} \\ &= \frac{g_0'(s_2)f_0'(s_1)}{g_0'(s_1)} \frac{g_0'(s_3)f_0'(s_2)}{g_0'(s_2)} \cdots \frac{g_0'(s_1)f_0'(s_q)}{g_0'(s_q)} \\ &= \prod_{k=1}^{q} f_0'(s_k) < 1\end{aligned}$$

The second assertion is proved similarly. □

We are now ready to explain some of what appears in Figure 3.1.

Theorem 3.12. *Let* $a > 0$.

(a) Except among solutions whose initial values satisfy (3.15) there are no positive solutions of (3.13) that are periodic with an odd period.

(b) If $a \geq 6.26$ *and (3.15) holds, then (3.13) has periodic solutions of all possible periods, including odd periods, as well as chaotic solutions in the sense of Li and Yorke.*

(c) Let $r_{-1}, r_0 > 0$ be given initial values and define t_0 by (3.7). Assume that $t_0 \neq e^{a/2}$ and the sequence of iterates $\{f_0^n(r_{-1})\}$ of the map f_0 converges to a minimal q-cycle $\{s_1, \ldots, s_q\}$. Then the corresponding solution $\{r_n\}$ of (3.13) converges to the cycle $\{s_1, g_0(s_1), \ldots, s_q, g_0(s_q)\}$ of minimal period $2q$ in the sense that

$$\lim_{k\to\infty} |r_{2(k+j)-1} - s_j| = \lim_{k\to\infty} |r_{2(k+j)} - g_0(s_j)| = 0 \quad \textit{for} \quad j = 1, \ldots, q \tag{3.21}$$

(d) If $\{s_1, \ldots, s_q\}$ in (c) satisfies (3.20) and $s_j \neq 1$ for $j = 1, \ldots, q$ then for intial values $r'_{-1} > 0$ and $r'_0 = g_0(r'_{-1})$ where $|r'_{-1} - r_{-1}|$ is sufficiently small, the sequence $\{f_0^n(r'_{-1})\}$ converges to $\{s_1, \ldots, s_q\}$ and (3.21) holds.

(e) Let $r_{-1}, r_0 > 0$ be given initial values and define t_0 by (3.7). If the sequence of iterates $\{f_0^n(r_{-1})\}$ of the map f_0 is non-periodic then (3.13) has a non-periodic solution.

Proof. (a) This statement is an immediate consequence of Lemma 3.7 since the number of points in a cycle must divide two, i.e. the number of curves g_0, g_1. An exception occurs when (3.15) holds and the curves g_0, g_1 coincide.

(b) Suppose that the initial values r_{-1}, r_0 satisfy (3.15). Then $g_0 = g_1 = g$ and the trace of g contains the orbits of (3.13) since the trace of g is invariant by Lemma 3.7. By Lemma 3.8 g has a period-three point if $a \geq 6.26$ and in this case, (3.13) has solutions with all possible periods in the state-space, including odd periods. In addition, iterates of g also exhibit chaos in the sense of [65]. For (3.13) this is manifested in the state-space on the trace of g if the initial point (r_{-1}, r_0) is on the trace of g. For arbitrary initial values, odd periods do not occur by (a) and chaotic behavior generally occurs on the pair of curves g_0, g_1; see the Remark following this proof.

(c) This is an immediate consequence of Lemmas 3.7 and 3.10.

(d) If $|r'_{-1} - r_{-1}|$ is sufficiently small then Lemma 3.11 implies that the sequence

$\{f_0^n(r'_{-1})\}$ converges to $\{s_1,\ldots,s_q\}$. Now, if $r'_0 = g_0(r'_{-1})$ then $r'_0/r'_{-1}e^{r'_{-1}} = t_0$ and thus, (3.21) holds by Part (c).

(e) This is clear from Lemmas 3.7 and 3.10. □

Remark 3.13. 1. Theorem 3.12 explains how qualitatively different solutions in Figure 3.1 are generated by different initial values. Changes in the initial value r_0 of (3.13) while r_{-1} is fixed result, by (3.7) in changes in the parameter value t_0 in the mapping f_0. The one-dimensional map f_0 generates different types of orbits with different values of t_0 in the conventional way that is explained by the basic theory. All of these orbits, combined with the iterates of the shadow map f_1 appear in the state-space of (3.13) as points on the aforementioned pair of curves.

2. Part (d) of Theorem 3.12 explains the sense in which the solutions of (3.13) are stable and therefore appear as shown in Figure 3.1. This is not local or linearized stability since if $r'_0 \neq g_0(r'_{-1})$ then

$$t'_0 = \frac{r'_0}{r'_{-1}e^{-r'_{-1}}} \neq t_0$$

and with the different parameter value t'_0, $\{f_0^n(r'_{-1})\}$ may not converge to $\{s_1,\ldots,s_q\}$ even if $|r'_{-1} - r_{-1}|$ is small enough to imply local convergence for the iterates of f_0 defined with the original value t_0.

3. In Parts (a) and (b) of Theorem 3.12 if the initial point is not on the trace of g then the occurrence of all possible even periods and chaotic behavior is observed for smaller values of a. In fact, since g involves $a/2$ but f_0 involves a it follows that f_0 actually has period 3 points for $a \geq 3.13$ if the initial values yield a sufficiently small value of t_0. In Figure 3.2 a stable three-cycle is identified for $a = 3.6$ and initial values satisfying $r_0 = r_{-1}e^{-r_{-1}}$ (so that $t_0 = 1$). Odd periods do not occur for (3.13) in this case but all possible even periods, as well as chaotic behavior (on curve-pairs) do occur.

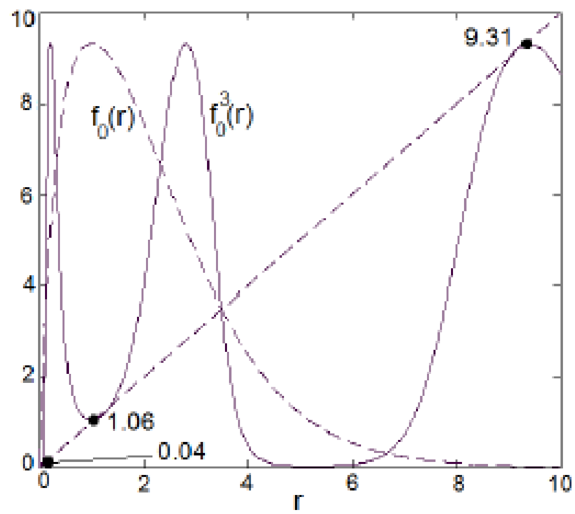

Figure 3.2: Occurrence of period three for the associated interval map

3.2.3 Convergence to two-cycles

The preceding results indicate that the solutions of (3.18) and (3.19) determine the solutions of (3.13). From Theorem 3.12 it is evident that complex behavior tends to occur when a is sufficiently large. Otherwise, the solutions of (3.13) tend to behave more simply as noted in Theorem 3.6. We now consider the occurrence of two-cycles for a range of values of a that are not too large but extend the range in Theorem 3.6, by examining the following first-order difference equation that is derived from (3.18) and (3.19)

$$r_{n+1} = r_n e^{a - r_n - \gamma r_n e^{-r_n}}, \quad \gamma > 0 \tag{3.22}$$

Lemma 3.14. *If $0 < a \leq 2$ then (3.22) has a unique positive fixed point $\bar{x}$.*

Proof. Existence: Let $\eta(x) = a - x - \gamma x e^{-x}$. The nonzero fixed points of (3.22) must satisfy $e^{\eta(x)} = 1$, i.e. $\eta(x) = 0$. Since $\eta(0) = a > 0$ and $\eta(a) = -\gamma a e^{-a} < 0$ there is a real number $\bar{x} \in (0, a)$ such that $\eta(\bar{x}) = 0$. This proves existence.

Uniqueness: Note that $\eta'(x) = -1 - \gamma e^{-x} + \gamma x e^{-x}$.

Case 1: $\gamma \leq e$; The function xe^{-x} is maximized on $(0, \infty)$ at $h(1) = e^{-1}$ so

$$\eta'(x) = -1 - \gamma e^{-x} + \gamma x e^{-x} \leq -1 + 1 - \gamma e^{-x} = -\gamma e^{-x} < 0$$

It follows that $\eta(x)$ is decreasing on $(0, \infty)$ for this case and has a unique zero that occurs at $\bar{x}$.

Case 2: $e < \gamma < e^2$; Consider the function $p(x) = x + \gamma x e^{-x}$. Now

$$p'(x) = 1 + \gamma e^{-x} - \gamma x e^{-x} = e^{-x}(e^x + \gamma - \gamma x)$$

The function $q(x) = e^x + \gamma - \gamma x$ attains a minimum value at $x = \ln(\gamma)$ since $q'(x) = e^x - \gamma$. Furthermore,

$$q(\ln(\gamma)) = 2\gamma - \gamma \ln(\gamma) = \gamma(2 - \ln(\gamma)) > 0$$

for $\gamma < e^2$. This implies that $p'(x) > 0$ on $(0, \infty)$ and therefore $p(x)$ is increasing on $(0, \infty)$. Since $\eta(x) = a - p(x)$, this implies that $\eta(x)$ is decreasing on $(0, \infty)$ and therefore it has a unique zero that occurs at $\bar{x}$.

Case 3: $\gamma > e^2$; In this case, we know that $\eta(x)$ is decreasing on $[0, 1]$ and $\eta(x) < 0$ for $x \in [d, \infty)$. Thus it remains to establish that $\eta(x) < 0$ on $(1, a)$.

$$\eta(x) = a - x - \gamma x e^{-x} < a - 1 - e^{2-x} < a - 2 \leq 0$$

Thus $\eta(x)$ has a unique zero that occurs at $\bar{x}$ and this completes the proof for all the above cases. $\square$

The above observations also indicate that $\eta(x) > 0$ for $x \in (0, \bar{x})$ and $\eta(x) < 0$ for $x \in (\bar{x}, \infty)$, which we will use in our further analysis. Before examining the stability profile of $\bar{x}$, we need to explore the properties of the function $f(x)$.

Since $f(x) = xe^{a-x-\gamma xe^{-x}} = xe^{\eta(x)}$, then $f'(x) = e^{\eta(x)} + x\eta'(x)e^{\eta(x)}$. By direct calculations, $f'(x)$ can be written as

$$f'(x) = e^{\eta(x)}(1-x)(1-\gamma xe^{-x})$$

It follows that f has critical points when $x = 1$ and $1 - \gamma xe^{-x} = 0$. Now we consider the function $\phi(x) = 1 - \gamma xe^{-x}$, which has a critical point at $x = 1$, since $\phi'(x) = \gamma e^{-x}(1-x)$. Hence it is decreasing on $(0,1)$ and increasing on $(1,\infty)$ and $\phi(1) = 1 - \frac{\gamma}{e}$ is the minimum of the function.

(i) When $\gamma < e$, then $\phi(1) > 0$, so $\phi(x) > 0$ on $(0,\infty)$, hence $f(x)$ has only one critical point at $x = 1$. When $\gamma = e, \phi(1) = 0$, and again, the only critical point of $f(x)$ occurs at $x = 1$. We further break down the case of $\gamma \leq e$ into the following subcases:

a. When $a < 1+\frac{\gamma}{e}$, $\eta(1) = a-1-\frac{\gamma}{e} < 0$, thus $\bar{x} < 1$. Moreover, $f(1) = a-1-\frac{\gamma}{e} < 1$, which lets us conclude that $f(x) < 1$ for all $x \in (0,\infty)$.

b. When $a \geq 1 + \frac{\gamma}{e}$, $\eta(1) = a - 1 - \frac{\gamma}{e} \geq 0$. This implies that $\bar{x} > 1$ if $a > 1 + \frac{\gamma}{e}$ and $\bar{x} = 1$ if $a = 1 + \frac{\gamma}{e}$.

(ii) When $\gamma > e, \phi(1) < 0$, so $f(x)$ has three critical points at $x' < 1, x' = 1, x'' > 1$.

On $(0, x')$, $1 - x > 0$ and $\phi(x) > 0$, so f is increasing. On $(x', 1)$, $1 - x > 0$ and $\phi(x) < 0$, so f is decreasing. On $(1, x'')$, $1 - x < 0$ and $\phi(x) < 0$, so f is increasing. On (x'', ∞), $1 - x < 0$ and $\phi(x) > 0$, so f is decreasing. By the above observations, it follows that x', x'' are local maxima and 1 is a minimum point. Next observe that

$$f(1) = e^{a-1-\frac{\gamma}{e}} < 1$$

Given that $\gamma x' e^{-x'} = \gamma x'' e^{-x''} = 1$,

$$f(x') = x' e^{a - x' - \gamma x' e^{-x'}} = x' e^{a - x' - 1} < x' e^{2 - x' - 1} = x' e^{1 - x'}$$

Similarly, $f(x'') < x'' e^{1-x''}$. Now, the function $s(x) = xe^{1-x}$ attains its maximum at $x = 1$, since $s'(x) = (1-x)e^{1-x}$. Since $s(1) = 1$, $s(x) < 1$ for all $x \neq 1, x > 0$. This implies that $f(x'), f(x'') < 1$ as well, thus for this case $f(x) < 1$ for all $x \in (0, \infty)$.

Now we establish the global stability of $\bar{x}$.

Lemma 3.15. *If $0 < a \leq 2$ then every solution to (3.22) from positive initial values converges to $\bar{x}$.*

Proof. We establish convergence to $\bar{x}$ by showing that $|f(x) - \bar{x}| < |x - \bar{x}|$ for $x \neq \bar{x}$. This is equivalent to

$$x < f(x) < 2\bar{x} - x \quad \text{for} \quad x < \bar{x} \tag{3.23a}$$

$$x > f(x) > 2\bar{x} - x \quad \text{for} \quad x > \bar{x} \tag{3.23b}$$

The first inequalities in (3.23a-3.23b) are straightforward to establish: since $\eta(x) > 0$ for $x < \bar{x}$ and $\eta(x) < 0$ for $x > \bar{x}$, then $f(x) = xe^{\eta(x)} > x$ if $x < \bar{x}$ and $f(x) = xe^{\eta(x)} < x$ if $x > \bar{x}$.

To establish the second inequalities in (3.23a)-(3.23b), let

$$t(x) = f(x) + x - 2\bar{x}$$

Notice that $t(0) = -2\bar{x} < 0$ and $t(\bar{x}) = 0$. We now show that the inequalities $f(x) < 2\bar{x} - x$ for $x < \bar{x}$ and $f(x) > 2\bar{x} - x$ for $x > \bar{x}$ are equivalent to $t(x) < 0$ for $x < \bar{x}$ and $t(x) > 0$ for $x > \bar{x}$, respectively. We establish this by showing that $t(x)$ is strictly increasing on $(0, \infty)$, i.e.

$$t'(x) = f'(x) + 1 > 0 \;\text{ for }\; x > 0$$

We consider two cases: *Case 1*: $\gamma \leq e$; recall that $f(x)$ is maximized at the unique critical point $x = 1$. Thus $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$. We also showed that $1 - \gamma x e^{-x} > 0$ for $x > 0$. Thus for all $x > 1$, since $d \leq 2$

$$\begin{aligned}
|f'(x)| &\leq e^{2-x-\gamma x e^{-x}}(x-1)(1-\gamma x e^{-x}) \\
&= (x-1)e^{1-x}e^{1-\gamma x e^{-x}}(1-\gamma x e^{-x}) \\
&< e^{-1}e^{1-\gamma x e^{-x}}(1-\gamma x e^{-x}) \\
&= e^{-\gamma x e^{-x}}(1-\gamma x e^{-x}) < 1
\end{aligned}$$

i.e. $t'(x) > 0$ for $x > 0$ and inequalities in (3.23a)-(3.23b) follow.

Case 2: $\gamma > e$; in this case, $f(x)$ has three critical points occurring at $x' < 1$, 1 and $x'' > 1$, where x' and x'' are maxima and 1 is a minimum. Thus

$$\begin{aligned}
&f'(x) > 0 \;\text{ and }\; 1 - \gamma x e^{-x} > 0 \;\text{ for }\; x \in (0, x') \\
&f'(x) < 0 \;\text{ and }\; 1 - \gamma x e^{-x} < 0 \;\text{ for }\; x \in (x', 1) \\
&f'(x) > 0 \;\text{ and }\; 1 - \gamma x e^{-x} < 0 \;\text{ for }\; x \in (1, x'') \\
&f'(x) < 0 \;\text{ and }\; 1 - \gamma x e^{-x} > 0 \;\text{ for }\; x \in (x'', \infty)
\end{aligned}$$

Thus $f'(x) < 0$ if either $x < 1$ and $1 - \gamma x e^{-x} < 0$ or $x > 1$ and $1 - \gamma x e^{-x} > 0$. If

$x < 1$ and $1 - \gamma x e^{-x} < 0$, then

$$\begin{aligned}
|f'(x)| &\leq e^{2-x-\gamma x e^{-x}}(1-x)(\gamma x e^{-x} - 1) \\
&= (\gamma x e^{-x} - 1)e^{1-\gamma x e^{-x}} e^{1-x}(1-x) \\
&< e^{-1} e^{1-x}(1-x) \\
&= e^{-x}(1-x) < 1
\end{aligned}$$

If $x > 1$ and $1 - \gamma x e^{-x} > 0$, then

$$\begin{aligned}
|f'(x)| &\leq e^{2-x-\gamma x e^{-x}}(x-1)(1-\gamma x e^{-x}) \\
&= (x-1)e^{1-x}(1-\gamma x e^{-x})e^{1-\gamma x e^{-x}} \\
&< e^{-1} e^{1-\gamma x e^{-x}}(1-\gamma x e^{-x}) \\
&= e^{-\gamma x e^{-x}}(1-\gamma x e^{-x}) < 1
\end{aligned}$$

In either case, if $f(x)$ is decreasing then $-1 < f'(x) < 0$, thus $t'(x) = f'(x) + 1 > 0$, thus $t(x)$ is increasing for $x > 0$, from which the second inequalities in (3.23a)-(3.23b) follow. □

By Lemmas 3.7 and 3.15, the even and odd terms of (3.13) converge to $M = \bar{x}_{t_0} > 0$ and $m = \bar{x}_{t_1} > 0$, proving the existence and stability of a two-cycle in the sense described in Theorem 3.12(c). Since M and m must satisfy

$$m = Me^{a-M-m} \text{ and } M = me^{d-m-M}$$

and

$$Mm = mMe^{2a-2(M+m)} \quad \text{i.e.} \quad e^{2a-2(M+m)} = 1$$

we conclude that $M + m = a$. Thus the following extension of Theorem 3.6 is obtained.

Theorem 3.16. *Let $0 < a \leq 2$. Then every non-constant, positive solution of (3.13)* *converges, in the sense of Theorem 3.12(c), to a two-cycle* $\{\rho_1, \rho_2\}$ *that satisfy* $\rho_1 + \rho_2 = a$*, i.e. the mean value of the limit cycle is the fixed point* $\bar{r} = a/2$*.*

3.2.4 A concluding remark on multistability

We finally mention a feature of (3.13) that may make its multistable nature less surprising. Consider the following class of nonautonomous first-order equations

$$x_{n+1} = x_n e^{\gamma_n - \theta_n x_n}$$

where γ_n, θ_n are given sequences of period 2 with $\theta_n > 0$ for all n. The change of variable $u_n = \theta_n x_n$ reduces this equation to

$$u_{n+1} = u_n e^{c_n - u_n}, \quad c_n = \gamma_n + \ln \frac{\theta_{n+1}}{\theta_n} \tag{3.24}$$

This equation can be written as

$$u_{n+1} = u_{n-1} e^{c_{n-1} + c_n - u_{n-1} - u_n}$$

Since c_n has period 2, the sum $c_{n-1} + c_n = a$ is a constant and (3.13) is obtained.

If $r_{-1} = u_0$ and $r_0 = u_1 = u_0 e^{c_0 - u_0}$ then the corresponding solution of (3.13) is the solution of (3.24) with the arbitrary initial value u_0. Therefore, all solutions of (3.24) appear among the solutions of (3.13) but not conversely. In fact, if c_n' is any other sequence of period 2 such that $c_n' + c_{n-1}' = d$ then while

$$u_{n+1} = u_n e^{c_n' - u_n}$$

is a different equation than (3.24), it yields exactly the same second-order equation (3.13). Hence, the following assertion is justified:

Proposition 3.17. *The solutions of (3.13) include the solutions of all first-order equations of type (3.24) with* $c_n + c_{n-1} = a$.

The coexistence of solutions of so many different first-order equations among the solutions of (3.13) is a further indication of the diversity of solutions that the latter may exhibit.

3.3 Periodic coefficients: the case where $p > 1$

In this section, we assume that the coefficients $\{a_n\}$ in

$$r_{n+1} = r_{n-1}e^{a_n - r_{n-1} - r_n} \tag{3.25}$$

are periodic with minimal period $p > 1$.

As might be expected, (3.25) has periodic solutions, which we establish below. But the range of variation, or amplitude of a_n, as well as whether the period of a_n is even or odd also play decisive roles. If the values of a_n are sufficiently large then (3.25) has both periodic and non-periodic solutions that are stable in a sense to be described below. Equation (3.25) thus has an abundance of qualitatively different, multistable solutions if a_n has a sufficiently large amplitude. We also show that the two cases where sequence $\{a_n\}$ has even period or odd lead to fundamentally different types of behaviors for the solutions of (3.25).

Recall from the previous section that (3.25) admits a semiconjugate factorization given by

$$t_{n+1} = \frac{e^{a_n}}{t_n}, \quad t_0 = \frac{r_0}{r_{-1}e^{-r_{-1}}} \tag{3.26}$$

$$r_{n+1} = t_{n+1}r_n e^{-r_n} \tag{3.27}$$

and that for arbitrary sequence of real numbers $\{a_n\}$ and $t_0 \neq 0$, the general solution of (3.26) is given by

$$t_n = t_0^{(-1)^n} e^{(-1)^n s_n}, \quad n = 1, 2, ... \tag{3.28}$$

where

$$s_n = \sum_{j=1}^{n} (-1)^j a_{j-1} \tag{3.29}$$

For every solution $\{t_n\}$ of (3.26), $t_{n+1}t_n = e^{a_n}$ for all n, so the even terms of the sequence $\{r_n\}$ are

$$r_{2n+2} = r_{2n} \exp\left(a_{2n+1} - r_{2n} - t_{2n+1}r_{2n}e^{-r_{2n}}\right) \tag{3.30}$$

and the odd terms are

$$r_{2n+1} = r_{2n-1} \exp\left(a_{2n} - r_{2n-1} - t_{2n}r_{2n-1}e^{-r_{2n-1}}\right) \tag{3.31}$$

Following the definition in (3.29) we let

$$\sigma = s_p = \sum_{j=1}^{p} (-1)^j a_{j-1} \tag{3.32}$$

Our next result lists a special case for σ in the equation (3.32) that makes the sequence $\{t_n\}$ in (3.9) periodic.

Lemma 3.18. *Let σ be defined by (3.32) and assume that $\{a_n\}$ is periodic with minimal period p. If $\sigma = 0$ and $t_0 = 1$, then $\{t_n\}$ is periodic with period p.*

Proof. If $\sigma = 0$, then by (3.9) and (3.10) in Lemma 3.3 we have:

$$t_p = t_0^{(-1)^p} e^{(-1)^p s_p} = e^{(-1)^p \sigma} = 1 = t_0$$

and

$$t_{n+p} = t_0^{(-1)^{n+p}} e^{(-1)^{n+p} s_{n+p}}.$$

Now

$$s_{n+p} = \sum_{j=1}^{n+p} (-1)^j a_{j-1} = \sum_{j=1}^{p} (-1)^j a_{j-1} + \sum_{j=p+1}^{n+p} (-1)^j a_{j-1} = \sigma + \sum_{j=p+1}^{n+p} (-1)^j a_{j-1}$$

If p is even, then

$$\sum_{j=p+1}^{n+p} (-1)^j a_{j-1} = -a_p + a_{p+1} + \cdots + (-1)^{n+p} a_{n+p-1} = -a_0 + a_1 + \cdots + (-1)^n a_{n-1} = s_n$$

so

$$t_{n+p} = t_0^{(-1)^{n+p}} e^{(-1)^{n+p} s_{n+p}} = t_0^{(-1)^n} e^{(-1)^n s_n} = t_n.$$

If p is odd, then

$$\sum_{j=p+1}^{n+p} (-1)^j a_{j-1} = a_p - a_{p+1} + \cdots + (-1)^{n+p} a_{n+p-1} = a_0 - a_1 + \cdots - (-1)^n a_{n-1} = -s_n$$

so

$$t_{n+p} = t_0^{(-1)^{n+p}} e^{(-1)^{n+p} s_{n+p}} = t_0^{(-1)^n} e^{-(-1)^n (-s_n)} = t_0^{(-1)^n} e^{(-1)^n s_n} = t_n.$$

and the proof is complete. □

The next result is based on an assumption that the sequence $\{t_n\}$ is periodic. We use the cofactor equation (3.27) to establish the following.

Lemma 3.19. *Let $\{r_n\}$ be a solution of (3.25) with initial values $r_{-1}, r_0 > 0$ and assume that $\{t_n\}$ given by (3.9)-(3.10) is periodic with period $q \geq 1$.Define*

$$g_k(r) = t_k r e^{-r},\ k = 0, 1, \ldots, q-1$$

Also define

$$h_k = g_k \circ g_{k-1} \circ \cdots \circ g_0, \quad k = 0, 1, \ldots, q-1$$

$$f = h_{q-1} = g_{q-1} \circ g_{q-2} \circ \cdots \circ g_1 \circ g_0$$

Then $\{r_n\}$ is determined by the q sequences

$$r_{qm+k} = h_k \circ f^m(r_{-1}), \quad k = 0, 1, \ldots, q-1 \tag{3.33}$$

that are obtained by iterations of one-dimensional maps of the interval $(0, \infty)$, with f^0 being the identity map.

Proof. Given the initial values $r_{-1}, r_0 > 0$ the definition of t_0 and (3.27) imply that

$$r_0 = t_0 r_{-1} e^{-r_{-1}} = g_0(r_{-1}) = h_0(r_{-1})$$

$$r_1 = t_1 r_0 e^{-r_0} = g_1(r_0) = g_1 \circ g_0(r_{-1}) = h_1(r_{-1})$$

and so on:

$$r_k = h_k(r_{-1}), \quad k = 0, 1, \ldots, q-2$$

Thus (3.33) holds for $m = 0$. Further, $r_{q-1} = h_{q-1}(r_{-1}) = f(r_{-1})$. Inductively, we suppose that (3.33) holds for some $m \geq 0$ and note that for $k = 0, 1, \ldots, q-2$

$$h_{k+1} = g_{k+1} \circ g_k \circ \cdots \circ g_0 = g_{k+1} \circ h_k$$

Now since $\{t_n\}$ is periodic and by (3.27), we have

$$\begin{aligned} r_{q(m+1)-1} &= t_{qm+q-1} r_{qm+q-2} e^{-r_{qm+q-2}} \\ &= t_{q-1} h_{q-2} \circ f^m(r_{-1}) e^{-h_{q-2} \circ f^m(r_{-1})} \\ &= g_{q-1} \circ h_{q-2} \circ f^m(r_{-1}) \\ &= h_{q-1} \circ f^m(r_{-1}) \\ &= f^{m+1}(r_{-1}) \end{aligned}$$

So (3.33) holds for $k = q - 1$ by induction. Further, again by (3.27), the preceding equality and periodicity of $\{t_n\}$ imply that

$$\begin{aligned} r_{q(m+1)} &= t_{qm+q} r_{qm+q-1} e^{-r_{qm+q-1}} \\ &= t_0 f^{m+1}(r_{-1}) e^{-f^{m+1}(r_{-1})} \\ &= g_0 \circ f^{m+1}(r_{-1}) \\ &= h_0 \circ f^{m+1}(r_{-1}) \end{aligned}$$

Similarly,

$$\begin{aligned} r_{q(m+1)+1} &= t_{q(m+1)+1} r_{q(m+1)} e^{-r_{q(m+1)}} \\ &= t_1 h_0 \circ f^{m+1}(r_{-1}) e^{-h_0 \circ f^{m+1}(r_{-1})} \\ &= g_1 \circ h_0 \circ f^{m+1}(r_{-1}) \\ &= h_1 \circ f^{m+1}(r_{-1}) \end{aligned}$$

Repeating this calculation $q-2$ times establishes (3.33) and completes the induction step and the proof. $\square$

Lemma 3.20. *Suppose that $\{t_n\}$ is periodic with period $q \geq 1$.*

(a) If the map f in Lemma 3.19 has a periodic point of minimal period r then

there is a solution of (3.25) with period rq.

