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An Isomorphism Theorem for Graphs

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An Isomorphism Theorem for Graphs

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

AN ISOMORPHISM THEOREM FOR GRAPHS

By Laura Jean Culp, Master of Science.

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In the 1970's, L. Lovász proved that two graphs G and H are isomorphic if and only if for every graph X, the number of homomorphisms from $X \to G$ equals the number of homomorphisms from $X \to H$. He used this result to deduce cancellation properties of the direct product of graphs.

We develop a result analogous to Lovász's theorem, but in the class of graphs without loops and with weak homomorphisms. We apply it prove a general cancellation property for the strong product of graphs.

Preliminaries

In this chapter we present some introductory Graph theory that is required for later chapters. Many elementary definitions are presented without reference, but may be found in one or more of the texts listed in the Bibliography, especially that by Diestel [1]. In the final section of this chapter, we prove a version of a theorem by Lovász in a new context. The collection of lemmas and propositions presented in the final section of this chapter will be used as a template for the proof of our main result in Chapter 2.

1.1 Elementary Definitions

DEFINITION 1.1. A **graph** $G = (V(G), E(G))$ is a pair of sets $V(G)$ and $E(G)$ such that $V(G)$ is nonempty and $E(G)$ is the subset of unordered pairs of elements in $V(G)$. The elements of $V(G)$ are called **vertices (or nodes or points)**. The elements of $E(G)$ are called **edges (or lines)**. A graph has a visual interpretation in which vertices are points and edges are arcs joining points.

Note that we abbreviate the edge $\{a, b\}$ as ab or ba.

DEFINITION 1.2. The number of vertices of a graph G is its **order** and is written as $|G|$. Graphs are $finite$, $infinite$, $countable$ and so on according to their order. All graphs under discussion in this thesis will be of finite order.

EXAMPLE 1.3. Figure 1.1 shows one rendering of the graph G with vertex set $V(G) = \{ 1,2,3,4,5,6,7,8,9 \}$ and edge set $E = \{ \{1,2\}, \{1,3\} , \{2,3\} , \{4,5\} , \{6,7\} , \{6,8\} \}$.

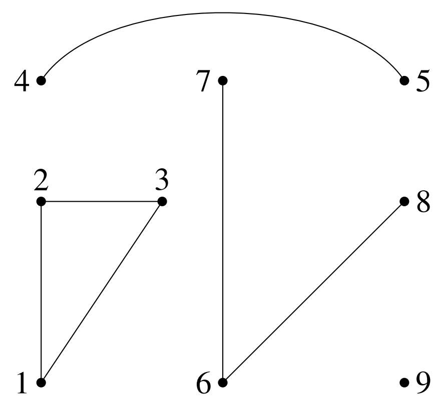


Figure 1.1: An example of a Graph

DEFINITION 1.4. In a graph G, two **vertices** are said to be **adjacent** if there is an edge between them and two **edges** are **adjacent** if they share a common vertex.

DEFINITION 1.5. A **loop** is an edge that joins a vertex to itself.

Figure 1.2 depicts a graph with four loops.

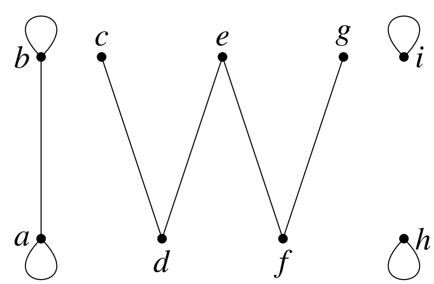


Figure 1.2: An example of a graph with four loops

This brings us to an important property that divides graphs into two broad classifications according to whether or not loops are admitted.

Definition 1.6. The class of graphs which may contain loops is denoted as Γ_0 while the class of graphs in which there are no loops is denoted as Γ.

This distinction has consequences regarding the definition of the structure-preserving mappings between graphs, as we shall see in what follows.

1.2 Partitions and Quotients of Graphs

Partitions and quotients will play a major role in the proof of Lovász's Theorem. We therefore present a detailed analysis of both concepts here.

Definition 1.7. A **partition Θ of a set** V is a set of subsets of V, such that the union of all the subsets equals V and the intersection of any two subsets is $\emptyset$.

Definition 1.8. Suppose we have a graph $G = (V(G), E(G)) \in \Gamma_0$ and a partition Θ of $V(G)$. The **quotient of** G **by** Θ taken in Γ_0 is another graph, which we denote as G/Θ. It is defined as follows. Its vertex set is $V(G/\Theta) = \Theta$. The edge set is $E(G/\Theta) = \{XY : X, Y \in \Theta$ and there is an edge $xy \in E(G)$ with $x \in X$, $y \in Y\}$.

We now illustrate this concept with a graph G with $V(G) = \{ a, b, c, d, e, f, g \}$ and $E(G) = \{ ab, ac, bd, cd, ce, cf, eg, gf, fd \}$ as shown in Figure 1.3.

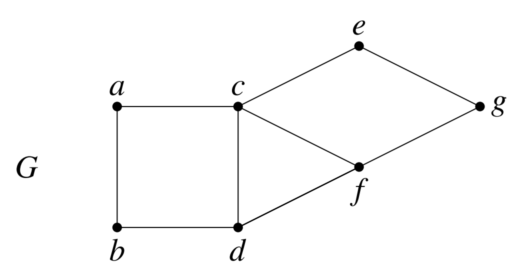


Figure 1.3: The Graph G

Consider a partition Θ of $V(G)$, where $\Theta = \{\{a\},\{b\},\{c,d,f\},\{e,g\}\}$. So G/Θ has four vertices, one for each element of Θ. Label the vertices as follows.

$$A = \{a\} \quad B = \{b\} \quad C = \{c,d,f\} \quad D = \{e,g\}$$

Let's look at an example using the same G and Θ in the previous example. By definition of the quotient in Γ_0, we have

$$V(G/\Theta) = \{A,B,C,D\}$$
$$E(G/\Theta) = \{AB,AC,BC,CC,CD,DD\}$$

Figure 1.4 shows the graph G alongside the graph G/Θ for the above partition Θ. Note that there is a loop at C and at D due to the edges $cd,cf,df \in E(G)$ and the edge $eg \in E(G)$ whose vertices belong to the same subsets, $C = \{\ c,d,f\ \}$ and $D = \{\ e,g\ \}$ in the partition Θ.

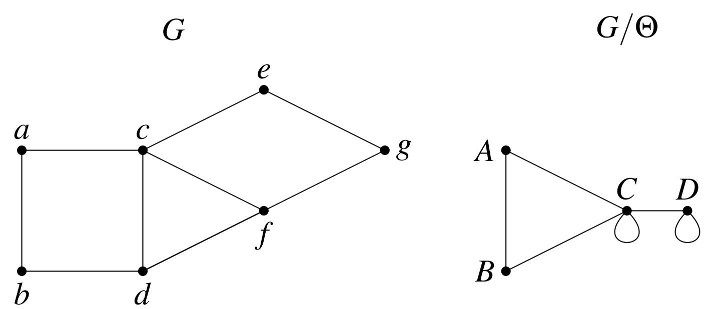


Figure 1.4: The graphs G and G/Θ in Γ_o

Now let us look at the definition of a quotient graph in Γ. The definition is basically the same as the previous definition except that we disallow loops.

Definition 1.9. Suppose we have a graph $G = (V(G),E(G)) \in \Gamma$ and a partition Θ of $V(G)$. The **quotient of G by Θ** taken in Γ is another graph, which we denote as G/Θ. It is defined as follows. Its vertex set is $V(G/\Theta) = \Theta$. The edge set is $E(G/\Theta) = \{XY : X,Y \in \Theta$ and there is an edge $xy \in E(G)$ with $x \in X$, $y \in Y$ and $X \neq Y\}$

By definition of the quotient in Γ, we have

$$V(G/\Theta) = \{A,B,C,D\}$$

$$E(G/\Theta) = \{AB,AC,BC,CD\}$$

Figure 1.5 shows G alongside G/Θ in Γ. Note there is no loop at C or D.

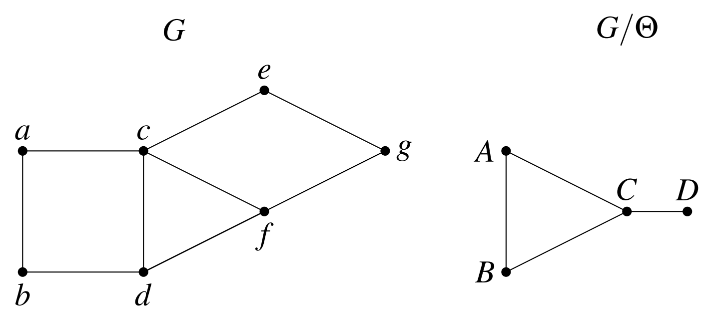


Figure 1.5: The graph G and G/Θ in Γ

1.3 Binary Operations on Graphs

Graph products will play an important role for us as applications of our theorems. There are three principal types of graph products–Cartesian, direct, and strong–whose definitions are given below.

DEFINITION 1.10. The **Cartesian Product** of two graphs G and H is the graph $G\Box H$ with vertex set $V(G\Box H) = \{(g,h) \mid g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G\Box H) = \{(g,h)(g',h') \mid (gg' \in E(G) \text{ and } h = h') \text{ or } (hh' \in E(H) \text{ and } g = g')\}$.

Figure 1.6 pictures an example of the Cartesian product of two graphs G and H. In this first picture of product graphs, the vertices are labeled to illustrate the naming of vertices.

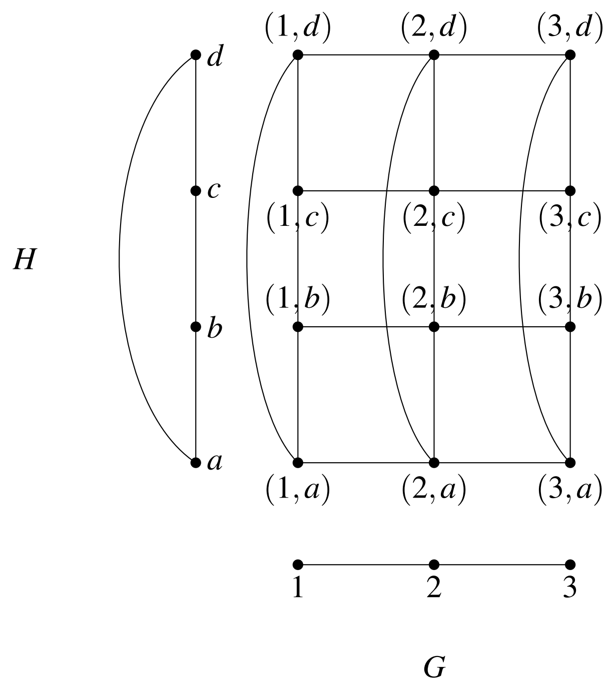


Figure 1.6: The Cartesian Product: $G \,\square\, H$

DEFINITION 1.11. The **direct product** of two graphs G and H is the graph $G \times H$ with vertex set $V(G \times H) = \{(g,h) \mid g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G \times H) = \{(g,h)(g',h') \mid (gg' \in E(G) \text{ and } hh' \in E(H)\}$.

