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Cancellation Properties of Direct Products of Digraphs

Katherine Toman
Virginia Commonwealth University

College of Humanities and Sciences
Virginia Commonwealth University

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Richard Hammack, Department of Mathematics

Wes Cain, Department of Mathematics

Micol Hammack, Focused Inquiry Faculty

John F. Berglund, Graduate Chair, Mathematics and Applied Mathematics

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Fred Hawkridge, Dean, College of Humanities and Sciences

Dr. F. Douglas Boudinot, Graduate Dean

Date

Cancellation Properties of Direct Products of Graphs

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

Katherine Elaine Toman
Master of Science

Director: Richard Hammack, Associate Professor
Department of Mathematics and Applied Mathematics

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Abstract

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By Katherine Elaine Toman, Master of Science.

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This paper discusses the direct product cancellation of digraphs. We define the exact conditions on G such that $G \times K \cong H \times K$ implies $G \cong H$. We focus first on simple equations such as $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$ where $\overrightarrow{K_2}$ denotes a single arc and then extend this to the more general situation, $G \times K \cong H \times K$. Our results are acheived by using a "factorial" operation on graphs, which is in some sense analogous to the factorial of an integer.

Introduction

This thesis will discuss the conditions on a digraph H such that given digraphs G and K, $G \times K \cong H \times K$ implies $G \cong H$. However we must start with a few definitions. A solid introduction to the ideas and definitions will make the later sections more understandable. We begin with the main structure discussed throughout this paper, which is a graph.

1.1 Background

DEFINITION 1.1. A finite graph, denoted G, is a nonempty set of points, called vertices, and a subset of unordered pairs of points, called edges. The set of vertices in G is called the vertex set of G and is denoted $V(G)$ while the set containing all unordered pairs of $V(G)$ is called the edge set of G and is denoted $E(G)$. The number of elements in the vertex set or the edge set is called the order of $V(G)$ and/or $E(G)$ and is denoted $|V(G)|$ and $|E(G)|$, respectively.

A *vertex* of G is labeled with a single letter or number such as g or 1. An *edge* of G is denoted gg' where g and g' are vertices of G and there is an edge connecting g and g'.

Two vertices are *adjacent* if there is an edge connecting them directly. We simply reference a graph G when refering to both the edges and the vertices of the graph, however we will discuss the *vertex set of G* and the *edge set of G* separately as well. Figure 1.1 shows three examples of graphs.

Notice in Figure 1.1, the first two graphs have the same vertex set, $V(G) = V(H) = \{a,b,c,d\}$, however the edge sets differ greatly with $E(G) = \{ab,bc,cd,da\}$ and $E(H) =$

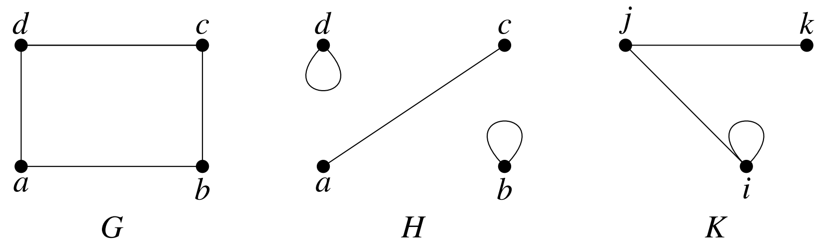


Figure 1.1: Three examples of graphs G, H and K

$\{ac, bb, dd\}$. We should observe that the second graph is also not *connected*. This means it is not possible to trace along the edges and connect every pair of vertices using only edges. Connectedness will not be an important graph characteristic in this paper however. For a more in-depth discussion of connectness, refer to G. Chartrand and L. Lesniak's Fourth Edition of Graphs and Digraphs [1].

At this time you may realize there can be an edge that begins and ends at the same point, as in the second and third graphs of Figure 1.1. This is called a *loop*. The formal definition is below.

DEFINITION 1.2. Given a graph G, a loop is an edge that connects a vertex $g \in V(G)$ to itself and is written gg.

In this paper we will focus our study on a particular generalization of graphs, refered to as digraphs. A digraph is similar to a graph in that it has vertices and edges, however the edges are given a direction.

DEFINITION 1.3. A digraph G consists of a vertex set, $V(G)$, and an arc set, $E(G)$. An element of $E(G)$ is denoted gg' and the order in which the vertices are written indicates a direction from g to g'. So, if $gg' \in E(G)$ then there is an arc (or arrow) pointing from g to g'.

Refer to Figure 1.2 for examples of a digraph. Observe that $V(G) = \{a, b, c\}$ and $E(G) = \{ac, ca, ab\}$ while $V(H) = \{i, j, k\}$ and $E(H) = \{ik, ji, jk, jj\}$. Also note that for

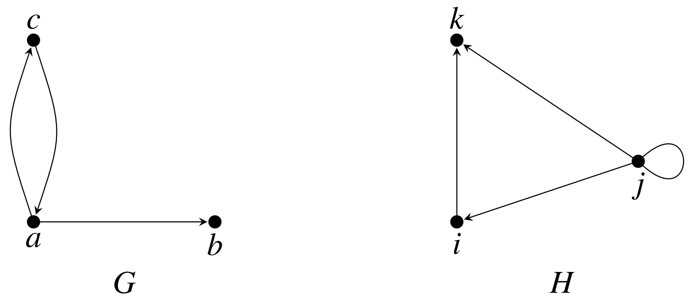


Figure 1.2: Two examples of digraphs G and H

digraphs it is possible that $ab \in E(G)$, but ba is not contained in $E(G)$, as in Figure 1.2.

If $ab \in E(G)$ for some digraph G, then it is said that a is *incident to* b and b is *incident from* a. Throughout the later sections the graphs mentioned can be assumed to be digraphs and characteristics of digraphs will be important to note within proofs. One characteristic of digraphs that should be mentioned is the indegree and outdegree of a vertex in G.

DEFINITION 1.4. The number of vertices incident from a vertex $g \in V(G)$ is the outdegree of g and the number of vertices incident to g is the indegree of g. The total degree of a vertex g is the number of vertices incident to g added to the number of vertices incident from g. A loop $gg \in E(G)$ adds 1 to the indegree of g and 1 to the outdegree of g.

Another way to consider the degree of a vertex focuses on the arcs of G. The outdegree of a vertex $g \in V(G)$ is the number of arcs that begin at g and conversely, the indegree of g is the number of arcs ending at g. Also, the total degree of a vertex g is the number of arcs beginning or ending at g. We denote outdegree of g as $\text{od}(g)$ and indegree of g as $\text{id}(g)$. The total degree of g is denoted $\deg(g) = \text{od}(g) + \text{id}(g)$.

Please note that the numbers in Figure 1.3 represent the total degree of each vertex. In this example, the vertex with total degree 4 and no loops has outdegree 2 and indegree 2, and the vertex with total degree 4 and a loop has indegree 3 and outdegree 1.

DEFINITION 1.5. A vertex with a total degree of zero is called an isolated vertex.

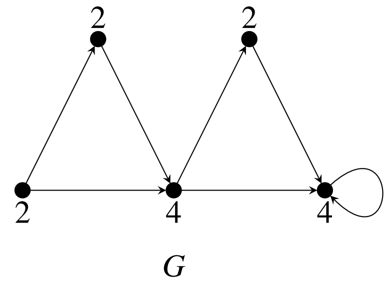


Figure 1.3: Example of a digraph with the total degree of all vertices

Along with considering the degree of vertices in $V(G)$, we can look at properties of digraphs that incorporate both vertices and arcs, such as bipartiteness.

DEFINITION 1.6. A digraph G is said to be bipartite if it is possible to partition the vertex set of G into two subsets, V_1 and V_2 such that every arc in $E(G)$ connects a vertex in V_1 with a vertex in V_2.

In other words, arcs in a bipartite digraph G either begin at a vertex in V_1 and end at a vertex in V_2 or begin at a vertex in V_2 and end at a vertex in V_1. For an example, see Figure 1.4.

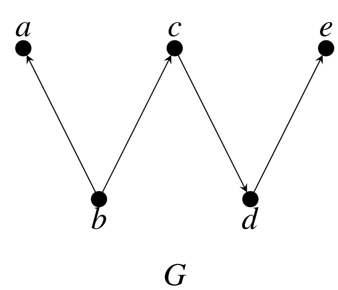
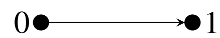


Figure 1.4: A bipartite graph G

In Figure 1.4, notice that $V_1 = \{a, b, c\}$ and $V_2 = \{b, d\}$ and each arc of G connects a vertex in V_1 to a vertex in V_2 or a vertex in V_2 to a vertex in V_1. Observe that any graph containing a loop will not be bipartite. An example of a simple bipartite graph is $\overrightarrow{K_2}$ and is shown in Figure 1.5. We will be using this particular bipartite graph throughout this thesis. Please note its use in concrete examples leading up to the theory portion of this paper.



$\overrightarrow{K_2}$

Figure 1.5: The graph of a simple bipartite digraph

Bipartiteness is a definition that will come into play when we discuss the direct product of two graphs, one being $\overrightarrow{K_2}$.

1.2 The Direct Product

The direct product of two graphs is a way of “multiplying” two graphs together in such a way that we get another graph.

DEFINITION 1.7. The direct product of two digraphs G and H is a digraph $G \times H$ and is defined as follows: $V(G \times H) = V(G) \times V(H)$ and $(g,h)(g',h') \in E(G \times H)$ if and only if $gg' \in E(G)$ and $hh' \in E(H)$. Thus elements in $V(G \times H)$ are ordered pairs (g,h) such that $g \in V(G)$ and $h \in V(H)$.