(b) If the map f in Lemma 3.19 has a non-periodic point then (3.25) has a non-periodic solution.

Proof. (a) By hypothesis, there is a number $s \in (0, \infty)$ such that $f^{n+r}(s) = f^n(s)$ for all $n \geq 0$. We may assume that the number t_0 is fired since f is defined on the basis of the numbers t_k for $k = 0, 1, \ldots, q-1$. Let $r_{-1} = s$ and define $r_0 = h_0(s)$. By Lemma 3.19 the solution r_n corresponding to these initial values follows the track shown below:

$$
\begin{array}{rrcr}
r_{-1} = s \rightarrow & r_0 = h_0(s) \rightarrow & \cdots \rightarrow & r_{q-2} = h_{q-2}(s) \rightarrow \\
r_{q-1} = h_{q-1}(s) = f(s) \rightarrow & r_q = h_0(f(s)) \rightarrow & \cdots \rightarrow & r_{2q-2} = h_{q-2}(f(s)) \rightarrow \\
r_{2q-1} = h_{q-1}(f(s)) = f^2(s) \rightarrow & r_{2q} = h_0(f^2(s)) \rightarrow & \cdots \rightarrow & r_{3q-2} = h_{q-2}(f^2(s)) \rightarrow \\
\vdots & \vdots & \vdots & \vdots \\
r_{rq-1} = h_{q-1}(f^{r-1}(s)) = f^r(s) = s \rightarrow & r_{rq} = h_0(s) \rightarrow & \cdots \rightarrow & r_{q(r+1)-2} = h_{q-2}(s) \rightarrow \cdots
\end{array}
$$

The pattern in above list evidently repeats after rq entries. So $r_{rq+n} = r_n$ for $n \geq 0$ and it follows that the solution $\{r_n\}$ of (3.25) has period rq.

(b) Suppose that $\{f^n(r_{-1})\}$ is a non-periodic sequence for some $r_{-1} > 0$. Then by Lemma 3.19 the solution $\{r_n\}$ of (3.25) with initial values r_{-1} and $r_0 = g_0(r_{-1})$ has the non-periodic subsequence

$$r_{qn-1} = f^n(r_{-1})$$

It follows that $\{r_n\}$ is non-periodic. $\square$

3.3.1 The odd period case

When $\{a_n\}$ is periodic with minimal odd period, the sequence $\{t_n\}$ itself is periodic, as we show in the next lemma.

Lemma 3.21. *Suppose that $\{a_n\}$ is sequence of real numbers with minimal odd period*

$p \geq 1$ and let $\{t_n\}$ be a solution of (3.5). Then $\{t_n\}$ has period $2p$ with a complete cycle $\{t_0, t_1, \ldots, t_{2p-1}\}$ where t_k is given by (3.9) with

$$s_k = \begin{cases} \sum_{j=1}^{k}(-1)^j a_{j-1}, & \text{if } 1 \leq k \leq p \\ \sum_{j=k}^{2p-1}(-1)^j a_{j-p}, & \text{if } p+1 \leq k \leq 2p-1 \end{cases} \tag{3.34}$$

Proof. Let $\{a_0, a_1, \ldots, a_{p-1}\}$ be a full cycle of a_n and define σ as in (3.32) to be

$$\sigma = \sum_{j=1}^{p}(-1)^j a_{j-1} = -a_0 + a_1 - a_2 + \ldots - a_{p-1}$$

Since a full cycle of a_n has an odd number of terms, expanding s_n in (3.10) yields a sequence with alternating signs in terms of σ

$$s_n = \sigma - \sigma + \cdots + (-1)^{m-1}\sigma + (-1)^m \sum_{j=1}^{i}(-1)^j a_{j-1}$$

for integers i, m such that $n = pm + i$, $m \geq 0$ and $1 \leq i \leq p$. If m is even then for $i = 1, 2, \ldots, p$

$$s_n = \sum_{j=1}^{i}(-1)^j a_{j-1} = \begin{cases} -a_0 & n = pm+1 \quad \text{(odd)} \\ -a_0 + a_1 & n = pm+2 \quad \text{(even)} \\ \vdots & \vdots \\ -a_0 + a_1 \ldots - a_{p-1} & n = pm+p \quad \text{(odd)} \end{cases}$$

Similarly, if m is odd then for $i = 1, 2, \ldots, p$

$$s_n = \sigma - \sum_{j=0}^{i}(-1)^j a_j = \begin{cases} \sigma + a_0 & n = pm+1 \quad \text{(even)} \\ \sigma + a_0 - a_1 & n = pm+2 \quad \text{(odd)} \\ \vdots & \vdots \\ \sigma + a_0 - a_1 + \ldots - a_{p-1} & n = pm+p \quad \text{(even)} \end{cases}$$

The above list repeats for every consecutive pair of values of m and indicates a complete cycle for $\{s_n\}$. In particular, for $m = 0$ we obtain for $i = 1, 2, \ldots, p$

$$s_n = \sum_{j=1}^{i} (-1)^j a_{j-1} = \begin{cases} -a_0 & n = 1 \\ -a_0 + a_1 & n = 2 \\ \vdots & \vdots \\ -a_0 + a_1 \ldots - a_{p-1} & n = p \end{cases}$$

and for $m = 1$ we obtain for $i = 1, 2, \ldots, p - 1$

$$\begin{aligned} s_n = \sigma - \sum_{j=0}^{i} (-1)^j a_j &= \begin{cases} a_1 - a_2 + \ldots - a_{p-1} & n = p + 1 \\ -a_2 + \ldots - a_{p-1} & n = p + 2 \\ \vdots & \vdots \\ -a_{p-1} & n = 2p - 1 \end{cases} \\ &= \sum_{j=p+1}^{2p-1} (-1)^j a_{j-p} \end{aligned}$$

This proves the validity of (3.34) and shows that the sequence $\{s_n\}$ has period $2p$. Now (3.9) implies that $\{t_n\}$ also has period $2p$ and the proof is complete. $\square$

For $p = 1$, Lemma 3.21 implies that $\{t_n\}$ is the two-cycle that we encountered before:

$$\left\{ t_0, \frac{e^a}{t_0} \right\}$$

where a is the constant value of the sequence $\{a_n\}$. For $p = 3$, $\{t_n\}$ is the six-cycle

$$\left\{ t_0, \frac{e^{a_0}}{t_0}, t_0 e^{a_1 - a_0}, \frac{e^{a_2 - a_1 + a_0}}{t_0}, t_0 e^{a_1 - a_2}, \frac{e^{a_2}}{t_0} \right\}.$$

The next result establishes a special case where $\{t_n\}$ is periodic with odd period p.

Lemma 3.22. *Let $\{a_n\}$ be periodic with minimal odd period p and let σ be defined as in (3.32). If $\sigma \neq 0$ and $t_0 = e^{-\sigma/2}$, then $\{t_n\}$ is periodic with period p.*

Proof. If $\sigma \neq 0$ and p is odd, then

$$t_p = t_0^{(-1)^p} e^{(-1)^p s_p} = e^{\sigma/2-\sigma} = e^{-\sigma/2} = t_0$$

and since in Lemma 3.18 it was shown that $s_{n+p} = \sigma - s_n$, then

$$\begin{aligned} t_{n+p} &= t_0^{(-1)^{n+p}} e^{(-1)^{n+p} s_{n+p}} \\ &= e^{(-1)^n \sigma/2} e^{-(-1)^n (\sigma - s_n)} \\ &= e^{(-1)^n \sigma/2 + (-1)^n s_n} = t_0^{(-1)^n} e^{(-1)^n s_n} = t_n \end{aligned}$$

□

Theorem 3.23. *Suppose that $\{a_n\}$ is periodic with minimal odd period $p \geq 1$ and let f be the interval map in Lemma 3.19 where $t_0 > 0$ is a fixed real number and t_k is given by (3.9)-(3.10) for $k \geq 1$.*

(a) If s is a periodic point of f with period q then all solutions of (3.25) with initial values $r_{-1} = s$ and $r_0 = t_0 s e^{-s}$ (i.e. (r_{-1}, r_0) is on the curve g_0) have period $2pq$.

(b) If $\sigma = 0$, $t_0 = 1$ and s is a periodic point of f with period q, then all solutions of (3.25) with initial values $r_{-1} = s$ and $r_0 = t_0 s e^{-s}$ have period pq.

(c) If $\sigma \neq 0$, $t_0 = e^{\sigma/2}$ and s is a periodic point of f with period q, then all solutions of (3.25) with initial values $r_{-1} = s$ and $r_0 = t_0 s e^{-s}$ have period pq.

(d) If the map f has a non-periodic point, then (3.25) has a non-periodic solution.

(e) If f has a period-three point then (3.25) has periodic solutions of period $2pn$ for all positive integers n as well as chaotic solutions in the sense of Li-Yorke [65].

Proof. Parts (a)-(d) follow directly by Lemma 3.20 combined with Lemmas 3.18, 3.22 and 3.21.

(e) By [65], if f has a period three point, then f has periodic points of every period $n > 0$, as well as aperiodic, chaotic solutions in the sense of Li-Yorke. Therefore, by parts (a) and (b), (3.25) has periodic solutions of period $2pn$, as well as chaotic solutions. □

3.3.2 The even period case

When $\{a_n\}$ periodic with minimal even period p the next result shows that the sequence $\{t_n\}$ is *not* periodic with the exception of a boundary case. Once again, for convenience we define the quantity σ by (3.32).

Lemma 3.24. *Suppose that $\{a_n\}$ is a sequence of real numbers with minimal even period $p \geq 2$ and let $\{t_n\}$ be a solution of (3.5). Then*

$$t_n = \left(t_0 e^{d_n \sigma + \gamma_n}\right)^{(-1)^n} \tag{3.35}$$

where the integer divisor $d_n = [n - n(\operatorname{mod} p)]/p$ is uniquely defined by each n and

$$\gamma_n = \begin{cases} \sum_{j=1}^{n(\operatorname{mod} p)} (-1)^j a_{j-1} & \text{if } n(\operatorname{mod} p) \neq 0 \\ 0 & \text{if } n(\operatorname{mod} p) = 0 \end{cases} \tag{3.36}$$

In particular, $\{t_n\}$ is periodic with period p iff $\sigma = 0$, i.e.

$$a_0 + a_2 + \cdots a_p = a_1 + a_3 + \cdots + a_{p-1}. \tag{3.37}$$

Proof. Let $\{a_0, a_1, \ldots, a_{p-1}\}$ be a full cycle of a_n with an even number of terms. Since

$n = pd_n + n(\mathrm{mod}\, p)$ for $n \geq 1$, expand s_n in (3.10) to obtain

$$s_n = d_n\sigma + \sum_{j=1}^{n(\mathrm{mod}\, p)} (-1)^j a_{j-1}$$

if $n(\mathrm{mod}\, p) \neq 0$. If p divides n so that $n(\mathrm{mod}\, p) = 0$ then we assume that the sum is 0 and $s_n = d_n\sigma$. Thus $s_n = d_n\sigma + \gamma_n$ where γ_n is as defined in (3.36).

The σ terms have uniform signs in this case since there are an even number of terms in each full cycle of a_n. Now (3.9) yields

$$t_n = t_0^{(-1)^n} e^{(-1)^n s_n} = t_0^{(-1)^n} e^{(-1)^n (d_n\sigma + \gamma_n)}$$

which is the same as (3.35).

Next, if $\sigma \neq 0$ then $d_n\sigma$ is unbounded as n increases without bound so $\{t_n\}$ is not periodic. But if $\sigma = 0$ then (3.35) reduces to

$$t_n = (t_0 e^{\gamma_n})^{(-1)^n} \tag{3.38}$$

Since the sequence γ_n has period p, the expression on the right hand side of (3.38) has period p with a full cycle

$$t_1 = \frac{e^{a_0}}{t_0}, t_2 = t_0 e^{-a_0+a_1}, t_3 = \frac{e^{a_0-a_1+a_2}}{t_0}, \ldots, t_p = t_0 e^{-a_0+a_1+\cdots+(-1)^p a_{p-1}} = t_0.$$

$\square$

By the preceding result,

$$\begin{aligned} t_{2m} &= t_0 e^{\gamma_{2m}} e^{d_{2m}\sigma} && \text{if } n = 2m \text{ is even} \\ t_{2m+1} &= \frac{1}{t_0} e^{-\gamma_{2m+1}} e^{-d_{2m+1}\sigma} && \text{if } n = 2m+1 \text{ is odd} \end{aligned}$$

Suppose that $\sigma \neq 0$. If $\sigma > 0$ then since $\lim_{n\to\infty} d_n = \infty$ it follows that t_{2m} is unbounded but t_{2m+1} converges to 0, and the reverse is true if $\sigma < 0$. Therefore,

$$\lim_{m\to\infty} t_{2m} = \infty, \quad \lim_{m\to\infty} t_{2m+1} = 0, \quad \text{if } \sigma > 0, \tag{3.39}$$

$$\lim_{m\to\infty} t_{2m} = 0, \quad \lim_{m\to\infty} t_{2m+1} = \infty, \quad \text{if } \sigma < 0. \tag{3.40}$$

Lemma 3.25. *Suppose that $\{a_n\}$ is a sequence of real numbers with minimal even period $p \geq 2$ and let $\{r_n\}$ be a solution of (3.25) with initial values $r_{-1}, r_0 > 0$. Then $\lim_{n\to\infty} r_{2n+1} = 0$ if $\sigma > 0$ and $\lim_{n\to\infty} r_{2n} = 0$ if $\sigma < 0$.*

Proof. Assume first that $\sigma > 0$ but $\lim_{n\to\infty} r_{2n+1} \neq 0$ for some choice of initial values $r_{-1}, r_0 > 0$. Then by (3.31) there is a number $c > 0$ and a subsequence r_{n_k} such that for all $k = 1, 2, 3, \ldots$

$$r_{2n_k-1} \exp\left(a_{2n_k} - r_{2n_k-1} - t_{2n_k} r_{2n_k-1} e^{-r_{2n_k-1}}\right) = r_{2n_n+1} \geq c > 0 \tag{3.41}$$

We show that this inequality leads to a contradiction. By Theorem 3.1 and the boundedness of a_{2n_k} there is $M > 0$ such that $0 < r_{2n_k-1} \exp(a_{2n_k}) \leq M$ for all k. Thus, by (3.41)

$$\begin{aligned} r_{2n_k-1} \exp\left(a_{2n_k}\right) \exp(-r_{2n_k-1} - t_{2n_k} r_{2n_k-1} e^{-r_{2n_k-1}}) &\geq c \\ \exp(-r_{2n_k-1}) \exp(-t_{2n_k} r_{2n_k-1} e^{-r_{2n_k-1}}) &\geq \frac{c}{M} \\ \exp(-t_{2n_k} r_{2n_k-1} e^{-r_{2n_k-1}}) &> \frac{c}{M} \end{aligned}$$

Further, $ue^{-u} \leq 1/e$ for all $u \geq 0$ so that

$$\exp\left(\frac{-t_{2n_k}}{e}\right) \geq \exp(-t_{2n_k} r_{2n_k-1} e^{-r_{2n_k-1}}) > \frac{c}{M} > 0$$

However, by (3.39) $\lim_{n\to\infty} t_{2n_k} = \infty$ so the left hand side of the above chain converges to zero and we arrive at a contradiction. Hence, $\lim_{n\to\infty} r_{2n+1} = 0$ if $\sigma > 0$ as claimed.

If $\sigma < 0$ then a similar argument using (3.30) and (3.40) yields $\lim_{n\to\infty} r_{2n} = 0$ to complete the proof. □

Lemma 3.25 clearly indicates a marked difference between the case where a_n has an even period and the case where it has an odd period. Unlike the odd period case, half of the terms of every solution $\{r_n\}$ of (3.25) converge to 0 in the even period case if $\sigma \neq 0$. We now examine the *other* half of the terms of each solution $\{r_n\}$ of (3.25).

If $\sigma > 0$ then $\lim_{n\to\infty} t_{2n+1} = 0$ by (3.39) and (3.30) reduces to the equation

$$u_{2n+2} = u_{2n} e^{a_{2n+1} - u_{2n}} \tag{3.42}$$

If $\sigma < 0$ then $\lim_{n\to\infty} t_{2n} = 0$ by (3.40) and (3.31) reduces to

$$u_{2n+1} = u_{2n-1} e^{a_{2n} - u_{2n-1}} \tag{3.43}$$

Remark 3.26. Lemma 3.25 and equations (3.42), (3.43) indicate another significant difference between the odd and even period cases in the behaviors of solutions of (3.25). Specifically, if $\sigma \neq 0$ then **the asymptotic behavior in the even period case does not depend on the initial values**. Because t_n tends to either 0 or ∞, the number t_0 and thus the initial values, do not affect the limit set of the solution.

If $\sigma = 0$, the sequence $\{t_n\}$ is periodic, so the behavior of the iterates of (3.25) is similar to the case when p is odd. Similar to the odd case, we state the following result:

Theorem 3.27. *Suppose that $\{a_n\}$ is periodic with minimal even period $p > 1$ and let f be the interval map in Lemma 3.19 where $t_0 > 0$ is a fixed real number and t_k is given by (3.9)-(3.10) for $k \geq 1$. Further assume that $\sigma = 0$.*

(a) If s is a periodic point of f with period q, then all solutions of (3.25) with initial values $r_{-1} = s$, $r_0 = t_0 s e^{-s}$ have period pq.

(b) If the map f has a non-periodic point, then (3.25) has a non-periodic solution.

(c) If f has a period-three point, the (3.25) has periodic solutions of period pn for all $n > 0$, as well as chaotic solutions in the sense of Li and Yorke.

Proof. The proof is similar to that of Theorem 3.23. □

3.4 Concluding remarks, open problems and conjectures

The results obtained for the nonautonomous equation (3.25) with periodic coefficients $\{a_n\}$ show substantial differences of the behavior of the solutions depending on whether the period p is even or odd. In particular, the solutions of (3.25) exhibit dependence on initial conditions when p is odd, whereas it is not seen for the case when p is even, unless $\sigma = 0$. In previous sections we derived the mechanism that demonstrates why this is the case. Nonetheless, a number of questions remain to fully explain the behavior of the solutions of (3.25) with periodic coefficients. We conclude this chapter with the following open problems and conjectures that we leave for future research.

Conjecture 3.28. *Let $\{a_n\}$ be periodic with minimal period p and assume that $\{t_n\}$ is periodic with period q. If $0 < a_n < 2$, then the solution of (3.25) from initial values $r_{-1}, r_0 > 0$ converges to some s-cycle Γ, where s is the least common multiple of p and q. Further, if $r'_0, r'_{-1} > 0$ and*

$$\frac{r'_0}{r'_{-1} e^{-r'_{-1}}} = \frac{r_0}{r_{-1} e^{-r_{-1}}}$$

then the solution of (3.25) corresponding to r'_0, r'_{-1} converges to the same s-cycle Γ.

If the above conjecture is true, then the following can be shown for the case when p is odd.

Conjecture 3.29. *Let $\{a_n\}$ be periodic with minimal odd period p and further assume that $0 < a_i < 2$ for $0 \leq i = 0 \leq p-1$. Then for each pair of initial values $r_{-1}, r_0 > 0$,*

(a) the solution of (3.25) converge to a cycle Γ with length $2p$ that is determined by the values of $r_{-1}, r_0 > 0$ as stated in Conjecture 3.28.

(b) Furthermore, if either

$$\sigma = 0,\ t_0 = 1 \quad or \quad \sigma \neq 0,\ t_0 = e^{-\frac{\sigma}{2}}$$

then Γ is also a periodic cycle with period p.

A similar result can be shown for the case when p is even, and $\sigma = 0$.

Conjecture 3.30. *Suppose $\{a_n\}$ is periodic with minimal even period $p \geq 2$ and $\sigma = 0$. If $0 < a_n < 2$, then each solution of (3.25) from initial values $r_{-1}, r_0 > 0$ converges to a cycle of length p that is determined by $r_{-1}, r_0 > 0$ as stated in Conjecture 3.28.*

Conjecture 3.31. *Let $\{a_n\}$ be periodic with minimal even period $p \geq 2$. If $\sigma \neq 0$, then (3.25) has a globally attracting periodic solution $\{\bar{r}_n\}$ with period p (i.e. the p-cycle is not dependent on initial values).*

Problem 3.32. *Explore the behavior of the solutions of (3.25) when a_n are outside the range (0,2).*

In particular,

Conjecture 3.33. *Let $\{a_n\}$ be periodic with minimal period $p > 1$. Show that for certain values of $\{a_n\}$ outside the range $(0, 2)$, the equation (3.25) has chaotic solutions in the sense of Li and Yorke.*

Problem 3.34. *What specific results are possible in answering Problem 3.32 and Conjecture 3.33 if $p = 2$ so that (3.30) and (3.31) is autonomous in each case?*

A generalization of (3.2) given by

$$x_{n+1} = x_{n-1}e^{a_n - b_n x_n - c_n x_{n-1}} \text{ where } b_n \neq c_n \tag{3.44}$$

is a natural choice for future studies. In addition, exponential equation of the type

$$x_{n+1} = x_n e^{a_n - b_n x_n - c_n x_{n-1}} \tag{3.45}$$

has not been well-explored and may be of interest for future investigation. Since equations in (3.44) and (3.45) do not admit semiconjugate factorization and monotone function techniques generally do not apply, their study will involve alternative and possibly new methods of analysis.

CHAPTER IV

Folding of a Rational Planar System

In this chapter, we use the folding method to study the linear-rational planar system given by

$$x_{n+1} = a_n x_n + b_n y_n + c_n \tag{4.1a}$$
$$y_{n+1} = \frac{a'_n x_n + b'_n y_n + c'_n}{a''_n x_n + b''_n y_n + c''_n} \tag{4.1b}$$

where all parameters are sequences of real numbers.[1] The system in (4.1) is a natural choice for application of the folding method, since one of the equations in the system is linear and the non-linearity is confined solely in the second equation that is in linear-fractional form. The autonomous case of the system in (4.1) given by

$$x_{n+1} = a x_n + b y_n + c \tag{4.2a}$$
$$y_{n+1} = \frac{a' x_n + b' y_n + c'}{a'' x_n + b'' y_n + c''} \tag{4.2b}$$

and is a special case of the system initiated by [17] where both equations in (4.2) are of the form given in (4.2b).

[1]The content of this chapter, unless otherwise noted, is from [61] and [56].

We derive general conditions for uniform boundedness and convergence of solutions of (4.1) to the origin. We next study the autonomous case of (4.2) where the parameters are assumed to be constant. We investigate conditions on parameter values that guarantee the existence and uniqueness of a fixed point in the positive quadrant and show that under several broad assumptions on parameter values, this fixed point is a non-repeller. Using the folding equation, we then derive sufficient conditions for global convergence of the solutions of (4.2) to the fixed point, as well as occurrence of periodic solutions.

We then find special cases of (4.2) via its folding with some negative parameter values that exhibit chaos in the sense of Li-Yorke within the positive quadrant of the plane.The occurrence of chaotic orbits for (4.2) is far from obvious. It is well-known that a system of linear difference equations with constant coefficients does not have chaotic orbits. On the other hand, if one of the equations of the system is a polynomial of degree greater than 1 then the system may possess chaotic orbits within a bounded invariant set, as in the case of the familiar logistic map on the real line or the Henon map in the plane; see, e.g., [30], [33].

Prior studies of linear-fractional equations and systems (see [51] and references therein) have not been focused on demonstrating the occurrence of chaos or coexisting cycles and recent works [43], [73] that investigate homogeneous rational systems did not consider chaotic behavior. Studies of chaos in rational or planar systems generally do exist in the literature as indicated in the references below; see, e.g. [13]-[14] and [87]. In particular, in [87] the occurrence of chaos in homogeneous rational systems in the plane is established.

Since (4.2b) is discontinuous on the plane (unlike polynomial equations) the existence of solutions is guaranteed for (4.2) only if division by zero is avoided at every step of the iteration. In typical studies of rational systems it is generally assumed that all nine parameters and the initial values are non-negative (we refer to this as the *positive case*) to avoid possible occurrence of singularities in the positive quadrant $(0,\infty)^2 = (0,\infty)\times(0,\infty)$ of the plane. This quadrant is also the part of the plane that is naturally of greatest interest in modeling applications such as the aforementioned adult-juvenile model. But the type of nonlinearity exhibited by linear-fractional equations is of a particular kind that tends to be mild in nature away from singularities. This may be one reason for the relatively well-behaved orbits in the positive case rather than complex orbits that tend to be associated with rapid rates of change.

To be more precise, we show that in the positive case any fixed point $(\bar{x}, \bar{y})$ of (4.2) in the positive quadrant ($\bar{x}, \bar{y} > 0$) must be non-repelling, i.e., it is not true that both of the eigenvalues of the system's linearization at $(\bar{x}, \bar{y})$ have modulus greater than 1. This implies that $(\bar{x}, \bar{y})$ is not a snap-back repeller for this case.

We then consider cases where some of the 9 system parameters are negative and allow singularities to occur in the positive quadrant $(0,\infty)^2$. For instance, if $a''b'' < 0$ then the straight line $a''x+b''y+c'' = 0$ which is part of the singularity or forbidden set of the system in this case, crosses the positive quadrant so if any point (x_n, y_n) of an orbit of (4.2) falls on this line then division by zero occurs. With negative parameters it is necessary to either determine the forbidden sets or find a way of avoiding them. Determination of forbidden sets has been done for some higher order equations; see, e.g., [27], [74], [82]. But this is a difficult task for systems like (4.2). To identify special cases of (4.2) where orbits avoid such singularities we fold the system, i.e., transform it into a second-order quadratic-fractional equation and then find special cases in which the occurrence of Li-Yorke type chaos can be established *in the positive quadrant*. As a bonus, we find special cases of (4.2) that have periodic solutions of all possible periods in the positive quadrant. Obtaining these results would be quite difficult without folding.

4.1 Folding the system

Assuming that $b_n \neq 0$ for all $n \geq 0$ we solve (4.2a) for y_n to obtain

$$y_n = \frac{1}{b_n}(x_{n+1} - a_n x_n - c_n) \tag{4.3}$$

To avoid reductions to linear systems or to triangular systems, we may assume that for all $n \geq 0$

$$b_n \neq 0, \quad |a'_n| + |a''_n|, |a''_n| + |b''_n|, |a'_n| + |b'_n| + |c'_n| > 0. \tag{4.4}$$

We fold the above system as follows:

$$x_{n+2} = a_{n+1}x_{n+1} + b_{n+1}y_{n+1} + c_{n+1} = c_{n+1} + a_{n+1}x_{n+1} + \frac{b_{n+1}(a'_n x_n + b'_n y_n + c'_n)}{a''_n x_n + b''_n y_n + c''_n}$$

Using (4.1b) and (4.3) to eliminate y_n yields

$$x_{n+2} = c_{n+1} + a_{n+1}x_{n+1} + \frac{b_{n+1}[a'_n x_n + (b'_n/b_n)(x_{n+1} - a_n x_n - c_n) + c'_n]}{a''_n x_n + (b''_n/b_n)(x_{n+1} - a_n x_n - c_n) + c''_n}$$

Combining terms and simplifying we obtain the rational, second-order equation

$$x_{n+2} = a_{n+1}x_{n+1} + \frac{\sigma_{1,n}x_{n+1} + \sigma_{2,n}x_n + \sigma_{3,n}}{b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n}} \tag{4.5}$$

$$\sigma_{1,n} = b_{n+1}b'_n + c_{n+1}b''_n, \quad \sigma_{2,n} = b_{n+1}D'_{ab,n} + c_{n+1}D''_{ab,n}$$

$$\sigma_{3,n} = b_{n+1}D'_{cb,n} + c_{n+1}D''_{cb,n}$$

where

$$D'_{ab,n} = a'_n b_n - a_n b'_n, \quad D''_{ab,n} = a''_n b_n - a_n b''_n, \quad D'_{cb,n} = b_n c'_n - b'_n c_n, \quad D''_{cb,n} = b_n c''_n - b''_n c_n \tag{4.6}$$

The pair of equations (4.3) and (4.5) constitute a *folding* of (4.1). If (x_0, y_0) is an initial point for an orbit of (4.1) then the corresponding solution of the core equation (4.5) with initial values

$$x_0 \text{ and } x_1 = a_0 x_0 + b_0 y_0 + c_0 \tag{4.7}$$

yields the x-component of the orbit $\{(x_n, y_n)\}$ and the y-component is given (passively) by (4.3). Orbits of (4.1) are related to the solutions of (4.5), as seen next.