Figure 1.7 pictures an example of the Direct product of the same two graphs G and H used in the example of a Cartesian product.

DEFINITION 1.12. The **strong product** of two graphs G and H is the graph $G \boxtimes H$ with vertex set $V(G \times H) = \{(g,h) \mid g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G \boxtimes H) = E(G \times H) \bigcup E(G \,\square\, H)$.

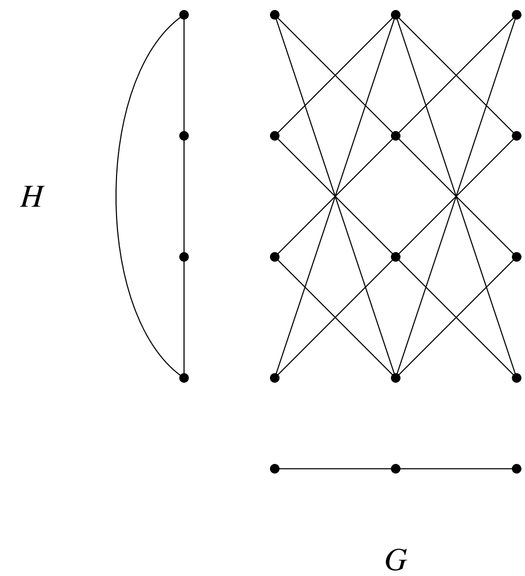


Figure 1.7: The direct product: $G \times H$

Figure 1.8 pictures an example of the Strong product of the same two graphs G and H used in the examples of a Cartesian and of a Direct product.

1.4 Structure-preserving Mappings

Now we turn to definitions of structure-preserving mappings. There are three types under consideration for this thesis.

DEFINITION 1.13. A **homomorphism** $f : G \to H$ from a graph G to a graph H is a map $f : V(G) \to V(H)$ for which $xy \in E(G)$ implies $f(x)f(y) \in E(H)$.

DEFINITION 1.14. A **weak homomorphism** $f : G \to H$ from a graph G to a graph H is a map $f : V(G) \to V(H)$ for which $xy \in E(G)$ implies $f(x)f(y) \in E(H)$ or $f(x) = f(y)$.

We now turn to an example of a homomorphism. Let G and H be the graphs in Figure 1.9 and consider a map $f : V(G) \to V(H)$ as defined in the Table 1.1. The effect of f (i.e. $f(a) = j, f(b) = j$, etc.) on edges is shown in Table 1.2.

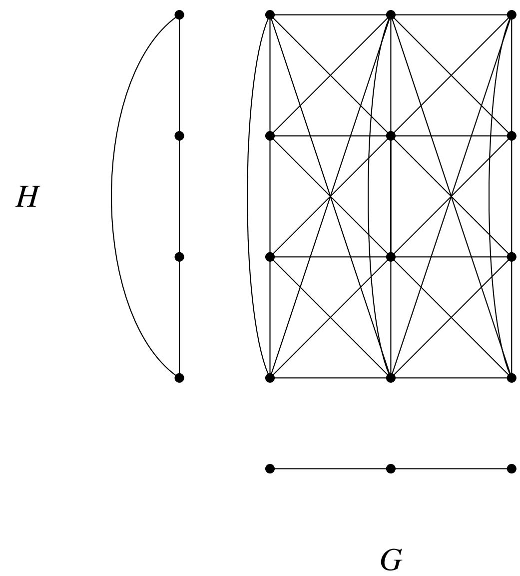


Figure 1.8: The strong product: $G \boxtimes H$

Table 1.1: Homomorphic Vertex Mapping

Graph	Vertex Mappings						
G	a	b	c	d	e	f	g
H	A	B	C	C	D	C	D

The edge mappings corresponding the vertex mappings in Table 1.1 are shown in Table 1.2.

Table 1.2: Homomorphic Edge Mapping

Graph	Edge Mappings								
G	ab	bd	ac	cd	cf	df	ce	fg	eg
H	AB	BC	AC	CC	CC	CC	CD	CD	DD

The mapping in Table 1.1 is illustrated in Figure 1.9.

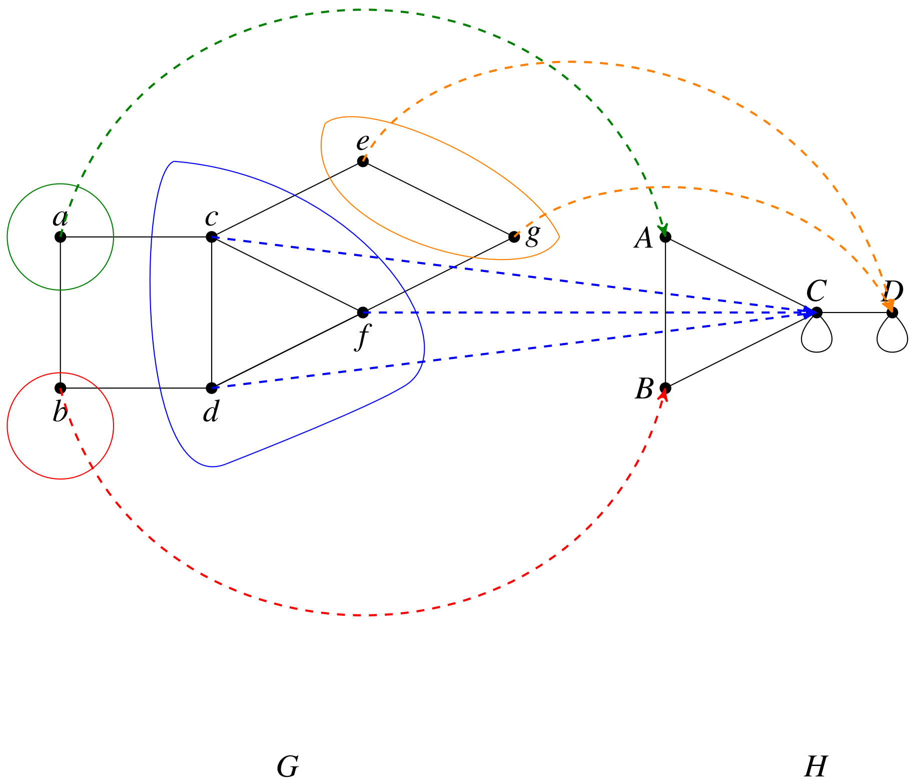


Figure 1.9: A homomorphism $f : G \to H$

DEFINITION 1.15. Given graphs G and H, we denote the set of all homomorphisms from G to H by $\mathrm{Hom}(G,H)$ and the set of all weak homomorphisms from G to H by $\mathrm{Hom}_{\mathrm{w}}(G,H)$.

DEFINITION 1.16. Given graphs G and H, we denote the number of weak homomorphisms from G to H by $\mathbf{hom_w(G,H)}$, the number of homomorphisms from G to H by $\mathbf{hom(G,H)}$ and the number of injective homomorphisms from G to H $\mathbf{inj(G,H)}$. Note that we show that any injective weak homomorphism is a homomorphism in Lemma 2.1, so there is no necessity for defining $\mathbf{inj_w(G,H)}$.

For example, consider the following equations. (The first two are pictured in Figures

1.10 and 1.11):

$$\mathrm{hom}(K_2, K_3) = 6$$
$$\mathrm{hom_w}(K_2, K_3) = 9$$
$$\mathrm{hom}(K_3, K_2) = 0$$
$$\mathrm{hom_w}(K_3, K_2) = 8$$

In general, the number of weak homomorphisms will exceed the number of homomorphisms. This is due to a relaxing of the requirement that an edge map to an edge–with weak homomorphisms, we can collapse two vertices into one without the necessity of a loop at that vertex. Thus, there are no homomorphisms from K_3 to K_2. However, the number of weak homomorphisms is equal to the number of ways all three vertices in K_3 can be mapped to two vertices. The set of all weak homomorphisms from K_3 to K_2 is pictured in Figure 1.12.

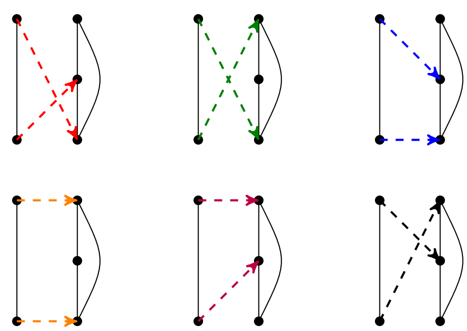


Figure 1.10: All homomorphisms from K_2 to K_3

In addition to the homomorphisms pictured in Figure 1.10, the number of weak homomorphisms from K_2 to K_3 include those pictured in Figure 1.11.

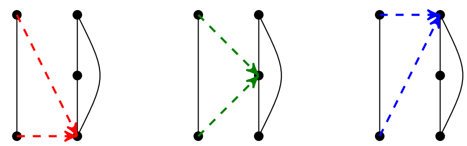


Figure 1.11: Additional weak homomorphisms from K_2 to K_3

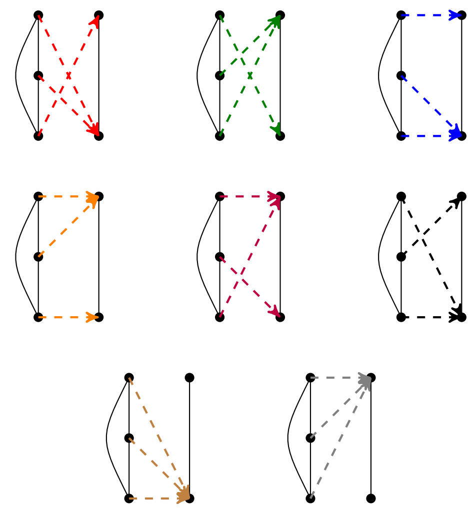


Figure 1.12: Weak homomorphisms from K_3 to K_2

DEFINITION 1.17. Two graphs G and H in Γ are **isomorphic** if there is a bijection $f : V(G) \to V(H)$ satisfying $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. The function f is called an **isomorphism** from G to H.

EXAMPLE 1.18. The two graphs in Figure 1.13 are isomorphic. An isomorphism $f : G \to H$ is defined by the vertex mapping in Table 1.3. The corresponding edge mappings are shown in Table 1.4.

Table 1.3: Isomorphic Vertex Mapping

Graph	Vertex Mappings							
G	a	b	c	d	e	f	g	h
H	i	j	n	m	l	k	o	p

The edge mappings corresponding to the vertex mappings in Table 1.3 are shown in Table 1.4.

Table 1.4: Isomorphic Edge Mapping

Graph	Edge Mappings											
G	ab	ad	ae	bf	bc	fg	ef	eh	gh	cd	cg	dh
H	ij	im	il	jk	jn	ko	lk	lp	op	nm	no	mp

Figure 1.13 illustrates the isomorphism defined in Tables 1.3 and 1.4.