This definition can be difficult to grasp with only words and therefore please refer to the following examples for clarification.

In Figure 1.6, notice that $V(G) = \{1,2,3\}$ and $V(H) = \{a,b\}$ and therefore, using Definition 1.7, $V(G \times H) = \{(1,a),(2,a),(3,a),(1,b),(2,b),(3,b)\}$. Also, $E(G) = \{12,32,33\}$ and $E(H) = \{01\}$ so by Definition 1.7 $E(G \times H) = \{(1,a)(2,b),(3,a)(2,b),(3,a)(3,b)\}$.

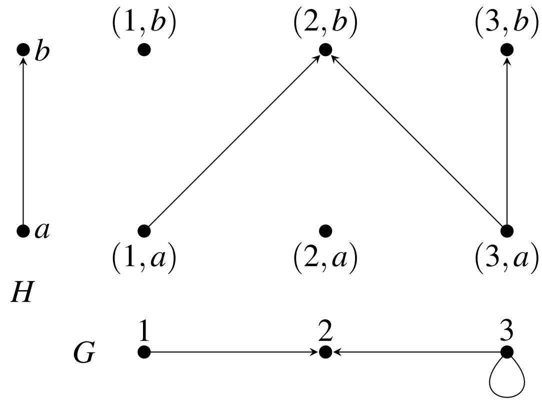


Figure 1.6: The first example of a direct product $G \times H$.

In this first example of $G \times H$, you can see that each element of $V(G \times H)$ is an ordered pair with first coordinate from $V(G)$ and the second coordinate from $V(H)$. Also, only when $gg' \in E(G)$ and $hh' \in E(H)$ will $(g,h)(g',h') \in E(G \times H)$. For example, $12 \in E(G)$ and $ab \in E(H)$, therefore $(1,a)(2,b) \in E(G \times H)$. Another observation worth noticing is that Figure 1.6 is an example of a direct product that is bipartite. Observe that $V_1 = \{(1,a),(2,a),(3,a)\}$ while $V_2 = \{(1,b),(2,b),(3,b)\}$. The bipartite nature of this graph is due to H in the product, given that H is bipartite.

REMARK 1.8. Let H be a bipartite graph with partite sets V_1 and V_2. Then there can be no arc of $G \times H$ jointing vertices in $V(G) \times V_1$ and likewise there can be no arcs connecting vertices in $V(G) \times V_2$. To support this claim, consider that an arc $(g,v)(g',v')$ with $v,v' \in V_1$ would imply that $gg' \in E(G)$ and $vv' \in E(V_1)$. However, since H is bipartite, we know that there are no arcs between two vertices in V_1 and so vv' is not in $E(G \times V_1)$. Therefore each arc in $E(G \times H)$ must connect a vertex in $V(G) \times V_1$ and a vertex in $V(G) \times V_2$. So, $G \times H$ is also bipartite.

As you may notice in Figure 1.7, G is the same graph as Figure 1.6, however computing the direct product of G and K creates a completely different graph than $G \times H$.

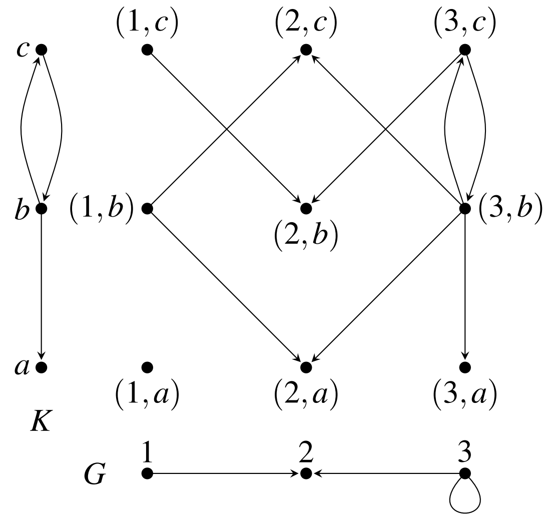


Figure 1.7: Second Example of a direct product $G \times K$.

This last example of a direct product, Figure 1.8, shows the possiblity for a loop in the product. Though it is not as common, in many examples there will be a loop in the product of G and J if there is a loop in both G and J.

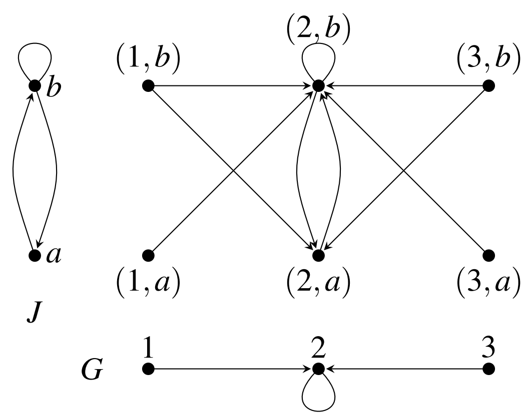


Figure 1.8: Third example of a direct product $G \times J$.

The majority of this paper will focus on the direct products of graphs and the existence of a homomorphism or an isomorphism between them.

1.3 Homomorphisms and Isomorphisms

DEFINITION 1.9. A homomorphism φ from a digraph G to a digraph H is a map $\varphi : V(G) \to V(H)$ satisfying $gg' \in E(G)$ implies $\varphi(g)\varphi(g') \in E(H)$.

Figure 1.9 illustrates the definition of a homomorphism from a graph G to a graph H. Notice that $V(G) = \{1,2,3,4\}$ and $V(H) = \{a,b,c\}$. Let $\varphi : V(G) \to V(H)$ be defined as follows: $\varphi(1) = a$, $\varphi(2) = b$, $\varphi(3) = c$ and $\varphi(4) = a$. Since $E(G) = \{12,23,34\}$ we can see that $\varphi(1)\varphi(2) = ab \in E(H)$ and $\varphi(2)\varphi(3) = bc \in E(H)$ and $\varphi(3)\varphi(4) = ca \in E(H)$. Therefore φ is a homomorphism from G to H.

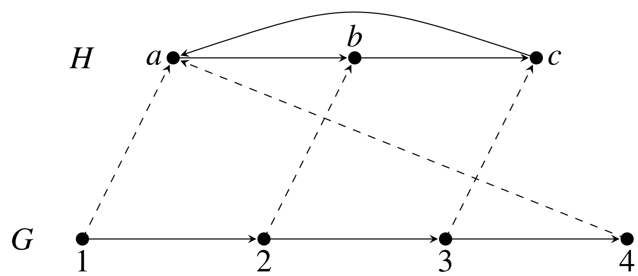


Figure 1.9: Example of a homomorphism φ between G and H.

We denote the *number of homomorphisms* from G to H as $\hom(G,H)$. Also, the number of *injective homomorphisms* from G to H is denoted $\mathrm{inj}(G,H)$.

DEFINITION 1.10. Two digraphs G and H are isomorphic if the exists a bijective mapping $\varphi : V(G) \to V(H)$ such that $gg' \in E(G)$ if and only if $\varphi(g)\varphi(g') \in E(H)$. such a map is called an isomorphism.

Observe that Figure 1.9 is **not** an example of an isomorphism between G and H because φ is not a bijection between $V(G)$ and $V(H)$ shown by $\varphi(1) = \varphi(4)$ when $1 \neq 4$. However, Figure 1.10 is an example of an isomorphism φ from G to H.

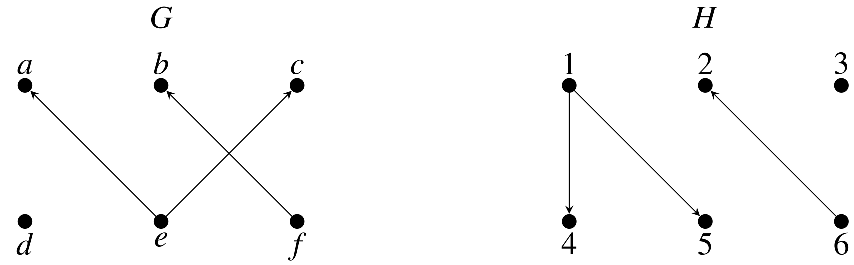


Figure 1.10: Example of two isomorphic graphs G and H.

Observe that $V(G) = \{a,b,c,d,e,f\}$, $V(H) = \{1,2,3,4,5,6\}$, $E(G) = \{ea,ec,fb\}$ and $E(H) = \{14,15,62\}$. If $\varphi : V(G) \to V(H)$ is defined as $\varphi(a) = 4$, $\varphi(b) = 2$, $\varphi(c) = 5$, $\varphi(d) = 3$, $\varphi(e) = 1$, and $\varphi(f) = 6$ then we can verify that φ is indeed an isomorphism between G and H. This means $gg' \in E(G)$ if and only if $\varphi(g)\varphi(g') \in E(H)$ and such is the case for Figure 1.10.

Many observations have been made about homomorphisms and isomorphisms between graphs and we should discuss a few of them that will be relevant in this paper. Firstly, please refer to Remark 1.11 to understand why the number of homomorphisms from $\overrightarrow{K_2}$ to any graph K is equal to the number of edges in K.