Theorem 4.1. *(a) Let $\{x_n\}$ be a solution of (4.5) with initial values (4.7). If $\{y_n\}$ is given by (4.3) then the sequence $\{(x_n, y_n)\}$ is an orbit of (4.1).*

(b) Let $\{(x_n, y_n)\}$ be an orbit of (4.1) from an initial point (x_0, y_0). Then $\{x_n\}$ is a solution of (4.5).

Proof. (a) Assume $\{x_n\}$ is a solution of (4.5) with initial values (4.7). Then by (4.3)

$$y_{n+1} = \frac{1}{b_{n+1}}(x_{n+2} - a_{n+1}x_{n+1} - c_{n+1}) \tag{4.8}$$

Substituting the expression from (4.5) into (4.8) yields

$$\begin{aligned}
y_{n+1} &= \frac{1}{b_{n+1}}(x_{n+2} - a_{n+1}x_{n+1} - c_{n+1}) \\
&= \frac{\sigma_{1,n}x_{n+1} + \sigma_{2,n}x_n + \sigma_{3,n}}{b_{n+1}(b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n})} - \frac{c_{n+1}}{b_{n+1}} \\
&= \frac{\sigma_{1,n}x_{n+1} + \sigma_{2,n}x_n + \sigma_{3,n} - c_{n+1}(b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n})}{b_{n+1}(b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n})} \\
&= \frac{(\sigma_{1,n} - c_{n+1}b''_n)x_{n+1} + (\sigma_{2,n} - c_{n+1}D''_{ab,n})x_n + \sigma_{3,n} - c_{n+1}D''_{cb,n}}{b_{n+1}(b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n})}
\end{aligned}$$

Expanding the terms for $\sigma_{1,n}, \sigma_{2,n}, \sigma_{3,n}$ and simplifying reduces the above to

$$y_{n+1} = \frac{b'_n x_{n+1} + D'_{ab,n}x_n + D'_{cb,n}}{b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n}}$$

On the other hand, since

$$
\begin{aligned}
\frac{a'_n x_n + b'_n y_n + c'_n}{a''_n x_n + b''_n y_n + c''_n} &= \frac{a'_n x_n + b'_n (x_{n+1} - a_n x_n - c_n)/b_n + c'_n}{a''_n x_n + b''_n (x_{n+1} - a_n x_n - c_n)/b_n + c''_n} \\
&= \frac{b'_n x_{n+1} + a'_n b_n x_n - a_n b'_n x_n - b'_n c_n + b_n c'_n}{b''_n x_{n+1} + a''_n b_n x_n - a_n b''_n x_n - b''_n c_n + b_n c''_n} \\
&= \frac{b'_n x_{n+1} + D'_{ab,n} x_n + D'_{cb,n}}{b''_n x_{n+1} + D''_{ab,n} x_n + D''_{cb,n}} \\
&= y_{n+1}
\end{aligned}
$$

the proof of (a) is complete.

(b) Assume $\{(x_n, y_n)\}$ is an orbit of (4.1) with initial point (x_0, y_0). Then by (4.1a)

$$x_{n+2} = a_{n+1} x_{n+1} + b_{n+1} y_{n+1} + c_{n+1} \tag{4.9}$$

and it is necessary to show that the right hand side of the above equality matches that in (4.5). By (4.1b) and above calculations

$$
\begin{aligned}
b_{n+1} y_{n+1} + c_{n+1} &= b_{n+1} \frac{a'_n x_n + b'_n y_n + c'_n}{a''_n x_n + b''_n y_n + c''_n} + c_{n+1} \\
&= b_{n+1} \frac{b'_n x_{n+1} + D'_{ab,n} x_n + D'_{cb,n}}{b''_n x_{n+1} + D''_{ab,n} x_n + D''_{cb,n}} + c_{n+1} \\
&= \frac{b_{n+1}(b'_n x_{n+1} + D'_{ab,n} x_n + D'_{cb,n}) + c_{n+1}(b''_n x_{n+1} + D''_{ab,n} x_n + D''_{cb,n})}{b''_n x_{n+1} + D''_{ab,n} x_n + D''_{cb,n}} \\
&= \frac{(b_{n+1} b'_n + c_{n+1} b''_n) x_{n+1} + (b_{n+1} D'_{ab,n} + c_{n+1} D''_{ab,n}) x_n + b_{n+1} D'_{cb,n} + c_{n+1} D''_{cb,n}}{b''_n x_{n+1} + D''_{ab,n} x_n + D''_{cb,n}} \\
&= \frac{\sigma_{1,n} x_{n+1} + \sigma_{2,n} x_n + \sigma_{3,n}}{b''_n x_{n+1} + D''_{ab,n} x_n + D''_{cb,n}} = x_{n+2} - a_{n+1} x_{n+1}
\end{aligned}
$$

Therefore, (4.9) implies (4.5) and the proof is complete. $\square$

Even when all the parameter sequences in (4.1) are non-negative the coefficients in (4.5) may be negative. Similarly, the coefficients in (4.5) may be positive even when some of the system parameters are negative. So the next result is worth highlighting.

Theorem 4.2. *(a) Assume that all the parameter sequences in (4.1) are non-negative with a''_n, b''_n, c''_n not simultaneously zero for every n. Then every orbit of (4.2) with $x_0, y_0 > 0$ is well-defined and in the positive quadrant so every solution of (4.5) with initial values $x_0 > 0$ and $x_1 = a_0x_0 + b_0y_0 + c_0$ is well-defined and positive.*

(b) Assume that all the coefficients in (4.5) are non-negative with $b''_n, D''_{ab,n}, D''_{cb,n}$ not simultaneously zero for every n. Then every solution of (4.5) with initial values $x_0, x_1 > 0$ is well-defined and positive so every orbit of (4.1) with $x_0 > 0$ and $y_0 = (x_1 - a_0x_0 - c_0)/b_0$ is well-defined and lies in the right half-plane.

4.2 Uniform boundedness and convergence to zero

In this section we obtain sufficient conditions for the uniform boundedness and permanence of the system (4.1) and for its folding (4.5). We also derive sufficient conditions for the global convergence of all non-negative solutions of (4.5) to its zero solution and the implication of this for the system.

4.2.1 Uniform boundedness of the system's orbits

In this section we assume that all parameters in (4.1) are non-negative. It is clear that in this case orbits $\{(x_n, y_n)\}$ of (4.2) whose initial points (x_0, y_0) in the positive quadrant $[0, \infty)^2$ remain there, i.e., $[0, \infty)^2$ is an invariant set of the system. We are interested in conditions that imply the uniform boundedness of all orbits in the positive quadrant.

Theorem 4.3. *Assume that all parameters in (4.2) are non-negative, let $\{b_n\}$ and $\{c_n\}$ be bounded sequences and $c''_n > 0$ for all $n \geq 0$.*

(a) Suppose that $\limsup_{n\to\infty} a_n < 1$ and there is $M > 0$ such that for all $n \geq 0$

$$a'_n \leq Ma''_n, \quad b'_n \leq Mb''_n, \quad c'_n \leq Mc''_n. \tag{4.10}$$

Then each orbit $\{(x_n, y_n)\}$ of (4.1) with initial point $(x_0, y_0) \in [0, \infty)^2$ is uniformly bounded from above.

(b) If in addition to the hypotheses in (a), $\liminf_{n\to\infty} b_n > 0$ *and there is* $L \in (0, M)$ *such that*

$$a'_n \geq La''_n, \quad b'_n \geq Lb''_n, \quad c'_n \geq Lc''_n \tag{4.11}$$

then there are $L', M' > 0$ such that for all sufficiently large n, $(x_n, y_n) \in [L', M'] \times [L, M]$ for each orbit of (4.1) with initial point $(x_0, y_0) \in [0, \infty)^2$.

Proof. Since $c''_n > 0$ it follows that y_n is defined for all $n \geq 0$. By (4.10) for all $n \geq 0$,

$$y_{n+1} = \frac{a'_n x_n + b'_n y_n + c'_n}{a''_n x_n + b''_n y_n + c''_n} \leq \frac{Ma''_n x_n + Mb''_n y_n + Mc''_n}{a''_n x_n + b''_n y_n + c''_n} = M$$

Hence the y-component is bounded. If $\limsup_{n\to\infty} a_n < 1$ then there is $a \in (0, 1)$ such that $a_n \leq a$ for all sufficiently large values of n. Also $\{b_n\}$ and $\{c_n\}$ are bounded so there is $\mu > 0$ such that $b_n, c_n \leq \mu$ for all n. Therefore, for n large enough,

$$x_{n+1} \leq ax_n + b_n M + c_n \leq ax_n + \mu(M + 1) \tag{4.12}$$

If N is a positive integer such that (4.12) holds for $n \geq N$ then in particular, $x_{N+1} \leq ax_N + \mu(M + 1)$ and

$$x_{N+2} \leq ax_{N+1} + \mu(M + 1) \leq a^2 x_N + \mu(M + 1)(1 + a)$$

Proceeding this way inductively we obtain

$$x_{n+N} \leq a^{n-1} x_N + \mu(M + 1)(1 + a + \cdots + a^{n-1}) = \frac{\mu(1 + M)}{1 - a} + a^n \left[x_N - \frac{\mu(1 + M)}{1 - a} \right]$$

As $n \to \infty$ the last term of the above expression approaches zero; in particular, for all n sufficiently large

$$a^{n-1}\left[x_N - \frac{\mu(1+M)}{1-a}\right] \leq \frac{a}{1-a}.$$

Therefore, for all n sufficiently large

$$x_n \leq \frac{\mu(1+M)}{1-a} + \frac{a}{1-a} = \frac{\mu(1+M)+a}{1-a}. \tag{4.13}$$

(b) By (4.11) for all $n \geq 0$,

$$y_{n+1} \geq \frac{La''_n x_n + Lb''_n y_n + Lc''_n}{a''_n x_n + b''_n y_n + c''_n} = L$$

Hence, $L \leq y_n \leq M$ for all $n \geq 0$. Since $\liminf_{n\to\infty} b_n > 0$ there is $b > 0$ such that $b_n \geq b$ for all large n so

$$x_{n+1} \geq ax_n + b_n L + c_n \geq ax_n + bL \tag{4.14}$$

If N is a positive integer such that (4.14) holds for $n \geq N$ then in particular, $x_{N+1} \geq ax_N + bL$ and

$$x_{N+2} \geq ax_{N+1} + bL \geq a^2 x_N + bL(1+a)$$

Proceeding this way inductively we obtain

$$x_{n+N} \geq a^{n-1}x_N + bL(1+a+\cdots+a^{n-1}) = \frac{bL}{1-a} + a^n\left[x_N - \frac{bL}{1-a}\right]$$

It follows that for all large n

$$x_n \geq \frac{bL+a}{1-a} \tag{4.15}$$

Define M' to be the right hand side of (4.13) and L' to be the right hand side of

(4.15). Then

$$0 < L' < \frac{bM + a}{1 - a} \leq \frac{\mu M + a}{1 - a} < M'$$

and the proof is complete. $\square$

The next consequence of Theorem 4.3 also applies to the autonomous version of (4.1) where all sequences are constants.

Corollary 4.4. *Assume that* $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ *be bounded sequences of non-negative real numbers. If* $a'', b'', c'' > 0$ *and* $\limsup_{n\to\infty} a_n < 1$ *then there are* $M, M' > 0$*, such that for all sufficiently large* n*,* $(x_n, y_n) \in [0, M'] \times [0, M]$ *for each orbit of the following system*

$$x_{n+1} = a_n x_n + b_n y_n + c_n \tag{4.16a}$$

$$y_{n+1} = \frac{a'_n x_n + b'_n y_n + c'_n}{a'' x_n + b'' y_n + c''} \tag{4.16b}$$

with initial point $(x_0, y_0) \in [0, \infty)^2$.

Proof. If we define

$$M = \max\left\{\frac{1}{a''}\sup_{n\geq 0} a'_n, \frac{1}{b''}\sup_{n\geq 0} b'_n, \frac{1}{c''}\sup_{n\geq 0} c'_n, \right\}$$

then (4.10) holds and the proof is concluded by applying Part (a) of Theorem 4.3. $\square$

Corollary 4.5. *Let* $0 \leq a < 1$*,* $c \geq 0$ *and* $b, a', b', c', a'', b'', c'' > 0$*. Then there are* M_i, L_i*,* $i = 1, 2$ *such that* $0 < L_i < M_i$ *and for all sufficiently large* n*,* $(x_n, y_n) \in [L_1, M_1] \times [L_2, M_2]$ *for each orbit of the autonomous system*

$$x_{n+1} = a x_n + b y_n + c \tag{4.17a}$$

$$y_{n+1} = \frac{a' x_n + b' y_n + c'}{a'' x_n + b'' y_n + c''} \tag{4.17b}$$

with initial point $(x_0, y_0) \in [0,\infty)^2$.

Proof. The existence of M_1, M_2 is established in Corollary 4.4 with all sequences being constants. Further, if

$$L = \min\left\{\frac{a'}{a''}, \frac{b'}{b''}, \frac{c'}{c''}\right\}.$$

then (4.11) holds and the proof is conlcuded by applying Theorem 4.3 and defining $L_1 = L'$, $M_1 = M'$, $L_2 = L$, $M_2 = M$. ☐

The condition $c''_n > 0$ ensures the existence of solutions ($\liminf_{n\to\infty} c''_n = 0$ is admissible). It can be replaced by a number of other conditions that we will not discuss. The next result applies to certain systems that do not satisfy some the hypotheses of Theorem 4.3 or its corollaries. In particular, the parameter b''_n is arbitrary.

Theorem 4.6. *Assume that all parameters in (4.1) are non-negative, let* $\{b_n\}$ *and* $\{c_n\}$ *be bounded sequences and* $c''_n > 0$ *for all* $n \geq 0$*. Also suppose that there is* $M > 0$ *such that* $a'_n \leq Ma''_n$, $c'_n \leq Mc''_n$ *for all* $n \geq 0$*.If*

$$\limsup_{n\to\infty} a_n < 1, \qquad \limsup_{n\to\infty} \frac{b'_n}{c''_n} < 1$$

then each orbit $\{(x_n, y_n)\}$ *of (4.1) with initial point* $(x_0, y_0) \in [0,\infty)^2$ *is uniformly bounded from above.*

Proof. By the hypotheses, there is $\rho \in (0,1)$ such that $b'_n/c''_n \leq \rho$ for all sufficiently large n. Thus,

$$y_{n+1} = \frac{a'_n x_n + b'_n y_n + c'_n}{a''_n x_n + b''_n y_n + c''_n} \leq \frac{b'_n y_n + Ma''_n x_n + Mc''_n}{a''_n x_n + c''_n} \leq \frac{b'_n}{c''_n} y_n + M \leq \rho y_n + M$$

Using an argument similar to that in the proof of Part (a) of Theorem 4.3 we conclude that for all large n

$$y_n \leq \frac{M+\rho}{1-\rho}.$$

Since this proves that the y-component is bounded, again following the line of reasoning in the proof of Theorem 4.3 the proof is concluded. □

Corollary 4.7. *Let* $0 \leq a < 1$*,* $c, a', b', c', a'' \geq 0$ *and* $b, c'', a' + b' + c' > 0$*. If* $b'/c'' < 1$ *Then there are* $M, M' > 0$ *such that for all sufficiently large* n*,* $(x_n, y_n) \in [0, M'] \times [0, M]$ *for each orbit of the autonomous system*

$$x_{n+1} = ax_n + by_n + c$$
$$y_{n+1} = \frac{a'x_n + b'y_n + c'}{a''x_n + c''}$$

with initial point $(x_0, y_0) \in [0, \infty)^2$.

4.2.2 Uniform boundedness of the folding's solutions

The results in the preceding section establish the boundedness of solutions when the system parameters are non-negative even when certain folding parameters are negative. In this section, we study the boundedness of solutions when the folding parameters are non-negative even when certain system parameters are negative.

The following is a general result on the uniform boundedness of all solutions of (4.5).

Theorem 4.8. *Assume that* $b''_n, D''_{ab,n} \geq 0$ *and* $D''_{cb,n} > 0$ *for all* $n \geq 0$*, let* L, M *be real numbers such that* $0 \leq L < M$ *and for all* $n \geq 0$

$$Lb''_n \leq \sigma_{1,n} \leq Mb''_n, \; LD''_{ab,n} \leq \sigma_{2,n} \leq MD''_{ab,n}, \; LD''_{cb,n} \leq \sigma_{3,n} \leq MD''_{cb,n}. \tag{4.18}$$

If $\limsup_{n\to\infty} a_n < 1$ *then for all* n *sufficiently large,* $L \leq x_n \leq (M + a)/(1 - a)$ *for every solution* $\{x_n\}$ *of (4.5) with non-negative initial values.*

Proof. If (4.18) holds then $\sigma_{i,n} \geq 0$ for all n. In particular,

$$x_2 = a_1 x_1 + \frac{\sigma_{1,0} x_1 + \sigma_{2,0} x_0 + \sigma_{3,0}}{b_0'' x_1 + D_{ab,0}'' x_0 + D_{cb,0}''} \geq 0.$$

By induction it follows that $x_n \geq 0$ for all n. Further, by (4.18)

$$x_{n+2} \leq a_{n+1} x_{n+1} + \frac{b_n'' M x_{n+1} + D_{ab,n}'' M x_n + D_{cb,n}'' M}{b_n'' x_{n+1} + D_{ab,n}'' x_n + D_{cb,n}''} \leq a_{n+1} x_{n+1} + M \tag{4.19}$$

If $\limsup_{n\to\infty} a_n < 1$ then there is $a \in (0,1)$ such that $a_n \leq a$ for all sufficiently large values of n. If N is an integer large enough that (4.19) holds for $n \geq N$ then

$$x_{N+2} \leq a x_{N+1} + M$$

It follows that

$$x_{N+3} \leq a x_{N+2} + M \leq a^2 x_{N+1} + M(1+a)$$

Proceeding this way inductively we obtain for $n > N$

$$x_n \leq a^{n-N-1} x_{N+1} + M(1 + a + \cdots + a^{n-N-2}) = \frac{M}{1-a} + a^{n-N-1}\left[x_1 - \frac{M}{1-a}\right]$$

As $n \to \infty$ the last term of the above expression approaches zero; in particular, for all n sufficiently large

$$a^{n-N-1}\left[x_1 - \frac{M}{1-a}\right] \leq \frac{a}{1-a}.$$

Therefore, for all n sufficiently large

$$x_n \leq \frac{M}{1-a} + \frac{a}{1-a} = \frac{M+a}{1-a}$$

Finally, the proof is completed by observing that for all n

$$x_{n+2} \geq \frac{\sigma_{1,n}x_{n+1} + \sigma_{2,n}x_n + \sigma_{3,n}}{b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n}} \geq \frac{Lb''_n x_{n+1} + LD''_{ab,n}x_n + LD''_{cb,n}}{b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n}} \geq L.$$

□

The preceding result implies that (4.5) is *permanent* if $L > 0$; i.e., every solution is bounded and has no subsequence that converges to zero. The next result applies Theorem 4.8 to the original system.

Theorem 4.9. *Assume that (4.18) holds and* $|c_n| \leq \delta|b_n|$ *for some* $\delta > 0$ *and all* $n \geq 0$. *Then each orbit of (4.2) with* $x_0 > 0$ *and* $a_0x_0 + b_0y_0 + c_0 > 0$ *is bounded and contained in the right half-plane. Further, if* $L > 0$ *then no such solution approaches a point on the* y*-axis.*

Proof. By Theorem 4.8 and (4.3),

$$|y_n| \leq \left|\frac{x_{n+1} - a_nx_n - c_n}{b_n}\right| \leq \frac{x_{n+1} + a_nx_n}{b} + \left|\frac{c_n}{b_n}\right| \leq \frac{(1+a)(M+a)}{b(1-a)} + \delta.$$

The first conclusion now follows. If $L > 0$ then by Theorem 4.8 $x_n \geq L$ so the second conclusion also follows. □

The next result, in which the parameters are constants (independent of n) also follows from Theorem 4.8.

Corollary 4.10. *Assume* $0 \leq a < 1$, $b'', D''_{ab}, D''_{cb} > 0$, $\sigma_1 + \sigma_2 + \sigma_3 > 0$. *Then all solutions of the autonomous equation*

$$x_{n+2} = ax_{n+1} + \frac{\sigma_1 x_{n+1} + \sigma_2 x_n + \sigma_3}{b''x_{n+1} + D''_{ab}x_n + D''_{cb}}$$

with $x_0, x_1 \geq 0$ *are uniformly bounded. If in addition,* $\sigma_1, \sigma_2, \sigma_3 > 0$ *then all such solutions are bounded from below by a positive number.*

Proof. Define

$$M = \max\left\{\frac{\sigma_1}{b''}, \frac{\sigma_2}{D''_{ab}}, \frac{\sigma_3}{D''_{cb}}\right\} > 0.$$

Then the upper bound inequalities in (4.18) hold and Theorem 4.8 may be applied to conclude the proof. If additionally, all three parameters $\sigma_1, \sigma_2, \sigma_3$ are positive then define

$$L = \min\left\{\frac{\sigma_1}{b''}, \frac{\sigma_2}{D''_{ab}}, \frac{\sigma_3}{D''_{cb}}\right\} > 0$$

to satisfy the lower bound inequalities in (4.18) and apply Theorem 4.8 again to complete the proof. □

4.2.3 Global exponential stability of the zero solution

In this section we discuss conditions that lead to the exponential convergence of all non-negative solutions of (4.5) to the zero solution and what this means for the system. We assume that $\sigma_{3,n} = 0$ for all n in this section so that (4.5) reduces to

$$x_{n+2} = a_{n+1}x_{n+1} + \frac{\sigma_{1,n}x_{n+1} + \sigma_{2,n}x_n}{b''_n x_{n+1} + D''_{ab,n}x_n + D''_{cb,n}} \tag{4.20}$$

All parameters in the above equation are assumed to be non-negative. If $D''_{cb,n} > 0$ for all n then the above equation has a zero solution $x_n = 0$. Under certain conditions, this trivial solution is exponentially stable and attracts all non-negative solutions. To prove the main result of this section we need a general result from [85] that we state as a lemma.

Lemma 4.11. *Let $\alpha \in (0,1)$ and assume that the functions $f_n : \mathbb{R}^{k+1} \to \mathbb{R}$ satisfy the inequality*

$$|f_n(u_0, \ldots, u_k)| \leq \alpha \max\{|u_0|, \ldots, |u_k|\} \tag{4.21}$$

for all $n \geq 0$*. Then for every solution* $\{x_n\}$ *of*

$$x_{n+1} = f_n(x_n, x_{n-1}, \dots, x_{n-k}) \tag{4.22}$$

the following is true

$$|x_n| \leq \alpha^{n/(k+1)} \max\{|x_0|, |x_{-1}| \dots, |x_{-k}|\}. \tag{4.23}$$

We note that (4.21) implies that $x_n = 0$ is a constant solution of (4.22) and further, (4.23) implies that this solution is stable.

Theorem 4.12. *Assume that all parameters in (4.20) are non-negative and* $D''_{cb,n} > 0$ *for all* $n \geq 0$*. If* $\limsup_{n\to\infty} a_n < 1$ *and there is* $\mu \in (0,1)$ *such that for all large values of* n

$$\sigma_{1,n} + \sigma_{2,n} \leq (\mu - a_{n+1}) D''_{cb,n} \tag{4.24}$$

then every solution of (4.20) with initial values $x_0, x_1 \geq 0$ *converges to zero exponentially.*

Proof. Define

$$f_n(u_0, u_1) = a_{n+1} u_0 + \frac{\sigma_{1,n} u_0 + \sigma_{2,n} u_1}{b''_n u_0 + D''_{ab,n} u_1 + D''_{cb,n}}$$

Then

$$\begin{aligned} f_n(u_0, u_1) &\leq a_{n+1} u_0 + \frac{\sigma_{1,n} u_0 + \sigma_{2,n} u_1}{D''_{cb,n}} \\ &\leq \left(a_{n+1} + \frac{\sigma_{1,n} + \sigma_{2,n}}{D''_{cb,n}} \right) \max\{u_0, u_1\} \end{aligned}$$

By hypothesis $\limsup_{n\to\infty} a_n < 1$ so there is $\delta \in (0,1)$ such that $a_n < \delta$ for all large

n. If (4.24) holds and we define $\alpha = \max\{\mu, \delta\}$ then for all large n

$$\sigma_{1,n} + \sigma_{2,n} \leq (\alpha - a_{n+1})D''_{cb,n}$$

which may be written as

$$a_{n+1} + \frac{\sigma_{1,n} + \sigma_{2,n}}{D''_{cb,n}} \leq \alpha.$$

Thus (4.21) holds and the proof concludes by applying Lemma 4.11. $\square$

The following is the special case of Theorem 4.12 for the autonomous equation

$$x_{n+2} = ax_{n+1} + \frac{\sigma_1 x_{n+1} + \sigma_2 x_n}{b'' x_{n+1} + D''_{ab} x_n + D''_{cb}} \tag{4.25}$$

Corollary 4.13. *Assume that all parameters in (4.25) are non-negative and $D''_{cb} > 0$. If $a < 1$ and*

$$\sigma_1 + \sigma_2 < (1 - a)D''_{cb} \tag{4.26}$$

then (4.25) has no positive fixed points and 0 is globally exponentially stable with respect to $[0, \infty)$.

Proof. First, note that if (4.26) holds and we define

$$\mu = a + \frac{\sigma_1 + \sigma_2}{D''_{cb}} = \frac{\sigma_1 + \sigma_2 + aD''_{cb}}{D''_{cb}} < 1$$

then $\mu \in (a, 1)$ and $\sigma_1 + \sigma_2 \leq (\mu - a)D''_{cb}$. Thus, in light of Theorem 4.12 it only remains to show that (4.25) has no positive fixed points. A fixed point $\bar{x}$ of (4.25) satisfies the equation

$$\bar{x} = a\bar{x} + \frac{\sigma_1 \bar{x} + \sigma_2 \bar{x}}{b'' \bar{x} + D''_{ab} \bar{x} + D''_{cb}}$$

So if $\bar{x} \neq 0$ then the above equality yields

$$(1-a)(b'' + D''_{ab})\bar{x} = \sigma_1 + \sigma_2 - (1-a)D''_{cb}$$

Since $(1-a)(b'' + D''_{ab}) \geq 0$ if (4.26) holds then $\sigma_1 + \sigma_2 - (1-a)D''_{cb} < 0$ so $\bar{x}$ cannot be positive. $\square$

The next result applies Theorem 4.12 to the system (4.2).

Corollary 4.14. *Assume that the hypotheses of Theorem 4.12 are true with* $\sigma_{3,n} = 0$ *for all* n.

(a) Every orbit of (4.2) with $x_0 > 0$ *and* $ax_0 + b_0 y_0 + c_0 > 0$ *is in the right half-plane* $x > 0$ *and converges to the sequence* $\{(0, -c_n/b_n)\}$.

(b) If $b_n = b \neq 0$ *and* $c_n = c$ *are constants then the orbit* $\{(x_n, y_n)\}$ *of (4.2) in (a) lies in the right half-plane above the line* $ax + by + c = 0$ *as it converges to the fixed point* $(0, -c/b)$ *of (4.2) on the y-axis.*

Inequality (4.24) may hold when all folding parameters are non-negative but does it hold also when all *system* parameters in (4.2) are non-negative? To answer this question note that

$$\begin{aligned}
\sigma_1 + \sigma_2 - (1-a)D''_{cb} &= bb' + cb'' + bD'_{ab} + cD''_{ab} - (1-a)(bc'' - b''c) \\
&= bb' + cb'' + b(a'b - ab') + c(a''b - ab'') - (1-a)bc'' + (1-a)b''c \\
&= (1-a)bb' + a'b^2 + a''bc + 2(1-a)b''c - (1-a)bc''
\end{aligned}$$

We assume that $0 \leq a < 1$ and $b > 0$. Then the last inequality above may be negative only if $c'' > 0$. Now, $\sigma_3 = bD'_{cb} + cD''_{cb} = 0$ implies

$$c > 0 \Rightarrow bc'' = b''c + bb' - \frac{b^2 c'}{c}$$

which yields

$$\begin{aligned}\sigma_1+\sigma_2-(1-a)D''_{cb} &= (1-a)bb'+a'b^2+a''bc+2(1-a)b''c-(1-a)\left(b''c+bb'-\frac{b^2c'}{c}\right)\\ &= a'b^2+a''bc+(1-a)\left(b''c+\frac{b^2c'}{c}\right)>0\end{aligned}$$

Hence, (4.24) does not hold if $c>0$. Assume that $c=0$. Then $\sigma_3=0$ implies that $b^2c'=0$ so $c'=0$ and

$$\sigma_1+\sigma_2-(1-a)D''_{cb}=(1-a)bb'+a'b^2-(1-a)bc''$$

To satisfy (4.24) we set the above quantity to be negative and obtain

$$\begin{aligned}(1-a)b'+a'b-(1-a)c''&<0\\ (1-a)(b'-c'')+a'b&<0\end{aligned}$$

This leads to the following consequence of Corollary 4.13 for the system.