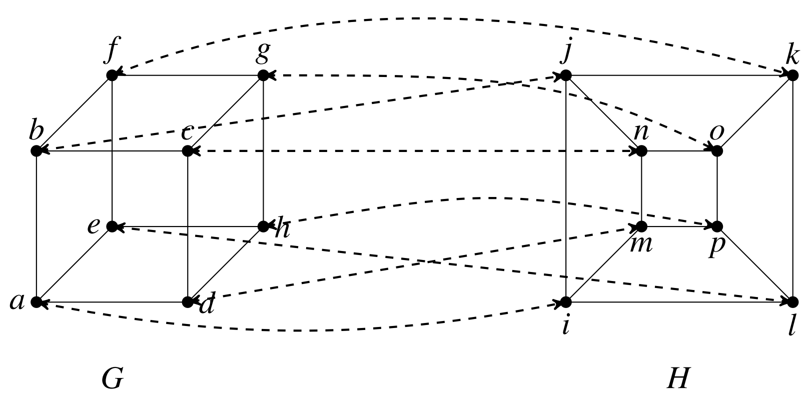


Figure 1.13: Isomorphic graphs G and H

Example 1.19. The two graphs A and B in Figure 1.14 are isomorphic where the isomorphism $f : A \to B$ is defined as defined in Table 1.5 with corresponding edge mappings as shown in Table 1.6.

Table 1.5: Isomorphic Vertex Mapping

Graph	Vertex Mappings					
A	a	b	c	d	e	f
B	g	i	k	h	j	l

Table 1.6: Isomorphic Edge Mapping

Graph	Edge Mappings								
A	ad	ae	af	bd	be	bf	cf	ce	cd
B	gh	gj	gl	ih	ij	il	kl	kj	kh

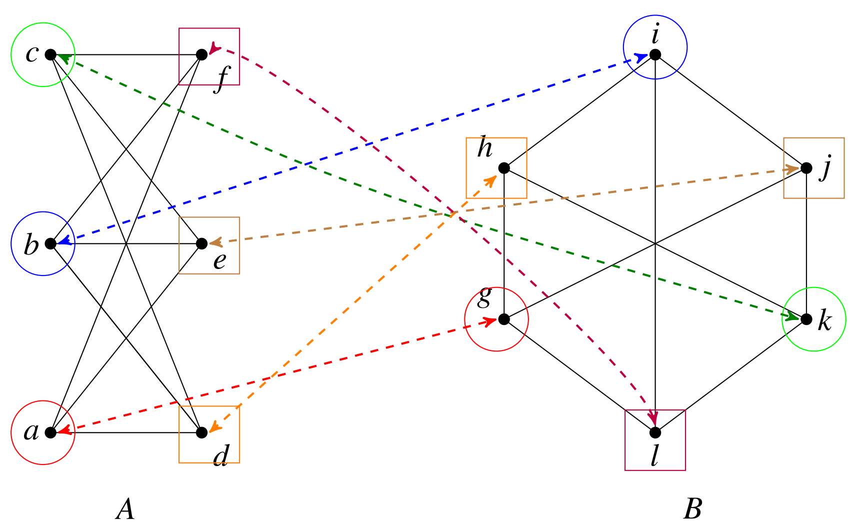


Figure 1.14: Isomorphic graphs A and B

1.5 Development of Lovász's Theorem

We will now prove a celebrated Theorem of Lovász which states that $G \cong H$ if and only if $\hom(X,G) = \hom(X,H)$ for all graphs X in Γ_0.

Before proving the theorem, we need some lemmas and propositions. In Chapter 2, we will examine this same sequence of lemmas and propositions in the context of Γ and prove a version of Lovász's Theorem where weak homomorphisms replace homomorphisms..

Recall that two graphs G and H are isomorphic if and only if there is a one-to-one correspondence between the vertices and edges and the graphs are structurally the same, i.e., there exists vertex mappings from G to H and H to G such that adjacency is preserved. For example, in Figure 1.15, the graphs G and H have the same number of vertices and edges, but the graphs are not isomorphic due to the fact that structurally they are very different. To see this, note that G has a loop while H does not and that G is disconnected while H is connected.

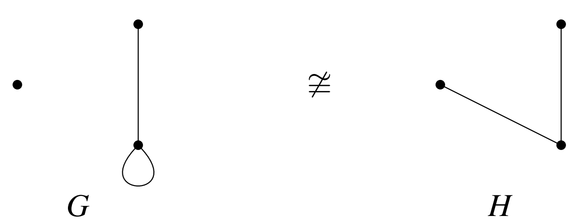


Figure 1.15: Non-isomorphic Graphs

Intuitively it seems reasonable that if the number of homomorphisms from some graph X is different for the two fixed graphs G and H whose number of vertices and edges are the same, then there is something structurally different. Using the same graphs in Figure 1.15, we can demonstrate this difference quite easily using $X = K_3$ as shown in 1.16 for the graph G and in 1.17 for the graph H.

Figure 1.17 illustrates an instance where there are no homomorphisms from K_3 to H

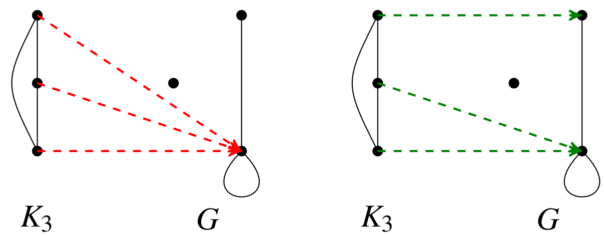
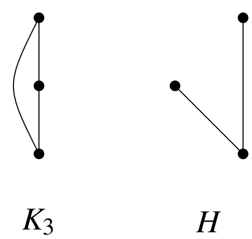


Figure 1.16: Homomorphisms between K_3 and G



no homomorphisms!

Figure 1.17: Homomorphisms from K_3 to H

Thus we come to the the overall strategy whereby we first establish certain properties of injective homomorphisms and then the relationships that exist between counts of homomorphisms and counts of injective homomorphisms. Once this has been accomplished, all the tools necessary for proving the Lovász's Theorem for undirected graphs are in place and we can proceed with the proof.

1.5.1 Properties of Injective Homomorphisms in the class Γ_0

In this section, lemmas are presented and proven which establish the necessary conditions under which an injective homomorphism implies isomorphism.

Lemma 1.20. Suppose G and H are graphs in Γ_0 for which $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$. Moreover, suppose there is a homomorphism, $f : G \to H$, which is injective. Then $G \cong H$.

Proof. We are given that f is a homomorphism so for all $xy \in E(G)$, it follows that $f(x)f(y) \in E(H)$. We begin by showing f is an onto function by assuming the contrary. Let $h \in V(H)$ such that $h \notin f(G)$. Since f is also injective, this implies that $|V(G)| < |V(H)|$ which contradicts our assumption that $|V(G)| = |V(H)|$ so f must be an onto function. Now since f is onto and injective $f^{-1} : H \to G$ exists and is also injective. We claim f is an isomorphism. Suppose f is not an isomorphism. Then there exists $ab \in E(H)$ such that $f^{-1}(a)f^{-1}(b) \notin E(G)$. This implies $|E(G)| < |E(H)|$ which contradicts our assumption that $|E(G)| = |E(H)|$. Therefore $f^{-1}(a)f^{-1}(b) \in E(G)$. Now f is a bijection satisfying $g_1g_2 \in E(G)$ if and only if $f(g_1)f(g_2) \in E(H)$. Thus $f : G \to H$ is an isomorphism and $G \cong H$. □

LEMMA 1.21. Suppose G and H are graphs in Γ_0. If $\text{inj}(G,H) > 0$ and $\text{inj}(H,G) > 0$, then $G \cong H$.

Proof. If we can show (1) that $|V(G)| = |V(H)|$ and (2) that $|E(G)| = |E(H)|$, then by Lemma 1.20, $G \cong H$ since $\text{inj}(G,H) > 0$ implies that there exists at least one injective homomorphism from G into H.

(1) Note that $\text{inj}(H,G) > 0$ implies there exists at least one injective homomorphism from H into G. This, in turn, implies that $|V(H)| \leq |V(G)|$ and that $|E(H)| \leq |E(G)|$. To see that $|E(H)| \leq |E(G)|$, observe that $xy \in E(H)$ implies $f(x)f(y) \in E(G)$ since f is a homomorphism. So each edge in H is mapped to an edge in G which implies that $|E(H)| \leq |E(G)|$.

(2) Also note that $\text{inj}(G,H) > 0$ implies there exists at least one injective homomorphism from G into H. This, in turn, implies that $|V(G)| \leq |V(H)|$ and that $|E(G)| \leq |E(H)|$. To see that $|E(G)| \leq |E(H)|$, observe that $xy \in E(G)$ implies $f(x)f(y) \in E(H)$ since f is a homomorphism. So each edge in G is mapped to an edge in H which implies that $|E(G)| \leq |E(H)|$.

From (1) and (2), we may conclude that $|V(G)| = |V(H)|$ and $|E(H)| = |E(G)|$ and the proof is complete. $\square$

1.5.2 Injections and Partitions in the class Γ_0

In this section, we explore the relationships between homomorphisms, partitions, and injective homomorphisms. The lemmas presented here will be critical to the final proof of our theorem.

LEMMA 1.22. Let $G \in \Gamma_0$ and let Θ be a partition of G. Let $\lambda_\Theta : G \to G/\Theta$ be the function that sends each vertex $x \in V(G)$ to the set $X \in \Theta$ that contains x. Then λ_Θ is a homomorphism.

Proof. Let Θ be a partition of G and $xy \in E(G)$. We need to show that $\lambda_\Theta(x)\lambda_\Theta(y) \in E(G/\Theta)$. Let $x \in X$ and let $y \in Y$ where $X, Y \in \Theta$. Then $\lambda_\Theta(x)\lambda_\Theta(y) = XY \in E(G/\Theta)$ so $xy \in E(G)$ implies $XY \in E(G/\Theta)$. Thus λ_Θ is a homomorphism. $\square$

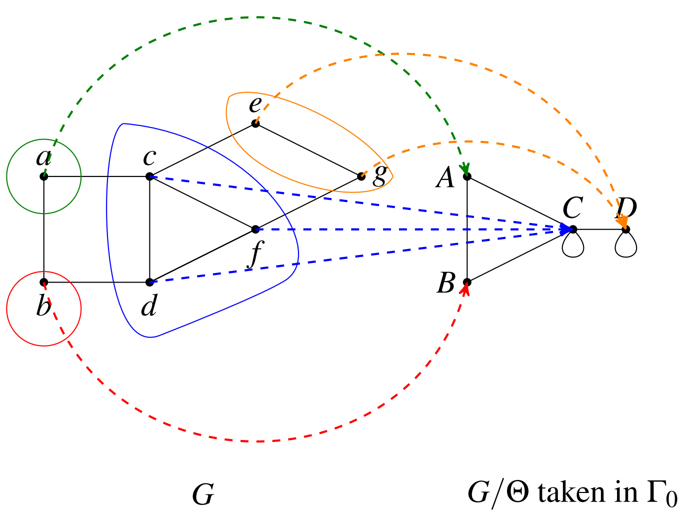


Figure 1.18: Homomorphism from G into G/Θ in Γ_0

LEMMA 1.23. Let $f : G \to H$ be a homomorphism and consider the partition of $V(G)$, $\Theta = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$. Notice that there is a well-defined map $f^* : G/\Theta \to H$ defined as $f^*(X) = f(x)$ for any $x \in X$. Then f^* is an injective homomorphism.