REMARK 1.11. If K is a digraph with at least one edge, then hom($\overrightarrow{K_2}$,K)=$\mid E(K) \mid$. This is made clear given that there is a unique homomorphism $\varphi : \overrightarrow{K_2} \to K$ such that $\varphi(0) = k$ and $\varphi(1) = k'$ for each $kk' \in E(K)$. Conversely, any homomorphism $\varphi : \overrightarrow{K_2} \to K$ is necessarily of this form. Therefore, the number of homomorphisms from $\overrightarrow{K_2}$ to K is equal to the number of edges in $E(K)$.

Another fascinating observation about homomorphisms is that $\hom(X, G \times H) = \hom(X,G) \cdot \hom(X,H)$ for any digraphs G, H and X. This is proven below in form of the proof provided in Hell [3]

PROPOSITION 1.12. For any digraphs G, H and X, $\hom(X, G \times H) = \hom(X,G) \cdot \hom(X,H)$.

Proof. First we will show that every pair of homomorphisms $g : X \to G$ and $h : X \to H$ corresponds to a unique homomorphism $k : X \to G \times H$. Let g and h be such homomorphisms. Let $k : V(X) \to V(G \times H)$ be defined as $k(x) = (g(x), h(x))$ for each $x \in V(X)$. Now observe that k is a homomorphism. Suppose $xx' \in E(X)$. Since g and h are homomorphisms, $g(x)g(x') \in E(G)$ and $h(x)h(x') \in E(H)$ so we know that $(g(x), h(x))(g(x'), h(x')) = k(x)k(x') \in E(G \times H)$. Therefore, k is a homomorpihsm and so it follws that $\hom(X, G \times H) \geq \hom(X, G) \cdot \hom(X, H)$.

Conversely, we want to show $\hom(X, G \times H) \leq \hom(X, G) \cdot \hom(X, H)$. In other words, we need to show that any homomorphism $k : X \to G \times H$ is of the form $k(x) = (g(x), h(x))$ where $g : X \to G$ and $h : X \to H$ are homomorphisms. Consider the projections $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$. Observe that these are homomorphisms. Consider that any homomorphism $k : X \to G \times H$ can be written as $k(x) = (\pi_G \circ k(x), \pi_H \circ k(x))$. Note that a composition of homomorphisms is again a homomorphism. Therefore, let $g = \pi_G \circ k(x)$ and $h = \pi_H \circ k(x)$. So for any homomorphism k, it follows that $k(x) = (g(x), h(x))$ for homomorphisms g and h. So $\hom(X, G \times H) \leq \hom(X, G) \cdot \hom(X, H)$.

Therefore, $\hom(X, G \times H) = \hom(X, G) \cdot \hom(X, H)$. $\square$

1.4 Lovász's Theorem

Another important observation about the homomorphisms between isomorphic graphs was made by Lovász in 1971. The well-known result in his paper, "On the Cancellation Law Among Finite Relational Structures" [4] considered the condtions on K such that $G \times K \cong H \times K$ implies $G \cong H$. Though this paper focuses on the structure of G, the observations made by Lovász on the homomorphisms between graphs is very helpful for our discussion. Before we discuss his conclusions, we must state another definition that will be used in the proof.

DEFINITION 1.13. Suppose X is a digraph and $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_k\}$ is a partition of $V(X)$. The quotient of X by σ, denoted X/σ, is a digraph that is defined as follows: $V(X/\sigma) = \{\sigma_1, \sigma_2, ..., \sigma_k\}$ and $E(X/\sigma) = \{\sigma_i\sigma_j : X$ has an arc pointing from σ_i to $\sigma_j\}$.

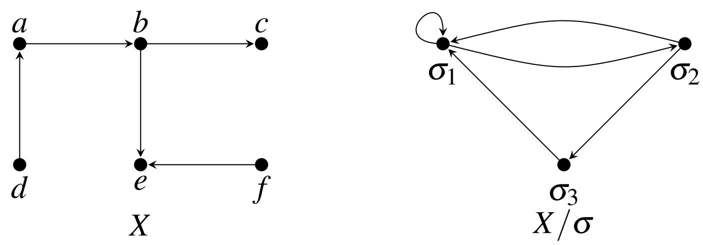


Figure 1.11: First Example of X/σ.

Note that in Figure 1.11, $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$, $\sigma_1 = \{a, d, e\}$, $\sigma_2 = \{b\}$ and $\sigma_3 = \{c, f\}$. Therefore, $V(X/\sigma) = \{\sigma_1, \sigma_2, \sigma_3\}$. Also, $E(X/\sigma) = \{\sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_1\}$.

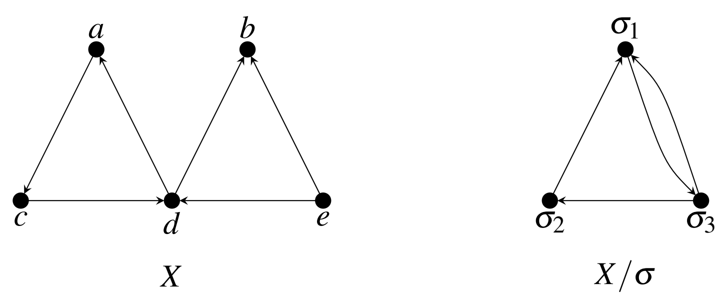


Figure 1.12: Second Example of X/σ.

In Figure 1.12, $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$, $\sigma_1 = \{a, b\}$, $\sigma_2 = \{d\}$ and $\sigma_3 = \{c, e\}$. According to Definition 1.13, we know that $V(X/\sigma) = \{\sigma_1, \sigma_2, \sigma_3\}$ and $E(X/\sigma) = \{\sigma_1\sigma_3, \sigma_3\sigma_1, \sigma_3\sigma_2, \sigma_2\sigma_1\}$. Along with this new definition, Lovász uses the below remark on injective mappings in his cancellation work.

REMARK 1.14. For digraphs G and H, if $\mathrm{inj}(G,H) \neq 0$ and $\mathrm{inj}(H,G) \neq 0$ then $G \cong H$.

Proof. Let G and H be digraphs and let $\mathrm{inj}(G,H) \neq 0$ and $\mathrm{inj}(H,G) \neq 0$. Therefore there are at least one $\mu : G \to H$ and $\lambda : H \to G$ such that both are injective mappings. So $|E(G)| \leq |E(H)|$ and $|E(H)| \leq |E(G)|$ and so $|E(G)| = |E(H)|$. Also, note that $|V(G)| = |V(H)|$.

Now consider the bijective mapping $\mu : V(G) \to V(H)$. Since μ is a homomorphism, each $gg' \in E(G)$ implies $\mu(g)\mu(g') \in E(H)$ and given that $|E(G)| = |E(H)|$, we know that $gg' \in E(G)$ if and only if $\mu(g)\mu(g') \in E(H)$.

Therefore, $G \cong H$. □

Also, the below observations are discussed in more detail in [1].

REMARK 1.15. For any digraphs X and G and any homomorphism $F : X \to G$, there is a partition of $V(X)$ denoted $\sigma_F = \{F^{-1}(g) : g \in V(G)\}$.

REMARK 1.16. There exists an injective homomorphism $\widehat{F} : X/\sigma_F \to G$ defined as

$\widehat{F}(F^{-1}(g)) = g$ for all $g \in V(G)$. Conversely, given any partition σ of $V(X)$ and any homomorphism $\widehat{F} : X/\sigma \to G$, there is a homomorphism $F : X \to G$ for which $\sigma = \sigma_F$.

These two remarks lead to the fact that $\mathrm{hom}(X,G) = \sum_{\sigma \in \mathcal{P}} \mathrm{inj}(X/\sigma, G)$ where $\mathcal{P}$ denotes the set of all partitions of $V(G)$. As mentioned, this is discussed in more depth in [1].

Lovász stated that two digraphs G and H were isomorphic if and only if the number of homomorphisms from a digraph X to G were equal to the number of homomorphisms from X to H for all X. The proof is below in the style of Graphs and Digraphs [1].

THEOREM 1.17. Given digraphs G and H, $G \cong H$ if and only if $\mathrm{hom}(X,G) = \mathrm{hom}(X,H)$ for any graph X.

Proof. It is enough to show that $\mathrm{inj}(X,G)$=$\mathrm{inj}(X,H)$ because if $X = H$ then there is one injective homomorphism from H to G and if $X = G$ then there is an injective homomorphism from G to H and therefore there is an isomorphism between G and H, by Remark 1.16.

Now, let $\hom(X,G) = \hom(X,H)$. Therefore, by the above remarks, we know that $\sum_{\sigma \in \mathcal{P}} \operatorname{inj}(X/\sigma, G) = \sum_{\sigma \in \mathcal{P}} \operatorname{inj}(X/\sigma, H)$ for all σ that are partitions of $V(G)$. Consider that there is an identity partition, t, that puts each vertex of G, or H, into its own partition, so that $G/t \cong G$ and $H/t \cong H$. So we can now write the following:

$$\operatorname{inj}(X/\mathrm{t}, G) + \sum_{\sigma \in \mathcal{P} \smallsetminus \{\mathrm{t}\}} \operatorname{inj}(X/\sigma, G) = \operatorname{inj}(X/\mathrm{t}, H) + \sum_{\sigma \in \mathcal{P} \smallsetminus \{\mathrm{t}\}} \operatorname{inj}(X/\sigma, H)$$

where t is the identity partition in the set of all partitions, $\mathcal{P}$. We can observe that this is equivalent to $\operatorname{inj}(X,G) + \sum_{\sigma \in \mathcal{P} \smallsetminus \{\mathrm{t}\}} \operatorname{inj}(X/\sigma, G) = \operatorname{inj}(X,H) + \sum_{\sigma \in \mathcal{P} \smallsetminus \{\mathrm{t}\}} \operatorname{inj}(X/\sigma, H)$. By induction it is clear that $\sum_{\sigma \in \mathcal{P} \smallsetminus \{\mathrm{t}\}} \operatorname{inj}(X/\sigma, G) = \sum_{\sigma \in \mathcal{P} \smallsetminus \{\mathrm{t}\}} \operatorname{inj}(X/\sigma, H)$ and therefore it follows that $\operatorname{inj}(X,G) = \operatorname{inj}(X,H)$.