Corollary 4.15. *In (4.2) assume that $c=c'=0$, $0\le a<1$, $a'',b''\ge 0$ and $a',b,c''>0$. If*

$$a'b<(1-a)(c''-b')$$

then every orbit of (4.2) with initial point in $[0,\infty)^2$ converges to the origin; i.e., the origin is a global attractor of all orbits in $[0,\infty)^2$.

4.3 Folding to an autonomous equation

Suppose that there are constants $a, A, B, C, \sigma_1, \sigma_2, \sigma_3$ such that for all $n \geq 0$

$$a_n = a, \quad b''_n = A, \quad D''_{ab,n} = B, \quad D''_{cb,n} = C \tag{4.27a}$$

$$b_{n+1}b'_n + c_{n+1}b''_n = \sigma_1, \tag{4.27b}$$

$$b_{n+1}D'_{ab,n} + c_{n+1}D''_{ab,n} = \sigma_2, \tag{4.27c}$$

$$b_{n+1}D'_{cb,n} + c_{n+1}D''_{cb,n} = \sigma_3, \tag{4.27d}$$

The above conditions trivially hold if (4.2) is autonomous; however, they also hold for many types of nonautonomous systems. If (4.27) holds then (4.5) reduces to the autonomous equation

$$x_{n+2} = ax_{n+1} + \frac{\sigma_1 x_{n+1} + \sigma_2 x_n + \sigma_3}{Ax_{n+1} + Bx_n + C} \tag{4.28}$$

The 7 equations in (4.27) determine as many of the 9 system parameters. The following leaves two of the system parameters arbitrary and gives the values of the remaining parameters in terms of these two.

Theorem 4.16. *Suppose that (4.62) and (4.27) hold. Then:*

(a) $|A| + |B| > 0$; *i.e., A and B are not both zeros.*

(b) If $\{b_n\}$ and $\{c_n\}$ are arbitrary sequences such that $b_n \neq 0$ for all $n \geq 0$ then

$$a''_n = \frac{B + Aa}{b_n}, \quad c''_n = \frac{C + Ac_n}{b_n}, \quad a'_n = \frac{\sigma_2 + \sigma_1 a - (B + Aa)c_{n+1}}{b_n b_{n+1}}$$
$$b'_n = \frac{\sigma_1 - Ac_{n+1}}{b_{n+1}}, \quad c'_n = \frac{\sigma_3 + \sigma_1 c_n - (Ac_n + C)c_{n+1}}{b_n b_{n+1}}$$

Proof. (a) If $A \neq 0$ then we are done. Suppose that $A = 0$, i.e., $b''_n = 0$ for all n. Then by (4.27a) and (4.6), $a''_n b_n = B$. If $B = 0$ then $a''_n = 0$ for all n since by (4.62), $b_n \neq 0$ for all n. But now $|a''_n| + |b''_n| = 0$ which contradicts a hypothesis in (4.62). Hence,

$B \neq 0$ and (a) follows.

(b) From (4.27a) and (4.6) we obtain

$$B = a_n'' b_n - a_n b_n'' = a_n'' b_n - aA, \quad C = c_n'' b_n - c_n b_n'' = c_n'' b_n - c_n A$$

Solving these equations for a_n'' and c_n'', respectively, yields the stated results. Next, from (4.27b) we have

$$\sigma_1 = b_{n+1} b_n' + c_{n+1} b_n'' = b_{n+1} b_n' + c_{n+1} A$$

which may be solved for b_n' to yield the stated result. Similarly, (4.27c) yields

$$\begin{aligned}\sigma_2 &= b_{n+1} D_{ab,n}' + c_{n+1} D_{ab,n}'' = b_{n+1}(a_n' b_n - a b_n') + c_{n+1} B \\ &= b_n b_{n+1} a_n' - a(\sigma_1 - A c_{n+1}) + c_{n+1} B\end{aligned}$$

which we can solve for a_n'. Finally, (4.27d) implies

$$\begin{aligned}\sigma_3 &= b_{n+1} D_{cb}' + c_{n+1} D_{cb}'' = b_{n+1}(b_n c_n' - b_n' c_n) + c_{n+1} C \\ &= b_{n+1} b_n c_n' - (\sigma_1 - A c_{n+1}) c_n + c_{n+1} C\end{aligned}$$

which can be solved for c_n' to yield the stated result. □

Although in dealing with (4.28) we usually assume that its parameters are all non-negative, the next result shows that solutions may be non-negative even when a parameter is negative.

Theorem 4.17. *Let* $A, B, C, \sigma_2, \sigma_3 \geq 0$ *in (4.28). If*

$$\sigma_1 > -\left(Ca + 2\sqrt{Aa\sigma_3}\right) \tag{4.29}$$

then $x_n > 0$ *for all* n *if* $x_0, x_1 > 0$.

Proof. If (4.29) holds then since $A + B > 0$ by Theorem 4.16,

$$x_2 \geq ax_1 + \frac{\sigma_1 x_1 + \sigma_2 x_0 + \sigma_3}{Ax_1 + Bx_0 + C} \geq \frac{Aax_1^2 + (Ca + \sigma_1)x_1 + \sigma_3}{Ax_1 + Bx_0 + C} > \frac{(\sqrt{Aa}x_1 - \sqrt{\sigma_3})^2}{Ax_1 + Bx_0 + C} \geq 0.$$

It follows that by induction that $x_n > 0$ for all n. □

4.3.1 Fixed points in the positive quadrant

The fixed points of (4.2) satisfy the following equations:

$$\bar{x} = a\bar{x} + b\bar{y} + c \tag{4.30a}$$

$$\bar{y} = \frac{a'\bar{x} + b'\bar{y} + c'}{a''\bar{x} + b''\bar{y} + c''} \tag{4.30b}$$

From (4.30a),

$$\bar{y} = \frac{(1-a)\bar{x} - c}{b} \tag{4.31}$$

Before calculating the values of the x- and y-components we note the following facts about the solutions of the system (4.2).

Lemma 4.18. *Assume that all system parameters are non-negative and satisfy (4.62), i.e.,*

$$b > 0, \quad a' + a'', a'' + b'', a' + b' + c' > 0. \tag{4.32}$$

(a) If there is a fixed point $(\bar{x}, \bar{y})$ *of the system in the positive quadrant (i.e.,* $\bar{x}, \bar{y} > 0$*) then* $0 \leq a < 1$ *and* $\bar{x} > c/(1-a)$.

(b) If $a > 1$ *then every orbit of (4.2) in the positive quadrant is unbounded.*

Proof. (a) Let $(\bar{x}, \bar{y})$ be a fixed point of the system in the positive quadrant. Then by (4.30a)

$$(1-a)\bar{x} = b\bar{y} + c > c \geq 0 \tag{4.33}$$

since $b, \bar{y} > 0$ by hypothesis and (4.32). Since $\bar{x} > 0$ it follows that $1-a > 0$ or $a < 1$.

(b) From (4.2a) it follows that for all n

$$x_{n+1} = ax_n + by_n + c \geq ax_n$$

By induction, $x_n \geq a^n x_0$ for all n and it follows that the orbit is unbounded if $x_0 > 0$. □

Now, to calculate the fixed points, from (4.31) and (4.30b) we obtain

$$\begin{aligned}
\frac{(1-a)\bar{x} - c}{b} &= \frac{a'\bar{x} + b'[(1-a)\bar{x} - c]/b + c'}{a''\bar{x} + b''[(1-a)\bar{x} - c]/b + c''} \\
&= \frac{a'b\bar{x} + b'(1-a)\bar{x} - b'c + bc'}{a''b\bar{x} + b''(1-a)\bar{x} - b''c + bc''} \\
&= \frac{(D'_{ab} + b')\bar{x} + D'_{cb}}{(D''_{ab} + b'')\bar{x} + D''_{cb}}
\end{aligned}$$

Multiplying and rearranging the terms yields a quadratic equation in $\bar{x}$ given by

$$d_1\bar{x}^2 - d_2\bar{x} - d_3 = 0 \tag{4.34}$$

where

$$d_1 = (1-a)(b'' + D''_{ab})$$

$$d_2 = b(D'_{ab} + b') + c(D''_{ab} + b'') - (1-a)D''_{cb} = \sigma_1 + \sigma_2 - (1-a)D''_{cb}$$

$$d_3 = bD'_{cb} + cD''_{cb} = \sigma_3$$

Depending on whether some of the last 3 parameters are zeros or not, a number of possibilities for fixed points occur. Since we are only interested in the fixed points in the positive quadrant, it is relevant to point out that

$$b'' + D''_{ab} = b'' + a''b - ab'' = a''b + (1-a)b'' > 0 \quad \text{by (4.32)}$$

so by Lemma 4.18 $d_1 > 0$. Assuming that

$$d_2^2 + 4d_1d_3 \geq 0 \tag{4.35}$$

to ensure the existence of real solutions for (4.34), we calculate the roots:

$$\begin{aligned}\bar{x} &= \frac{d_2 \pm \sqrt{d_2^2 + 4d_1d_3}}{2d_1}\\ &= \frac{b(D''_{ab}+b') + c(D''_{ab}+b'') - (1-a)D''_{cb}}{2(1-a)(D''_{ab}+b'')} \pm \\ &\qquad \frac{\sqrt{[(b(D''_{ab}+b') + c(D''_{ab}+b'') - (1-a)D''_{cb}]^2 + 4(1-a)(D''_{ab}+b'')(cD''_{cb}+bD'_{cb})}}{2(1-a)(D''_{ab}+b'')}\end{aligned}$$

These roots can be expressed more succinctly using the parameters of the folding. We use the notation $\bar{x}$ for the root with the positive sign

$$\bar{x} = \frac{\sigma_1 + \sigma_2 - (1-a)D''_{cb} + \sqrt{[\sigma_1 + \sigma_2 - (1-a)D''_{cb}]^2 + 4(1-a)(b'' + D''_{ab})\sigma_3}}{2(1-a)(b'' + D''_{ab})} \tag{4.37}$$

with $\bar{y}$ given by (4.31) and use $\tilde{x}$ to denote the root with the negative sign

$$\tilde{x} = \frac{\sigma_1 + \sigma_2 - (1-a)D''_{cb} - \sqrt{[\sigma_1 + \sigma_2 - (1-a)D''_{cb}]^2 + 4(1-a)(b'' + D''_{ab})\sigma_3}}{2(1-a)(b'' + D''_{ab})} \tag{4.38}$$

with $\tilde{y}$ again given by (4.31). It is an interesting fact that of the two fixed points above only one of them can be in the positive quadrant.

Lemma 4.19. *Let all system parameters in (4.2) be non-negative and satisfy (4.32). If (4.2) has a fixed point in* $(0,\infty)^2$ *then that fixed point is* $(\bar{x}, \bar{y})$ *and it is unique with*

$\bar{x}$ given by (4.37) and $\bar{y}$ given by (4.31).

Proof. Lemma 4.18 and the above discussion indicate that a necessary condition for the existence of fixed points in the positive quadrant is that $0 \leq a < 1$ holds. We found two possible fixed points given by (4.37) and (4.38) plus (4.31). Both of these are well defined if and only if (4.35) holds. Now, again by Lemma 4.18 the fixed point $(\bar{x}, \bar{y})$ is in the positive quadrant if $\bar{x} > c/(1-a)$, i.e.,

$$\begin{aligned} d_2 + \sqrt{d_2^2 + 4d_1d_3} &> \frac{2cd_1}{1-a} \\ \sqrt{d_2^2 + 4(1-a)(b'' + D''_{ab})\sigma_3} &> 2c(b'' + D''_{ab}) - d_2 \\ (1-a)\sigma_3 &> c^2(b'' + D''_{ab}) - cd_2 \end{aligned} \tag{4.39}$$

Similarly for $\tilde{x}$ it is required that

$$\begin{aligned} d_2 - \sqrt{d_2^2 + 4d_1d_3} &> \frac{2cd_1}{1-a} \\ \sqrt{d_2^2 + 4(1-a)(b'' + D''_{ab})\sigma_3} &< d_2 - 2c(b'' + D''_{ab}) \\ (1-a)\sigma_3 &< c^2(b'' + D''_{ab}) - cd_2. \end{aligned} \tag{4.40}$$

The preceding covers all possible fixed points in the first quadrant under the hypotheses of the lemma. We now show that (4.40) cannot hold, thus leaving $(\bar{x}, \bar{y})$ as the only possible fixed point in the first quadrant. Note that

$$\begin{aligned} cd_2 - c^2(b'' + D''_{ab}) &= c[\sigma_1 + \sigma_2 - (1-a)D''_{cb}] - c^2b'' - c^2D''_{ab} \\ &= c[bb' + b''c + b(a'b - ab') + cD''_{ab} - (1-a)D''_{cb}] - c^2b'' - c^2D''_{ab} \\ &= (1-a)bb'c + a'b^2c - c(1-a)D''_{cb} \end{aligned}$$

Therefore,

$$\begin{aligned}(1-a)\sigma_3 + cd_2 - c^2(b'' + D''_{ab}) &= (1-a)bD'_{cb} + (1-a)bb'c + a'b^2c \\ &= (1-a)b(c'b - b'c) + (1-a)bb'c + a'b^2c \\ &= (1-a)b^2c' + a'b^2c\end{aligned}$$

Since the last quantity is non-negative under the hypotheses, it follows that (4.40) does not hold and the proof is complete. □

4.4 Non-existence of repellers

We see in the proof of Lemma 4.19 that $(\bar{x}, \bar{y})$ exists in the positive quadrant if (4.35) and (4.39) both hold. Of particular interest to us is whether $(\bar{x}, \bar{y})$ can be repelling under the hypotheses of Lemma 4.19. We recall that a fixed point is *repelling* if all eigenvalues of the linearization of the system at that point have modulus greater than 1.

Theorem 4.20. *Let all system parameters in (4.2) be non-negative and satisfy (4.32). If (4.2) has a fixed point in $(0, \infty)^2$ then it is uniquely $(\bar{x}, \bar{y})$ and this is not a repelling fixed point.*

Proof. The first assertion follows from Lemma 4.19. To show that $(\bar{x}, \bar{y})$ is not repelling we examine the eigenvalues of the linearization of (4.2) at $(\bar{x}, \bar{y})$. The Jacobian matrix of (4.2) evaluated at the fixed point $(\bar{x}, \bar{y})$ is

$$J(\bar{x}, \bar{y}) = \begin{pmatrix} a & b \\ p & q \end{pmatrix}$$

where

$$p = \frac{a'(a''\bar{x} + b''\bar{y} + c'') - a''(a'\bar{x} + b'\bar{y} + c')}{(a''\bar{x} + b''\bar{y} + c'')^2},$$
$$q = \frac{b'(a''\bar{x} + b''\bar{y} + c'') - b''(a'\bar{x} + b'\bar{y} + c')}{(a''\bar{x} + b''\bar{y} + c'')^2}$$

Since by (4.30b)

$$a'\bar{x} + b'\bar{y} + c' = \bar{y}(a''\bar{x} + b''\bar{y} + c'') \tag{4.41}$$

the above expressions for p and q reduce to

$$p = \frac{a' - a''\bar{y}}{a''\bar{x} + b''\bar{y} + c''}, \quad q = \frac{b' - b''\bar{y}}{a''\bar{x} + b''\bar{y} + c''}. \tag{4.42}$$

The characteristic equation of the above Jacobian is

$$\lambda^2 - (a + q)\lambda - (bp - aq) = 0 \tag{4.43}$$

where

$$a + q = a + \frac{b' - b''\bar{y}}{a''\bar{x} + b''\bar{y} + c''} = \frac{aa''\bar{x} - (1 - a)b''\bar{y} + ac'' + b'}{a''\bar{x} + b''\bar{y} + c''} \tag{4.44}$$

and

$$bp - aq = \frac{a'b - a''b\bar{y} - ab' + ab''\bar{y}}{a''\bar{x} + b''\bar{y} + c''} = \frac{D'_{ab} - D''_{ab}\bar{y}}{a''\bar{x} + b''\bar{y} + c''} \tag{4.45}$$

Let

$$\alpha = a + q, \quad \beta = bp - aq$$

and write (4.43) as

$$\lambda^2 - \alpha\lambda - \beta = 0. \tag{4.46}$$

The roots of (4.46) are the eigenvalues, i.e.,

$$\lambda_1 = \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2}, \quad \lambda_2 = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2}.$$

When $\alpha^2 + 4\beta < 0$ (or $\beta < -\alpha^2/4$) the eigenvalues λ_1, λ_2 are complex and their common modulus is $-\beta$. So both eigenvalues have modulus greater than 1 if and only if

$$\beta < -1. \tag{4.47}$$

If $\alpha^2 + 4\beta \geq 0$ then both eigenvalues are real with $\lambda_1 \leq \alpha/2 \leq \lambda_2$. By considering the 3 possible cases

$$\lambda_1, \lambda_2 < -1, \quad \lambda_1, \lambda_2 > 1 \quad \text{or} \quad \lambda_1 < -1, \lambda_2 > 1$$

routine calculations show that both eigenvalues have modulus greater than 1 if and only if

$$2 < |\alpha| < 1 - \beta \quad \text{or} \quad \beta > 1 + |\alpha|. \tag{4.48}$$

With regard to (4.47) note that by (4.31) $\bar{x} - b\bar{y} = a\bar{x} + c$ so

$$\begin{aligned}
\beta + 1 &= \frac{D'_{ab} - D''_{ab}\bar{y} + a''\bar{x} + b''\bar{y} + c''}{a''\bar{x} + b''\bar{y} + c''} \\
&= \frac{a'b - ab' + a''(\bar{x} - b\bar{y}) + ab''\bar{y} + b''\bar{y} + c''}{a''\bar{x} + b''\bar{y} + c''} \\
&= \frac{a'b - ab' + a''c + a(a''\bar{x} + b''\bar{y}) + b''\bar{y} + c''}{a''\bar{x} + b''\bar{y} + c''}
\end{aligned}$$

By (4.41)

$$a''\bar{x} + b''\bar{y} = \frac{a'\bar{x} + b'\bar{y} + c'}{\bar{y}} - c'' = \frac{a'\bar{x} + c'}{\bar{y}} + b' - c''$$

so

$$\beta + 1 = \frac{a'b + a''c + a(a'\bar{x} + c')/\bar{y} + (1-a)c'' + b''\bar{y}}{a''\bar{x} + b''\bar{y} + c''} \geq 0.$$

It follows that (4.47) does not hold and further, $1 - \beta \le 2$ so that the first of the inequalities in (4.48) also does not hold. To check the remaining inequality $\beta > 1 + |\alpha|$ it is more convenient if we rewrite the expressions for α, β in terms of the folding parameters, using (4.31) to eliminate $\bar{y}$

$$\alpha = \frac{[(2a-1)b'' + aD''_{ab}]\bar{x} + \sigma_1 + aD''_{cb}}{(b'' + D''_{ab})\bar{x} + D''_{cb}} \tag{4.49}$$

$$\beta = \frac{\sigma_2 - (1-a)D''_{ab}\bar{x}}{(b'' + D''_{ab})\bar{x} + D''_{cb}} \tag{4.50}$$

Note that

$$\begin{aligned}(b'' + D''_{ab})\bar{x} + D''_{cb} &> \frac{c[a''b + (1-a)b'']}{1-a} + bc'' - b''c \\ &= \frac{a''bc + (1-a)b''c + (1-a)(bc'' - b''c)}{1-a} \\ &= \frac{a''bc + (1-a)bc''}{1-a} \ge 0\end{aligned}$$

so $\beta > 1 - \alpha$ if and only if

$$[(2a-1)b'' + aD''_{ab}]\bar{x} + \sigma_1 + aD''_{cb} > D''_{cb} + (b'' + D''_{ab})\bar{x} - \sigma_2 + (1-a)D''_{ab}\bar{x}$$

which reduces to

$$2(1-a)(b'' + D''_{ab})\bar{x} < \sigma_1 + \sigma_2 - (1-a)D''_{cb} \tag{4.51}$$

However, (4.37) implies that

$$2(1-a)(b'' + D''_{ab})\bar{x} \ge \sigma_1 + \sigma_2 - (1-a)D''_{cb}.$$

So (4.51) is false and thus, $\alpha \le 1 - \beta$, or equivalently, $\beta \le 1 - \alpha \le 1 + |\alpha|$. Hence, $(\bar{x}, \bar{y})$ is not repelling in the positive quadrant. $\square$

The above theorem shows that any fixed point of the system in the positive quadrant is non-repelling if all system parameters are non-negative; in particular, there are no snap-back repellers in the positive case (though unstable saddle fixed points exist for some parameter values).

4.5 Global stability and periodic solutions

In this section, we obtain several sufficient conditions for convergence of the solutions of (4.2) to the positive fixed point $\bar{x}$. Notice that the results obtained in Chapter 2 are directly applicable for this case, so the results below follow as corollaries.

Corollary 4.21. *Let the parameters of (4.2) satisfy*

$$0 \leq a < 1 \;\; b > 0, \;\; c,b'' \geq 0, \;\; bb' > -cb'' \tag{4.52a}$$

$$a'b > ab' \;\; a''b > ab'', \;\; c'b > cb' \;\; c''b > cb'' \tag{4.52b}$$

Then the system in (4.2) has a unique fixed point $(\bar{x}, \bar{y}) \in (0, \infty)^2$ where $\bar{x}$ is given by (4.37) and

$$\bar{y} = \frac{(1-a)\bar{x}}{b} \tag{4.53}$$

Proof. The conditions in (4.52) are sufficient to ensure that the parameters in the folding (4.5) as defined by (4.27) satisfy the conditions (2.2) in Chapter 2which implies that $\bar{x}$ given by (4.37) and $\bar{y}$ given by (4.53) are strictly positive. □

Corollary 4.22. *Let (4.52) hold. If the parameters of (4.2) satisfy either of the conditions below:*

(i)

$$D''_{ab}(bb' + cb'' \leq b''(bD'_{ab} + cD''_{ab}) + 2aD''_{ab}$$

$$b''(bD'_{cb} + cD''_{cb}) \leq a + bb' + cb''$$

$$D''(bD'_{cb} + cD''_{cb}) \leq bD'_{ab} + cD''_{ab}$$

(ii)

$$\frac{1}{b''}(bb' + cb'') \geq bD'_{cb} + cD''_{acb} \geq \frac{1}{D''_{cb}}(bD'_{ab} + cD''_{ab})$$

then all solutions of (4.2) from initial values $(x_0, y_0) \in [0, \infty)^2$ *converge to* $(\bar{x}, \bar{y}) \in (0, \infty)^2$.

Proof. The conditions in (4.52) are sufficient to ensure that the parameters in the folding (4.5) as defined by (4.27) satisfy the conditions (2.2) in Chapter 2 and that if $x_0, y_0 \geq 0$, then $x_0, x_1 = ax_0 + by_0 + c \geq 0$. Conditions in (i) and (ii) satisfy the hypotheses of Corollaries 2.8 and 2.17 in Chapter 2 from which the result follows. □

Example 4.23. Consider the following system

$$x_{n+1} = 0.2x_n + y_n + 1 \tag{4.54a}$$

$$y_{n+1} = \frac{1.2x_n + y_n + 2}{x_n + 0.8y_n + 1} \tag{4.54b}$$

The system in (4.54) folds into

$$y_n = x_{n+1} - 0.2x_n - 1 \tag{4.55a}$$

$$x_{n+2} = 0.2x_{n+1} + \frac{2x_{n+1} + 1.8x_n + 2}{x_{n+1} + 0.8x_n + 1} \tag{4.55b}$$

Routine calculations show that the parameters of (4.54) satisfy the conditions (ii) in Corollary 4.52, which implies that all solutions of (4.54) from nonnegative initial values converge to the fixed point in the positive quadrant.

Corollary 4.24. *Let (4.52) hold. Then (4.2) has a prime period two solution in* $[0, \infty)^2$ *if and only if*

$$2a''\bar{x} < a'b + a''c - ac'' - (1+a)(b'' + c'') \tag{4.56}$$

Proof. The conditions in (4.52) are sufficient to ensure that the parameters in the folding (4.5) as defined by (4.27) satisfy (2.2) in Chapter 2 and that if $x_0, y_0 \geq 0$, then $x_0, x_1 = ax_0 + by_0 + c \geq 0$. Straightforward algebraic calculations show that the condition in (2.16) can be expressed with respect to parameters of (4.2) as (4.56). By Lemma 2.4 the fixed point $(\bar{x}, \bar{y})$ is a saddle and the proof follows from Theorem 2.22. □

Example 4.25. Consider the system

$$x_{n+1} = 0.01x_n + y_n + 0.1 \tag{4.57a}$$

$$y_{n+1} = \frac{5x_n + 2y_n + 1}{0.1x_n + y_n + 1} \tag{4.57b}$$

The system in (4.57) folds into

$$y_n = x_{n+1} - 0.01x_n - 0.1 \tag{4.58a}$$

$$x_{n+2} = 0.01x_{n+1} + \frac{2.1x_{n+1} + 4.989x_n + 0.89}{x_{n+1} + 0.09x_n + 0.9} \tag{4.58b}$$

Routine calculations show that the system in (4.57) satisfies the conditions in Corollary (4.24), which lets us conclude that (4.57) has a prime period two solution.

Example 4.26. Consider the following system

$$x_{n+1} = 0.6x_n + 0.1y_n + 0.2 \tag{4.59a}$$

$$y_{n+1} = \frac{-0.1x_n - 0.1y_n + 0.1}{x_n + 0.05y_n + 2} \tag{4.59b}$$

The system in (4.59) folds into

$$y_n = 10x_{n+1} - 6x_n - 2 \tag{4.60a}$$

$$x_{n+2} = 0.6x_{n+1} + \frac{0.019x_n + 0.041}{0.05x_{n+1} + 0.07x_n + 0.19} \tag{4.60b}$$

By routine calculations, one may further show that the parameters in (4.59) satisfy the conditions (ii) in Corollary (4.22) which lets us conclude that all solutions from nonnegative initial values (x_0, y_0) converge to the positive fixed point in the first quadrant.

4.6 Cycles and chaos in the positive quadrant

If $a = 0$ then (4.5) reduces to the linear-fractional equation

$$x_{n+2} = \frac{\sigma_1 x_{n+1} + \sigma_2 x_n + \sigma_3}{b'' x_{n+1} + D''_{ab} x_n + D''_{cb}} \tag{4.61}$$

This type of linear-fractional equation has been studied extensively under the assumption of non-negative parameters; see, e.g., [51]. Although many questions remain to be answered about (4.61), chaotic solutions for it have not been found. To assure the occurrence of limit cycles and chaos and to avoid reductions to linear systems or to triangular systems where one of the equations is single-variable, we assume that

$$a, b, a', b'' \neq 0. \tag{4.62}$$

If some of the parameters in (4.5) are negative then even the existence and boundedness of solutions are nontrivial issues. Our aim here is to show that special cases of (4.5) with some negative coefficients exhibit Li-Yorke chaos in the positive quadrant. We note that (4.5) reduces to a first-order difference equation if

$$D'_{ab} = D''_{ab} = 0. \tag{4.63}$$

In this case, we define $r_n = x_{n+1}$ and $r_0 = x_1 = ax_0 + by_0 + c$ to obtain

$$r_{n+1} = ar_n + \frac{\sigma_1 r_n + \sigma_3}{b'' r_n + D''_{cb}} \tag{4.64}$$

The theory of one-dimensional maps may be applied to (4.64). To simplify calculations we assume in addition to (4.63) that

$$D''_{cb} = 0, \quad D'_{cb} \neq 0 \tag{4.65}$$

which reduce (4.64) to

$$r_{n+1} = ar_n + q + \frac{s}{r_n}, \tag{4.66}$$
$$\text{where } q = c + \frac{bb'}{b''}, \; s = \frac{bD'_{cb}}{b''}$$

Note that if $D'_{cb} = 0$ also then (4.66) is affine and as such, it does not have chaotic solutions.

A comprehensive study of Equation (4.66) appears in [25]. The following is a consequence of the results in [25]. We point out that if p is the minimal period of a solution $\{r_n\}$ of (4.66) with $r_0 > 0$ then the sequence $\{x_n\}$ also has minimal period p and by (4.3) $\{y_n\}$ has period p. It follows that the orbit $\{(x_n, y_n)\}$ has minimal period p.