Proof. Let $X,Y \in G/\Theta$. Suppose $f^*(X) = f^*(Y) = h$. Then $f^{-1}(h) \subseteq X$ and $f^{-1}(h) \subseteq Y$. So $X \bigcap Y \neq \emptyset$ and therefore $X = Y$. So f^* is injective. To see that f^* is a homomorphism, let $XY \in E(G/\Theta)$. Then $xy \in E(G)$ for some $x \in X$ and $y \in Y$. So $f^*(X)f^*(Y) = f(x)f(y) \in E(H)$ since f is a homomorphism and thus f^* is a homomorphism. $\square$

EXAMPLE 1.24. To illustrate Lemma 1.23, let G and H be the graphs in Figure 1.19 and let the homomorphism $f : G \to H$ be defined by the Table 1.7. Observe that $f : G \to H$ induces a partition $\Theta = \{\ \{x_1,x_2\}\ \ \{x_3\}\ \{x_4,x_5\}\ \ \{x_6\}\ \}$ and that the function $f^* : G/\Theta \to H$ is then one to one.

Table 1.7: Homomorphic Vertex Mapping

Graph	Vertex Mappings					
G	x_1	x_2	x_3	x_4	x_5	x_6
H	y_1	y_1	y_2	y_3	y_3	y_4

LEMMA 1.25. For every partition Θ of $V(G)$ and injective homomorphism $f^* : G/\Theta \to H$ there is a homomorphism $f : G \to H$ defined as $f(x) = f^*(X)$ where $x \in X \in G/\Theta$ and $\Theta = \{\ f^{-1}(h) : h \in V(H)\ \}$.

Proof. Let Θ be a partition of $V(G)$ and $f^* : G/\Theta \to H$ an injective homomorphism. Let $f : G \to H$ be defined as $f(x) = f^*(X)$ where $x \in X \in G/\Theta$. Now let $xy \in E(G)$. So $XY \in E(G/\Theta)$ where $x \in X \in \Theta$ and $y \in Y \in \Theta$. We need to show that $f(x)f(y) \in E(H)$. By definition, $f(x)f(y) = f^*(X)f^*(Y) \in E(H)$ since f^* is given to be a homomorphism.

Now we will show that $\Theta = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$. Suppose $X \in \Theta$ and say $f^*(X) = h$. For any $x \in X$ we have $f(x) = f^*(X) = h$ which means $x \in f^{-1}(h)$, so $X \subseteq$

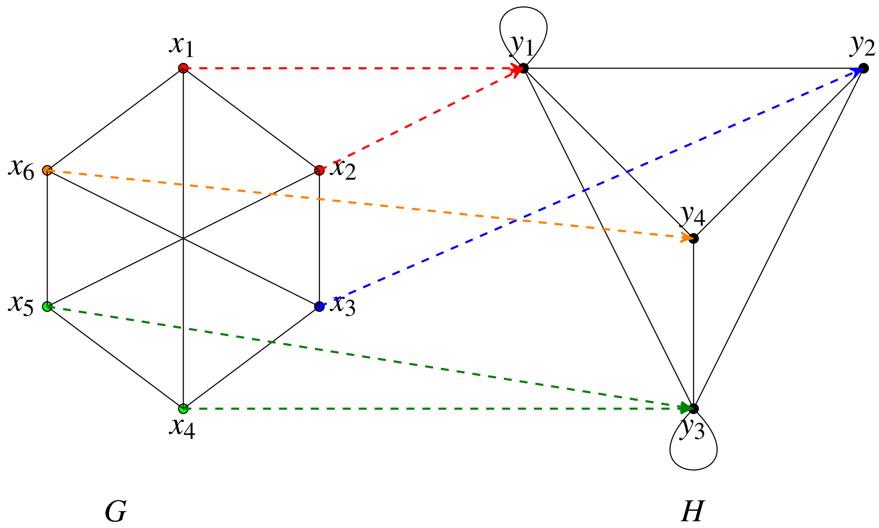


Figure 1.19: The graphs G and H in Example 1.24

$f^{-1}(h)$. On the other hand, suppose $x \in f^{-1}(h)$. This means $f(x) = h$. Suppose $x \in Y \in \Theta$. Then $f^*(Y) = f(x) = h$. Thus $f^*(X) = f^*(Y)$ so $Y = X$ by injectivity of f^*. Thus $x \in Y = X$. We have shown $x \in f^{-1}(h)$ implies $x \in X$ so $f^{-1}(h) \subseteq X$. Therefore $X = f^{-1}(h)$ and $\Theta = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$.

Thus for each partition Θ of $V(G)$ and $f^* : G/\Theta \to H$, there is a homomorphism f and the proof is complete. $\square$

PROPOSITION 1.26. Let G and H be graphs in Γ and suppose $f : G \to H$ is a homomorphism. Then there is a unique pair (Θ_f, f^*) where Θ_f is a partition of $V(G)$ and $f^* : G/\Theta \to H$ is an injective homomorphism for which $f = f^*\lambda_{\Theta_f}$ where $\lambda_{\Theta_f} : G \to G/\Theta_f$ is as defined in Lemma 1.22.

Proof. Suppose $f : G \to H$ is a homomorphism. Let $\Theta_f = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$. Define $f^* : G/\Theta \to H$ as $f^*(X) = f(x)$ where $x \in X$. By Lemma 1.23 this is well-defined and an injective homomorphism and satisfies $f = f^*\lambda_{\Theta_f}$ where $\lambda_{\Theta_f} : G \to G/\Theta_f$.

Now we confirm uniqueness. Suppose there is a pair (Θ, g) for which $g : G/\Theta \to H$ is an injective homomorphism, and $f = g\lambda_\Theta$. Observe that $(\Theta, g) = (\Theta_f, f^*)$, as follows. Using the facts that $f = g\lambda_\Theta$ and g is injective, we see that two vertices x, y are in the same class of Θ_f if and only if $f(x) = f(y)$, if and only if $g\lambda_\Theta(x) = g\lambda_\Theta(y)$, if and only if $\lambda_\Theta(x) = \lambda_\Theta(y)$, if and only if x and y are in the same class in Θ. Thus $\Theta_f = \Theta$. To confirm $f^*(X) = g(X)$ for any X, take $x \in X$ and note $f^*(X) = f(x) = g\lambda_{\Theta_f}(x) = g\lambda_\Theta(x) = g(X)$. □

PROPOSITION 1.27. Suppose that G and H are graphs in Γ_0 and that $\mathcal{P}$ is the set of all partitions of $V(G)$. Then

$$\hom(G,H) = \sum_{\Theta \in \mathcal{P}} \operatorname{inj}(G/\Theta, H).$$

Proof. Let $\operatorname{Hom}(G,H)$ be the set of all homomorphisms from G to H, so its cardinality is $\hom(G,H)$. Let $\Upsilon = \{(\Theta, f^*) : \Theta \in \mathcal{P},\ f^* \in \operatorname{Inj}(G/\Theta, H)\}$, so $|\Upsilon| = \sum_{\Theta \in \mathcal{P}} \operatorname{inj}(G/\Theta, H)$. Let $\beta : \operatorname{Hom}(G,H) \to \Upsilon$ be defined as $\beta(f) = (\{f^{-1} : h \in f(G) \subseteq V(H)\}, f^* : G/\Theta \to H))$ where $f^*(X) = f(x)$ for $x \in X$.

By Proposition 1.26, any $f \in \operatorname{Hom}(G,H)$ is associated with a unique pair $(\Theta, f^*) \in \Upsilon$, where $\Theta_f = \{f^{-1}(h) : h \in V(H)\}$ and $f^* : G/\Theta_f \to H$ is defined as $f^*(X) = f(x)$ for $X \in G/\Theta$. Thus we have a injection $\beta : \operatorname{Hom}(G,H) \to \Upsilon$ defined as $\beta(f) = (\Theta, f^*)$.

To see that β is surjective, take any $(\Theta, f^*) \in \Upsilon$. By Lemma 1.25, there is a $f \in \operatorname{Hom}(G,H)$ with $\beta(f) = (\Theta, f^*)$.

Now we have shown $\beta : \operatorname{Hom}(G,H) \to \Upsilon$ is one to one and onto, so $\beta(f) = (\Theta, f^*)$ and therefore the cardinality of the sets $\operatorname{Hom}(G,H)$ and (Θ, f^*) is the same. Thus $\hom(G,H) = \sum_{\Theta \in \mathcal{P}} \operatorname{inj}(G/\Theta, H)$. □

1.5.3 Lovász's Isomorphism Theorem

We are now ready for the proof of the Lovász's Isomorphism Theorem for undirected graphs in the class Γ_0.

THEOREM 1.28. Let G and H be fixed graphs in Γ_0 and X any graph in Γ_0. Then $G \cong H$ if and only if $\hom(X,G) = \hom(X,H)$ for every graph $X \in \Gamma$.

Proof. Let X be any graph and let G and H be fixed graphs. We begin by showing that $G \cong H$ implies $\hom(X,G) = \hom(X,H)$. By definition, $G \cong H$ implies there exists an isomorphism from G onto H, so let $k : G \to H$ be an isomorphism. Since k is a bijection, it is also true that $k^{-1} : H \to G$ is an isomorphism. Now let $g : X \to G$ be any homomorphism from X into G and let $h : X \to H$ be any homomorphism from X into H. Then the composition of functions $kg : X \to H$ and functions $k^{-1}h : X \to G$ are both homomorphisms. To see this, note that g maps an edge in X to and edge in G and k maps an edge to an edge , so kg is a homomorphism. A corresponding argument applies to $k^{-1}h$. Note also that $kg : X \to H$ and $kh : X \to G$, implies there is a one-to-one correspondence between any homomorphism from X into G with any homomorphism from X into H. Thus, $\hom(X,G) = \hom(X,H)$.

To show that $\hom(X,G) = \hom(X,H)$ implies $G \cong H$ we begin by showing a preliminary result: that $G \cong H$ if $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$ for every graph X. To see that this implies our result, let $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$ for every graph X. Since G and H are graphs, we may substitute G and H for X so that $\mathrm{inj}(G,G) = \mathrm{inj}(G,H)$ and $\mathrm{inj}(H,G) = \mathrm{inj}(H,H)$. Let $g_{\mathrm{I}} : G \to G$ be the identity map from G to G and $h_{\mathrm{I}} : H \to H$ be the identity map from H to H. Then both I_g and I_h are injective automorphisms. So $\mathrm{inj}(G,G) = \mathrm{inj}(G,H) > 0$ implies $\mathrm{inj}(X,G) > 0$ and $\mathrm{inj}(H,H) = \mathrm{inj}(H,G) > 0$ implies $\mathrm{inj}(X,H) > 0$. By Lemma 1.21, $G \cong H$.

Now we assume $\hom(X,G) = \hom(X,H)$ for every graph X and show that $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$ using induction on $|V(X)|$.

If $|V(X)| = 1$ then $\hom(X,G) = \mathrm{inj}(X,G) = |V(G)|$ and $\hom(X,H) = \mathrm{inj}(X,H) = |V(H)|$ because a function cannot fail to be injective if its domain has just one vertex. So, $\hom(X,G) = \hom(X,H)$ implies that $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$.