Therefore $G \cong H$ if and only if $\hom(X,G)$ =$\hom(X,H)$ for any digraph X. $\square$

Using both Propostion 1.12 and Lovász's Theorem 1.17, we can now conclude that if $G \times K \cong H \times K$ and there exists at least one homomorphism from a digraph M to K, then we know that $G \times M \cong H \times M$. This result is also due to Lovász's work on the cancellation law among finite structures [4].

PROPOSITION 1.18. If $G \times K \cong H \times K$ and there exists a homomorphism $f : M \to K$ for some digraph M, then $G \times M \cong H \times M$.

Proof. Suppose $G \times K \cong H \times K$. Therefore $\hom(X, G \times K) = \hom(X, H \times K)$ for any digraph X, by Theorem 1.17. So, $\hom(X,G) \cdot \hom(X,K) = \hom(X,H) \cdot \hom(X,K)$ by Proposition 1.12.

Case One: Let $\hom(X,K) \neq 0$. Then $\hom(X,G) = \hom(X,H)$ and so $\hom(X,G) \cdot \hom(X,M) = \hom(X,H) \cdot \hom(X,M)$ because there exists a homomorphism f from M to K. So $\hom(X, G \times M) = \hom(X, H \times M)$.

Case Two: Let $\hom(X,K) = 0$. Then if there was a homomorphism $g : X \to M$ then the composition function $f \circ g : X \to M \to K$ would be a homomorphism from X to K. However there does not exist such a homomorphism from X to M because $\hom(X,K) = 0$ and therefore, $\hom(X,M) = 0$. So consider that $\hom(X, G \times M) = \hom(X,G) \cdot \hom(X,M) = 0 = \hom(X,H) \cdot \hom(X,M) = \hom(X, H \times M)$. So, $\hom(X, G \times M) = \hom(X, H \times M)$ for all digraphs X, so $G \times M \cong H \times M$ by Theorem 1.17.

So, if $G \times K \cong H \times K$ and there exists a homomorphism $f : M \to K$ for some digraph M, then $G \times M \cong H \times M$. $\square$

Using this introduction as our basis, our goal for this paper will be to discover the exact conditions needed on a digraph G such that $G \times K \cong H \times K$ implies $G \cong H$. Lovász called a graph K a *zero divisor* if there exists two nonisomorphic graphs G and H such that $G \times K \cong H \times K$. In his paper [4], he defines a particular homomorphism on K such that its absence guarantees that $G \times K \cong H \times K$ implies that $G \cong H$. For example, Figure 1.13 shows that K is a zero divisor. Clearly $A \ncong B$, yet $A \times K \cong B \times K$. Note that both products are isomorphic to three copies of K.

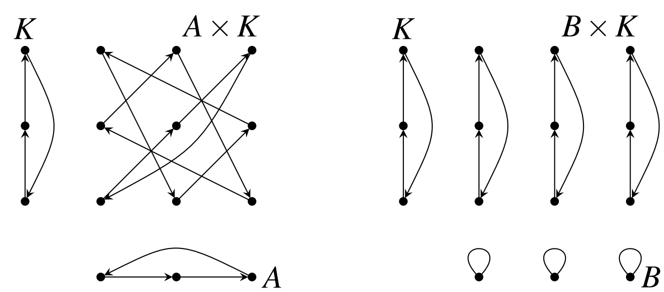


Figure 1.13: Example of a zero divisor

The following is the main result concerning zero divisors.

THEOREM 1.19. A digraph C is a zero divisor if and only if there is a homomorphism $\varphi : C \to \overrightarrow{C_{p_1}} + \overrightarrow{C_{p_2}} + \overrightarrow{C_{p_3}} + \cdots + \overrightarrow{C_{p_k}}$ for prime numbers $p_1, p_2, \ldots, p_k$.

Also, the following corollary was addressed by Lovász in his paper.

COROLLARY 1.20. A graph K with at least one edge is a zero divisor if and only if K is bipartite.

Theorem 1.19 and Corollary 1.20 can be regarded as cancellation laws for the direct product. They give exact conditions on K (namely the absence of a homomorphism $\varphi : K \to \overrightarrow{C_{p_1}} + \overrightarrow{C_{p_2}} + \cdots + \overrightarrow{C_{p_k}}$) under which $A \times K \cong B \times K$ necessarily implies $A \cong B$. Refer to Lovász's paper [4] for a more comprehensive expanation.

The focus of this paper is to instead focus on the type of graph H is, with respect to G, such that the above statement holds. We would like to take any G and know whether or not $G \cong H$ if $G \times K \cong H \times K$ for a digraph K. We will pin down specific properties of H so that we can always say $G \times K \cong H \times K$ implies $G \cong H$.

Main Results

In this chapter we will describe the conditions required for a digraph G so that $G \times K \cong H \times K$ implies $G \cong H$ for any digraph K. Considering that this question involves both direct products of digraphs and just digraphs, we must observe the relationship between isomorphic direct products of digraphs G and H.

Given digraphs G and K, we will see that the digraphs H for which $G \times K \cong H \times K$ are closely linked to the permutations of $V(G)$.

2.1 Permutations of $V(G)$ and the Construction of G^π

DEFINITION 2.1. Given a digraph G, the set of permutations of $V(G)$ is denoted $\text{Perm}(V(G))$. Thus, elements $\pi \in \text{Perm}(V(G))$ are bijections $\pi : V(G) \to V(G)$. Recall that for a graph with n vertices there are $n!$ unique permutations on $V(G)$.

DEFINITION 2.2. Given a digraph G and $\pi \in \text{Perm}(V(G))$, define the digraph G^π as follows. The vertx set is G^π is $V(G^\pi) = V(G)$ and the edges set of G^π is $E(G^\pi) = \{g\pi(g') : gg' \in E(G)\}$.

Refer to Figure 2.1 for an illustration of digraphs G and G^π where $\pi = \begin{pmatrix} a & b & c & d \\ c & b & d & a \end{pmatrix}$. Notice that $V(G) = \{a,b,c,d\} = V(G^\pi)$ and $E(G) = \{ab,bc,cd,ca\}$ and $E(G^\pi) = \{a\pi(b), b\pi(c), c\pi(d), c\pi(a)\} = \{ab,bd,ca,cc\}$.

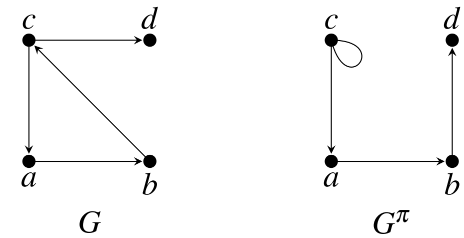


Figure 2.1: Example of G and G^π.

Figure 2.2, we see another example of a digraph G and G^π. For this example, let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ and according to Definition 2.2, the edge set of G^π is $E(G^\pi) = \{1\pi(4), 4\pi(3), 3\pi(2)\} = \{14, 41, 33\}$.

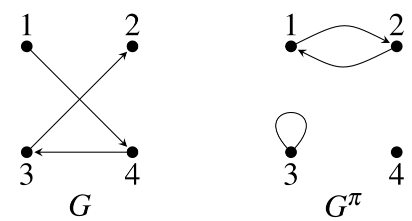


Figure 2.2: A second example illustrating G and G^π.

It is clear that $G \not\cong G^\pi$ in both Figure 2.1 and Figure 2.2. Note that in Figure 2.2, G contains no loops, while G^π contains one loop. Though G and G^π may not be isomorphic for every $\pi \in \mathrm{Perm}(V(G))$, it may be the case that $G \times K$ and $G^\pi \times K$ are isomorphic graphs. An example of this effect is illustrated in Figure 2.3. In this example we allow K to be $\overrightarrow{K_2}$ and we have used the same G and G^π which were not isomorphic graphs in Figure 2.2.

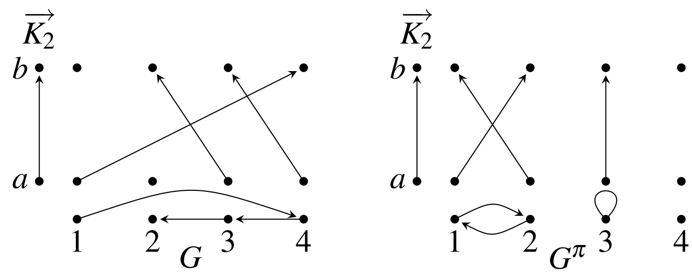


Figure 2.3: An example of $G \times K \cong G^\pi \times K$.

Also, considering a particularly simple graph K may make $G \times K$ and $H \times K$ clearer. As in many areas of mathematics, beginning with a simpler case allows for expansion later on in research.