Theorem 4.27. *Assume that conditions (4.62), (4.63) and (4.65) hold with the (normalized) values* $a = 1$, $b'' = bD'_{cb}$ *and define* $q = c + bb'/b''$.

(a) If $-2 < q < 0$ *then all orbits of (4.2) with* $x_0 + by_0 + c > 0$ *are well-defined and bounded. If also* $b'/b'' > 0$ *then these orbits are contained in* $(0, \infty)^2$.

(b) If $-\sqrt{2} < q < 0$ *then all orbits of (4.2) with* $x_0 + by_0 + c > 0$ *converge to the unique fixed point* $(\bar{x}, \bar{y}) = (-1/q, -c/b)$ *of (4.2).*

(c) If $-\sqrt{5/2} < q < -\sqrt{2}$ *then (4.2) has an asymptotically stable 2-cycle* $\{(x_1, y_1), (x_2, y_2)\}$ *where* y_i *is given by (4.3) and*

$$x_1 = \frac{-q - \sqrt{q^2 - 2}}{2}, \quad x_2 = \frac{-q + \sqrt{q^2 - 2}}{2}$$

(d) If $q = -\sqrt{3}$ *and* $x_0 + by_0 + c = 2\,(1 + \cos\pi/9)\,/\sqrt{3}$ *then the points* (x_i, y_i), $i = 1, 2, 3$ *constitute a stable orbit of period 3 for (4.2) where* y_i *is given by (4.3) and*

$$x_1 = \frac{2}{\sqrt{3}}\left(1 + \cos\frac{\pi}{9}\right), \quad x_2 = x_1 - \sqrt{3} + \frac{1}{x_1}, \quad x_3 = x_2 - \sqrt{3} + \frac{1}{x_2}$$

(e) If $-2 < q \le -\sqrt{3}$ *then orbits of (4.2) with* $x_0 + by_0 + c > 0$ *include cycles of all possible periods.*

(f) For $-2 < q < -\sqrt{3}$ *orbits of (4.2) with* $x_0 + by_0 + c > 0$ *are bounded and exhibit chaotic behavior.*

Proof. Statements (a)-(f) follow largely from Theorems 4-6 in [25]. It only remains to show that orbits whose initial points satisfy $x_0 + by_0 + c > 0$ are contained in $(0, \infty)^2$ and to determine the unique fixed point. Since $x_n = r_{n-1} > 0$ for all $n \ge 1$, (4.3) and

(4.66) imply under the assumptions in (a) that

$$\begin{aligned} y_n &= \frac{1}{b}(r_n - ar_{n-1} - c) \\ &= -\frac{c}{b} + \frac{1}{b}\left(q + \frac{1}{r_{n-1}}\right) \\ &= -\frac{c}{b} + \frac{c}{b} + \frac{b'}{b''} + \frac{1}{br_{n-1}} \\ &= \frac{b'}{b''}\left(1 + \frac{1}{r_{n-1}}\right) \end{aligned}$$

If $b'/b'' > 0$ then it follows that $y_n > 0$ for all $n \geq 1$ and the proof of (a) is complete. Finally, in (b) we see that the fixed point of (4.2) when $a = 1$ is determined from (4.30), (4.63) and (4.65) as

$$\left(\frac{-b''}{bb' + b''c}, -\frac{c}{b}\right) = \left(-\frac{1}{q}, -\frac{c}{b}\right)$$

which is in the positive quadrant if $q, c/b < 0$. □

Example 4.28. To illustrate Theorem 4.27, consider the following special case of (4.2)

$$\begin{aligned} x_{n+1} &= x_n + 2y_n - 2 \\ y_{n+1} &= \frac{0.75x_n + 1.5y_n}{3x_n + 6y_n - 6} \end{aligned}$$

which satisfies Part (c) of Theorem 4.27 ($q = -1.5$) and there exists an asymptotically stable 2-cycle $\{(1, 0.75), (0.5, 1.25)\}$ (a limit cycle) for this system. Different parameter values yield the following system which satisfies Parts (e) and (f) of Theorem 4.27

with $q \approx -1.83$

$$x_{n+1} = x_n + 2y_n - 2$$
$$y_{n+1} = \frac{0.25x_n + 0.5y_n + 1}{3x_n + 6y_n - 6}$$

This special case of (4.2) has periodic orbits of all periods (depending on initial points) and exhibits Li-Yorke type chaos. This fact is far from obvious and even the existence of cycles in the first quadrant for these equations is quite difficult to prove without folding.

We also mention that $b'/b'' > 0$ in both of the above systems so every orbit whose initial point (x_0, y_0) satisfies $x_0 + by_0 + c > 0$ is contained in the positive quadrant $(0, \infty)^2$.

The hypotheses of Theorem 4.27 are sufficient but not necessary for the occurrence of complex behavior in the positive quadrant. In fact, due to the continuity of rational expressions in terms of their parameters, the conclusions of Theorem 4.27 hold if the quantities $D'_{ab}, D''_{ab}, D''_{cb}$ are sufficiently small but not necessarily zero. Numerical simulations indicate Li-Yorke chaos persists in the positive quadrant if the parameters in the last system above are slightly perturbed. Caution is needed though because if we deviate too much from the conditions of Theorem 4.27 then the nontrivial nature of the singularity set must be taken into account before a claim of the occurrence of chaos can be verified.

4.7 Concluding remarks

In this chapter, we studied the dynamics of a linear-rational planar system and derived general conditions for uniform boundedness and convergence of solutions to the origin for the general, nonautonomous case. By folding the system into a second order quadratic rational equation, we derived several sufficient conditions for global convergence of the solutions to the positive fixed point, as well as occurrence of period two solutions. We showed that these conditions can hold for broader nonautonomous cases that fold into an equation with constant coefficients. In addition, we showed that these results can hold even if some of the parameters in the system are negative. We then used the folding to find special cases with some negative parameter values where the system has chaotic solutions within the positive quadrant of the plane.

Our use of the folding method is not standard in the published literature and leads to results that would have otherwise been difficult to establish. Since this method has not been systematically used in the study of systems (both in continuous and discrete time), further exploration of the method and its applicability is of great interest. In particular, the question of whether there are certain patterns or regularities in foldability of systems and their subsequent foldings are worth investigating.

CHAPTER V

Applications to Biological Models of Species Populations

Difference equations have been increasingly used in the study of species populations in biology, as seen in [89]. Systems of difference equations are used to capture interactions of two or more species, or of a single species where the members of population are differentiated by age or gender. Models with differentiation between adult (reproducing) and juvenile (nonreproducing) members are also known as stage- or age-structured models. In this chapter, we study the dynamics of a planar system that generalizes many common stage-structured population models in discrete time.[1]

Discrete time stage-structured models of single-species populations with lowest dimension are expressed as planar systems of difference equations. For a general expression of these models consider the system

$$A(t+1) = s_1(t)\sigma_1(c_{11}(t)J(t), c_{12}(t)A(t))J(t) + s_2(t)\sigma_2(c_{21}(t)J(t), c_{22}(t)A(t))A(t) \tag{5.1a}$$

$$J(t+1) = b(t)\phi(c_1(t)J(t), c_2(t)A(t))A(t) \tag{5.1b}$$

from [21] in which $J(t)$ and $A(t)$ are numbers (or densities) of juveniles and adults,

[1] The content of this chapter, unless otherwise indicated, is from [57] and [60].

respectively, remaining after t (juvenile) periods. The vital rates s_i and b (survival and inherent fertility rates) as well as the competition coefficients c_i and c_{ij} in (5.1) may be density dependent, i.e. they may depend on J and A and also explicitly on time, i.e. the system may be non-autonomous.

The models such as (5.1) are known as matrix models, in the sense that they can be expressed as

$$z(t+1) = P(t, z(t))z(t)$$

where $z(t) = [A(t), J(t)]'$ and $P(t, z(t))$ is the projection matrix of vital rates that may or may not be time or density dependent. Early examples of matrix models used to model species population dynamics can be found in [11],[62], [63], [64] and their comprehensive treatment is provided in [18].

Under certain constraints on the various functions, including periodic vital rates and competition coefficients having the same common period p, sufficient conditions for global convergence to zero (extinction) as well as the existence of periodic orbits for (5.1) are established in [21]. If μ is the mean fertility rate (the mean value of $b(t)$ above) then it is also shown that orbits of period p appear when μ exceeds a critical value μ_c while global convergence to the origin, or extinction occurs when $\mu < \mu_c$. On the other hand, conditions under which the species survives (i.e. *permanence*) were studied in [19] and [50].

In this chapter we study the following abstraction of the matrix model (5.1):

$$x_{n+1} = \sigma_{1,n}(x_n, y_n)y_n + \sigma_{2,n}(x_n, y_n)x_n \tag{5.2a}$$

$$y_{n+1} = \phi_n(x_n, y_n)x_n \tag{5.2b}$$

where for each time period $n \geq 0$ the functions $\sigma_{1,n}, \sigma_{2,n}, \phi_n : [0, \infty)^2 \to [0, \infty)$ are bounded on the compact sets in $[0, \infty)^2$. This feature allows for (0,0) to be a fixed point of the system and it is true if, e.g. $\sigma_{1,n}, \sigma_{2,n}, \phi_n$ are continuous functions for every n. Under biological constraints on the parameters, we may think of $x_n = A(n)$ and $y_n = J(n)$ as in (5.1).

System (5.2) includes typical stage-structured models in the literature. For instance, the tadpole-adult model for the green tree frog *Hyla cinerea* population that is proposed in [2] may be expressed as

$$x_n = \frac{y_n}{a + k_1 y_n} + \frac{x_n}{c + k_2 x_n} \tag{5.3a}$$

$$y_n = b_n x_n \tag{5.3b}$$

This is a system of type (5.2) with Beverton-Holt type functions σ_1 and σ_2. Competition in (5.3) occurs separately among juveniles and adults but not between the two classes, as they feed on separate resources; thus σ_1 and σ_2 do not depend on both juvenile and adult numbers and ϕ is independent of both numbers. Two cases are analyzed in [2]: (i) continuous breeding with constant $b_n = b$ so that (5.3) is autonomous, and (ii) seasonal breeding where b_n is periodic. In addition to considering extinction and survival in the autonomous case, it is shown that seasonal breeding may be deleterious (relative to continuous breeding) for populations with high birth rates, but it can be beneficial with low birth rates.

Another system of type (5.2) is the autonomous stage-structured model with harvesting that is discussed in [68] and [92], which may be written as

$$x_{n+1} = (1 - h_j)s_j y_n + (1 - h_a)s_a x_n \tag{5.4a}$$

$$y_{n+1} = x_n f((1 - h_a)x_n) \tag{5.4b}$$

The numbers $h_j, h_a \in [0, 1]$ denote the harvest rates of juveniles and adults, respectively. The stock-recruitment function $f : [0, \infty) \to [0, \infty)$ may be compensatory (e.g. Beverton-Holt [12]) or overcompensatory (e.g. Ricker [77]). Compensatory recruitment is used in populations where recruitment increases with increase in densities before reaching an asymptote, while in overcompensatory models recruitment declines as density increases as shown in Figure 5.1 (see [92] and [35]). A thorough analysis of the dynamics of (5.4) with the Ricker function appears in [68]. The results in [68] and [92] clarify many issues with regard to the effects of harvesting in stage-structured models. These results include global convergence to zero and the existence of a stable survival equilibrium, as well as the so-called *hydra effect* for different harvesting scenarios and with different recruitment functions. The latter refers to the counter-intuitive situation where an increase in the harvest or mortality rate results in a corresponding increase in the total population (for example, see [1], [42], [69]).

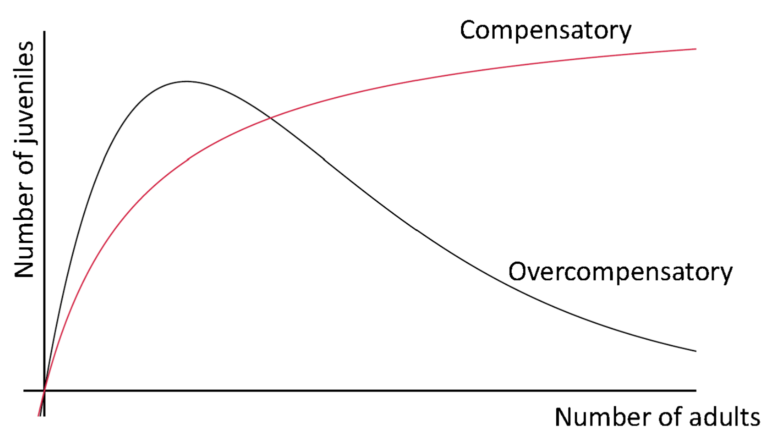

Figure 5.1: Compensatory and overcompensatory recruitment functions

Also studied in [68] is the occurrence of periodic and non-periodic attractors and chaotic behavior for certain parameter ranges.

Next, the model in [39] studies the harvesting and predation of sex and age structured populations. Although the added stage for two sexes results in a three-dimensional model, the existence of an attracting, invariant planar manifold reduces the study of the asymptotic behavior of the system to that of the planar system

$$x_{n+1} = ps_Y y_n + sx_n \tag{5.5a}$$

$$y_{n+1} = x_n f(y_n + x_n/p) \tag{5.5b}$$

where the density-dependent per capita reproductive rate f may be compensatory or overcompensatory (e.g. Beverton-Holt or Ricker), similarly to f in (5.4b). Here x_n is the number of females and y_n is the number of juveniles (the male population is a fixed proportion of the females).

We also mention the adult-juvenile model

$$x_{n+1} = s_1 y_n \tag{5.6a}$$

$$y_{n+1} = x_n f(x_n, y_n) \tag{5.6b}$$

in which all adults are removed through harvesting, predation, migration or just dying after one period, as in the case of semelparous species, i.e. organisms that reproduce only once before death. In [37] conditions for the global attractivity of the positive fixed point and the occurrence of two-cycles for (5.6) are obtained. A significant difference between (5.5) and (5.6) and the systems (5.3) and (5.4) is that y_{n+1} in (5.5b) or in (5.6b) may depend on both x_n and y_n.

We study the qualitative properties of the orbits of (5.2) such as uniform boundedness and global convergence to the origin under minimal restrictions on time-dependent parameters. Biological constraints may be readily imposed to obtain special cases relevant to population models.

We also investigate convergence to zero with periodic parameters (extinction in a periodic environment). In particular, we show that convergence to zero occurs even if the mean value of $\sigma_{2,n}$ exceeds 1, a situation that cannot occur if $\sigma_{2,n}$ is constant in n; see Remark 5.16 below.

In the final sections we study the dynamics of two special cases of (5.2) given by rational (Beverton-Holt) and exponential (Ricker) functions. Sufficient conditions for the global asymptotic stability of a fixed point in the positive quadrant $[0,\infty)^2$ are obtained, as well as conditions for the occurrence of orbits of prime period two. For the Beverton-Holt case, we establish that a sufficiently high level of interspecies competition tends to destabilize the survival fixed point and result in periodic oscillations. The dynamics of the Ricker case include examples of chaotic behavior that occurs in a variety of scenarios. In particular, chaotic behavior can occur both when the vital rates are constant, as well as periodic. In biological contexts, the periodicity can be thought of as seasonal fluctuations in vital rates.

5.1 Uniform boundedness of orbits

Conditions under which the orbits of (5.2) are bounded are not transparent. In this section we obtain general results about the uniform boundedness of orbits of (5.2) in the positive quadrant $[0,\infty)^2$. We begin with a simple, yet useful lemma.

Lemma 5.1. *Let $\alpha > 0$, $0 < \beta < 1$ and $x_0 \geq 0$. If for all $n \geq 0$*

$$x_{n+1} \leq \alpha + \beta x_n \tag{5.7}$$

then for every $\varepsilon > 0$ and all sufficiently large values of n

$$x_n \leq \frac{\alpha}{1-\beta} + \varepsilon.$$

Proof. Let $u_0 = x_0$ and note that every solution of the linear, first-order equation

$u_{n+1} = \alpha + \beta u_n$ converges to its fixed point $\alpha/(1-\beta)$. Further,

$$x_1 \leq \alpha + \beta x_0 = \alpha + \beta u_0 = u_1$$

$$x_2 \leq \alpha + \beta x_1 \leq \alpha + \beta u_1 = u_2$$

and by induction, $x_n \leq u_n$. Since $u_n \to \alpha/(1-\beta)$ for every $\varepsilon > 0$ and all sufficiently large n

$$x_n \leq u_n \leq \frac{\alpha}{1-\beta} + \varepsilon.$$

□

Theorem 5.2. *Let $\sigma_{1,n}, \sigma_{2,n}, \phi_n$ be bounded on the compact sets in $[0,\infty)^2$ for each $n = 0, 1, 2, \ldots$ and suppose that for some $r, M > 0$*

$$\sup_{(u,v)\in[0,r]^2} \sigma_{2,n}(u,v) \leq M \quad \textit{for all } n \geq 0 \tag{5.8}$$

i.e. the sequence of functions $\{\sigma_{2,n}\}$ is uniformly bounded on the square $[0,r]^2$. If there are numbers $M_0, M_1 > 0$ and $\bar{\sigma} \in (0,1)$ such that uniformly for all n

$$u\phi_n(u,v) \leq M_0 \quad \textit{if } (u,v) \in [0,\infty)^2, \tag{5.9}$$

$$\sigma_{1,n}(u,v) \leq M_1 \quad \textit{if } (u,v) \in [0,\infty) \times [0, M_0], \tag{5.10}$$

$$\sigma_{2,n}(u,v) \leq \bar{\sigma} \quad \textit{if } (u,v) \in (r,\infty) \times [0, M_0], \tag{5.11}$$

then all orbits of (5.2) are uniformly bounded and for all sufficiently large values of n

$$0 \leq x_n \leq \frac{M_0 M_1 + rM + \bar{\sigma}}{1-\bar{\sigma}}, \quad y_n \leq M_0. \tag{5.12}$$

Proof. By (5.2b) and (5.9) $y_n \leq M_0$ for $n \geq 1$ so by (5.2a) and (5.10)

$$0 \leq x_{n+1} \leq M_0 M_1 + \sigma_{2,n}(u,v) x_n.$$

By (5.8) and (5.11)

$$0 \le x_{n+1} \le M_0 M_1 + \max\{\bar{\sigma} x_n, Mr\} \le \bar{\sigma} x_n + M_0 M_1 + rM$$

Next, applying Lemma 5.1 with $\varepsilon = \bar{\sigma}/(1-\bar{\sigma})$ we obtain for all (large) n

$$0 \le x_n \le \frac{M_0 M_1 + rM}{1-\bar{\sigma}} + \varepsilon = \frac{M_0 M_1 + rM + \bar{\sigma}}{1-\bar{\sigma}}$$

as stated. □

Corollary 5.3. *For functions $\sigma_{1,n}, \sigma_{2,n}, \phi_n$ defined on $[0,\infty)^2$ for $n = 0, 1, 2, \ldots$ assume that there are numbers $M_0, M_1 > 0$ and $\bar{\sigma} \in (0,1)$ such that for all $(u,v) \in [0,\infty)^2$*

$$u\phi_n(u,v) \le M_0, \quad \sigma_{1,n}(u,v) \le M_1, \quad \sigma_{2,n}(u,v) \le \bar{\sigma}$$

uniformly all n. Then all orbits of (5.2) are uniformly bounded and for all sufficiently large values of n

$$0 \le x_n \le \frac{M_0 M_1 + \bar{\sigma}}{1-\bar{\sigma}}, \quad y_n \le M_0.$$

Theorem 5.2 is more general than the preceding corollary. For instance, Corollary 5.3 does not apply to the system

$$\begin{aligned} x_{n+1} &= a x_n + \frac{b y_n^2}{1 + c x_n} \\ y_{n+1} &= \frac{\alpha x_n}{1 + \beta x_n + \gamma y_n} \end{aligned}$$

However, if $a \in (0,1)$, $b, \alpha, \beta > 0$ and $c, \gamma \ge 0$ then all orbits of this system with initial values in $[0,\infty)^2$ are uniformly bounded by Theorem 5.2.

5.2 Global attractivity of the origin

In this section we obtain general sufficient conditions for the convergence of all orbits of the system to (0,0). For population models these yield conditions that imply the extinction of species.

5.2.1 General results

Throughout this section we assume that $\sigma_{i,n}, \phi_n$ are all bounded functions for $i = 1, 2$ and every $n = 0, 1, 2, \ldots$ Then the following are well-defined sequences of real numbers

$$\bar{\sigma}_{i,n} = \sup_{u,v \geq 0} \sigma_{i,n}(u,v), \quad \bar{\phi}_n = \sup_{u,v \geq 0} \phi_n(u,v). \tag{5.13}$$

Theorem 5.4. *If the following inequality holds*

$$\limsup_{n\to\infty}(\bar{\sigma}_{1,n}\bar{\phi}_{n-1} + \bar{\sigma}_{2,n}) < 1 \tag{5.14}$$

then $\lim_{n\to\infty} x_n = 0$ *for every orbit* $\{(x_n, y_n)\}$ *of the planar system (5.2) in the positive quadrant* $[0,\infty)^2$*. If also either the sequence* $\{\bar{\phi}_n\}$ *is bounded, or the following inequality holds*

$$\liminf_{n\to\infty} \bar{\sigma}_{1,n} > 0, \tag{5.15}$$

then every orbit of (5.2) converges to (0,0).

Proof. By (5.14) there is $\delta \in (0,1)$ such that $\bar{\sigma}_{1,n}\bar{\phi}_{n-1} + \bar{\sigma}_{2,n} \leq \delta$ for all (large) n. From (5.2a)

$$y_n \leq \bar{\phi}_{n-1}x_{n-1}$$

so for all (large) n (5.2b) yields

$$x_{n+1} \leq \bar{\phi}_{n-1}\bar{\sigma}_{1,n}x_{n-1} + \bar{\sigma}_{2,n}x_n \leq \left(\bar{\sigma}_{1,n}\bar{\phi}_{n-1} + \bar{\sigma}_{2,n}\right)\max\{x_n, x_{n-1}\} \leq \delta \max\{x_n, x_{n-1}\}$$

Lemma 4.11 now implies that $\lim_{n\to\infty} x_n = 0$. Further, by hypothesis either there is a positive number μ such that $\bar{\phi}_n \leq \mu$ or by (5.15) there is a positive number ρ such that $\bar{\sigma}_{1,n} \geq \rho$ for all (large) n so that

$$\bar{\phi}_{n-1} \leq \frac{\delta - \bar{\sigma}_{2,n}}{\bar{\sigma}_{1,n}} \leq \frac{\delta}{\rho}$$

for all sufficiently large values of n. Now, if $M = \mu$ or $M = \delta/\rho$ as the case may be, then from (5.2b) in the planar system we see that

$$\lim_{n\to\infty} y_n \leq \lim_{n\to\infty} \bar{\phi}_{n-1} x_{n-1} \leq M \lim_{n\to\infty} x_{n-1} = 0$$

and the proof is complete. $\square$

Remark 5.5. 1. Theorem 5.4 is valid even if the separate sequences $\{\sigma_{1,n}\}$ or $\{\bar{\phi}_n\}$ are not bounded by 1 as long as for all n large enough, $\bar{\sigma}_{1,n}\bar{\phi}_{n-1} \leq \delta - \bar{\sigma}_{2,n}$.

2. If (5.14) is satisfied but $\{\bar{\phi}_n\}$ is unbounded and $\{\bar{\sigma}_{1,n}\}$ does not satisfy (5.15) then y_n may not converge to 0; see the example following Corollary 5.18 below.

We consider an application of Theorem 5.4 to "noisy" autonomous system next. Let $\varepsilon_n, \varepsilon_{i,n},\ i = 1, 2$ be bounded sequences of real numbers and let

$$\bar{\varepsilon} = \sup_{n\geq 1} \varepsilon_n, \quad \bar{\varepsilon}_i = \sup_{n\geq 1} \varepsilon_{i,n},\ i = 1, 2$$

Also let $\sigma_1, \sigma_2, \phi : [0, \infty)^2 \to [0, \infty)$ be bounded functions and denote their supremums over $[0, \infty)^2$ by $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\phi}$, respectively. If in (5.2) we have

$$\phi_n(x_n, y_n) = \phi(x_n, y_n) + \varepsilon_n, \quad \sigma_{i,n}(x_n, y_n) = \sigma_i(x_n, y_n) + \varepsilon_{i,n},\ i = 1, 2$$

then we refer to (5.2) as an autonomous system with low-amplitude disturbances or fluctuations in the rates σ_1, σ_2, ϕ, assuming that all three of these are positive functions and for all $u, v \geq 0$

$$|\bar{\varepsilon}| \leq \phi(u,v), \quad |\bar{\varepsilon}_i| \leq \sigma_i(u,v),\ i = 1,2.$$

These inequalities ensure that the functions ϕ_n and $\sigma_{i,n}$ are positive, as required for (5.2).

Corollary 5.6. *Suppose that (5.2) is an autonomous system with low-amplitude disturbances or fluctuations in the above sense. If*

$$(\bar{\sigma}_1 + \bar{\varepsilon}_1)(\bar{\phi} + \bar{\varepsilon}) + \bar{\sigma}_2 + \bar{\varepsilon}_2 < 1 \tag{5.16}$$

then the origin is the unique, globally asymptotically stable fixed point of (5.2) relative to the positive quadrant $[0, \infty)$.

Note that (5.16) holds for nontrivial sequences $\varepsilon_n, \varepsilon_{i,n}$ of real numbers if $\bar{\sigma}_1\bar{\phi} + \bar{\sigma}_2 < 1$.

Remark 5.7. Since in the above discussion the sequences $\epsilon_n, \epsilon_{i,n}, i = 1,2$ are arbitrary bounded sequences, they can also be sequences of random variables that are drawn from distributions with finite support. For example, $\epsilon_n, \epsilon_{i,n}$ can be drawn from a uniform distribution on some interval $[0, \theta]$. So long as

$$(\bar{\sigma}_1 + \theta)(\bar{\phi} + \theta) + \bar{\sigma}_2 + \theta < 1$$

Corollary 5.6 will hold, implying that the origin is globally attracting even in the presence of noise.

In the autonomous case where the three parameter functions $\sigma_{1,n}, \sigma_{2,n}, \phi_n$ do not depend on n at all, we have the following planar system

$$x_{n+1} = \sigma_1(x_n, y_n)y_n + \sigma_2(x_n, y_n)x_n \tag{5.17a}$$

$$y_{n+1} = \phi(x_n, y_n)x_n \tag{5.17b}$$

If in Corollary 5.6 we set $\bar{\varepsilon}_i, \bar{\varepsilon} = 0$ in (5.16) then we obtain the following result for the above autonomous system.

Corollary 5.8. *Assume that $\sigma_1, \sigma_2, \phi : [0,\infty)^2 \to [0,\infty)$ are bounded functions and the following inequality holds*

$$\bar{\sigma}_1 \bar{\phi} + \bar{\sigma}_2 < 1 \tag{5.18}$$

then the origin is the unique, globally asymptotically stable fixed point of (5.17) relative to the positive quadrant $[0,\infty)^2$.

Remark 5.9. For the autonomous system given by

$$x_{n+1} = \sigma_1 y_n + \sigma_2 x_n$$
$$y_{n+1} = \phi(x_n, y_n) x_n$$

the equation (5.14) reduces to

$$\phi(x_n, y_n) \frac{\sigma_1}{1 - \sigma_2} < 1$$

The left hand side of the above equation corresponds to the density-dependent net reproductive rate R_0 described in [24] and [22]. General autonomous stage-structured matrix models can be written as

$$\boldsymbol{z}_{n+1} = P \boldsymbol{z}_n$$

where $\boldsymbol{z}_n$ is a vector of m stages of the species. If the projection matrix $P = F + T$ is additively decomposed to the matrix $F = [f_{i,j}]$ of birth processes and the matrix $T = [t_{i,j}]$ of transition probabilities from one stage to another, the net reproductive rate can be defined, as in [22], as

$$R_0 = \sum_{i=1}^{m} f_{1,j} \prod_{j=0}^{i-1} \frac{t_{j,j-1}}{1 - t_{j,j}}$$

In [22] it was shown that the species population growth rate r and the net reproductive rate R_0 are on the same side of 1. While for nonautonomous matrix systems, the definition of R_0 is not straightforward (see, for example, [23] for the case where the matrix P is periodically forced), the quantity

$$\bar{\phi}_{n-1} \frac{\bar{\sigma}_{1,n}}{1 - \bar{\sigma}_{2,n}}$$

can be thought of as the net reproductive rate at each period n. Since this implies that the population growth rate at each period is less than one, the biological interpretation of the result in Theorem 5.4 is not surprising.