Now we let X be a graph with $|V(X)| = n > 1$ and assume that $\mathrm{inj}(X',G) = \mathrm{inj}(X',H)$ for all graphs X' with $|V(X')| < n$. By Proposition 1.27, the equation $\hom(X,G) = \hom(X,H)$ implies that

$$\sum_{\Theta\in\mathcal{P}} \mathrm{inj}(X/\Theta,G) = \sum_{\Theta\in\mathcal{P}} \mathrm{inj}(X/\Theta,H)$$

where $\mathcal{P}$ is the set of all partitions of $V(X)$. Let $t \in \mathcal{P}$ be the trivial partition of $V(X)$ such that each set in t contains a single vertex, so that $X/t \cong X$. Then

$$\sum_{\Theta\in\mathcal{P}} \mathrm{inj}(X/\Theta,G) = \sum_{\Theta\in\mathcal{P}} \mathrm{inj}(X/\Theta,H)$$

can be rewritten as

$$\mathrm{inj}(x/t,G) + \sum_{\Theta-\{\mathrm{t}\}\in\mathcal{P}} \mathrm{inj}(X/\Theta,G) = \mathrm{inj}(x/t,H) + \sum_{\Theta-\{\mathrm{t}\}\in\mathcal{P}} \mathrm{inj}(X/\Theta,H).$$

Since $|V(X/\Theta)| < n$ for $\Theta - \{\mathrm{t}\} \in \mathcal{P}$, by the inductive hypothesis we have

$$\sum_{\Theta-\{\mathrm{t}\}\in\mathcal{P}} \mathrm{inj}(X/\Theta,G) = \sum_{\Theta-\{\mathrm{t}\}\in\mathcal{P}} \mathrm{inj}(X/\Theta,H)$$

Therefore, $\mathrm{inj}(X/t,G) = \mathrm{inj}(X,G) = \mathrm{inj}(X/t,H) = \mathrm{inj}(X,H)$ and the proof is complete. $\square$

1.6 Applications of Lovász's Theorem to the Direct Product

We will now apply Lovász's Theorem to deduce a cancellation property for the direct product.

LEMMA 1.29. Consider the projection mappings π_1 and π_2:

$$\pi_1 : V(G \times H) \to V(G) \text{ defined as } \pi_1(g,h) = g$$

$$\pi_2 : V(G \times H) \to V(G) \text{ defined as } \pi_2(g,h) = h.$$

Then π_1 is a homomorphism from $G \times H$ into G and π_2 is a homomorphism from $G \times H$ into H.

Proof. Let $(g_1,h_1)(g_2,h_2) \in E(G \times H)$. By the definition of the direct product, both $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. So $g_1 g_2 \in E(G)$ implies

$$\pi_1((g_1,h_1)(g_2,h_2)) = \pi_1(g_1,h_1)\pi_1(g_2,h_2) = g_1 g_2 \in E(G).$$

Therefore π_1 is a homomorphism. The corresponding reasoning applies to π_2. □

LEMMA 1.30. The composition of two homomorphisms is again a homomorphism.

Proof. Let X, G, H be graphs in Γ_0 and let $f : X \to G$ and $h : G \to H$ be homomorphisms. Then $hf : X \to H$. Let $xy \in E(X)$. Then $h(f(x)f(y))$ where $f(x)f(y) \in E(G)$ since f is a homomorphism. So $h(f(x)f(y)) = h(f(x))h(f(y)) \in E(H)$ since h is a homomorphism and thus hf is a homomorphism. □

LEMMA 1.31. Given homomorphisms $g : V(X) \to V(G)$ and $h : V(X) \to V(H)$, the map $f : V(X) \to V(G \times H)$ defined as $f(x) = (g(x), h(x))$ is a homomorphism.

Proof. Let $x_1 x_2 \in E(X)$ and note $f(x_1 x_2) = (g_1,h_1)(g_2,h_2)$. We need to show that $(g_1,h_1)(g_2,h_2) \in E(G \times H)$. Now $(g_1,h_1)(g_2,h_2) \in E(G \times H)$ whenever $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. Since g is a homomorphism, $g(x_1 x_2) = g(x_1)g(x_2) \in E(G)$. Since h is a homomorphism, $h(x_1x_2) = h(x_1)h(x_2) \in E(H)$. Therefore

$$f(x_1)f(x_2) = (g(x_1),h(x_1))(g(x_2),h(x_2)) \in E(G \times H)$$

and thus f is a homomorphism. $\square$

PROPOSITION 1.32. For any graphs X, G, and H, $\hom(X, G \times H) = \hom(X,G) \cdot \hom(X,H)$

Proof. Let $f : X \to G \times H$ be a homomorphism and let x_1x_2 be any edge in X. If we let $g = \pi_1 f$ and $h = \pi_2 f$, then any homomorphism $f : X \to G \times H$ becomes an ordered pair of homomorphisms from X into G and from X into H, defined as $f(x) = (g(x),h(x))$. It is also true that any ordered pair of homomorphisms from X into G and from X into H constitute a homomorphism from X into $G \times H$. Thus,

$$\begin{aligned}\hom(X, G \times H) &= |\{\ (g,h) : g : X \to G \text{ and } h : X \to H \text{ are homomorphisms }\}| \\ &= \hom(X,G) \cdot \hom(X,H)\end{aligned}$$

and the proof is complete. $\square$

We can now state our cancellation theorem.

THEOREM 1.33. Let A, B, C be graphs in Γ_0 where C has a loop. Then $A \times C \cong B \times C$ implies that $A \cong B$

Proof. By Theorem 2.10 $A \times C \cong B \times C$ implies that $\hom(X, A \times C) = \hom(X, B \times C)$. So $\hom(X,A) \cdot \hom(X,C) = \hom(X,B) \cdot \hom(X,C)$ by Proposition 1.32.

Now, $\hom(X,C) > 0$ since if C is a graph with a loop in Γ_0 and X is any graph, then there is always at least one homomorphism $f : X \to C$. To see this, let $c \in V(C)$ where $cc \in E(C)$ is a loop. Then, for all $x_1x_2 \in E(X)$, the mapping $f : X \to C$, defined as $f(x) = c$ for all $x \in V(X)$ is a homomorphism since $f(x_1x_2) = f(x_1)f(x_2) = cc$. Thus $\hom(X,C) \neq 0$.

So $\hom(X,A) \cdot \hom(X,C) = \hom(X,B) \cdot \hom(X,C)$ implies $\hom(X,A) = \hom(X,B)$ and therefore $A \cong B$ by Theorem 1.28 $\square$

Isomorphism Theorem for Graphs With No Loops

What follows is the development of the principal result of this paper that shows the two graphs G and H in Γ to be isomorphic if, and only if, the number of weak homomorphisms between any graph X and two graphs G and H are equal. We are thus showing that Lovász's Theorem holds for graphs in Γ when we replace homomorphism by weak homomorphisms. For the remainder of this paper, all graphs are in Γ, the class of graphs without loops.

2.1 Preliminaries

In this section, we establish that an injective weak homomorphism is in fact a homomorphism.

Lemma 2.1. Given graphs $G, H \in \Gamma$ and an injective weak homormorphism $f : G \to H$, then f is a homomorphism.

Proof. Let $f : G \to H$ be an injective weak homomorphism. Then for all $xy \in E(G)$, $f(x)f(y) \in E(H)$ or $f(x) = f(y)$. Since f is injective, $f(x) \neq f(y)$ for each $x, y \in V(G)$ and $x \neq y$. Further, $xy \in E(G)$ implies $x \neq y$ since $G \in \Gamma$. Therefore, $xy \in E(G)$ implies $f(x)f(y) \in E(H)$ and thus f is a homomorphism. $\square$

2.2 Properties of Injective Weak Homomorphisms

Here lemmas are presented and proven which establish the necessary conditions under which an injective weak homomorphism is an isomorphism.

LEMMA 2.2. Suppose G and H are graphs in Γ for which $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$. Moreover, suppose there is a weak homomorphism, $f : G \to H$, which is injective. Then $G \cong H$.

Proof. By Lemma 2.1, we know that if f is an injective weak homomorphism then for all $xy \in E(G)$, it follows that $f(x)f(y) \in E(H)$. We begin by showing f is an onto function by assuming the contrary. Let $h \in V(H)$ such that $h \notin f(G)$. This implies that $|V(G)| < |V(H)|$ since f is injective. This contradicts our assumption that $|V(G)| = |V(H)|$ so f must be an onto function. Now since f is onto and injective $f^{-1} : H \to G$ exists and is also injective. We claim f is an isomorphism. Suppose f is not an isomorphism. Then there exists $ab \in E(H)$ such that $f^{-1}(a)f^{-1}(b) \notin E(G)$. This implies $|E(G)| < |E(H)|$ which contradicts our assumption that $|E(G)| = |E(H)|$. Therefore $f^{-1}(a)f^{-1}(b) \in E(G)$. Now f is a bijection satisfying $g_1g_2 \in E(G)$ if and only if $f(g_1)f(g_2) \in E(H)$. Thus $f : G \to H$ is an isomorphism and $G \cong H$. $\square$

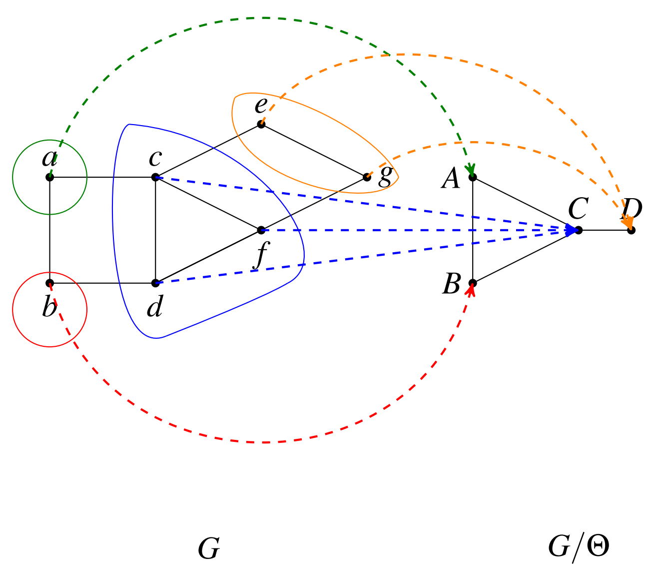


Figure 2.1: Weak homomorphism from G to G/Θ in Γ

By Lemma 2.1 there is no need to reprove Lemma 1.21 so we shall only restate it here.

LEMMA 2.3. If G and H are graphs and $\mathrm{inj}(G,H) > 0$ and $\mathrm{inj}(H,G) > 0$, then $G \cong H$.

2.3 Injections and Partitions

In this section, we explore the relationships between weak homomorphisms, partitions, and injective weak homomorphisms. The lemmas presented here will be critical to the final proof of our theorem.

LEMMA 2.4. Let $G \in \Gamma$ and let Θ be a partition of G. Let $\lambda_\Theta : G \to G/\Theta$ be the function that sends each vertex $x \in V(G)$ to the set $X \in \Theta$ that contains x. Then λ_Θ is a weak homomorphism.