2.2 Simple Products of Digraphs

In fact, Lemma 2.3 considers a relationship between the direct product of G and $\overrightarrow{K_2}$ and the direct product of G^π and $\overrightarrow{K_2}$, given an arbitrary $\pi \in \mathrm{Perm}(V(G))$ and a digraph G.

LEMMA 2.3. If G is a digraph and $\pi \in \mathrm{Perm}(V(G))$, then $G \times \overrightarrow{K_2} \cong G^\pi \times \overrightarrow{K_2}$.

Proof. Let G be a digraph and let $\pi \in \mathrm{Perm}(V(G))$. Define a function $\varphi : V(G \times \overrightarrow{K_2}) \to V(G^\pi \times \overrightarrow{K_2})$ in the following way:

$$\varphi((a,b)) = \begin{cases} (a,b) & \text{if } b = 0 \\ (\pi(a),b) & \text{if } b = 1 \end{cases}$$

.

We will now show that φ is an isomorphism.

We begin by showing that φ is onto. Let $(c,d) \in E(G^\pi \times \overrightarrow{K_2})$. The current question is now: what does φ map to (c,d)? By definition of $\overrightarrow{K_2}$, either $d = 0$ or $d = 1$.

Case One: If $d = 0$, then $\varphi((c,0)) = (c,0) = (c,d)$.

Case Two: If $d = 1$, then $\varphi((\pi^{-1}(c),1)) = (\pi(\pi^{-1}(c)),1) = (c,1) = (c,d)$.

Therefore, φ is onto.

Next we must show that φ is one-to-one. Let $\varphi((a,b)) = \varphi((a',b'))$ where $(a,b),(a',b') \in V(G \times \overrightarrow{K_2})$. Note that the second component of $\varphi((a,b))$ is always b. So $b = b' = 0$ or $b = b' = 1$.

Case One: If $b = b' = 0$, then $\varphi((a,0)) = \varphi((a',0))$. So, by definition of φ, we know $(a,0) = (a',0)$. Therefore, $(a,b) = (a',b')$.

Case Two: If $b = b' = 1$, then, $\varphi((a,1)) = \varphi((a',1))$. So $(\pi(a),1) = (\pi(a'),1)$ which implies $\pi(a) = \pi(a')$. Since π is a permutation, $a = a'$ and so $(a,1) = (a',1)$. So, $(a,b) = (a',b')$.

Therefore, φ is one-to-one.

In order to show that φ is an isomorphism, we must show that $(a,b)(a',b') \in E(G \times \overrightarrow{K_2})$ if and only if $\varphi(a,b)\varphi(a',b') \in E(G^\pi \times \overrightarrow{K_2})$.

Firstly, let $(a,b)(a',b') \in E(G \times \overrightarrow{K_2})$. By definition, $aa' \in E(G)$ and $bb' \in E(\overrightarrow{K_2})$. Observe that $bb' = 01$ by definition of $\overrightarrow{K_2}$. Also, $a\pi(a') \in E(G^\pi)$ as we have defined G^π. So, $(a,b)(\pi(a'),b') \in E(G^\pi \times \overrightarrow{K_2})$ and since $b = 0$ and $b' = 1$, $\varphi((a,b)) = (a,b)$ and $\varphi((a',b')) = (\pi(a'),b')$. So, $\varphi((a,b))\varphi((a',b')) = (a,b)(\pi(a'),b') \in E(G^\pi \times \overrightarrow{K_2})$.

Now, let $(a,b)(a',b') \in E(G^\pi \times \overrightarrow{K_2})$. So $aa' \in E(G^\pi)$ and $bb' \in E(\overrightarrow{K_2})$. Note that $a' = \pi(\pi^{-1}(a'))$ and $a\pi^{-1}(a') \in E(G)$. Also, note that $b = 0$ and $b' = 1$ since $bb' \in E(\overrightarrow{K_2})$. So, $(a,0)(\pi^{-1}(a'),1) \in E(G \times \overrightarrow{K_2})$. This implies $\varphi((a,0))\varphi((\pi^{-1}(a'),1)) = (a,0)\varphi((\pi^{-1}(a'),1)) = (a,0)(a',1) = (a,b)(a',b')$. So, if $(a,b)(a',b') \in E(G^\pi \times \overrightarrow{K_2})$ then we know that there is an edge $(a,0)(\pi^{-1}(a'),1) \in E(G \times \overrightarrow{K_2})$ that φ sends to $(a,0)(a',1') \in E(G^\pi \times \overrightarrow{K_2})$.

So now, $\varphi(a,b)\varphi(a',b') \in E(G^\pi \times \overrightarrow{K_2})$ if and only if $(a,b)(a',b') \in E(G \times \overrightarrow{K_2})$. Therefore, φ is an isomorphism.

In conclusion, $(G \times \overrightarrow{K_2}) \cong (G^{\pi} \times \overrightarrow{K_2})$ for any $\pi \in \mathrm{Perm}(V(G))$. $\square$

With this result we can tell that even nonisomorphic graphs could have isomorphic products when crossed with the $\overrightarrow{K_2}$. Is it possible that if $G \times K \cong H \times K$ then H must be something of the form G^{π}? If Lemma 2.3 could be reversed and still hold true we could conclude that for a digraph G and arbitrary $\pi \in \mathrm{Perm}(V(G))$, $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$ if and only if $H \cong G^{\pi}$. This is a subcase of our original question and proving this statement will guide us in the correct direction. The below proof verifies the converse of Lemma 2.3.

LEMMA 2.4. If $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$, then $H \cong G^{\pi}$ for some $\pi \in \mathrm{Perm}(V(G))$.

Proof. Let $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$. Then there exists an isomorphism $\varphi : V(G \times \overrightarrow{K_2}) \to V(H \times \overrightarrow{K_2})$. Notice that $G \times \overrightarrow{K_2}$ and $H \times \overrightarrow{K_2}$ are bipartite digraphs since $\overrightarrow{K_2}$ has exactly one edge. Therefore all $(g,0) \in V(G \times \overrightarrow{K_2})$ will have an in-degree of zero, regardless of its out-degree, as will all $(h,0) \in V(H \times \overrightarrow{K_2})$. Also, for all $(g,1) \in V(G \times \overrightarrow{K_2})$ and $(h,1) \in V(H \times \overrightarrow{K_2})$ the out-degree will be zero regardless of the in-degree.

Define $A = \{(g,0) \in V(G \times \overrightarrow{K_2})\}$ and $B = \{(g,1) \in V(G \times \overrightarrow{K_2})\}$ and $C = \{(h,0) \in V(H \times \overrightarrow{K_2})\}$ and define $D = \{(h,1) \in V(H \times \overrightarrow{K_2})\}$.

Notice that $|A| = |B|$ and $|C| = |D|$ because we have crossed G with $\overrightarrow{K_2}$ and H with $\overrightarrow{K_2}$. Notice that $|A| = |C|$ and $|B| = |D|$ because $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$ and so all elements of A will have in-degree zero and all elements of C will also have zero in-degree. The same arguement applies for $|B| = |D|$, given that all elements of B will have out-degree zero and all elements with out-degree zero will be contained in D as well.

Since $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$, for each element in A, we know that there is a corresponding element in C with the same out-degree. Therefore, A and C contain the same number of isolated points. Let the number of isolated points in A be n. Also, B and D contain the same number of isolated points given that for each element in B, there is a corresponding element in D with the same in-degree. Let the number of isolated points in B be m. So $\{(a_1,0),(a_2,0),...,(a_n,0)\}$ is the set of isolated points in A, $\{(c_1,0),(c_2,0),...,(c_n,0)\}$ is the set of isolated points in C, $\{(b_1,1),(b_2,1),...,(b_m,1)\}$ is the set of isolated points in B, and $\{(d_1,1),(d_2,1),...,(d_m,1)\}$ is the set of isolated points in D. Consider the function $\widehat{\varphi}: V(G\times\overrightarrow{K_2}) \to V(H\times\overrightarrow{K_2})$ as defined below,

$$\widehat{\varphi}((g,k)) = \begin{cases} \varphi((g,k)) & \text{if} \quad \deg(g,k) \geq 1 \qquad \text{and } k=0,1 \\ (c_i,k) & \text{if} \quad g=a_i, 1\leq i\leq n \quad \text{and } k=0 \\ (d_i,k) & \text{if} \quad g=b_i, 1\leq i\leq m \quad \text{and } k=1 \end{cases}$$

.