Inequality (5.18) may be explicitly related to the local asymptotic stability of the origin for (5.17) when the functions σ_1, σ_2, ϕ are smooth. Consider the associated mapping

$$F(u, v) = (u\sigma(u, v) + v\sigma_1(u, v), u\phi(u, v))$$

whose linearization at (0,0) has eigenvalues

$$\lambda^{\pm} = \frac{\sigma_2(0, 0) \pm \sqrt{\sigma_2(0, 0)^2 + 4\sigma_1(0, 0)\phi(0, 0)}}{2}$$

These are real and a routine calculation shows that $|\lambda^{\pm}| < 1$ if

$$\sigma_1(0, 0)\phi(0, 0) + \sigma_2(0, 0) < 1.$$

Under suitable differentiability hypotheses, this inequality is implied by (5.18), and is equivalent to it if the suprema of σ_2 and $\sigma_1\phi$ occur at (0,0).

5.2.2 Folding the system

In the next, and later sections it will be convenient to fold the system (5.2) to a second order equation. System (5.2) in general folds as follows: Substitute for y_{n+1} from (5.2b) into (5.2a) to obtain

$$x_{n+2} = \sigma_{1,n+1}(x_{n+1}, \phi_n(x_n, h_n(x_n, x_{n+1}))x_n)\phi_n(x_n, h_n(x_n, x_{n+1}))x_n + \\ \sigma_{2,n+1}(x_{n+1}, \phi_n(x_n, h_n(x_n, x_{n+1}))x_n)x_{n+1} \tag{5.19}$$

where

$$h_n(x_n, x_{n+1}) = y_n \tag{5.20}$$

is derived by solving (5.2a) for y_n. Although an explicit formula for h_n is not feasible in general, it is readily obtained in typical cases; for instance, suppose that $\sigma_{2,n}(u, v) = \sigma_{2,n}(u)$ and $\sigma_{1,n}(u, v) = \sigma_{1,n}(u)$ are both independent of (or constant in) v for all n; note that the systems (5.3), (5.4), (5.5) and (5.6) are all of this type. In this case it is clear that

$$y_n = h_n(x_n, x_{n+1}) = \frac{x_{n+1} - \sigma_{2,n}(x_n)x_n}{\sigma_{1,n}(x_n)} \tag{5.21}$$

and further, (5.19) reduces to

$$x_{n+2} = \sigma_{1,n+1}(x_{n+1})\phi_n\left(x_n, \frac{x_{n+1} - \sigma_{2,n}(x_n)x_n}{\sigma_{1,n}(x_n)}\right)x_n + \sigma_{2,n+1}(x_{n+1})x_{n+1} \tag{5.22}$$

The pair of first-order equations (5.21) and (5.22) represent a folding of (5.2). Note that with positive parameter functions, each pair $x_0, y_0 \geq 0$ generates an orbit $\{(x_n, y_n)\}$ of (5.2) that is in $[0, \infty)^2$ for all n. So we have $x_{n+1}, x_n \geq 0$ and also by (5.20) $h_n(x_n, x_{n+1}) \geq 0$ so $\phi_n(x_n, h_n(x_n, x_{n+1}))$ is well defined for every such orbit of (5.2).

Remark 5.10. An even simpler reduction than the above is possible if $\phi_n(u, v) = \phi_n(u)$

is independent of (or constant in) v. In this case,

$$x_{n+2} = \sigma_{1,n+1}(x_{n+1}, \phi_n(x_n)x_n)\phi_n(x_n)x_n + \sigma_{2,n+1}(x_{n+1}, \phi_n(x_n)x_n)x_{n+1} \tag{5.23}$$

and it is not necessary to solve (5.2a) for y_n implicitly (i.e. the system folds without inversions). Special cases of this type include systems (5.3) and (5.4).

5.2.3 Global convergence to zero with periodic parameters

The results in this section show that global convergence to zero may occur even if (5.14) does not hold; see Remark 5.16 below. Recall from the proof of Theorem 5.4 that

$$x_{n+1} \leq \sigma_{1,n}\bar{\phi}_{n-1}x_{n-1} + \bar{\sigma}_{2,n}x_n. \tag{5.24}$$

The right hand side of the above inequality is a linear expression. Consider the linear difference equation

$$u_{n+1} = a_n u_n + b_n u_{n-1}, \qquad a_{n+p_1} = a_n,\ b_{n+p_2} = b_n \tag{5.25}$$

where the coefficients a_n, b_n are non-negative and their periods p_1, p_2 are positive integers with least common multiple $p = \operatorname{lcm}(p_1, p_2)$; we say that the linear difference equation (5.25) is periodic with period p. In this study we assume that

$$a_n, b_n \geq 0, \quad n = 0, 1, 2, \ldots \tag{5.26}$$

By Lemma 4.11 every solution of (5.25) converges to zero if $a_n + b_n < 1$ for all n. However, it is known that convergence to zero may occur even when $a_n + b_n$ exceeds 1 (for infinitely many n in the periodic case). We use the approach in [86] to examine the consequences of this issue when the planar system has periodic parameters. The following result is an immediate consequence of Theorem 13 in [86].

Lemma 5.11. *Assume that α_j, β_j for $j = 1, 2, \ldots, p$ are obtained by iteration from (5.25) from the real initial values*

$$\alpha_0 = 0,\ \alpha_1 = 1; \quad \beta_0 = 1,\ \beta_1 = 0 \tag{5.27}$$

Suppose that the quadratic polynomial

$$\alpha_p r^2 + (\beta_p - \alpha_{p+1})r - \beta_{p+1} = 0 \tag{5.28}$$

is proper, i.e. not $0 = 0$ and has a real root $r_1 \neq 0$. If the recurrence

$$r_{n+1} = a_n + \frac{b_n}{r_n} \tag{5.29}$$

generates nonzero real numbers $r_2, \ldots, r_p$ then $\{r_n\}_{n=1}^{\infty}$ is periodic with preiod p and yields a triangular system of first order equations that is equivalent to (5.25) as follows:

$$t_{n+1} = -\frac{b_n}{r_n} t_n, \quad t_1 = u_1 - r_1 u_0 \tag{5.30}$$

$$u_{n+1} = r_{n+1} u_n + t_{n+1}. \tag{5.31}$$

The system (5.30)-(5.31) is also known as a *semiconjugate factorization* of (5.25); see [83] for an introduction to this concept. The sequence $\{r_n\}$ that is generated by (5.29) is said to be a *(unitary) eigensequence* of (5.25). Eigenvalues are essentially constant eigensequences for if $p = 1$ in Lemma 5.11 then Equation (5.28) reduces to

$$\alpha_1 r^2 + (\beta_1 - \alpha_2)r - \beta_2 = 0$$

$$r^2 - a_1 r - b_1 = 0$$

and the latter equation is the standard characteristic equation of (5.25) with constant coefficients; see [86] for more details on the semiconjugate factorization of linear difference equations.

Each of the equations (5.30) and (5.31) readily yields a solution by iteration as follows

$$t_n = t_1(-1)^{n-1}\left(\frac{b_1 b_2 \cdots b_{n-1}}{r_1 r_2 \cdots r_{n-1}}\right), \tag{5.32}$$

$$\begin{aligned} u_n &= r_n r_{n-1} \cdots r_2 u_1 + r_n r_{n-1} \cdots r_3 t_2 + \cdots r_n t_{n-1} + t_n \\ &= r_n r_{n-1} \cdots r_2 r_1 u_0 + \sum_{i=1}^{n-1} r_n r_{n-1} \cdots r_{i+1} t_i + t_n \end{aligned} \tag{5.33}$$

Lemma 5.12. *Suppose that the numbers α_n and β_n are defined as in Lemma 5.11 though we do not assume that (5.25) is periodic here. Then*

(a) $\beta_n = 0$ for all $n \geq 2$ if and only if $b_1 = 0$.

(b) If (5.26) holds then for all $n \geq 2$

$$\alpha_n \geq a_1 a_2 \cdots a_{n-1}, \quad \beta_n \geq b_1 a_2 \cdots a_{n-1} \tag{5.34}$$

$$\alpha_{2n-1} \geq b_2 b_4 \cdots b_{2n-2}, \quad \beta_{2n} \geq b_1 b_3 \cdots b_{2n-1} \tag{5.35}$$

Proof. (a) Let $b_1 = 0$. Then $\beta_2 = b_1 = 0$ and since $\beta_1 = 0$ by definition it follows that $\beta_3 = 0$. Induction completes the proof that $\beta_n = 0$ if $n \geq 2$. The converse is obvious since $b_1 = \beta_2$.

(b) Since $\alpha_2 = a_1$ and $\beta_2 = b_1$ the stated inequalities hold for $n = 2$. If (5.34) is true for some $k \geq 2$ then

$$\alpha_{k+1} = a_k \alpha_k + b_k \alpha_{k-1} \geq a_k \alpha_k \geq a_1 a_2 \cdots a_{k-1} a_k$$

$$\beta_{k+1} = a_k \beta_k + b_k \beta_{k-1} \geq a_k \beta_k \geq b_1 a_2 \cdots a_{k-1} a_k$$

Now, the proof is completed by induction. The proof of (5.35) is similar since

$$\alpha_3 = a_2\alpha_2 + b_2\alpha_1 \geq b_2 \quad \text{and} \quad \beta_4 = a_3\beta_3 + b_3\beta_2 \geq b_3 b_1$$

and if (5.35) holds for some $k \geq 2$ then

$$\alpha_{2k+1} \geq b_{2k}\alpha_{2k-1} \geq b_2 b_4 \cdots b_{2k-2} b_{2k}$$

$$\beta_{2k+2} \geq b_{2k+1}\beta_{2k} \geq b_1 b_3 \cdots b_{2k-1} b_{2k+1}$$

which establishes the induction step. □

Lemma 5.13. *Assume that (5.26) holds with $a_i > 0$ for $i = 1, \ldots, p$ and (5.25) is periodic with period $p \geq 2$. Then*

(a) Equation (5.25) has a positive (hence unitary) eigensequence $\{r_n\}$ of period p.

(b) If $b_i > 0$ for $i = 1, \ldots, p$ then

$$r_1 r_2 \cdots r_p = \frac{1}{2}\left(\alpha_{p+1} + \beta_p + \sqrt{(\alpha_{p+1} - \beta_p)^2 + 4\alpha_p\beta_{p+1}}\right) \tag{5.36}$$

Hence, $r_1 r_2 \cdots r_p < 1$ if

$$\alpha_p\beta_{p+1} < (1 - \alpha_{p+1})(1 - \beta_p) \tag{5.37}$$

(c) If $b_i < 1$ for $i = 1, \ldots, p$ then $r_1 r_2 \cdots r_p > b_1 b_2 \cdots b_p$.

Proof. (a) Lemma 5.12 shows that $\alpha_i > 0$ for $i = 2, \ldots, p+1$. Now, either (i) $b_1 > 0$ or (ii) $b_1 = 0$. In case (i), the root r^+ of the quadratic polynomial (5.28) is positive since by Lemma 5.12 $\beta_{p+1} > 0$ and thus

$$r^+ = \frac{\alpha_{p+1} - \beta_p + \sqrt{(\alpha_{p+1} - \beta_p)^2 + 4\alpha_p\beta_{p+1}}}{2\alpha_p} > \frac{\alpha_{p+1} - \beta_p + |\alpha_{p+1} - \beta_p|}{2\alpha_p} \geq 0.$$

If $r_1 = r^+$ then from (5.29) $r_i = a_{i-1} + b_{i-1}/r_{i-1} \geq a_{i-1} > 0$ for $i = 2, \ldots, p+1$. Thus by Lemma 5.11, (5.25) has a unitary (in fact, positive) eigensequence of period p. If $b_1 = 0$ then by Lemma 5.12 $\beta_p = \beta_{p+1} = 0$ and (5.28) reduces to

$$\alpha_p r^2 - \alpha_{p+1} r = 0$$

which has a root $r^+ = \alpha_{p+1}/\alpha_p > 0$. As in the previous case it follows that (5.25) has a positive eigensequence of period p.

(b) To estalish (5.36), let $r_1 = r^+$ and note that (5.28) can be written as

$$r_1 = \frac{\alpha_{p+1} r_1 + \beta_{p+1}}{\alpha_p r_1 + \beta_p} \tag{5.38}$$

Since $\{r_n\}$ has period p, $r_{p+1} = r_1$ so from (5.29) and the definition of the numbers α_n and β_n it follows that

$$\begin{aligned} a_p + \frac{b_p}{r_p} = r_{p+1} = \frac{\alpha_{p+1} r_1 + \beta_{p+1}}{\alpha_p r_1 + \beta_p} &= \frac{(a_p \alpha_p + b_p \alpha_{p-1}) r_1 + a_p \beta_p + b_p \beta_{p-1}}{\alpha_p r_1 + \beta_p} \\ &= \frac{a_p(\alpha_p r_1 + \beta_p) + b_p(\alpha_{p-1} r_1 + \beta_{p-1})}{\alpha_p r_1 + \beta_p} \\ &= a_p + \frac{b_p}{(\alpha_p r_1 + \beta_p)/(\alpha_{p-1} r_1 + \beta_{p-1})} \end{aligned}$$

Since $b_p \neq 0$ it follows that

$$r_p = \frac{\alpha_p r_1 + \beta_p}{\alpha_{p-1} r_1 + \beta_{p-1}}$$

We claim that if $b_i \neq 0$ for $i = 1, \ldots, p$ then

$$r_{p-j} = \frac{\alpha_{p-j} r_1 + \beta_{p-j}}{\alpha_{p-j-1} r_1 + \beta_{p-j-1}}, \quad j = 0, 1, \ldots, p-2 \tag{5.39}$$

This claim is easily seen to be true by induction; we showed that it is true for

$j = 0$ and if (5.39) holds for some j then by (5.29)

$$\begin{aligned}
a_{p-j-1} + \frac{b_{p-j-1}}{r_{p-j-1}} = r_{p-j} &= \frac{(a_{p-j-1}\alpha_{p-j-1} + b_{p-j-1}\alpha_{p-j-2})r_1 + (a_{p-j-1}\beta_{p-j-1} + b_{p-j-1}\beta_{p-j-2})}{\alpha_{p-j-1}r_1 + \beta_{p-j-1}} \\
&= \frac{a_{p-j-1}(\alpha_{p-j-1}r_1 + \beta_{p-j-1}) + b_{p-j-1}(\alpha_{p-j-2}r_1 + \beta_{p-j-2})}{\alpha_{p-j-1}r_1 + \beta_{p-j-1}} \\
&= a_{p-j-1} + \frac{b_{p-j-1}(\alpha_{p-j-2}r_1 + \beta_{p-j-2})}{\alpha_{p-j-1}r_1 + \beta_{p-j-1}}
\end{aligned}$$

from which it follows that

$$r_{p-j-1} = \frac{\alpha_{p-j-1}r_1 + \beta_{p-j-1}}{\alpha_{p-j-2}r_1 + \beta_{p-j-2}}$$

and the induction argument is complete. Now, using (5.39) we obtain

$$r_p r_{p-1} \cdots r_2 r_1 = \frac{\alpha_p r_1 + \beta_p}{\alpha_{p-1}r_1 + \beta_{p-1}} \frac{\alpha_{p-1}r_1 + \beta_{p-1}}{\alpha_{p-2}r_1 + \beta_{p-2}} \cdots \frac{\alpha_2 r_1 + \beta_2}{\alpha_1 r_1 + \beta_1} r_1 = \alpha_p r_1 + \beta_p \qquad (5.40)$$

Given that $r_1 = r^+$ (5.40) implies that

$$\begin{aligned}
r_1 r_2 \cdots r_p &= \alpha_p \frac{\alpha_{p+1} - \beta_p + \sqrt{(\alpha_{p+1} - \beta_p)^2 + 4\alpha_p\beta_{p+1}}}{2\alpha_p} + \beta_p \\
&= \frac{1}{2}\left(\alpha_{p+1} + \beta_p + \sqrt{(\alpha_{p+1} - \beta_p)^2 + 4\alpha_p\beta_{p+1}}\right)
\end{aligned}$$

and (5.36) is obtained. Hence, $r_1 r_2 \cdots r_p < 1$ if

$$\alpha_{p+1} + \beta_p + \sqrt{(\alpha_{p+1} - \beta_p)^2 + 4\alpha_p\beta_{p+1}} < 2$$

Upon rearranging terms and squaring:

$$(\alpha_{p+1} - \beta_p)^2 + 4\alpha_p\beta_{p+1} < 4 - 4(\alpha_{p+1} + \beta_p) + (\alpha_{p+1} + \beta_p)^2$$

which reduces to (5.37) after straightforward algebraic manipulations.

(c) First, assume that p is odd. Then by (5.35)

$$\alpha_p\beta_{p+1} = (b_2b_4\cdots b_{p-1})(b_1b_3\cdots b_p) = b_1b_2\cdots b_p$$

so from (5.36)

$$r_1r_2\cdots r_p > \sqrt{\alpha_p\beta_{p+1}} = \sqrt{b_1b_2\cdots b_p}$$

If $b_i < 1$ for $i = 1,\ldots,p$ then $b_1b_2\cdots b_p < 1$ so $\sqrt{b_1b_2\cdots b_p} > b_1b_2\cdots b_p$ as required. Now let p be even. Then from (5.36) and (5.35)

$$r_1r_2\cdots r_p > \frac{\alpha_{p+1}+\beta_p}{2} \geq \frac{b_2b_4\cdots b_p + b_1b_3\cdots b_{p-1}}{2}$$

If $b_i < 1$ for $i = 1,\ldots,p$ then $b_2b_4\cdots b_p \geq b_1b_2\cdots b_p$ and $b_1b_3\cdots b_{p-1} \geq b_1b_2\cdots b_p$ and the proof is complete. $\square$

Some of the numbers a_i may exceed 1 in Lemma 5.13 without affecting the conclusions of the lemma. Also not all the conditions in Lemma 5.13 are necessary. For instance, if $b_1 = 0$ then Lemma 5.13(c) holds trivially. Also, by Lemma 5.12(a) $\beta_n = 0$ for $n \geq 2$ so the following equality must hold instead of (5.36):

$$r_1r_2\cdots r_p = \alpha_{p+1}$$

This is in fact true because $r_1 = r^+ = \alpha_{p+1}/\alpha_p$ so, repeating the argument in the proof of Lemma 5.13(b) yields $r_{p-j} = \alpha_{p-j}/\alpha_{p-j-1}$ for $j = 0,1,\ldots,p-2$. Hence

$$r_pr_{p-1}\cdots r_2r_1 = \frac{\alpha_p}{\alpha_{p-1}}\frac{\alpha_{p-1}}{\alpha_{p-2}}\cdots\frac{\alpha_2}{\alpha_1}\frac{\alpha_{p+1}}{\alpha_p} = \alpha_{p+1}$$

as claimed. These observations establish the following version of Lemma 5.13.

Lemma 5.14. *Let $a_i > 0$ and $b_i \geq 0$ for $i = 1,\ldots,p$ with $b_1 = 0$. Then the linear* *equation (5.25) has a positive (hence unitary) eigensequence* $\{r_n\}$ *of period* p *given by*

$$r_1 = \frac{\alpha_{p+1}}{\alpha_p}, \quad r_j = \frac{\alpha_j}{\alpha_{j-1}},\ j = 2, \ldots, p$$

and $0 = b_1 b_2 \cdots b_p < r_1 r_2 \cdots r_p < 1$ *if* $\alpha_{p+1} < 1$.

In Lemma 5.14 some of the numbers a_i or b_i may exceed 1 without affecting the conclusions of the lemma.

Theorem 5.15. *Assume that (5.15) holds and the sequences and* $\{\bar{\sigma}_{1,n}\bar{\phi}_{n-1}\}$ *and* $\{\bar{\sigma}_{2,n}\}$ *have period* p *with* $\bar{\sigma}_{2,i} > 0$ *and* $\bar{\sigma}_{1,i}\bar{\phi}_{i-1} \geq 0$ *for* $i = 1, \ldots, p$. *Also let the numbers* α_n, β_n *be as previously defined with* $a_n = \bar{\sigma}_{2,n}$ *and* $b_n = \bar{\sigma}_{1,n}\bar{\phi}_{n-1}$. *All non-negative orbits of the planar system converge to (0,0) if either one of the following hold:*

(a) $0 < \bar{\sigma}_{1,i}\bar{\phi}_{i-1} < 1$ *and (5.37) holds;*

(b) $\bar{\sigma}_{1,1}\bar{\phi}_0 = 0$ *and* $\alpha_{p+1} < 1$.

Proof. Let $\{u_n\}$ be a solution of the linear equation (5.25) with $a_n = \bar{\sigma}_{2,n}$, $b_n = \bar{\sigma}_{1,n}\bar{\phi}_{n-1}$, $u_0 = x_0$ and $u_1 = x_1$. Then by (5.24)

$$x_2 \leq \bar{\sigma}_{1,1}\bar{\phi}_0 x_0 + \bar{\sigma}_{2,1} x_1 = \bar{\sigma}_{1,1}\bar{\phi}_0 u_0 + \bar{\sigma}_{2,1} u_1 = u_2$$

$$x_3 \leq \bar{\sigma}_{1,2}\bar{\phi}_2 x_2 + \bar{\sigma}_{2,2} x_2 \leq \bar{\sigma}_{1,2}\bar{\phi}_1 u_1 + \bar{\sigma}_{2,2} u_2 = u_3$$

By induction it follows that $x_n \leq u_n$. If (5.37) holds then by Lemma 5.13, $\lim_{n\to\infty} u_n = 0$ so $\{x_n\}$ converges to 0. Further, $\lim_{n\to\infty} y_n = 0$ as in the proof of Theorem 5.4 and the proof is complete. □

Remark 5.16. In Theorem 5.15 the individual sequences $\bar{\sigma}_{1,n}, \bar{\phi}_n$ need not be periodic or even bounded. Therefore, the theorem applies to (5.2a)-(5.2b) even if the system itself is not periodic as long as the combination $\bar{\sigma}_{1,n}\bar{\phi}_{n-1}$ of parameters is periodic along with $\bar{\sigma}_{2,n}$.

5.2.3.1 Stocking strategies that do not prevent extinction

Condition (5.37) involves the numbers α_j, β_j rather than the coefficients of (5.25) directly. In the case of period $p = 2$ the role of a_i and b_i is more apparent. Inequality (5.37) in this case is

$$\alpha_2\beta_3 < (1 - \alpha_3)(1 - \beta_2)$$
$$a_1a_2b_1 < (1 - b_2 - a_1a_2)(1 - b_1)$$

and simple manipulations reduce the last inequality to

$$a_1a_2 < (1 - b_1)(1 - b_2). \tag{5.41}$$

Inequality (5.41) holds even if $a_1 > 1$ or $a_2 > 1$ thus showing how global convergence to (0,0) my occur when (5.14) does not hold. Further, it is possible that (5.41) holds together with

$$\frac{a_1 + a_2}{2} > 1 \tag{5.42}$$

Note that (5.41) holds even with arbitrarily large mean value in (5.42) if, say $a_1 \to 0$ as $a_2 \to \infty$. In population models this implies that if (5.41) holds with $a_n = \bar{\sigma}_{2,n}$ and $b_n = \bar{\sigma}_{1,n}\bar{\phi}_{n-1}$ then extinction may still occur after restocking the adult population to raise the mean value of the composite parameter $\bar{\sigma}_{2,n}$ above 1 by a wide margin.

5.3 Dynamics of a Beverton-Holt type rational system

In this section we apply some of the preceding results and obtain some new ones to study boundedness, extinction and modes of survival in some rational special cases of (5.2). In population models these types of systems include the Beverton-Holt type interactions. Specifically, we consider the following non-autonomous system and some of its special cases

$$x_{n+1} = \frac{\alpha_{1,n} y_n}{1 + \beta_{1,n} x_n + \gamma_{1,n} y_n} + \frac{\alpha_{2,n} x_n}{1 + \beta_{2,n} x_n + \gamma_{2,n} y_n} \tag{5.43a}$$
$$y_{n+1} = \frac{b_n x_n}{1 + c_{1,n} x_n + c_{2,n} y_n} \tag{5.43b}$$

where we assume that for all $n \geq 0$ and $i = 1, 2$

$$\alpha_{1,n} > 0, \quad b_n, \alpha_{2,n}, \beta_{i,n}, \gamma_{i,n}, c_{i,n} \geq 0 \tag{5.44}$$
$$b_n > 0 \text{ for infinitely many } n$$

For example, if we think of α_i as the natural survival rates then the population model (5.3) is a special case of (5.43). If we allow α_i to include additional factors such as harvesting rates then (5.43) is an extension of the model in [92] (with a Beverton-Holt recruitment function) in the sense that the competition coefficients $\beta_{i,n}, \gamma_{i,n}, c_{i,n}$ may be nonzero as well as time-dependent.

5.3.1 Uniform boundedness and extinction

We now examine boundedness and global convergence to 0 (extinction) in (5.43). The next result is in part a consequence of Corollary 5.3.

Corollary 5.17. *Assume that (5.44) holds.*

(a) Let the sequence $\{\alpha_{1,n}\}$ be bounded and $\limsup_{n\to\infty} \alpha_{2,n} < 1$. *If there is $M_0 > 0$ such that $b_n \leq M_0 c_{1,n}$ for all n larger than a given positive integer then all orbits of (5.43) are uniformly bounded.*

(b) Let the sequence $\{b_n\}$ be bounded and suppose that there is $M > 0$ such that

$$\alpha_{1,n} \leq M\gamma_{1,n}, \quad \alpha_{2,n} \leq M\beta_{2,n} \tag{5.45}$$

for all n larger than a given positive integer. Then all orbits of (5.43) are uniformly bounded.

Proof. (a) By hypothesis, for all (large) n

$$\frac{b_n x_n}{1 + c_{1,n}x_n + c_{2,n}y_n} \leq \frac{M_0 c_{1,n} x_n}{1 + c_{1,n}x_n + c_{2,n}y_n} < M_0$$

Next, let

$$\sigma_{1,n}(u, v) = \frac{\alpha_{1,n}}{1 + \beta_{1,n}u + \gamma_{1,n}v}, \quad \sigma_{2,n}(u, v) = \frac{\alpha_{2,n}}{1 + \beta_{2,n}u + \gamma_{2,n}v}.$$

By hypothesis, there is $M_1 > 0$ and $\delta \in (0, 1)$ such that for all $u, v \geq 0$ and all sufficiently large values of n

$$\sigma_{1,n}(u, v) \leq \alpha_{1,n} \leq M_1, \quad \sigma_{2,n}(u, v) \leq \alpha_{2,n} \leq \delta.$$

Now an application of Corollary 5.3 completes the proof of (a).

(b) By (5.45) for all large n it follows that

$$\frac{\alpha_{1,n}y_n}{1 + \beta_{1,n}x_n + \gamma_{1,n}y_n} \leq \frac{M\gamma_{1,n}y_n}{1 + \beta_{1,n}x_n + \gamma_{1,n}y_n} < M$$

and likewise,

$$\frac{\alpha_{2,n}x_n}{1 + \beta_{2,n}x_n + \gamma_{2,n}y_n} \leq \frac{M\beta_{2,n}x_n}{1 + \beta_{2,n}x_n + \gamma_{2,n}y_n} < M$$

for all large n. Therefore, $x_n \leq 2M$. Next, if $\{b_n\}$ is bounded then $y_n \leq 2Mb_n$ is also bounded and the proof is complete. $\square$

The next result follows readily from Theorem 5.4.

Corollary 5.18. *The origin (0,0) attracts every orbit of (5.43) in* $[0,\infty)^2$ *if*

$$\limsup_{n\to\infty}(\alpha_{1,n}b_{n-1}+\alpha_{2,n})<1 \tag{5.46}$$

and either b_n *is bounded or* $\liminf_{n\to\infty}\alpha_{1,n}>0$.

The above corollary is false when (5.46) holds if b_n is unbounded and thus, $\alpha_{1,n}$ has a subsequence that converges to 0.