Proof. Let Θ be a partition of G and $xy \in E(G)$. We need to show that either $\lambda_\Theta(x)\lambda_\Theta(y) \in E(G/\Theta)$ or that $\lambda_\Theta(x) = \lambda_\Theta(y)$. There are two cases to consider: either x and y are elements of the same or of a different class in Θ.
In the first case we let $x, y \in X \in \Theta$. Then $\lambda_\Theta(x) = \lambda_\Theta(y) = X$. In the second case, we let $x \in X$ and let $y \in Y$ where $X,Y \in \Theta$ and $X \neq Y$. Then $\lambda_\Theta(x)\lambda_\Theta(y) = XY \in E(G/\Theta)$ by definition of G/Θ. So $xy \in E(G)$ implies $XY \in E(G/\Theta$. Thus $\lambda_\Theta(x)\lambda_\Theta(y) \in E(G/\Theta)$ or $\lambda_\Theta(x) = \lambda_\Theta(y)$ implies λ_Θ is a weak homomorphism. □

LEMMA 2.5. Let $f : G \to H$ be a weak homomorphism and consider the partition $\Theta = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$ of $V(G)$. Notice that there is a well-defined map $f^* : G/\Theta \to H$ defined as $f^*(X) = f(x)$ for any $x \in X$. Then f^* is an injective homomorphism.

Proof. Let $X,Y \in G/\Theta$. Suppose $f^*(X) = f^*(Y) = h$. Then $f^{-1}(h) \subseteq X$ and $f^{-1}(h) \subseteq Y$ which implies that $X \bigcap Y \neq \emptyset$ and thus $X = Y$. So f^* is injective. To see that f^* is a weak homomorphism, let $XY \in E(G/\Theta)$. Then $xy \in E(G)$ for some $x \in X$ and $y \in Y$. So $f^*(X)f^*(Y) = f(x)f(y) \in E(H)$ or $f^*(X) = f(x) = f(y) = f^*(Y)$ since f is a weak homomorphism and thus f^* is a weak homomorphism. Since f^* is also injective, by Lemma 2.1, we have that f^* is an injective homomorphism as required. $\square$

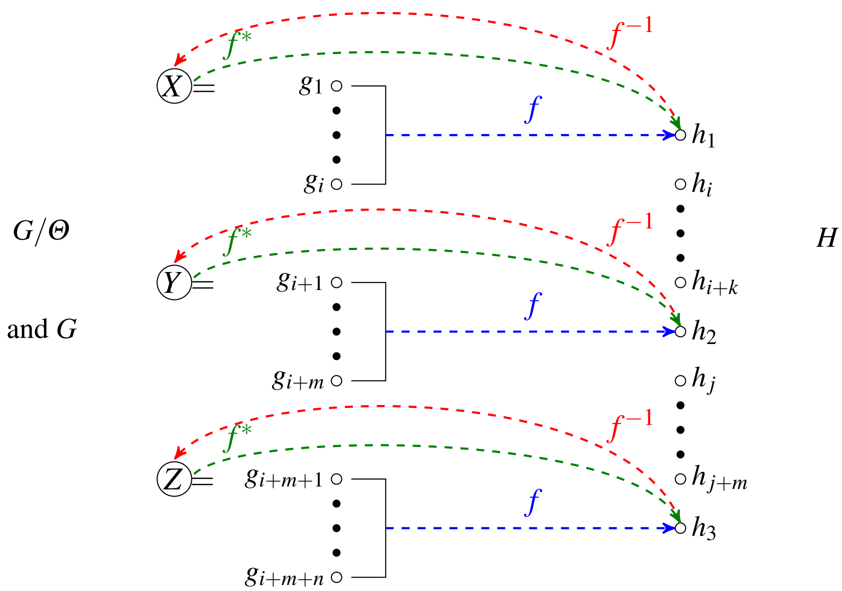


Figure 2.2: Illustration of Lemma 2.5

Lemma 2.6. For every partition Θ of $V(G)$ and injective homomorphism $f^* : G/\Theta \to H$ there is a weak homomorphism $f : G \to H$ defined as $f(x) = f(X)$ where $x \in X \in G/\Theta$ and $\Theta = \{ f^{-1}(h) : h \in f(G) \subseteq V(H) \}$.

Proof. Let Θ be a partition of $V(G)$ and $f^* : G/\Theta \to H$ an injective homomorphism. Let $f : G \to H$ be defined as $f(x) = f^*(X)$ where $x \in X \in G/\Theta$. Now let $xy \in E(G)$. We need to show that either $f(x)f(y) \in E(H)$ or $f(x) = f(y)$. If $x \in X$ and $y \in Y$ where $X \neq Y$, then by definition, $f(x)f(y) = f^*(X)f^*(Y) \in E(H)$ since f^* is given to be a homomorphism. Therefore f is a weak homomorphism.

Now we will show that $\Theta = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$. Suppose $X \in \Theta$ and say $f^*(X) = h$. For any $x \in X$ we have $f(x) = f^*(X) = h$ which means $x \in f^{-1}(h)$, so $X \subseteq$

$f^{-1}(h)$. On the other hand, suppose $x \in f^{-1}(h)$. This means $f(x) = h$. Suppose $x \in Y \in \Theta$. Then $f^*(Y) = f(x) = h$. Thus $f^*(X) = f^*(Y)$ so $Y = X$ by injectivity of f^*. Thus $x \in Y = X$. We have shown $x \in f^{-1}(h)$ implies $x \in X$ so $f^{-1}(h) \subseteq X$. Therefore $X = f^{-1}(h)$ and $\Theta = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$.

Thus for each partition Θ of $V(G)$ and $f^* : G/\Theta \to H$ there is a weak homomorphism f and the proof is complete. $\square$

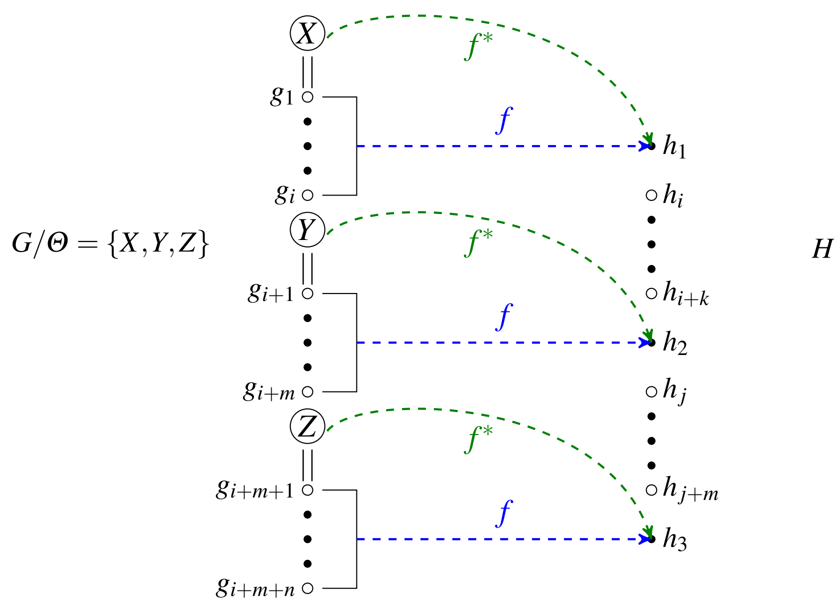


Figure 2.3: Illustration of Lemma 2.6

PROPOSITION 2.7. Let G and H be graphs in Γ and suppose $f : G \to H$ is a weak homomorphism. Then there is a unique pair (Θ_f, f^*) where Θ_f is a partition of $V(G)$ and $f^* : G/\Theta_f \to H$ is an injective homomorphism for which $f = f^*\lambda_{\Theta_f}$ where $\lambda_{\Theta_f} : G \to G/\Theta_f$ is defined by $\lambda_{\Theta_f}(x) = X$ such that $x \in X \in G/\Theta_f$ and $\Theta = \{\, f^{-1}(h) : h \in f(G) \subseteq V(H)\,\}$.

Proof. Suppose $f : G \to H$ is a weak homomorphism.

Let $\Theta_f = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$. Define $f^* : G/\Theta \to H$ as $f^*(X) = f(x)$ where

$x \in X$. This is well-defined and by Lemma 2.5 it is an injective homomorphism and thus satisfies $f = f^*\lambda_{\Theta_f}$ since $\lambda_{\Theta_f}(x) = X$ and $f^*(X) = f(x)$ where $x \in X \in G/\Theta_f$ and λ_{Θ_f} is a weak homomorphism by Lemma 2.4.

Now we confirm uniqueness. Suppose there is a pair (Θ, g) for which $g : G/\Theta \to H$ is an injective homomorphism, and $f = g\lambda_\Theta$. Observe that $(\Theta, g) = (\Theta_f, f^*)$, as follows. Using the facts that $f = g\lambda_\Theta$ and g is injective, we see that two vertices x, y are in the same class of Θ_f if and only if $f(x) = f(y)$, if and only if $g\lambda_\Theta(x) = g\lambda_\Theta(y)$, if and only if $\lambda_\Theta(x) = \lambda_\Theta(y)$, if and only if x and y are in the same class in Θ. Thus $\Theta_f = \Theta$. To confirm $f^*(X) = g(X)$ for any X, take $x \in X$ and note $f^*(X) = f(x) = g\lambda_{\Theta_f}(x) = g\lambda_\Theta(x) = g(X)$. $\square$

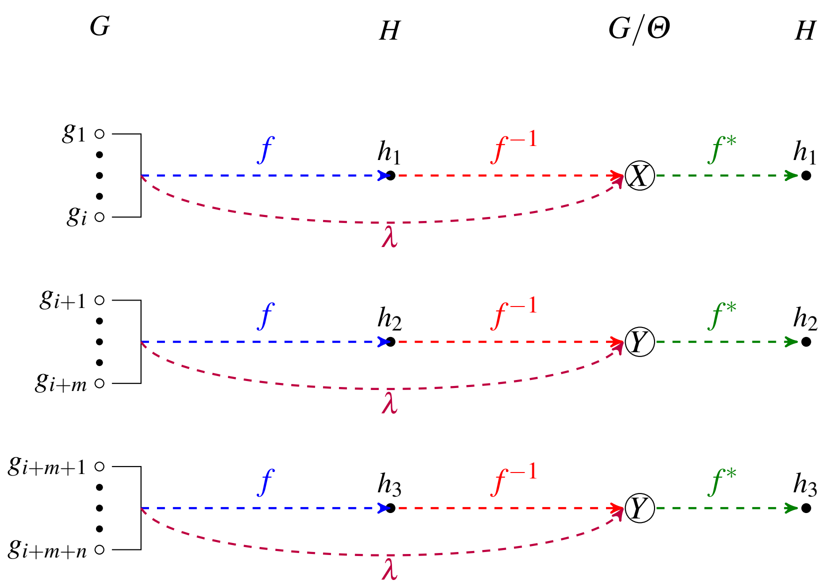


Figure 2.4: Illustration of Proposition 2.7

Lemma 2.8. Suppose G and H are graphs in Γ and that $\mathcal{P}$ is the set of all partitions of $V(G)$. Then

$$\hom_{\mathrm{w}}(G,H) = \sum_{\Theta \in \mathcal{P}} \operatorname{inj}(G/\Theta, H).$$