We know that $\widehat{\varphi}$ is an isomorphism because $(g,0)(g',1) \in E(G\times\overrightarrow{K_2})$ if and only if $\varphi((g,0))\varphi((g',1)) = \widehat{\varphi}((g,0))\widehat{\varphi}((g',1)) \in E(H\times\overrightarrow{K_2})$. This folows since, given any $(g,0)(g',1) \in E(G\times\overrightarrow{K_2})$ the endpoints $(g,0)$ and $(g',1)$ are clearly not isolated vertices. Therfore, by definition of $\widehat{\varphi}$, we have $\widehat{\varphi}(g,0)\widehat{\varphi}(g',1) = \varphi(g,0)\varphi(g',1)$ and this is an edge in $E(H\times\overrightarrow{K_2})$ because φ is an isomorphism. Note that the isomorphism $\widehat{\varphi}$ has the property that the second coordinate of $\widehat{\varphi}(g,b)$ is always b. Therefore, there are functions $\mu_0,\mu_1 : V(G)\to V(H)$ for which

$$\widehat{\varphi}((g,b)) = \begin{cases} (\mu_0(g),b) & \text{if } b=0 \\ (\mu_1(g),b) & \text{if } b=1 \end{cases}$$

Note that the fact that $\widehat{\varphi}$ is bijective forces μ_0 and μ_1 to be bijective. For example if $\mu_0(g)=\mu_0(g')$ then $(\mu_0(g),0)=(\mu_0(g'),0)$ and so $\widehat{\varphi}(g,0)=\widehat{\varphi}(g',0)$ and since $\widehat{\varphi}$ is injective, $(g,0)=(g',0)$ which implies $g=g'$. Likewise, observe that μ_0 is surjective. Now, observe that since μ_0 and μ_1 are bijections, $\pi=\mu_0^{-1}\mu_1$ with $\pi: V(G)\to V(G)$ is a permutaion in $\mathrm{Perm}(V(G))$. Note that by definition of G^π, we have $V(G)=V(G^\pi)$ and so

$\mu_0 : V(G^\pi) \to V(H)$. We want to show that μ_0 is an isomorphism between G^π and H. In other words, we want to show that $gg' \in E(G^\pi)$ if and only if $\mu_0(g)\mu_0(g') \in E(H)$. Observe that,

$$
\begin{array}{rcll}
gg' \in E(G^\pi) & \Leftrightarrow & g\pi^{-1}(g') \in E(G) & \text{by definition of } G^\pi \\
& \Leftrightarrow & (g,0)(\pi^{-1}(g'),1) \in E(G \times \overrightarrow{K_2}) & \\
& \Leftrightarrow & \widehat{\varphi}((g,0))\widehat{\varphi}((\pi^{-1}(g'),1)) \in G(H \times \overrightarrow{K_2}) & \text{since } G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2} \\
& \Leftrightarrow & (\mu_0(g),0)(\mu_1(\pi^{-1}(g'),1) \in E(H \times \overrightarrow{K_2}) & \text{by definition of } \widehat{\varphi} \\
& \Leftrightarrow & (\mu_0(g),0)(\mu_1\mu_1^{-1}\mu_0((g'),1) \in E(H \times \overrightarrow{K_2}) & \text{by definition of } \pi \\
& \Leftrightarrow & (\mu_0(g),0)(\mu_0(g'),1) \in E(H \times \overrightarrow{K_2}) & \\
& \Leftrightarrow & \mu_0(g)\mu_0(g') \in E(H). &
\end{array}
$$

Therefore, μ_0 is an isomorphism between G^π and H. In conclusion, $H \cong G^\pi$. □

Now, combining both Lemma 2.3 and Lemma 2.4 yields the following result.

PROPOSITION 2.5. If G and H are digraphs, then $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$ if and only if $H \cong G^\pi$ for some $\pi \in \mathrm{Perm}(V(G))$

Proof. Let G and H be digraphs and $\pi \in \mathrm{Perm}(V(G))$. First, let $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$. By Lemma 2.3, we know that $H \cong G^\pi$. Secondly, let $H \cong G^\pi$. By Lemma 2.4, we know that $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$.

Therefore, $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$ if and only if $H \cong G^\pi$ for some $\pi \in \mathrm{Perm}(V(G))$. □

Now that we have shown that $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$ if and only if $H \cong G^\pi$ for some $\pi \in \mathrm{Perm}(V(G))$, our goal is to determine what must hold about H if $G \times K \cong H \times K$, for some digraph K.

2.3 Replacing a Simple K with any K

The following proposition shows that given a digraph K such that $G \times K \cong H \times K$, it must be the case that $H \cong G^{\pi}$ for some $\pi \in \mathrm{Perm}(V(G))$. We assume K must have at least one arc because if K were without arcs, $G \times K$ and $H \times K$ would be digraphs with no arcs given any G and H and therefore $G \times K$ would always be isomorphic to $H \times K$, provided that $|V(G)| = |V(H)|$.

PROPOSITION 2.6. If $G \times K \cong H \times K$ where K has at least one edge and where G, H, K are digraphs, then $H = G^{\pi}$ for some $\pi \in \mathrm{Perm}(V(G))$.

Proof. Let G, H, K be digraphs such that K has at least one edge and let $G \times K \cong H \times K$. Since K has at least one edge, observe that $\hom(\overrightarrow{K_2}, K) = |E(K)| \geq 1$ for any such K. By Proposition 1.18, $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$. By Proposition 2.5, if $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$ then $H = G^{\pi}$ for some $\pi \in \mathrm{Perm}(V(G))$.

Therefore, if $G \times K \cong H \times K$ where K has at least one edge and where G, H, K are digraphs, then $H = G^{\pi}$ for some $\pi \in \mathrm{Perm}(V(G))$. □

With Proposition 2.6 concluded, it is reasonable to wonder if its converse holds true. The following remark shows that the converse is generally false.

REMARK 2.7. Figure 2.4 is a perfect example of how it is possible that $G \times K \ncong G^{\pi} \times K$. In this example, $K \cong \overrightarrow{K_2}$ and $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. Observe that $G \times K \ncong G^{\pi} \times K$.

If we can state that $H = G^{\pi}$ for some $\pi \in \mathrm{Perm}(V(G))$, when do we know that $G \times K \cong H \times K$ for some digraph K? In fact we need a homomorphism from K onto $\overrightarrow{K_2}$ in order to make use of our earlier proposition.

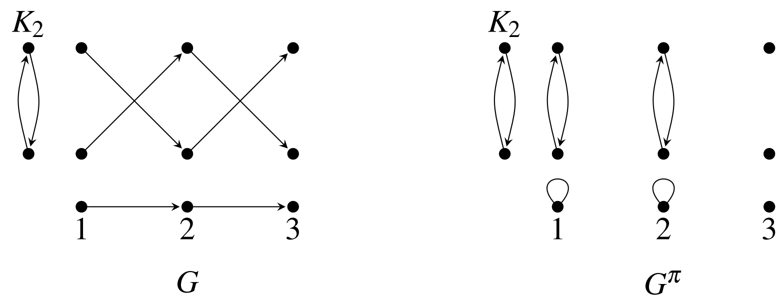


Figure 2.4: Example of $G \times K \ncong G^\pi \times K$

PROPOSITION 2.8. If $H = G^\pi$ for some $\pi \in \mathrm{Perm}(V(G))$, and there exists at least one homomorphism from K to $\overrightarrow{K_2}$, then $G \times K \cong H \times K$.

Proof. Let $H \cong G^\pi$ and let there exist at least one homomorphism $f \colon K \to \overrightarrow{K_2}$. Since $H \cong G^\pi$, $G \times \overrightarrow{K_2} \cong H \times \overrightarrow{K_2}$ by Proposition 2.5. Proposition 1.18 implies that $G \times K \cong H \times K$.

Therefore, $G \times K \cong H \times K$. $\square$

Note that in this proof we must have at least one homomorphism from K to $\overrightarrow{K_2}$, in order to conclude that $G \times K \cong H \times K$. Remark 2.7 illustrates the neccessity of the homomorphism as well since there is no homomorphism from K_2 to $\overrightarrow{K_2}$.

Keep in mind our final goal. We want to know conditions on G for which $G \times K \cong H \times K$ implies $G \cong H$. The following section introductes a necessary ingredient of our solution.

2.4 Factorial of a Digraph

DEFINITION 2.9. Given a digraph G, the factorial of G is another digraph, denoted $G!$, such that $V(G!) = \mathrm{Perm}(V(G))$ and an arc of $G!$ is defined as follows: if $\alpha, \beta \in V(G!)$, then $(\alpha)(\beta) \in E(G!)$ when $\alpha(g)\beta(g') \in E(G)$ if and only if $gg' \in E(G)$ for all $g, g' \in V(G)$. We denote arcs of $G!$ as $(\alpha)(\beta)$ instead of $\alpha\beta$ do avoid confusion with compositon of functions, given that α and β are actually functions.

A similar definition of $G!$ has been defined in Richard Hammack's paper "On Direct Product Cancellation of Graphs" [2] and was the starting point for the definition above.

As an example of $G!$, Figure 2.5 shows $K_2!$. For this example, please note that $\rho_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are the permutations of $V(K_2) = \{0,1\}$. Notice that $\rho_0(g)\rho_0(g') \in E(G)$ if and only if $gg' \in E(G)$ so there will be the loop $(\rho_0)(\rho_0) \in E(G!)$. Also, $\rho_1(g)\rho_1(g') \in E(G)$ if and only if $gg' \in E(G)$ so $(\rho_1)(\rho_1) \in E(G!)$. However, it is not true that $\rho_0(g)\rho_1(g') \in E(G)$ if any only if $gg' \in E(G)$, for all $g, g' \in V(G)$ so $(\rho_0)(\rho_1) \notin E(G!)$. Similarly, $(\rho_1)(\rho(0) \notin E(G!)$.

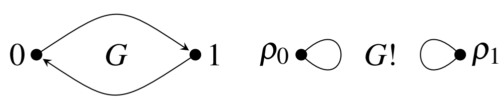


Figure 2.5: First example of G and $G!$

Figures 2.6, 2.7 and 2.8, show examples of factorial digraphs such that G has the vertex set $\{1,2,3\}$. Let $\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ and $\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ and $\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ be the permutations on $\{1,2,3\}$.