Example 5.19. Consider the system

$$\begin{aligned} x_{n+1} &= \alpha^{-n}y_n + sx_n \\ y_{n+1} &= \frac{\beta\alpha^n x_n}{1+cx_n} \end{aligned}$$

where $\alpha>1$, $\beta>0$, $0\le s<1$, $c\ge 0$, $\sigma_{1,n}=\alpha^{-n}$ and $b_n=\beta\alpha^n$. Then (5.46) is satisfied, so $\lim_{n\to\infty}x_n=0$. But y_n does not approach 0 for large enough α; this may be inferred from Lemma 4.11 which shows that x_n converges to 0 at an exponential rate $\delta^{n/2}$ where $\delta=s+\beta/\alpha\in(0,1)$. Thus

$$y_n=\frac{1}{\alpha^{-n}}(x_{n+1}-sx_n)=\alpha^n(x_{n+1}-sx_n)$$

will not converge to 0 if α is sufficiently large.

Corollary 5.17 takes a simpler form for the autonomous special case of (5.43), namely,

$$x_{n+1}=\frac{\alpha_1 y_n}{1+\beta_1 x_n+\gamma_1 y_n}+\frac{\alpha_2 x_n}{1+\beta_2 x_n+\gamma_2 y_n} \tag{5.47a}$$

$$y_{n+1}=\frac{bx_n}{1+c_1x_n+c_2y_n} \tag{5.47b}$$

with constant parameters

$$\alpha_1, b > 0, \quad \alpha_2, \beta_i, \gamma_i, c_i \geq 0. \tag{5.48}$$

The following result is applicable to (5.3) as well as special cases of (5.4) and (5.5) with rational f.

Corollary 5.20. *Assume that (5.48) holds. All orbits of (5.47) in* $[0,\infty)^2$ *are uniformly bounded if either one of the following conditions hold:*

(a) $\alpha_2 < 1$, $c_1 > 0$;

(b) $\gamma_1, \beta_2 > 0$.

It is noteworthy that if in Part (a) above $c_1 = 0$ then (5.47) may have unbounded solutions, as in, e.g. the system

$$\begin{aligned} x_{n+1} &= \alpha_1 y_n \\ y_{n+1} &= \frac{b x_n}{1 + c_2 y_n} \end{aligned}$$

where $\alpha_2 = c_1 = 0$ and the remaining parameters are positive. This system folds to the second-order rational equation

$$x_{n+2} = \frac{\alpha_1^2 b x_n}{\alpha_1 + c_2 x_{n+1}}$$

which is known to have unbounded solutions if $\alpha_1 b > 1$; see [51].

Corollary (5.18) likewise simplifies in the autonomous case.

Corollary 5.21. *Assume that (5.48) holds with* $\alpha_1 b + \alpha_2 < 1$. *Then the origin (0,0) is the globally asymptotically stable fixed point of (5.47) relative to* $[0,\infty)^2$.

5.3.2 Persistence and the role of competiton

We now explore the effects of competition in the autonomous system (5.47). There are 6 different competition coefficients and to reduce the number of different cases, we focus on the special case below where $\beta_i, \gamma_i = 0$

$$x_{n+1} = \alpha_1 y_n + \alpha_2 x_n \tag{5.49}$$

$$y_{n+1} = \frac{b x_n}{1 + c_1 x_n + c_2 y_n} \tag{5.50}$$

If α_i define the natural survival rates s_i, then this system is complementary to (5.3) and (5.4) in the sense that in both of those systems $c_2 = 0$.

By the last two corollaries, all orbits of the rational system (5.49)-(5.50) in $[0, \infty)^2$ are uniformly bounded if $c_1 > 0$ and $\alpha_2 < 1$ and they converge to the origin if $\alpha_1 b + \alpha_2 < 1$. We now examine this rational system in more detail using its folding, namely, the second-order rational equation

$$x_{n+2} = a x_{n+1} + \frac{\sigma x_n}{1 + A x_{n+1} + B x_n} \tag{5.51}$$

where

$$a = \alpha_2, \quad \sigma = \alpha_1 b, \quad A = \frac{c_2}{\alpha_1}, \quad B = \frac{1}{\alpha_1}(\alpha_1 c_1 - \alpha_2 c_2) \tag{5.52}$$

See (5.22); the y-component is given by (5.21), or calculated directly using (5.49) as

$$y_n = \frac{1}{\alpha_1}(x_{n+1} - \alpha_2 x_n). \tag{5.53}$$

With initial values x_0 and $x_1 = \alpha_1 y_0 + \alpha_2 x_0$ derived from $(x_0, y_0) \in [0, \infty)^2$, the x-component of the orbits $\{(x_n, y_n)\}$ of the system is obtained by iterating (5.51). The equation in (5.53) is passive in the sense that after the x-component of the orbit is generated by the core equation (5.51), the y-component is derived from (5.53) without any further iterations. This observation also establishes the nontrivial fact that solutions of (5.51) that correspond to the orbits of the system in $[0, \infty)^2$ are non-negative and well-defined, even for $B < 0$.

If $\alpha_1 b + \alpha_2 < 1$, i.e. $\sigma < 1 - a$, then zero is the only fixed point of (5.51). Corollary 5.21 establishes that in this case, zero is globally asymptotically stable relative to $[0, \infty)$. On the other hand, when $\alpha_1 b + \alpha_2 > 1$, i.e. $\sigma > 1 - a$, then 0 is no longer a stable fixed point of (5.51). By routine calculations, one can show that zero is a saddle point when $1 - a < \sigma < 1 + a$ and if $\sigma > 1 + a$ then zero is a repeller.

In addition, when $\sigma > 1 - a$ and $a = \alpha_2 < 1$, the system (5.49)-(5.50) also has a fixed point in $(0, \infty)^2$ given by

$$\bar{x} = \frac{\sigma - (1-a)}{(1-a)(A+B)} = \frac{\alpha_1(\alpha_1 b + \alpha_2 - 1)}{(1-\alpha_2)[\alpha_1 c_1 + (1-\alpha_2)c_2]}, \quad \bar{y} = \frac{(1-\alpha_2)}{\alpha_1}\bar{x} \tag{5.54}$$

We note that $\bar{x}$ is also a positive fixed point of the folding (5.51). Under certain conditions, $\bar{x}$ attracts all solutions of (5.51) with positive initial values, and it is thus a survival equilibrium.

Theorem 5.22. *Let $a < 1 < a + \sigma$, i.e., $\alpha_2 < 1 < \alpha_1 b + \alpha_2$. If the function*

$$f(u, v) = au + \frac{\sigma v}{Au + Bv + 1}$$

is nondecreasing in both arguments, then the fixed point $\bar{x}$ attracts all solutions of (5.51) with initial values in $(0, \infty)$.

Proof. If we let

$$h(t) = at + \frac{\sigma t}{1 + (A+B)t}$$

then the fixed point $\bar{x}$ is the solution of $h(t) = t$. For $t > 0$, we may write $h(t) = \phi(t)t$ where

$$\phi(t) = a + \frac{\sigma}{1 + (A+B)t} \quad \text{with} \quad \phi(\bar{x}) = \frac{h(\bar{x})}{\bar{x}} = 1$$

Now,

$$\phi'(t) = -\frac{\sigma(A+B)}{(1+(A+B)t)^2} < 0$$

for all $t > 0$, so $\phi(t)$ is strictly decreasing for all $t > 0$. Therefore,

$$t < \bar{x} \ \text{ implies that } \ h(t) = \phi(t)t > \phi(\bar{x})t = t,$$

$$t > \bar{x} \ \text{ implies that } \ h(t) = \phi(t)t < \phi(\bar{x})t = t$$

The rest of the proof follows from Lemma 2.6. □

Note that

$$f_u = a - \frac{A\sigma v}{(Au+Bv+1)^2} \ \text{ and } \ f_v = \frac{\sigma(Au+1)}{(Au+Bv+1)^2} > 0$$

If $\alpha_1 b + \alpha_2 > 1$ and $c_2 = 0$ then $A = 0$, so both $f_u, f_v > 0$. Therefore, by Theorem 5.22 $\bar{x}$ is globally asymptotically stable. However, if $c_2 > 0$ then f_u may not be positive, so the results of Theorem 5.22 may not apply to this case. The next result shows that orbits of the system may converge to $\bar{x}$ if $c_2 > 0$ but not too large.

Theorem 5.23. *Let $c_1 > 0$ and $a < 1 < a + \sigma$, i.e., $\alpha_2 < 1 < \alpha_1 b + \alpha_2$. Then there exists $c > 0$ such that for $c_2 \in [0, c]$ the fixed point $\bar{x}$ of (5.51) is globally asymptotically stable relative to $(0, \infty)$.*

Proof. Since

$$f_u = a - \frac{A\sigma v}{(Au+Bv+1)^2} = \frac{a(Au+Bv)^2 + 2Aau + a + (2aB - A\sigma)v}{(Au+Bv+1)^2}$$

to ensure that $f_u \geq 0$ it suffices for $2aB - A\sigma \geq 0$, i.e.

$$2\alpha_2(\alpha_1 c_1 - \alpha_2 c_2) - c_2 \alpha_1 b \geq 0$$

which is equivalent to

$$c_2 \leq \frac{2\alpha_1\alpha_2 c_1}{\alpha_1 b + 2\alpha_2^2} \doteq c$$

and the proof is complete. $\square$

If c_2 is sufficiently large then f_u is not positive on $(0, \infty)$. Furthermore, $\bar{x}$ also becomes unstable for large enough c_2, which we establish next by examining the linearization of (5.51) around $\bar{x}$.

The characteristic equation associated with the linearization of (5.51) at $\bar{x}$ is given by

$$\lambda^2 - p\lambda - q = 0 \tag{5.55}$$

where

$$p = f_u(\bar{x}, \bar{x}) = a - \frac{(1-a)A\bar{x}}{1 + (A+B)\bar{x}} \quad \text{and} \quad q = f_v(\bar{x}, \bar{x}) = \frac{\sigma - (1-a)B\bar{x}}{1 + (A+B)\bar{x}}$$

The roots of (5.55) are given by

$$\lambda_1 = \frac{p - \sqrt{p^2 + 4q}}{2}, \quad \lambda_2 = \frac{p + \sqrt{p^2 + 4q}}{2}$$

Since $f_v(u, v) > 0$ for all $u, v \in (0, \infty)$ it follows that $q > 0$ and both roots are real with $\lambda_1 < 0$ and $\lambda_2 > 0$. Further, $\lambda_2 < 1$ if

$$\frac{p + \sqrt{p^2 + 4q}}{2} < 1 \ \text{ i.e. } \ q < 1 - p$$

which is equivalent to

$$2(1-a)(A+B)\bar{x} > \sigma - (1-a)$$

This inequality holds, since $\bar{x} > 0$ under our assumptions on the parameters. Therefore, $\lambda_2 < 1$. On the other hand, $\lambda_1 > -1$ if and only if

$$\frac{p - \sqrt{p^2 + 4q}}{2} > -1 \text{ i.e. } p + 1 > q$$

which is equivalent to

$$2(Aa + B)\bar{x} > \sigma - (1 + a) \tag{5.56}$$

Note that when $(1-a) < \sigma < (1+a)$ this is trivially the case since $\bar{x} > 0$ under our assumptions on the parameters. Thus, $\bar{x}$ is locally asymptotically stable if $\sigma < 1 + a$.

Next, $\lambda_1 < -1$ if $\sigma > 1 + a$ and

$$2(Aa + B)\bar{x} < \sigma - (1 + a) \tag{5.57}$$

We summarize the above results in the following lemma.

Lemma 5.24. *Let $a < 1 < a + \sigma$, i.e., $\alpha_2 < 1 < \alpha_1 b + \alpha_2$. Then the fixed point $\bar{x}$ of (5.51) is:*

(a) locally asymptotically stable if and only if (5.56) holds. In particular, this is true if

$$1 - a < \sigma < 1 + a, \quad \textit{i.e.} \quad 1 - \alpha_2 < \alpha_1 b < 1 + \alpha_2.$$

(b) a saddle point if and only if (5.57) holds with $\sigma > 1 + a$, i.e. $\alpha_1 b > 1 + \alpha_2$.

The inequality (5.57) implies a range for c_2 that we now determine. Let

$$k = \frac{\sigma - (1 + a)}{\sigma - (1 - a)} < 1.$$

Then $k \in (0, 1)$ if $\sigma > 1 + a$

$$2(Aa + B)\bar{x} < \sigma - (1 + a) \Rightarrow \frac{2(Aa + B)}{A + B} < \frac{\sigma - (1 + a)}{\sigma - (1 - a)}(1 - a) = (1 - a)k \tag{5.58}$$

Since

$$2(Aa+B) = \frac{2}{\alpha_1}(c_2\alpha_2 + c_1\alpha_1 - c_2\alpha_2) = 2c_1 \quad \text{and:}$$
$$A+B = \frac{1}{\alpha_1}[c_1\alpha_1 + (1-\alpha_2)c_2]$$

(5.58) is equivalent to

$$\frac{2c_1\alpha_1}{c_1\alpha_1 + (1-\alpha_2)c_2} < (1-a)k = (1-\alpha_2)k$$

From the above inequality we obtain

$$c_2 > \frac{\alpha_1 c_1[2-(1-\alpha_2)k]}{(1-\alpha_2)^2 k} \doteq \bar{c}$$

Thus if $c_2 > \bar{c}$ then $\bar{x}$ is a saddle point and in particular, the fixed point $(\bar{x}, \bar{y})$ is unstable. These observations lead to the following that may be compared with Theorem 5.23.

Corollary 5.25. *Assume that (5.48) holds for the system (5.49)-(5.50) and* $\alpha_2 < 1 < \alpha_1 b + \alpha_2$. *Then the fixed point* $(\bar{x}, \bar{y})$ *is unstable if* $c_2 > \bar{c}$.

Our final result establishes that when $c_2 > 0$ is sufficiently large the system (5.49)-(5.50) can have a prime period two orbit which occurs as $\bar{x}$ becomes unstable. Existence of periodic orbits is established via the folding in (5.51).

The difference equation in (5.51) has a positive prime period two solution if there exist real numbers $m, M > 0, m \neq M$, such that

$$m = f(M,m) \;\text{ and }\; M = f(m,M)$$

i.e.

$$m = aM + \frac{\sigma m}{AM + Bm + 1} \;\text{ and }\; M = am + \frac{\sigma M}{Am + BM + 1}$$

from which we get

$$(m - aM)(AM + Bm + 1) = \sigma m \text{ and } (M - am)(Am + BM + 1) = \sigma M$$

i.e.

$$AmM + Bm^2 + m - AaM^2 - aBMm - aM = \sigma m \tag{5.59}$$

and

$$AmM + BM^2 + M - Aam^2 - aBMm - am = \sigma M \tag{5.60}$$

Taking the difference of the right and left sides of (5.59) and (5.60) yields

$$B(m^2 - M^2) + (m - M) - Aa(M^2 - m^2) - (M - m) = \sigma(m - M)$$
$$(B + Aa)(m - M)(m + M) = (\sigma - (1 + a))(m - M)$$

When $m \neq M$, we get

$$(B + Aa)(m + M) = \sigma - (1 + a)$$

and since the left side of the last equation is positive, this implies that $\sigma - (1+a) > 0$. Or stated differently, if $\sigma - (1 + a) < 0$, then (5.51) cannot have a positive prime period two solution.

Similarly, taking the sum of the right and left sides of (5.59) and (5.60) yields

$$2AmM + B(m^2 + M^2) + (m + M) - Aa(m^2 + M^2) - 2aBMm - a(m + M) = \sigma(m + M)$$

Adding and subtracting $2(B - Aa)$ to the LHS of the last expression yields

$$2(A - aB - B + Aa)Mm + (B - Aa)(m + M)^2 = (\sigma - (1 - a))(m + M)$$

i.e.

$$
\begin{aligned}
2(1+a)(A-B)Mm &= (\sigma-(1-a))(m+M)-(B-Aa)(m+M)^2\\
&= (m+M)(\sigma-(1-a)-(B-Aa)(m+M))\\
&= (m+M)(\sigma-(1-a)-\frac{(B-Aa)(\sigma-(1+a))}{B+Aa}\\
&= \frac{m+M}{Aa+B}\left[(B+Aa)(\sigma-(1-a))-(B-Aa)(\sigma-(1+a))\right]
\end{aligned}
$$

Simplifying the right hand side, it follows that

$$(1+a)(A-B)Mm=\frac{\sigma-(1+a)}{(Aa+B)^2}\left[Aa(\sigma-1)+aB\right] \tag{5.61}$$

Now, since we are assuming that $\sigma-(1+a)>0$, then $\sigma-1>0$, so the right side of (5.61) is positive, which implies that $A-B>0$. Stated differently, if $A<B$, then (5.51) has no positive prime period two solution.

From (5.61) we get

$$Mm=\frac{[\sigma-(1+a)]\left[Aa(\sigma-1)+aBc\right]}{(1+a)(A-B)(Aa+B)^2}:=Q$$

and let $m+M=P$, from which we obtain that $M=P-m$ and $m=P-M$. This means that

$$m(P-m)=Q \;\text{ and }\; M(P-M)=Q$$

i.e. m and M are the roots of the quadratic

$$S(t)=t^2-Pt+Q$$

where $P,Q>0$ and

$$t_{\pm}=\frac{P\pm\sqrt{P^2-4Q}}{2}$$

To ensure that m and M are real, the roots of $S(t)$ must be real, which is the case if and only if $P^2 - 4Q > 0$, i.e.

$$[\sigma - (1+a)]\left[(\sigma - (1+a) - \frac{4(Aa(\sigma-1)+aB)}{(1+a)(A-B)}\right] > 0$$

We summarize the above results as follows.

Theorem 5.26. *The second order difference equation in (5.51) has a positive prime period two solution if and only if all of the following conditions are satisfied:*

(a) $\sigma - (1+a) > 0$

(b) $A - B > 0$

(c) $(\sigma - (1+a))\left[(\sigma - (1+a) - \frac{4(Aa(\sigma-1)+aB)}{(1+a)(A-B)}\right] > 0$

The next result shows that a solution of period two appears when $\bar{x}$ loses its stability.

Corollary 5.27. *The second order difference equation in (5.51) has a positive prime period two solution if and only if $\bar{x}$ is a saddle point.*

Proof. Suppose $\bar{x}$ is a saddle point. Then by Theorem 5.24(b), $2(Aa+B)\bar{x} < \sigma-(1+a)$ from which we infer that $\sigma - (1+a) > 0$.

Now $2(Aa+B)\bar{x} < \sigma - (1+a)$ implies that

$$\frac{2(Aa+B)}{(1-a)(A+B)}(\sigma - (1-a)) < \sigma - (1+a)$$

which is true if and only if

$$2(Aa+B)(\sigma - (1-a)) < (1-a)(A+B)(\sigma - (1+a))$$

Adding and subtracting $(1+a)(A-B)(\sigma - (1+a))$ from the right side of the last expression yields:

$$
\begin{aligned}
&(1+a)(A-B)(\sigma-(1+a))+(\sigma-(1+a))((1-a)(A+B)-(1+a)(A-B)\\
&\qquad\qquad\qquad = (1+a)(A-B)(\sigma-(1+a))+(\sigma-(1+a))(2B-2Aa)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2(Aa+B)(\sigma-(1-a))+2(Aa-B)(\sigma-(1+a)) &< (1+a)(A-B)(\sigma-(1+a))\\
4(Aa(\sigma-1)+aB) &< (1+a)(A-B)(\sigma-(1+a))
\end{aligned}
$$

i.e.

$$(1+a)(A-B)(\sigma-(1+a))-4(Aa(\sigma-1)+aB)>0$$

from which we infer that $A-B>0$ and the roots of $S(t)$ are guaranteed to be real and positive. This satisfies all the conditions of Theorem 5.26 which completes the proof. □

Corollary 5.28. *Assume that (5.48) holds and further,* $\alpha_2 < 1 < \alpha_1 b + \alpha_2$ *and* $c_2 > \bar{c}$*. Then the system (5.49)-(5.50) has a cycle of period two in* $(0,\infty)^2$.

Figure 5.2 shows two orbits of the system (5.49)-(5.50) from initial points $(x_0, y_0) = (2.3, 1)$ and $(x_0, y_0) = (0.0001, 0.0001)$. Although both orbits converge to the period two cycle, a shadow of the stable manifold of the fixed point is also seen in the initial segments of the two orbits. If the initial points start exactly on the stable manifold of $\bar{x}$ then the solutions converge to $\bar{x}$.

5.4 Dynamics of a Ricker-type exponential system

The question of whether complex behavior occurs in stage-structured models is a pertinent one which has been discussed in the literature. We supplement the results

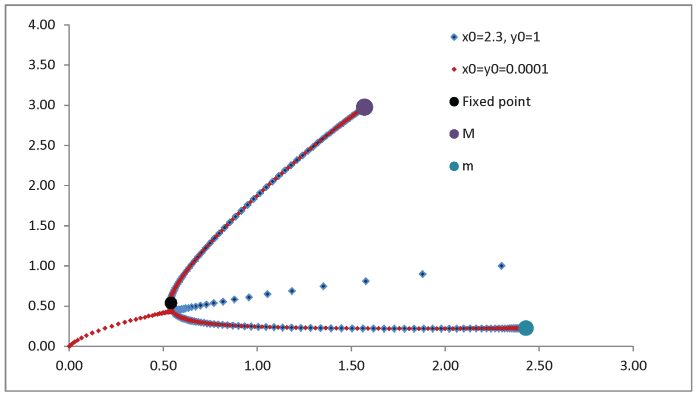

Figure 5.2: Orbits illustrating period two oscillations and the saddle point.

in [68] for (5.2) by examining cases not considered in [68] or elsewhere, that exhibits complex multistable behavior. Our results in this section complement the existing literature, e.g. [2], [39], [68] and [92].

To start, we consider the following non-autonomous system with a Ricker-type function for the juvenile fertility rate:

$$x_{n+1} = \sigma_{1,n} y_n + \sigma_{2,n} x_n \tag{5.62a}$$

$$y_{n+1} = \beta_n x_n e^{\alpha_n - c_{1,n} x_n - c_{2,n} y_n} \tag{5.62b}$$

where $\alpha_n, \beta_n, \sigma_{i,n}, c_{i,n}$ are non-negative numbers for $i = 1, 2$ and $n \geq 0$. This system has been used to model single-species, two-stage populations (e.g. juvenile and adult); see [21], [22], [24], [23], [29], [34], [35], [39], [68] and [92]. The exponential function that defines the time and density dependent fertility rate classifies the above system as a Ricker model. The coefficients $\sigma_{i,n}$ are typically composed of the natural survival rates s_i and possibly other factors. For example, they may include harvesting parameters, as in [68] and [92]:

$$\sigma_i = (1 - h_i)s_i, \quad \beta = (1 - h_1)b, \quad c_1 = (1 - h_1)\gamma, \quad c_2 = 0 \tag{5.63}$$

All parameters in (5.63) are assumed to be independent of n. In this case, $h_i, s_i \in [0, 1]$, $i = 1, 2$ denote harvest rates and natural survival rates, respectively. The study in [68] shows that the system (5.62a)-(5.62b) under (5.63) generates a wide range of different behaviors: the occurrence of periodic and chaotic behavior and phenomena such as bubbles and the counter-intuitive "hydra effect" (an increase in harvesing yields an increase in the over-all population) are established for the autonomous system

$$\begin{aligned} x_{n+1} &= (1 - h_1)s_1 y_n + (1 - h_2)s_2 x_n \\ y_{n+1} &= (1 - h_1)b x_n e^{\alpha - (1-h_1)\gamma x_n}. \end{aligned}$$

5.4.1 Uniform boundedness and extinction

We start with the following consequence of Corollary 5.3 and Theorem 5.14:

Corollary 5.29. *Assumed that* $\sigma_{1,n} > 0, \quad \sigma_{2,n}, \alpha_n, \beta_n, c_{1,n}, c_{2,n} \geq 0$ *and* $\beta_n > 0$ *for inifinitely many* n*, and further let* α_n *be bounded and* $\limsup_{n\to\infty} \sigma_{2,n} < 1$*, Then*
(a) If $\sigma_{1,n}$ *is bounded and there is* $M > 0$ *such that* $\beta_n \leq M c_{1,n}$ *for all* $n \geq 0$*, then every orbit of (5.62) in* $[0, \infty)^2$ *is uniformly bounded.*

(b) If β_n *is bounded and*

$$\limsup_{n\to\infty} (\sigma_{1,n}\beta_n e^{a_n}) < 1$$

then all orbits of (5.62) in $[0, \infty)^2$ *converge to* $(0, 0)$*.*

Proof. (a) For $u, v \geq 0$, and all $n \geq 0$, define

$$\phi_n(u,v) = \beta_n e^{\alpha_n - c_{1,n}u - c_{2,n}v}$$

If $c_{1,n} \neq 0$, for some n, then routine calculations yield

$$u\phi_n(u,v) \leq u\phi_n(\frac{1}{c_{1,n}}, 0) = \frac{\beta_n}{c_{1,n}} e^{a_n - 1}$$

If $c_{1,n} = 0$ for some n, then $\beta_n \leq Mc_{1,n}$ and $\phi_n(u,v) = 0$ for such n.

Next, by hypotheses, there are also numbers $M_1, M_2 > 0$ and $\bar{\sigma} \in (0,1)$ such that for all sufficiently large n

$$\sigma_{1,n} \leq M_1, \quad \alpha_n \leq M_2, \quad \sigma_{2,n} \leq \bar{\sigma}$$

Since $\beta_n \leq Mc_{1,n}$ it follows that for all n

$$u\phi_n(u,v) \leq Me^{M_2 - 1}$$

and the hypotheses of Corollary 5.3 are satisfied. Uniform boundedness follows.

(b) Let ϕ_n be as defined in (a) above. By hypotheses, the sequence

$$\bar{\phi_n} = \sup_{u,v \geq 0} \phi_n(u,v) = \beta_n e^{\alpha_n}$$

is bounded so by Theorem 5.14, all orbits of (5.62) in $[0,\infty)$ converge to $(0,0)$. $\square$

Remark 5.30. In Part (a) of the above corollary, it is less essential that β_n be bounded than to have $c_{1,n} \neq 0$. Indeed, unbounded solutions occur in the following autonomous linear system

$$x_{n+1} = \sigma_1 y_n + \sigma_2 x_n$$

$$y_{n+1} = \beta e^{\alpha} x_n$$

When folded to

$$x_{n+2} = \sigma_1 \beta e^{\alpha} x_n + \sigma_2 x_{n+1}$$

it is easy to see that unbounded solutions exist unless $\sigma_1 \beta e^{\alpha} x_n \leq 1 - \sigma_2$. This is a severe restriction resembling that in Part (b) of the above corollary.

5.4.2 Complex multistable behavior

To rigorously establish the occurrence of multiple stable orbits within the same state-space, we consider the reduced system

$$x_{n+1} = \sigma_{1,n} y_n \tag{5.64}$$

$$y_{n+1} = \beta_n x_n e^{\alpha_n - c_{1,n} x_n - c_{2,n} y_n} \tag{5.65}$$

where we assume that

$$\sigma_{1,n}, c_{1,n}, c_{2,n}, \beta_n > 0, \quad \alpha_n \geq 0. \tag{5.66}$$

In the context of stage-structured models the assumption $\sigma_{2,n} = 0$ applies in particular, to the case of a semelparous species, i.e. an organism that reproduces only once before death. Additional interpretations in terms of harvesting, migrations or other factors may be possible if $\sigma_{2,n}$ includes additional factors beyond the natural adult survival rate.

The system (5.64)-(5.65) with $c_{2,n} = 0$ has been studied in the literature; for instance, an autonomous version is discussed in [68] and [92]. The assumption $c_{2,n} > 0$, which adds greater inter-species competition into the stage-structured model, leads to theoretical issues that are not well-understood. We proceed by folding the system (5.64)-(5.65) to a second-order difference equation.

From (5.64) we obtain $y_n = x_{n+1}/\sigma_{1,n}$. Now using (5.64) and (5.65) we obtain:

$$x_{n+2} = \sigma_{1,n+1}\beta_n x_n e^{\alpha_n - c_{1,n}x_n - c_{2,n}y_n} = \sigma_{1,n}\beta_n x_n e^{\alpha_n - c_{1,n}x_n - (c_{2,n}/\sigma_{1,n})x_{n+1}}$$

This can be written more succinctly as

$$x_{n+1} = x_{n-1}e^{a_n - c_{1,n}x_{n-1} - (c_{2,n}/\sigma_{1,n})x_n} \tag{5.67}$$

where

$$a_n = \alpha_n + \ln(\beta_n \sigma_{1,n+1}).$$

5.4.2.1 Fixed points, global stability

It is useful to start by examining the fixed points of (5.67) when all parameters are constants, i.e. if (5.64)-(5.65) is an autonomous system. Then (5.67) takes the form of the autonomous difference equation:

$$x_{n+1} = x_{n-1}e^{a - c_1 x_{n-1} - (c_2/\sigma_1)x_n} \tag{5.68}$$

This equation clearly has a fixed point at 0. The following is consequence of Corollary 5.29(b).