Proof. Let $\mathrm{Hom_w}(G,H)$ be the set of all weak homomorphisms from G to H, so its cardinality is $\mathrm{hom_w}(G,H)$. Let $\Upsilon = \{(\Theta, f^*) : \Theta \in \mathcal{P},\ f^* \in \mathrm{Inj}(G/\Theta, H)\}$, so $|\Upsilon| = \sum\limits_{\Theta\in\mathcal{P}} \mathrm{inj}(G/\Theta, H)$. To see this, consider a partition Θ. Depending on $G/\mathit{\Theta}$ and H there will be $n \geq 0$ injective homomorphisms f^* from G/Θ to H. Notice that $n = \mathrm{inj}(G/\Theta, H)$ for a particular Θ. So if there are m partitions in $\mathcal{P}$, we have

$$|\Upsilon| = \mathrm{inj}(G/\Theta_1, H) + \cdots + \mathrm{inj}(G/\Theta_{\mathrm{m}}, H) = \sum_{\Theta\in\mathcal{P}} \mathrm{inj}(G/\Theta, H)$$

Let $\beta : \mathrm{Hom_w}(G,H) \to \Upsilon$ be defined as

$$\beta(f) = (\{f^{-1} : h \in f(G) \subseteq V(H)\}, f^* : G/\Theta \to H))$$

where $f^*(X) = f(x)$ for $x \in X$. By Proposition 2.7, any $f \in \mathrm{Hom_w}(G,H)$ is associated with a unique pair $(\Theta_f, f^*) \in \Upsilon$, where $\Theta_f = \{f^{-1}(h) : h \in f(G) \subseteq V(H)\}$ and $f^*(X) = f(x)$ for $x \in X$. By Lemma 2.6 there is a weak homomorphism $f : G \to H$ for every partition Θ_f and injective homomorphism $f^* : G/\Theta_f \to H$ defined by $f^*(X) = f(x)$ for $X \in G/\Theta_f$. Thus we have a injection $\beta : \mathrm{Hom_w}(G,H) \to \Upsilon$ defined as $\beta(f) = (\Theta_f, f^*)$.

To see that β is surjective, take any $(\Theta, f^*) \in \Upsilon$. By Lemma 2.4, there is a $f \in \mathrm{Hom_w}(G,H)$ with $\beta(f) = (\Theta, f^*)$.

We have shown $\beta : \mathrm{Hom_w}(G,H) \to (\Theta, f^*)$ is one to one and onto, so $\beta(f) = (\Theta, f^*)$ and therefore the cardinality of the sets $\mathrm{Hom_w}(G,H)$ and (Θ, f^*) is the same. Thus $\mathrm{hom_w}(G,H) = \sum\limits_{\mathit{\Theta}\in\mathcal{P}} \mathrm{inj}(G/\Theta, H)$. $\square$

$$\begin{array}{c} f \in \mathrm{Hom_w}(G,H) \\ \Big\updownarrow \beta \\ \Upsilon = (\Theta, f^*) \end{array}$$

where $\beta(f) = (\Theta = \{ f^{-1}(h) : h \in f(G) \subseteq V(H) \}, f^* : G/\Theta \to H)$

Figure 2.5: Diagram of β in Lemma 2.8

EXAMPLE 2.9. To illustrate Lemma 2.5, let G and H be the graphs in Figure 2.6 and let the homomorphism $f : G \to H$ be defined by the Table 2.1. Observe that $f : G \to H$ induces a partition $\Theta = \{ \{x_1, x_2\} \ \{x_3\} \ \{x_4, x_5\} \ \{x_6\} \}$ and that the function $f^* : G/\Theta \to H$ is then one to one.

Table 2.1: Homomorphic Mapping

Graph	Vertex Mappings					
A	x_1	x_2	x_3	x_4	x_5	x_6
B	y_1	y_1	y_2	y_3	y_3	y_4

2.4 Isomorphism Theorem in Γ

It is now time to prove the main result of this thesis and answer the question of whether or not a homomorphism that allows an edge to be collapsed into a single point will produce the same result as Lovász's Theorem.

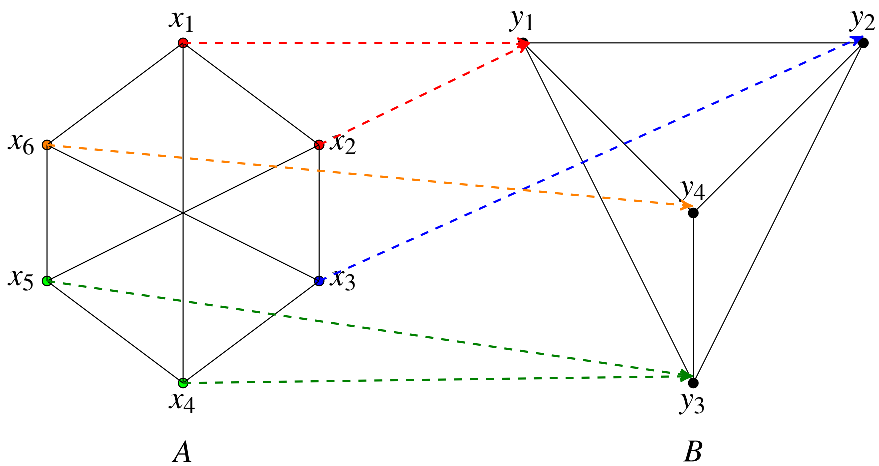


Figure 2.6: The graphs A and B in Example 2.9

THEOREM 2.10. Let G and H be fixed graphs in Γ and X any graph in Γ. Then $G \cong H$ if and only if $\hom_w(X,G) = \hom_w(X,H)$ for every graph $X \in \Gamma$.

Proof. Let X be a any graph and let G and H be fixed graphs. We begin by showing that $G \cong H$ implies $\hom_w(X,G) = \hom_{\mathrm{w}}(X,H)$. We know that $G \cong H$ implies there exists an isomorphism from G onto H, so let $k : G \to H$ be an isomorphism. Since k is a bijection, it is also true that $k^{-1} : H \to G$ is an isomorphism. Now let $g : X \to G$ be any weak homomorphism from X into G and let $h : X \to H$ be any weak homomorphism from X into H. Then the composition of functions $kg : X \to H$, and of functions $k^{-1}h : X \to G$ are both weak homomorphisms. To see this, note that g maps an edge in X to and edge in G or to a single vertex in G and k maps an edge to an edge and a single vertex to a single vertex, so kg is a weak homomorphism. A corresponding argument applies to $k^{-1}h$. Therefore for any $g \in \mathrm{Hom}_{\mathrm{w}}(X,G)$, we have $h = kg \in \mathrm{Hom}_{\mathrm{w}}(X,H)$ and for any $h \in \mathrm{Hom}_{\mathrm{w}}(X,H)$, we have $g = k^{-1}h \in \mathrm{Hom}_{\mathrm{w}}(X,G)$. Since k and k^{-1} are both bijections from G to H and H to G respectively, there is a one-to-one correspondence between any weak homomorphism from X into G with any weak homomorphism from X into H. Thus, $\hom_w(X,G) = \hom_{\mathrm{w}}(X,H)$.

To show that $\hom_{\mathrm{w}}(X,G) = \hom_{\mathrm{w}}(X,H)$ implies $G \cong H$ we begin by showing a preliminary result: that $G \cong H$ if $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$ for every graph X. To see this implies our result, let $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$ for every graph X. Since G and H are graphs, we may substitute G and H for X so that $\mathrm{inj}(G,G) = \mathrm{inj}(G,H)$ and $\mathrm{inj}(H,G) = \mathrm{inj}(H,H)$. Let $\mathrm{I_g} : G \to G$ be the identity map from G to G and $\mathrm{I_h} : H \to H$ be the identity map from H to H. Then both $\mathrm{I_g}$ and $\mathrm{I_h}$ are injective automorphisms. So $\mathrm{inj}(G,G) = \mathrm{inj}(G,H) > 0$ implies $\mathrm{inj}(X,G) > 0$ and $\mathrm{inj}(H,H) = \mathrm{inj}(H,G) > 0$ implies $\mathrm{inj}(X,H) > 0$. By Lemma 2.3, $G \cong H$.

Now we assume $\hom_{\mathrm{w}}(X,G) = \hom_{\mathrm{w}}(X,H)$ for every graph X and show that $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$ using induction on $|V(X)|$.

If $|V(X)| = 1$ then because a function cannot fail to be injective if its domain has just one vertex, $\hom_{w}(X,G) = \mathrm{inj}(X,G) = |V(G)|$ and $\hom_{w}(X,H) = \mathrm{inj}(X,H) = |V(H)|$. So, $\hom_{\mathrm{w}}(X,G) = \hom_{\mathrm{w}}(X,H)$ implies that $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$.

Now we let X be a graph with $|V(X)| = n > 1$ and assume that $\mathrm{inj}(X',G) = \mathrm{inj}(X',H)$ for all graphs X' with $|V(X')| < n$. By Lemma 2.8, the equation $\hom_{\mathrm{w}}(X,G) = \hom_{\mathrm{w}}(X,H)$ implies that $\sum_{\Theta\in\mathcal{P}} \mathrm{inj}(X/\Theta,G) = \sum_{\Theta\in\mathcal{P}} \mathrm{inj}(X/\Theta,H)$ where $\mathcal{P}$ is the set of all partitions of $V(X)$. Let $t \in \mathcal{P}$ be the trivial partition of $V(X)$ such that each set in t contains a single vertex, so that $X/t \cong X$. Then

$$\sum_{\Theta\in\mathcal{P}} \mathrm{inj}(X/\Theta,G) = \sum_{\Theta\in\mathcal{P}} \mathrm{inj}(X/\Theta,H)$$

can be rewritten as

$$\mathrm{inj}(x/t,G) + \sum_{\Theta-\{\mathrm{t}\}\in\mathcal{P}} \mathrm{inj}(X/\Theta,G) = \mathrm{inj}(x/t,H) + \sum_{\Theta-\{\mathrm{t}\}\in\mathcal{P}} \mathrm{inj}(X/\Theta,H).$$

Since $|V(X/\Theta)| < n$ for $\Theta - \{\mathrm{t}\} \in \mathcal{P}$, by the inductive hypothesis we have

$$\mathrm{inj}(X'/\Theta,G) = \mathrm{inj}(X'/\Theta,H)$$

Therefore, $\operatorname{inj}(X/t, G) = \operatorname{inj}(X, G) = \operatorname{inj}(X/t, H) = \operatorname{inj}(X, H)$ and the proof is complete. $\square$

2.5 Applications to the Strong Product

We will now apply the above result to obtaining a cancellation property for the strong product. Namely, we will show that $A \boxtimes C \cong B \boxtimes C$ implies $A \cong B$.

It is interesting to note that, unlike the direct product, a cancellation theorem for the strong product will not work in Γ_0. This problem is illustrated in Figure 2.7. In it we see that although $G \boxtimes C \cong H \boxtimes C$, the graphs G and H are not isomorphic. Therefore the cancellation theorem for graphs in Γ_0 does not apply to the strong product. However, we now have the tools to show a similar cancellation theorem in Γ.