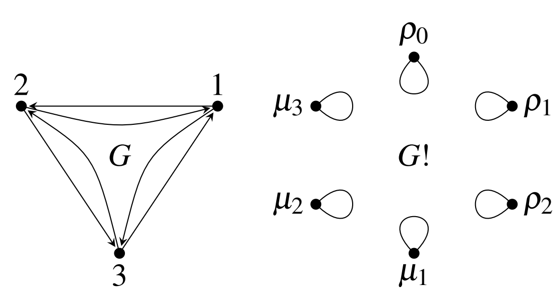


Figure 2.6: Second example of G and $G!$

In Figure 2.6, the only edges of $G!$ are the loops to each element of $V(G!)$. This is true because G is the complete graph, which means $E(G)$ contains all possible edges, excluding loops. Notice that each permutation π on $\{1,2,3\}$ is an automorphism of G, so if G is a complete graph, then, $gg' \in E(G)$ if and only if $\pi(g)\pi(g') \in E(G)$ for all $gg' \in E(G)$, and therefore $(\pi)(\pi) \in E(G!)$. Generally speaking, in a complete graph with n vertices, there will be $n!$ copies of a loop in $E(G!)$.

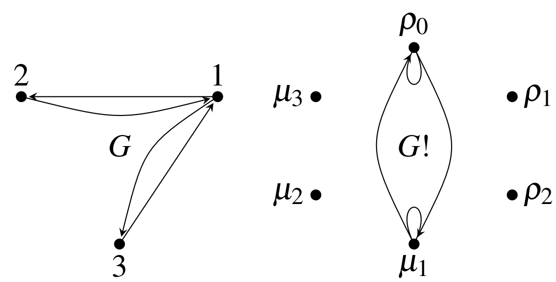


Figure 2.7: Third example of G and $G!$

In Figure 2.7, G is not a complete graph and therefore $G!$ is missing four loops at ρ_1, ρ_2, μ_3 and μ_4. For example, though $13 \in G$, $\mu_3(1)\mu_3(3) = 23$ is not an edge of G, so $(\mu_3)(\mu_3)$ is not in $E(G!)$.

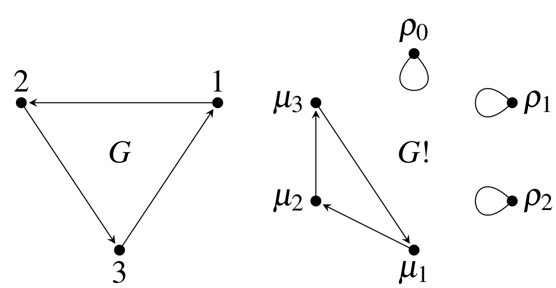


Figure 2.8: Last example of G and $G!$

Figure 2.8 shows a more interesting pattern for $G!$, which includes both loops and edges between elements of $V(G!)$. Each digraph G will have a unique factorial given that $G!$ is constructed using the edges and permutations of G.

2.5 An Equivalence Relation on Permutations

Now we can create a relation that compares permutations on $V(G)$ and encompasses the definition of $G!$. This will be the second component that will tell us exactly when $G^{\alpha} \cong G^{\beta}$.

DEFINITION 2.10. Given a digraph G, define a relation, $\sim$, on $\mathrm{Perm}(V(G))$ as follows: $\alpha \sim \beta$ if and only if $\alpha = \mu^{-1}\beta\lambda$ for some $(\mu)(\lambda) \in E(G!)$.

This relation definition gives us a chance to use characterisitcs of the permutation set of $V(G)$ and $G!$, both of which tell us a large deal about G. Keep in mind the end goal of determining constraints on G such that $G \times K \cong H \times K$ implies $G \cong H$. We will see that this relation narrows down the characteristics for G. Notice that as in Figures 2.5, 2.6, 2.7, 2.8 the identity permutation is labeled as ρ_0. In the remaining discussion, I will refer to the identity mapping as ρ_0.

PROPOSITION 2.11. The relation $\sim$, as defined in Definition 2.10, is an equivalence relation.

Proof. First we would like to show that $\alpha \sim \alpha$ for all $\alpha \in V(G!)$. Consider that for all $g, g' \in V(G)$, $gg' \in E(G)$ if and only if $\rho_0(g)\rho_0(g') \in E(G)$ since ρ_0 is the identity permutation, and therefore $(\rho_0)(\rho_0) \in E(G!)$. Also, note that $\alpha = \rho_0^{-1}(\alpha)\rho_0$ where $(\rho_0)(\rho_0) \in E(G!)$. So by definition of $\sim$, the reflexive property holds and $\alpha \sim \alpha$.

Secondly, we need to show that if $\alpha \sim \beta$, then $\beta \sim \alpha$. Let $\alpha \sim \beta$. Therefore $\alpha = \mu^{-1}\beta\lambda$ for some $(\mu)(\lambda) \in E(G!)$. Since $\alpha = \mu^{-1}\beta\lambda$, it follows that $\beta = \mu\alpha\lambda^{-1} = (\mu^{-1})^{-1}\alpha\lambda^{-1}$. Therefore, we need to show that $(\mu^{-1})(\lambda^{-1}) \in E(G!)$ or in other words we need to show that $gg' \in E(G)$ if and only if $\mu^{-1}(g)\lambda^{-1}(g') \in E(G)$. Observe that,

$$\begin{aligned} gg' \in E(G) &\Leftrightarrow \mu\mu^{-1}(g)\lambda\lambda^{-1}(g') \in E(G) \\ &\Leftrightarrow \mu^{-1}(g)\lambda^{-1}(g') \in E(G) \qquad \text{since } (\mu)(\lambda) \in E(G!). \end{aligned}$$

Therefore, $(\mu^{-1})(\lambda^{-1}) \in E(G!)$ while $\beta = (\mu^{-1})^{-1}(\alpha)(\lambda^{-1})$. So by definition of $\sim$, the symmetric property holds and $\beta \sim \alpha$.

Lastly we must show that if $\alpha \sim \beta$ and $\beta \sim \kappa$ then $\alpha \sim \kappa$. So let $\alpha \sim \beta$ and $\beta \sim \kappa$. So, $\alpha = \mu^{-1}\beta\lambda$ for some $(\mu)(\lambda) \in E(G!)$ and $\beta = \nu^{-1}\kappa\delta$ for some $(\nu)(\delta) \in E(G!)$. Therefore, $\alpha = \mu^{-1}\nu^{-1}\kappa\delta\lambda = (\nu\mu)^{-1}\kappa(\delta\lambda)$. We now need to show that $(\nu\mu)(\delta\lambda) \in E(G!)$ or other words we need to show that $gg' \in E(G)$ if and only if $\nu\mu(g)\delta\lambda(g') \in E(G)$. Obeserve that,

$$\begin{aligned} gg' \in E(G) &\Leftrightarrow \mu(g)\lambda(g') \in E(G) \qquad \text{because } (\mu)(\lambda) \in E(G!) \\ &\Leftrightarrow \nu\mu(g)\delta\lambda(g') \in E(G) \quad \text{because } (\nu)(\lambda) \in E(G!) \end{aligned}$$

Therefore $(\nu\mu)(\delta\lambda) \in E(G!)$ while $\alpha = (\nu\mu)^{-1}\kappa(\delta\lambda)$ so by definition of $\sim$, the transitive property holds and $\alpha \sim \kappa$.

In conclusion, $\sim$ is an equivalence relation. $\square$

Using this new equivalence relation we will now determine that two permutations on $V(G)$, call them α and β, are related as defined above if and only if $G^\alpha \cong G^\beta$. This will define a direct relation between permutations and products and will give us more information for the conditions we need for direct product cancellation.

THEOREM 2.12. For a digraph G and arbitrary $\alpha, \beta \in \mathrm{Perm}(V(G))$, it follows that $G^\alpha \cong G^\beta$ if and only if $\alpha \sim \beta$.

Proof. Let $G^\alpha \cong G^\beta$ where $\alpha, \beta \in \mathrm{Perm}(V(G))$. So $gg' \in E(G^\alpha)$ if and only if $\varphi(g)\varphi(g') \in E(G^\beta)$ for some isomorphism φ from G^α to G^β. We know that $gg' \in E(G^\beta)$ if and only if $\varphi^{-1}(g)\varphi^{-1}(g') \in E(G^{\alpha})$. Consider that $\beta = (\varphi^{-1})(\alpha)(\alpha^{-1}\varphi\beta)$ because φ, α, β are bijections. Thus, we only need to show that $(\varphi)(\alpha^{-1}\varphi\beta) \in E(G!)$. Now observe that,

$$\begin{aligned} \varphi(g)\alpha^{-1}\varphi\beta(g') \in E(G) &\Leftrightarrow \varphi(g)\varphi\beta(g') \in E(G^{\alpha}) && \text{by definition of } G^{\alpha} \\ &\Leftrightarrow (g)\beta(g') \in E(G^{\beta}) && \text{since } \varphi \text{ is an isomorphism} \\ &\Leftrightarrow gg' \in E(G) && \text{by definition of } G^{\beta}. \end{aligned}$$

So $\varphi(g)\alpha^{-1}\varphi\beta(g') \in E(G)$ if and only if $gg' \in E(G)$ for all $g, g' \in V(G)$.
So $(\varphi)(\alpha^{-1}\varphi\beta) \in E(G!)$. Thus, since $\beta = (\varphi^{-1})(\alpha)(\alpha^{-1}\varphi\beta)$, we know that $\alpha \sim \beta$.