Corollary 5.31. *Assume that the system (5.64)-(5.65) is autonomous, i.e. $\alpha_n = \alpha$, $\beta_n = \beta$, $\sigma_{1,n} = \sigma_1$, $c_{1,n} = c_1$ and $c_{2,n} = c_2$ are constants for all n.*

(a) If $a = \alpha + \ln(\beta\sigma_1) < 0$ then 0 is the unique fixed point of (5.68) in $[0, \infty)$ and all positive solutions of (5.68) converge to zero.

(b) The eigenvalues of the linearization of (5.68) at 0 are $\pm e^{a/2}$; thus, 0 is locally asymptotically stable if $a < 0$.

If $a > 0$ then (5.68) has exactly two fixed points: 0 and a positive fixed point

$$\bar{x} = \frac{a\sigma_1}{c_1\sigma_1 + c_2}.$$

Substituting $r_n = c_1 x_n$ in (5.68) yields

$$r_{n+1} = r_{n-1}e^{a-r_{n-1}-br_n}, \quad b = \frac{c_2}{\sigma_1 c_1} \tag{5.69}$$

The positive fixed point of this equation is

$$\bar{r} = \frac{a}{1+b} = c_1\bar{x}.$$

The next result is proved in [37] and can be stated in terms of the parameters in the reduced system in (5.64)-(5.65) and the equation in (5.69) as follows:

Theorem 5.32. *Let $a \in (0,1]$ (i.e. $0 < \alpha + ln(\beta\sigma_1) < 1$). If $b \in (0,1)$ (i.e. $c_2 < \sigma_1 c_1$) then the positive fixed point $\bar{r}$ of (5.69) is a global attractor of all of its positive solutions.*

Our last set of results pertain to the special case of the nonautonomous equation

$$r_{n+1} = r_{n-1}e^{a_n-r_{n-1}-b_n r_n} \tag{5.70}$$

and its autonomous counterpart in (5.69) where the coefficient $b_n = \frac{c_{2,n}}{\sigma_{1,n}c_{1,2}} = 1$,.i.e.

$$c_{2,n} = \sigma_{1,n}c_{1,n} \quad n = 0, 1, 2, \ldots \tag{5.71}$$

The semiconjugate factorization method that we used in Chapter 3 also applies to (5.70) if (5.71) holds. In this case, we substitute $r_n = c_{1,n}x_n$ in (5.67) to obtain

$$r_{n+1} = \frac{c_{1,n+1}}{c_{1,n-1}} r_{n-1} e^{a_n - r_{n-1} - r_n}$$

which can be written as

$$r_{n+1} = r_{n-1} e^{d_n - r_{n-1} - r_n} \tag{5.72}$$
$$d_n = a_n + \ln[c_{1,n+1}/c_{1,n-1}].$$

Note that if $c_{1,n}$ has period 2 or is constant then $c_{1,n+1} = c_{1,n-1}$ so $d_n = a_n$. In any case, a solution $x_n = r_n/c_{1,n}$ of (5.67) is derived in terms of a solution of (5.72) when (5.71) holds.

Equation (5.72) admits a semiconjugate factorization that splits it into two equations of order one.

$$t_{n+1} = \frac{e^{d_n}}{t_n}, \quad t_0 = \frac{r_0}{r_{-1} e^{-r_{-1}}} \tag{5.73}$$
$$r_{n+1} = t_{n+1} r_n e^{-r_n} \tag{5.74}$$

The results in Chapter 3 apply directly to the study of the system in (5.64)-(5.65), where some of the parameters are assumed to be constant, i.e.

$$\sigma_{1,n} = \sigma_1, \ \ \beta_n = \beta, \ \ \alpha_n = \alpha \tag{5.75}$$

Recall, that in Chapter 3, we showed that when $d_n = d$ is constant and $0 < d \leq 2$, then every non-constant solution of (5.72) corresponding to a given pair of initial values $r_{-1}, r_0 > 0$ converges to some two-cycle $\{\rho_1, \rho_2\}$, where $\rho_1 + \rho_2 = d$. Furthermore, we also showed that this cycle is dependent of initial values: if a different set of initial values r'_{-1}, r'_0 satisfies

$$\frac{r'_0}{r'_{-1}e^{-r'_{-1}}} = \frac{r_0}{r_{-1}e^{-r_{-1}}}$$

then the solution corresponding to r'_{-1}, r'_0 converges to the same two-cycle. We can now state the same result with respect to the system (5.64)-(5.65):

Theorem 5.33. *Assume that the parameters of (5.64)-(5.65) satisfy (5.66) and (5.75) and $c_2, n = \sigma_1 c_1, n$ for all $n > 0$, where $c_{1,n}$ has period two with $c_{1,2k-1} = \xi_1$ and $c_{1,2k} = \xi_2$, with $\xi_1, \xi_2 > 0$. If $\alpha + \ln(\sigma_1\beta) \in (0, 2]$, then*

(a) Every orbit $\{(x_n, y_n)\}$ is determined as

$$x_n = \frac{r_n}{c_{1,n}}, \quad y_n = \frac{r_{n+1}}{\sigma_1 c_{1,n+1}}$$

where $\{r_n\}$ is the solution of (5.72).

(b) The solution corresponding to the pair of initial values (x_0, y_0) converges to a two-cycle Γ

$$\left\{\left(\frac{\rho_1}{\xi_1}, \frac{\rho_2}{\sigma_1\xi_2}\right), \left(\frac{\rho_2}{\xi_2}, \frac{\rho_1}{\sigma_1\xi_1}\right)\right\}$$

where $\rho_i = \lim_{k\to\infty} r_{2k-i}$ for $i = 1, 2$ and $\rho_1 + \rho_2 = \alpha + \ln(\sigma_1\beta)$.

(c) If (x'_0, y'_0) are such that

$$\frac{y_0}{x_0 e^{-x_0}} = \frac{y'_0}{x'_0 e^{-x'_0}}$$

then the solution corresponding to (x'_0, y'_0) converges to the same two-cycle Γ as in (b).

Theorem 5.34. *Assume that the parameters of (5.64)-(5.65) satisfy (5.66) and (5.75) and $c_2, n = \sigma_1 c_1, n$ for all $n > 0$, where $c_{1,n} > 0$ has period two. If $a = \alpha + ln(\beta\sigma) \geq 6.26$ and the initial values $x_0, y_0 > 0$ satisfy*

$$\frac{y_0}{x_0 e^{-c_1 x_0}} = e^a$$

then (5.64)-(5.65) has periodic solutions of all possible periods, including odd periods, as well as chaotic solutions in the sense of Li and Yorke.

Notice that the results of the above theorem depends on initial values x_0, y_0, i.e. these initial values must be ordered pairs on the curve $y_0 = x_0 e^{a-c_1 x_0}$. While this assumption may be too restrictive from a biological standpoint, it does demonstrate possible periodic and chaotic behavior in species dynamics for infinitely many initial values x_0, y_0.

5.4.2.2 Oscillatory and complex behavior with periodic parameters

In the final section of this chapter, we turn our attention to the case where the vital rates of the system in (5.64)-(5.65) exhibit periodic fluctuations. In particular, we are interested in scenarios where the composite parameter

$$d_n = \alpha_n + ln(\beta_n \sigma_{1,n}) + ln[c_{1,n+1}/c_{1,n-1}]$$

is periodic. This assumption is broad enough that not all of the above parameters need to be periodic or be periodic of the same period. For example, one may allow for seasonal fluctuations in the fertility parameter α_n to account for high and low fertilities during warm and cold seasons of the year, while the rest of the parameters remain constant, in which case d_n will be of the same period of α_n. Alternatively, the period d_n may be determined by the common period of fluctuations of parameters that are periodic. Finally, time variant coefficients may not be periodic at all, but yield periodic fluctuations in d_n.

In Chapter 3, we showed a number of preliminary results for the equation in (5.72) with periodic d_n for the case when $c_{2,n} = \sigma_1 c_1, n$ for all $n > 0$. We ended Chapter 3 with a number of conjectures and open problems for future research. Given these conjectures, we turn to numerical simulations of the equation (5.72) to further show possible behavior of the dynamics in the species population in periodic environments.

Figure 5.3 shows convergence of solutions to cycles of periods six, three and four for cases where $d_n \in (0,2)$ is periodic with period $p = 3$ and $p = 4$. Figure 5.4 demonstrates the phenomenon of multistability, i.e. the dependence of cycles on initial values established for cases when p is odd, and when $p \geq 2$ is even with

$$\sigma = \sum_{j=1}^{p} (-1)^j d_{j-1} = 0.$$

In each of these cases, the periodic nature of solutions of (5.72) is expected. In contrast, Figure 5.5 shows the behavior for cases $p = 3$ and $p = 4$ when values of d_n are outside of the range $(0,2)$. The top left panel of Figure 5.5 shows a twelve-cycle, suggesting a possibility of period-doubling bifurcations that occur when values of d_n are sufficiently large. The behavior of the iterates in the top right panel is more unpredictable. In the latter case, values of all d_n are outside of the aforementioned range, whereas in the former case, only some of the d_n's are allowed to exceed 2. Similarly, the bottom two panels in Figure 5.5 show the behavior of the iterates for the case then $p = 4$. Unpredictable behavior is shown in the bottom left panel, where all of the d_ns exceed 2. In the case where some of the d_n's are less than two, we observe a stable four-cycle.

Finally, for the special case where $p = 2$, Figure 5.6 shows the behavior of the iterates, together with the orbits of even and odd indexed terms. In particular, the odd terms of the sequence $\{r_n\}$ are periodic with period 3, suggesting that (5.72) can have periodic solutions of all even periods.

In all of the above examples, numerical results demonstrate the complexity and the diversity of behavior that can occur in population dynamics. This behavior can depend on values and fluctuations of the vital rates, as well as on initial densities of adults and juveniles.

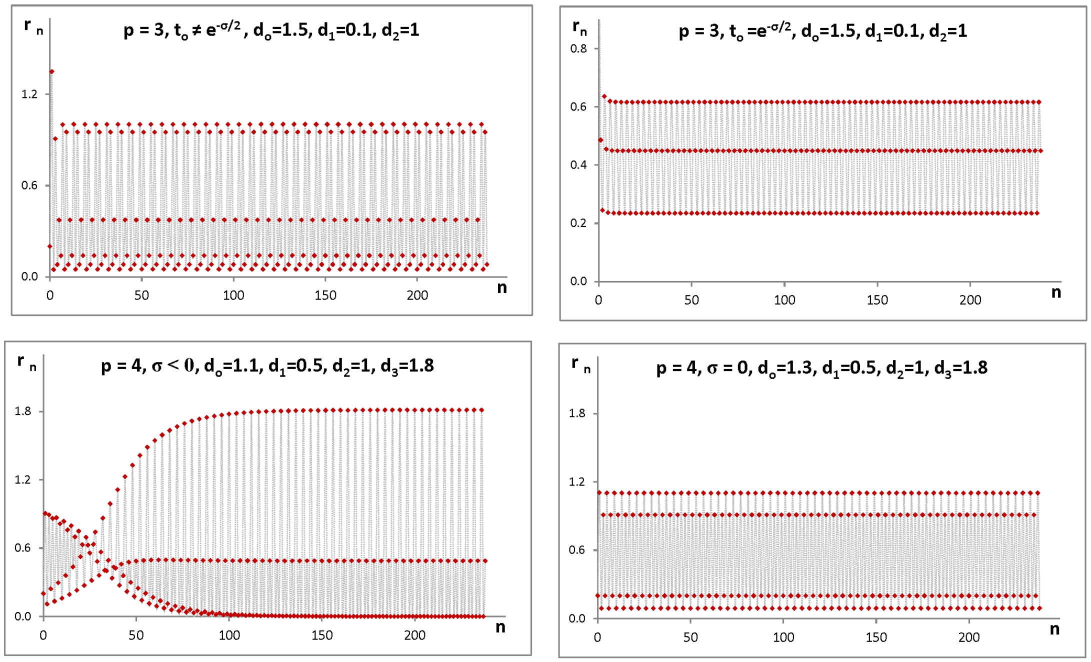

Figure 5.3: Periodic solutions for sufficiently small parameter values d_n.

5.5 Concluding remarks

We studied the dynamics of a general planar system that includes many common stage-structured population models that evolve in discrete time. We derived several results pertaining to extinction of the species for both autonomous and nonautonomous, as well as density dependent matrix models. These hypotheses are more general than what is typically assumed in population models and give us broader understanding of the mathematical properties of the system. Special cases of the model of Beverton-Holt and Ricker type were then considered to explore the role of

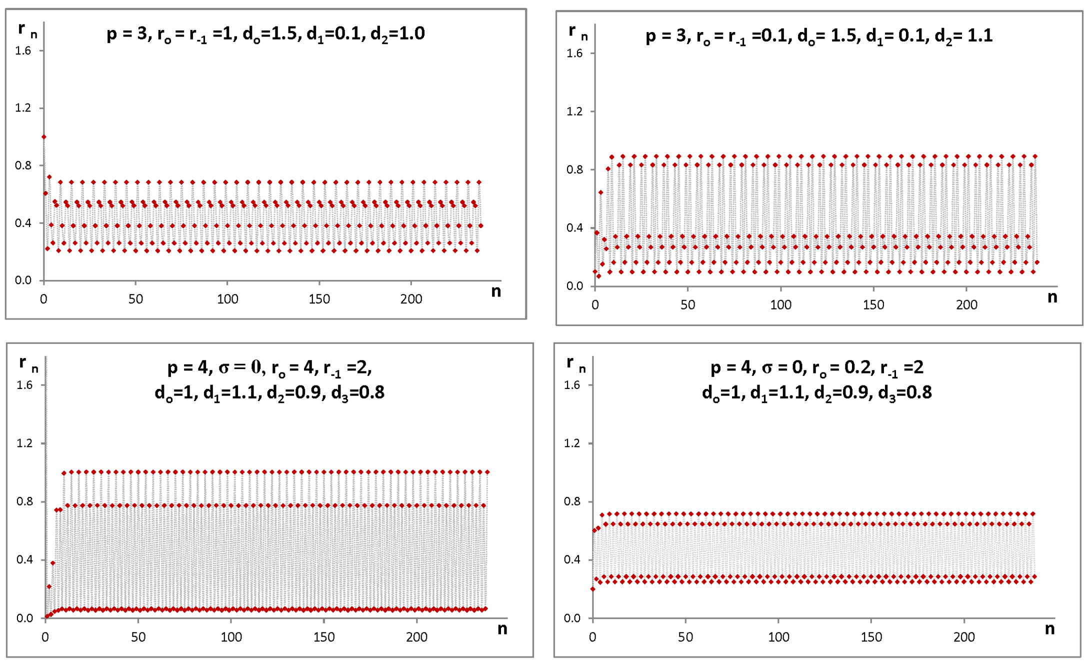

Figure 5.4: Dependence of solutions on initial values.

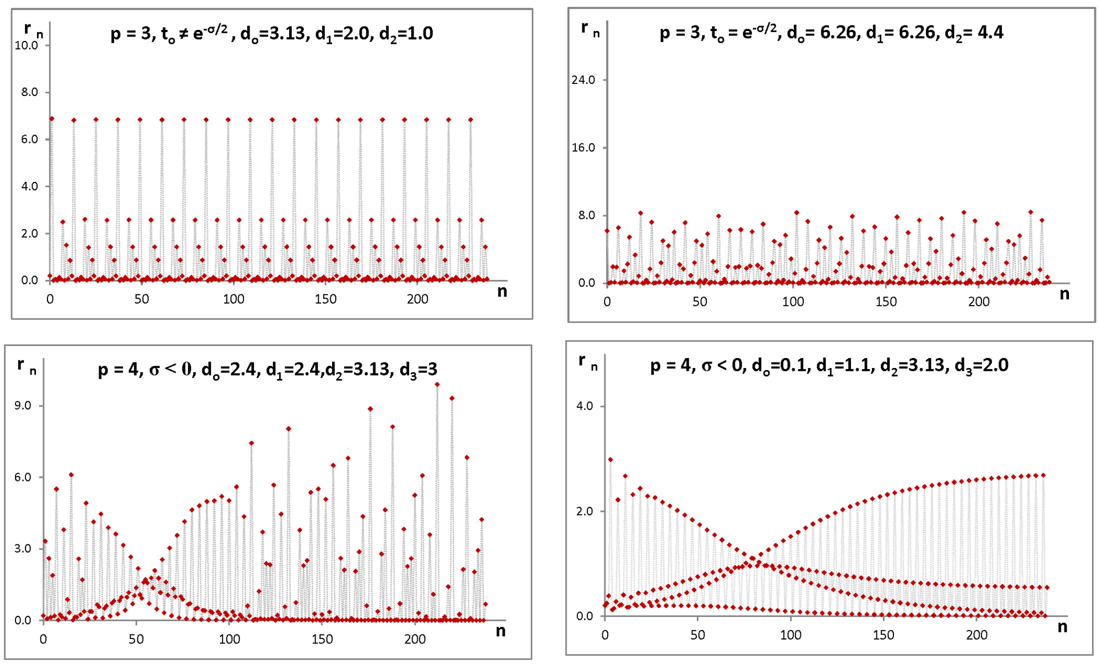

Figure 5.5: Complex behavior with sufficiently large values of d_n.

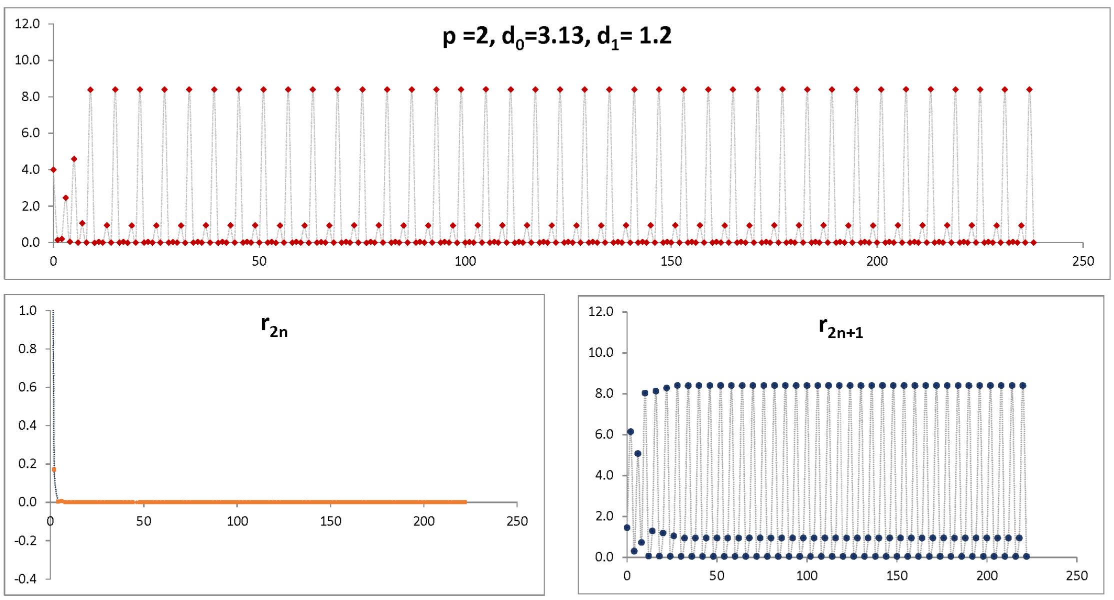

Figure 5.6: Period three solution of the odd terms for the case $p = 2$ for sufficiently large values of d_n.

intra-species competition, restocking strategies, as well as seasonal variations in the vital rates. For the system with Beverton-Holt type recruitment, we showed that sufficiently high level of competition can have destabilizing effect on the persistence equilibrium and lead to period-two oscillations. For the system with Ricker type recruitment, we showed occurrence of multistable periodic, as well as chaotic behavior. Instance of chaotic behavior were obtained for the autonomous system

$$x_{n+1} = \sigma_{1,n} y_n \tag{5.76}$$

$$y_{n+1} = \beta_n x_n e^{\alpha_n - c_{1,n} x_n - c_{2,n} y_n} \tag{5.77}$$

under the assumption that $c_{2,n} = \sigma_{1,n} c_{1,n}$. The case where $c_{2,n} \neq \sigma_{1,n} c_{1,n}$ can be studied next. In particular, the case where (5.76)-(5.77) can be folded into the the autonomous equation

$$r_{n+1} = r_{n-1} e^{d - b r_{n-1} - c r_n} \tag{5.78}$$

where $b, c > 0, b \neq c$ is of particular interest. In particular, we expect that mulitstable orbits will not occur although complex behavior is possible. There is currently no comprehensive study of the dynamics of (5.78) that we are aware of so obtaining significant details on the dynamics of this equation would be desirable.

CHAPTER VI

Conclusion

In this thesis, we studied planar systems of difference equations and their applications to biological models of species populations. These systems were studied via *folding* - the method of transforming systems of difference equations into a higher order scalar difference equations. When applicable, this method reduces systems of difference equations to scalar equations of higher order. For example, the planar system is transformed into a core second order difference equation and a passive non-dynamic equation.

We studied two classes of second order equations: quadratic fractional and exponential. These systems fold into second order quadratic fractional and exponential difference equations respectively. Besides being of great interest in the field of difference equations, rational and exponential equations have been widely used in applications to biological systems in general and in modeling species populations in particular.

In the study of the quadratic fractional equation

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{6.1}$$

we investigated the boundedness and persistence of solutions, uniqueness and the global stability of the positive fixed point and the occurrence of periodic solutions with non-negative parameters and initial values. We showed that when the function defining the difference equation is monotone in its arguments, the equation does not have any periodic solutions of period greater than two. In addition, we also established that under the above assumptions, in the absence of two-cycles, the solutions converge to the unique positive fixed point.

The above results were applied to the study of linear/rational systems of difference equations. Under common assumptions on initial values values and parameters, we derived several results on boundedness, global convergence to an equilibrium and the existence, or absence, of orbits with period two. These results allow some of the system parameters to be negative, instances not commonly considered in previous studies. Using the idea of folding, we also identified ranges of parameter values that provide sufficient conditions on existence of chaotic, as well as multiple stable orbits of different periods for the planar system.

We then studied the exponential difference equations with time varying parameters given by

$$x_{n+1} = x_{n-1}e^{a_n - x_n - x_{n-1}} \tag{6.2}$$

We obtained sufficient conditions for boundedness of solutions and global convergence to zero for a general nonautonomous case. We studied the special, autonomous case and showed occurrence of multistable periodic and nonperiodic orbits. For the nonautonomous case of periodic parameters, we showed that the nature of the solutions is qualitatively different depending on whether the period of the parameters is even or odd. In particular, cycles that occur when parameters are periodic with odd period are not unique, i.e. they are determined by the values of the initial conditions. This phenomenon, except for a limited special case, is absent when the period of the parameters is even.

The above results were then applied to the study of biological models of populations. Using various methods of analysis including folding, we investigated a broad class of planar systems that arise in the study of so-called stage-structured (adult-juvenile) single species populations, with and without time-varying parameters. In some cases, these systems are of the rational sort (e.g. the Beverton-Holt type), while in other cases the systems involve the exponential or Ricker function. In biological contexts, these results include conditions that imply extinction or survival of the species in some balanced form, as well as possible occurrence of complex and chaotic behavior, when a certain type of adult harvesting is implemented. We derived sufficient conditions for convergence of solutions to zero (species extinction) that are more general than what was considered in prior research, but can have an intuitive biological interpretation. We then considered special cases of the model to explore the role of inter-stage competition, restocking strategies, as well as seasonal fluctuations in the vital rates. We showed that in certain scenarios extinction may still occur even when restocking is present. In the rational special case of the system with Beverton-Holt type interactions, we showed that the persistence equilibrium in the positive quadrant may be globally attracting even in the presence of inter-stage competition. However, we also showed that with a sufficiently high level of competition, the persistence equilibrium becomes unstable and the system exhibits period-two oscillations. We then studied special cases of autonomous and nonautonomouse systems with Ricker type interactions to show the occurrence of chaotic and periodic solutions that vary greatly based on the amplitude and periodicity of the vital rates.

At the end of each chapter, we outlined open problems and conjectures for possible future research. In the study of the quadratic-rational second order equation in (6.1) we showed several sufficient conditions for convergence of solutions to a positive fixed point. These conditions require the function defining the second order equation to be monotone. Instances when this hypothesis fails were not addressed and could be investigated next.

Several open problems and conjectures were posed for the second order exponential equation (6.2) where parameters $\{a_n\}$ are periodic. A generalization of (6.2) given by

$$x_{n+1} = x_{n-1}e^{a_n - b_n x_n - c_n x_{n-1}} \quad \text{where} \quad b_n \neq c_n \tag{6.3}$$

is a natural choice for future studies. In addition, exponential equation of the type

$$x_{n+1} = x_n e^{a_n - b_n x_n - c_n x_{n-1}} \tag{6.4}$$

has not been well-explored and may be of interest for future investigation. Since equations in (6.3) and (6.4) do not admit semiconjugate factorization and monotone function techniques generally do not apply, their study will involve alternative and possibly new methods of analysis.

Finally, further exploration of the method of folding is also of interest. In this work we demonstrated how the folding method can greatly facilitate the analysis of planar systems. Since this method has not been used in a systematic study of higher dimensional systems, further identification of systems that could be analysed via folding may be of practical value. In addition, the question of whether there are certain patterns or regularities in foldability of systems and their subsequent foldings are worth investigating.

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Publications in mathematical journals

"Periodic and chaotic orbits of a discrete rational system," with H. Sedaghat, *Discrete Dynamics in Nature and Society,* vol. 2015, Article ID 519598, 2015, doi: 10/1155/2015/519598.

"Dynamics of planar systems that model stage-structured populations," with H. Sedaghat. *Discrete Dynamics in Nature and Society,* Special Issue Biological Models and Synchronization of Discrete Systems, vol. 2015, Article ID 137182, 2015. doi: 10.1155/2015/137182.

"Extinction, periodicity and multistability in a Ricker model of stage-structured populations," with H. Sedaghat. *Journal of Difference Equations and Applications,* to appear.

Federal Reserve publications

"The Prevalence of Apprenticeships in Germany and the US," with U. Neelakantan and D. Price, *Economic Brief,* Federal Reserve Bank of Richmond, 2014, 14-08.

"Foreclosure Crisis and Response: How Homeowners Fared in Reaching Out for Mortgage Assistance," with S. McKay and U. Neelakantan. *Community Scope*, Federal Reserve Bank of Richmond, 2013, 3(2).

Work in progress

Mathematics

"Global Stability and Periodic Solutions for a Second Order Rational Equation." with H. Sedaghat. *International Journal of Difference Equations.* Under review.

"Complex Orbits and Multistability in a Ricker Model of Stage-Structured Populations," with H. Sedaghat. In progress.

"Uniform Boundedness and Global Convergence in Higher Order Fractional Difference Equations" with H. Sedaghat. Working paper.

Economics and Statistics

"Portfolio Choice in a Two-Person Household," with U. Neelakantan, A. Lyons and C. Nelson. *Economic Inquiry*. Under review.

"Monetary Incentives and Mortgage Renegotiation Outcomes," with U. Neelakantan. In preparation.

"Using Regional Surveys to Gauge the National Economy" with S. Pinto. Working paper.

"The Impact of Foreclosure on Homelessness: Evidence from Greater Richmond Area," with U. Neelakantan and M. Ackermann. Unpublished manuscript.

"Nonlinear Dynamics in a Search and Matching Model," with T. Lubik, A. Wolman and T. Hursey. Working paper.

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Participation in conferences and seminars

Jan. 2016 — *"Periodic and chaotic orbits of a Ricker Model with Periodic Coefficients"* Invited talk at Joint Mathematics Meeting, Special Session on Difference Equations, to occur.

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Mar. 2015 — "*Global Dynamics and Periodic Solutions of a Quadratic-fractional Second Order Rational Difference Equation."* Invited talk at AMS Spring Southeastern Sectional Meeting, Washington, D.C.

Nov. 2014 — *"The Dynamics of a Planar System."* Invited talk at AMS Fall Southeastern Sectional Meeting, Greensboro, NC.

Sep. 2013 — "*Does Foreclosure Increase the Likelihood of Homelessness? Evidence from Greater Richmond Area."* Contributing talk at Collaborative Impact: The Case for Ending Homelessness, Federal Reserve of Richmond.

Apr. 2013 — *"The Impact of Foreclosure on Homelessness: Evidence from Greater Richmond Area."* Seminar presentation at the VCU Economics Department, Richmond VA.

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