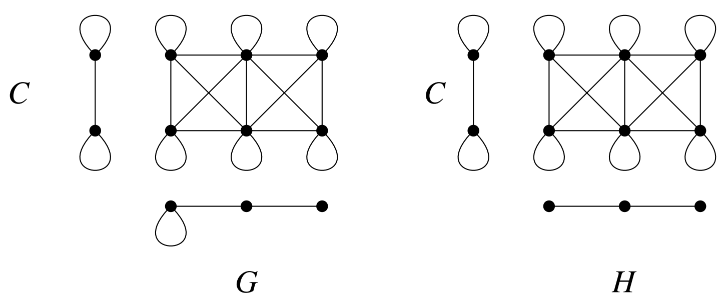


Figure 2.7: The graph products $G \boxtimes C$ and $H \boxtimes C$

Lemma 2.11. Consider the projection mappings π_1 and π_2:

$$\pi_1 : V(G \boxtimes H) \to V(G) \text{ defined as } \pi_1(g,h) = g$$

$$\pi_2 : V(G \boxtimes H) \to V(G) \text{ defined as } \pi_2(g,h) = h.$$

Then π_1 is a weak homomorphism from $G \boxtimes H$ into G and π_2 is a weak homomorphism from $G \boxtimes H$ into H.

Proof. Let $(g_1,h_1)(g_2,h_2) \in E(G \boxtimes H)$. Then by the definition of the strong product, either both $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$, $g_1g_2 \in E(G)$ and $h_1 = h_2$, or $h_1h_2 \in E(H)$and $g_1 = g_2$. If $g_1g_2 \in E(G)$ then $\pi_1((g_1,h_1)(g_2,h_2)) = \pi_1(g_1,h_1)\pi_1(g_2,h_2) = g_1g_2 \in E(G)$. If $g_1 = g_2$ then $\pi_1((g_1,h_1)(g_2,h_2)) = g_1 = g_2$. Therefore π_1 is a weak homomorphism. The corresponding reasoning applies to π_2. □

LEMMA 2.12. The composition of two weak homomorphisms is again a weak homomorphism.

Proof. Let X,G,H be graphs in Γ and let $f : X \to G$ and $h : G \to H$ be weak homomorphisms. Then $hf : X \to H$. Let $xy \in E(X)$. Then $h(f(x)f(y))$ where $f(x)f(y) \in E(G)$ or $f(x) = f(y)$. If $f(x)f(y) \in E(G)$ then $h(f(x)f(y)) = h(f(x))h(f(y)) \in E(H)$ or $h(f(x)) = h(f(y))$ and hf is a weak homomorphism. If $f(x) = f(y)$ then $h(f(x)f(y)) = h(f(x)) = h(f(y))$ and hf is a weak homomorphism. So, in all cases, hf is a weak homomorphism. □

LEMMA 2.13. Given weak homomorphisms $g : V(X) \to V(G)$ and $h : V(X) \to V(H)$, the map $f : V(X) \to V(G \boxtimes H)$ defined as $f(x) = (g(x),h(x))$ is a weak homomorphism.

Proof. Let $x_1x_2 \in E(X)$ and $f(x_1x_2) = (g_1,h_1)(g_2,h_2)$. For f to be a weak homomorphism, we need to show that $(g_1,h_1)(g_2,h_2) \in E(G \boxtimes H)$ or $(g_1,h_1) = (g_2,h_2)$. By definition $(g_1,h_1)(g_2,h_2) \in E(G \boxtimes H)$ whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$, or $g_1g_2 \in E(G)$ and $h_1 = h_2$, or $h_1h_2 \in E(H)$ and $g_1 = g_2$. Since g is a weak homomorphism, $g(x_1)g(x_2) = g_1g_2 \in E(G)$ or $g(x_1) = g_1 = g(x_2) = g_2$. Since h is a weak homomorphism, $h(x_1)h(x_2) = h_1h_2 \in E(H)$ or $h(x_1) = h(x_2)$. Therefore $f(x_1)f(x_2) = (g(x_1),h(x_1))(g(x_2),h(x_2)) \in E(G \boxtimes H)$ or $(g(x_1),h(x_1)) = (g(x_2),h(x_2))$ and thus f is a weak homomorphism. □

PROPOSITION 2.14. For any graphs X, G, and H, $\hom_{\mathrm{w}}(X, G \boxtimes H) = \hom_{\mathrm{w}}(X,G) * \hom_{\mathrm{w}}(X,H)$

Proof. Let $f : X \to G \boxtimes H$ be a weak homomorphism and let x_1x_2 be any edge in X. If we let $g = \pi_1 f$ and $h = \pi_2 f$, then any weak homomorphism $f : X \to G \boxtimes H$ becomes an ordered pair of homomorphisms from X into G and from X into H, defined as $f(x) = (g(x), h(x))$. It is also true that any ordered pair of weak homomorphisms from X into G and from X into H constitute a weak homomorphism from X into $G \boxtimes H$. Thus,

$$\begin{aligned}\hom(X, G \boxtimes H) &= |\{\ (g,h) : g : X \to G \text{ and } h : X \to H \text{ are weak homomorphisms }\}| \\ &= \hom(X,G) \cdot \hom(X,H).\end{aligned}$$

□

THEOREM 2.15. Let A, B, C be graphs in Γ. Then $A \boxtimes C \cong B \boxtimes C$ implies that $A \cong B$

Proof. By Theorem 2.10, $A \boxtimes C \cong B \boxtimes C$ implies that $\hom_w(X, A \boxtimes C) = \hom_w(X, B \boxtimes C)$. So $\hom_w(X,A) * \hom_w(X,C) = \hom_w(X,B) * \hom_w(X,C)$ by Proposition 2.14.

Now, $\hom_w(X,C) > 0$ since if C is a graph in Γ and X is any graph, then there is always at least one weak homomorphism. To see this, let $c \in V(C)$. Then, for all $x_1x_2 \in E(X)$, the mapping $f : X \to C$, defined as $f(x) = c$ for all $x \in V(X)$ is a weak homomorphism since $f(x_1x_2) = f(x_1) = f(x_2)$. So we can divide through by $\hom_w(X,C)$. The resulting equation $\hom_w(X,A) * \hom_w(X,C) = \hom_w(X,B) * \hom_w(X,C)$ implies $\hom_w(X,A) = \hom_w(X,B)$. By Theorem 2.10, $A \cong B$. □

We now examine how these results may be extended to a result involving the strong product of n graphs.

DEFINITION 2.16. Let $G_1, G_2, ..., G_k$ be graphs in Γ. Then their strong product is the graph $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k = \boxtimes_{i=1}^{k} G_i$ with vertex set $\{\ (a_1, a_2, ..., a_k) |\ a_i \in V(G_i)\ \}$, and for which two distinct vertices $(a_1, a_2, ..., a_k)$ and $(b_1, b_2, ..., b_k)$ are adjacent provided that either $a_ib_i \in E(G_i)$ or $a_j = b_j$ for each $i = 1, 2, ..., k$.

LEMMA 2.17. Consider the projection mappings π_i where $\pi_i : V(\boxtimes_{i=1}^{k} G_i) \to V(G_i)$ defined as $\pi_i(a_1, a_2, ..., a_k) = a_i \in V(G_i)$ for each $i = 1, 2, ..., k$. Then π_i is a weak homomorphism from $\boxtimes_{i=1}^{k} G_i$ into G_i for each $i = 1, 2, ..., k$.

Proof. Let $(a_1, a_2, ..., a_k)(b_1, b_2, ..., b_k) \in E(\boxtimes_{i=1}^{k} G_i)$. Then for each $i = 1, 2, ..., k$, either $a_i b_i \in E(G_i)$ or $a_i = b_i$. So $\pi_i(a_1, a_2, ..., a_k)\pi_i(b_1, b_2, ..., b_k) = a_i b_i$ implies that either $\pi_i(a_1, a_2, ..., a_k)\pi_i(b_1, b_2, ..., b_k) \in E(G_i)$ or that $\pi_i(a_1, a_2, ..., a_k) = \pi_i(b_1, b_2, ..., b_k)$. Therefore π_i is a weak homomorphism for each $i = 1, 2, .., k$. $\square$

LEMMA 2.18. Given weak homomorphisms $g_i : V(X) \to V(G_i)$ for each $i = 1, 2, ..., k$, $f : V(X) \to V(\boxtimes_{i=1}^{k} G_i)$ defined as $f(x) = (g_1(x), g_2(x), ..., g_k(x))$ is a weak homomorphism.

Proof. Let $x_1 x_2 \in E(X)$ and $f(x_1)f(x_2) = (a_1, a_2, ..., a_k)(b_1, b_2, ..., b_k)$. We need to show that $(a_1, a_2, ..., a_k)(b_1, b_2, ..., b_k) \in E(\boxtimes_{i=1}^{k} G)$ or $(a_1, a_2, ..., a_k) = (b_1, b_2, ..., b_k)$. So for each $i \in \{1, 2, ..., k\}$, g_i a weak homomorphism implies that $g_i(x_1 x_2) = g_i(x_1)g_i(x_2) \in E(G_i)$ or $g_i(x_1) = g_i(x_2)$. Thus, $x_1 x_2 \in E(X)$ implies

$$
\begin{aligned}
f(x_1 x_2) &= f(x_1)f(x_2) \\
&= (a_1, a_2, ..., a_k)(b_1, b_2, ..., b_k) \\
&= (g_1(x_1), g_2(x_1), ..., g_k(x_1))(g_1(x_2), g_2(x_2), ..., g_k(x_2))
\end{aligned}
$$

where $g_i(x_1)g_i(x_2) \in E(G_i)$ or $g_i(x_1) = g_i(x_2)$ for each $i \in \{1, 2, ...k\}$. Therefore
$(a_1, a_2, ..., a_k)(b_1, b_2, ..., b_k) \in E(\boxtimes_{i=1}^{k} G)$ or $(a_1, a_2, ..., a_k) = (b_1, b_2, ..., b_k)$.
So $f(x_1)f(x_2) \in E(\boxtimes_{i=1}^{k} G)$ or $f(x_1) = f(x_2)$ and thus f is a weak homomorphism. $\square$

PROPOSITION 2.19. For any graphs $X, G_1, G_2, ..., G_k$,

$$\hom_{\mathrm{w}}(X, \boxtimes_{i=1}^{k} G) = \hom_{\mathrm{w}}(X, G_1) * \hom_{\mathrm{w}}(X, G_2) * \cdots * \hom_{\mathrm{w}}(X, G_k)$$

Proof. Let $f : X \to G \boxtimes H$ be a weak homomorphism and let x_1x_2 be any edge in X. If we let $g_i = \pi_i f$, then any weak homomorphism $f : X \to \boxtimes_{i=1}^k G$ becomes an ordered tuple of weak homomorphisms, g_i, from X into $G_1, G_2, ..., G_k$ defined as $f(x) = (g_1(x), g_2(x), ..., g_k(x))$. It is also true that any ordered tuple of weak homomorphisms from X into G_i constitutes a weak homomorphism from X into $\boxtimes_{i=1}^k G$. Thus,

$$\begin{aligned}\hom(X, \boxtimes_{i=1}^k G) &= |\{\ (g_1, g_2, ..., g_k) \mid g_i : X \to G_i \text{ are weak homomorphisms }\}| \\ &= \hom(X, G_1) \cdot \hom(X, G_2) \cdot ... \cdot \hom(X, G_k)\end{aligned}$$

and the proof is complete. □

In conclusion, the original theorem as set forth by Lovász for directed graphs in Γ_0 holds for undirected graphs involving weak homomorphisms in the class Γ. In applying this result to product graphs, it became possible to prove a cancellation theorem involving the strong product.

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Vita

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