Now, let $\alpha \sim \beta$. So, $\alpha = \mu^{-1}\beta\lambda$ for some $(\mu)(\lambda) \in E(G!)$ Also, $\mu\alpha = \beta\lambda$ for the same $(\mu)(\lambda) \in E(G!)$. We will show that μ is an isomorphism from G^{α} to G^{β}. Note that we need to show that $\mu(g)\mu(g') \in E(G^{\beta})$ if and only if $gg' \in E(G^{\alpha})$. Note that if $\alpha = \mu^{-1}\beta\lambda$ then $\rho_0 = \mu^{-1}\beta\lambda\alpha^{-1}$. This will be used below. We know that,

$$\begin{aligned} gg' \in E(G^{\alpha}) &\Leftrightarrow g\alpha^{-1}(g') \in E(G) && \text{by definition of } G^{\alpha} \\ &\Leftrightarrow \mu(g)\lambda\alpha^{-1}(g') \in E(G) && \text{since } (\mu)(\lambda) \in E(G!) \\ &\Leftrightarrow \mu(g)\beta\lambda\alpha^{-1}(g') \in E(G^{\beta}) && \text{by definition of } G^{\beta} \\ &\Leftrightarrow \mu(g)\mu\mu^{-1}\beta\lambda\alpha^{-1}(g') \in E(G^{\beta}) && \text{since } \rho_0 = \mu\mu^{-1} \\ &\Leftrightarrow \mu(g)\mu\rho_0(g') \in E(G^{\beta}) && \text{since } \rho_0 = \mu^{-1}\beta\lambda\alpha^{-1} \\ &\Leftrightarrow \mu(g)\mu(g') \in E(G^{\beta}) \end{aligned}$$

.

In conclusion, $G^{\alpha} \cong G^{\beta}$. $\square$

2.6 Conclusion

We will now use the above facts and the following theorem to determine exactly when $G \times K \cong H \times K$ implies $G \cong H$.

REMARK 2.13. If G is such that $G \times K \cong H \times K$ implies $G \cong H$ then we can use Proposition 2.6 to conclude that $H \cong G^{\pi}$ for some $\pi \in \mathrm{Perm}(V(G))$ for any K with at least one edge. So $G \cong H \cong G^{\pi}$ which implies $G \cong G^{\pi}$ for any $\pi \in \mathrm{Perm}(V(G))$. So the definition of $\sim$ leads us to observe that $\rho_0 \sim \pi$ for any π. So $\sim$ must have one equivalence class if $G \times K \cong H \times K$ implies $G \cong H$.

THEOREM 2.14. Let Φ be defined as follows: $\Phi : E(G!) \to V(G!)$ such that $\Phi((\mu)(\lambda)) = \mu^{-1}\lambda$ for all $(\mu)(\lambda) \in E(G!)$. Then, Φ is a surjective mapping if and only if $G \times K \cong H \times K$ implies $G \cong H$.

Proof. Firstly, if Φ is a surjective mapping and defined as it is above, then for every $\delta \in V(G!)$, we have $\delta = \mu^{-1}\lambda$ for some $(\mu)(\lambda) \in E(G!)$. Clearly, $\delta = \mu^{-1}\rho_0\lambda$ also for every $\delta \in V(G!)$. Definition 2.10 implies that $\delta \sim \rho_0$ and by Theorem 2.12, $G^{\delta} \cong G^{\rho_0} \cong G$. So $G^{\delta} \cong G$ for all $\delta \in V(G!) = \mathrm{Perm}(V(G))$. Thus, we know $\sim$ has exactly one equivalence class. By Proposition 2.6, if $G \times K \cong H \times K$ and $\mid E(K) \mid \geq 1$, then $H \cong G^{\pi}$ for some $\pi \in \mathrm{Perm}(V(G))$. So $G \times K \cong H \times K$ implies $H \cong G^{\pi} \cong G$ and so $H \cong G$.

Conversely, if $G \times K \cong H \times K$ implies $G \cong H$ then by Remark 2.13, $\sim$ has one equivalence class. Hence, given any $\delta \in V(G!)$ we know $\delta \sim \rho_0$, which means there is an edge $(\mu)(\lambda) \in E(G!)$ with $\delta = \mu^{-1}\rho_0\lambda$. Thus, $\delta = \mu^{-1}\lambda = \Phi((\mu)(\lambda))$, and so Φ is surjective.

In conclusion, $G \times K \cong H \times K$ implies $G \cong H$ if and only if Φ is a surjective mapping.

$\square$

After such a large theorem, we must consider a few examples to make this theory more concrete.

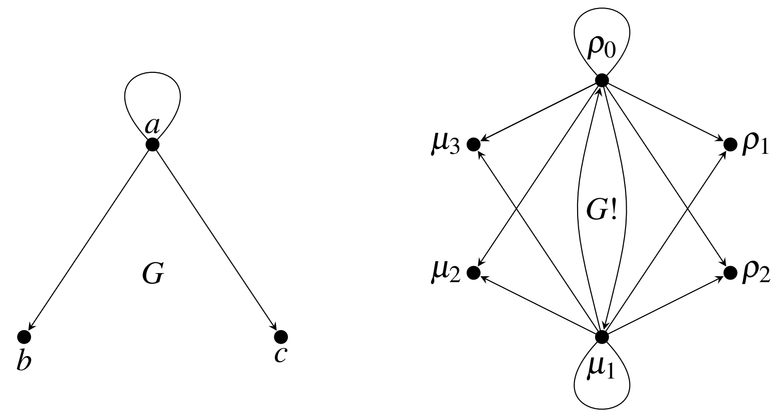


Figure 2.9: An Example of G and $G!$

Consider that Figure 2.9 is an example a digraph G for which $\Phi : E(G!) \to V(G!)$ which is surjective. Observe that $V(G) = \{a,b,c\}$ and $E(G) = \{aa,ab,ac\}$ and see that each arc begins at a, while $E(G!) = \{\rho_0\rho_0, \rho_0\rho_1, \rho_0\rho_2, \rho_0\mu_1, \rho_0\mu_2, \rho_0\mu_3, \mu_1\rho_0, \mu_1\rho_1, \mu_1\rho_2, \mu_1\mu_1, \mu_1\mu_2, \mu_1\mu_3\}$. Note that $V(G!) = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$ and consider that $\rho_0 = \Phi((\rho_0)(\rho_0))$, $\rho_1 = \Phi((\rho_0)(\rho_1))$, $\rho_2 = \Phi((\rho_0)(\rho_2))$, $\mu_1 = \Phi((\rho_0)(\mu_1))$, $\mu_2 = \Phi((\rho_0)(\mu_2))$, $\mu_3 = \Phi((\rho_0)(\mu_3))$. Therefore Φ is a surjective mapping.

Now, consider this next figure as an example when Φ is not surjective.

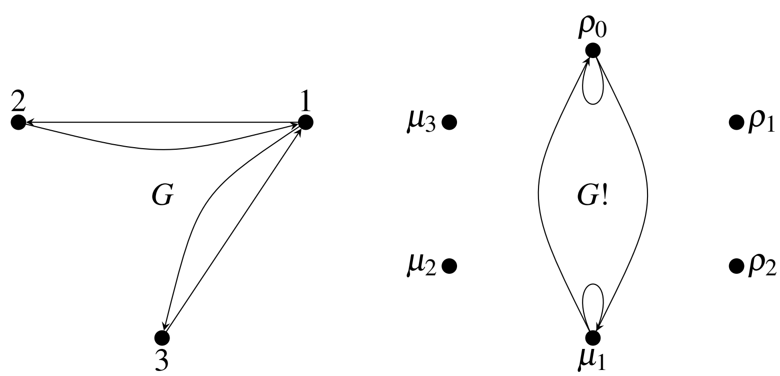


Figure 2.10: Third example of G and $G!$ for earlier section

This nonsurjective mapping Φ is deduced from Figure 2.10. Note that this is the same example of G and $G!$ as in Figure 2.7. Notice that $V(G) = \{1,2,3\}$ and $E(G) = \{12,21,13,31\}$ and just as in Figure 2.9, $V(G!) = \{\rho_0, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3\}$. However, $E(G!) = \{\rho_0\rho_0, \rho_0\mu_1, \mu_1\rho_0, \mu_1\mu_1\}$. This means that $\Phi : E(G!) \to V(G!)$ is not surjective. For justification, consider that $\rho_1 \in V(G!)$ but there is no element of $E(G!)$ such that Φ maps that element to ρ_1.

In conclusion, the mapping Φ from $E(G!)$ to $V(G!)$ must be surjective in order for $G \times K \cong H \times K$ to imply $G \cong H$. Note that only one such mapping must occur, however without such a mapping, cancellation will not hold true.

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Biography

Katherine Toman is a mathematics student with an appreciation for both the theory involving mathematics and the education of mathematics. She received her Bachelor of Arts Degree from St. Mary's College of Maryland in 2007 in Pure Mathematics after graduating from Poolesville High School in 2003. While at St. Mary's College, Katherine was given the chance to work with math students as a TA while being an undergraduate student herself. Also, as a member of Virginia Commonwealth Univeristy's Graduate Program she has spent her time enjoying classes focused in the algebraic fields whenever possible and teaching Math 141 and 131, which only made her passion for the subject grow. She was lucky to have found graph theory this past year while researching for her thesis requirement and hopes to continue the research after graduation. A deep love for all branches of mathematics and the success of her own students drive her desire to educate young adults in mathematics on the college level. When not in class, teaching or studying, she strives to capture the beauty in her world through photography and spends time appreciating the outdoors with her dogs and close friends, or relaxing at home with her cat. After teaching high school for a year or two, Katherine plans on returning to academia to receive her PhD and reach her goal of professorship.