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Coloring the Square of Planar Graphs Without 4-Cycles or 5-Cycles

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Coloring the Square of Planar Graphs Without 4-Cycles or 5-Cycles

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

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The famous Four Color Theorem states that any planar graph can be properly colored using at most four colors. However, if we want to properly color the square of a planar graph (or alternatively, color the graph using distinct colors on vertices at distance up to two from each other), we will always require at least $\Delta + 1$ colors, where Δ is the maximum degree in the graph. For all Δ, Wegner constructed planar graphs (even without 3-cycles) that require about $\frac{3}{2}\Delta$ colors for such a coloring.

To prove a stronger upper bound, we consider only planar graphs that contain no 4-cycles and no 5-cycles (but which may contain 3-cycles). Zhu, Lu, Wang, and Chen showed that for a graph G in this class with $\Delta \geq 9$, we can color G^2 using no more than $\Delta + 5$ colors. In this thesis we improve this result, showing that for a planar graph G with maximum degree

$\Delta \geq 32$ having no 4-cycles and no 5-cycles, at most $\Delta + 3$ colors are needed to properly color G^2. Our approach uses the discharging method, and the result extends to list-coloring and other related coloring concepts as well.

Chapter 1

Introduction

Within the mathematical field of graph theory, the topic and problems of graph coloring can be traced back to the mid-1800's. In the original context, the driving question had to do with coloring a map (e.g. of countries within a continent, or counties within a country): is it always possible, for any given map, to color the map using no more than four colors such that no two regions sharing some common boundary receive the same color? This question remained open for over a century before it was finally resolved, and many of the major developments in graph theory over that time can be tied back to attempts to solve (or at least understand the nature of) this problem.

A map can be easily transformed into a graph, where the vertices of the graph correspond to the regions of the map, and two vertices are joined by an edge whenever their corresponding regions share some boundary. In this way the problem of coloring a map can be translated into that of coloring the vertices of a particular type of graph. This problem can be widened to consider coloring the vertices of any given graph, with the restriction that two vertices which share an edge must not receive the same color. There are numerous applications of this concept, especially to problems involving scheduling and optimal resource allocation. In this thesis we consider this vertex coloring problem for a special class of graphs.

1.1 Basic Definitions

Throughout this thesis, we will let G denote a finite simple graph. This means that G consists of a finite set of vertices, denoted $V(G)$, along with a (possibly empty) set of edges, denoted $E(G)$. These edges are undirected, and G does not contain any loops or parallel edges, meaning elements in $E(G)$ are unordered pairs of distinct vertices. When two vertices u and v share an edge (i.e. $\{u, v\} \in E(G)$), we say that the vertices are *adjacent*. We will often abbreviate this edge as uv.

For a vertex $v \in V(G)$, we will denote by $N(v)$ the *neighborhood* of v, i.e. the set of all vertices sharing an edge with v. The *degree* of the vertex v is denoted $d(v)$, and is equal to the number of elements in $N(v)$. We will write a *k-vertex*, *k^+-vertex*, or *k^--vertex* to mean a vertex with degree equal to k, at least k, or at most k, respectively. The maximum degree of any vertex in G is denoted $\Delta(G)$, or simply Δ, while the minimum degree of any vertex in G is denoted $\delta(G)$.

A *k-cycle* in a graph G is a sequence $\{v_1, v_2, \ldots, v_k\}$ of k sequentially adjacent vertices in $V(G)$, where the first and last vertices are also adjacent. That is, $v_i v_{i+1} \in E(G)$ for each i such that $1 \leq i \leq k-1$, and also $v_1 v_k \in E(G)$. A *complete graph* is a graph in which every pair of distinct vertices is adjacent. For a graph G and a set $S \subseteq V(G)$, the *restriction of G to S*, denoted by $G[S]$, is a graph having vertex set S and with edge set $E = \{uv \in E(G) \mid u \in S \text{ and } v \in S\}$. A *clique* in a graph G is a set of mutually adjacent vertices in G, i.e. a set S such that $G[S]$ is a complete graph.

A *planar graph* is a graph that can be drawn in the plane without any edges crossing each other. Such a graph drawn in this way is called a *plane embedding* of the graph, or alternately a *plane graph*. It is then possible to talk about the *faces* of a plane graph: intuitively, these are just regions enclosed by some set of edges with no other edges going through them. More formally, the faces are the distinct connected regions of the plane that remain after removing the points corresponding to the vertices and edges of a plane graph. Every plane graph has a single unbounded exterior face. If G is a plane graph, let $F(G)$

denote the set of faces of G, and for a face $f \in F(G)$, let $d(f)$ denote the number of edges enclosing f. Let *k-face*, *k^+-face*, and *k^--face* denote a face f where $d(f)$ is equal to k, at least k, or at most k, respectively.

1.2 Vertex Coloring and Squares of Graphs

For a graph G, let $\phi : V(G) \to \{1, 2, \ldots, k\}$ be a function assigning distinct values to the endpoints of each edge in G, i.e. whenever $uv \in E(G)$, then $\phi(u) \neq \phi(v)$. In this case, we say ϕ is a *proper vertex coloring* of G, and in particular, is a *proper k-coloring.* The *chromatic number* of G, denoted $\chi(G)$, is the smallest value k such that G has a proper k-coloring.

The problem of determining the chromatic number of different classes of graphs has been a driving factor behind much of the development of graph theory for over a century. The most famous result in this area is the Four Color Theorem, which finally answered the question posed in the beginning of the Introduction. The theorem is stated here in terms of planar graphs rather than maps, but the translation from maps to graphs laid out in the Introduction always produces a planar graph, hence this theorem answers the original question about coloring maps.

Theorem 1.1 (Appel and Haken [1, 2]). *If G is a planar graph, then $\chi(G) \leq 4$.*

When certain structures do not appear in a planar graph G, it may be possible to construct a proper coloring of G using fewer than four colors. In particular, it was shown in [9] that a planar graph that does not contain any 3-cycles has chromatic number at most three. In [11] it was conjectured that a planar graph G without any 4-cycles or 5-cycles would also have $\chi(G) \leq 3$. A slightly stronger conjecture was put forth in [5], namely that the same result would be true if G had no 5-cycles and no adjacent 3-cycles. It was shown that if this conjecture were true, it would be best possible in the sense that planar graphs with chromatic number four exist which satisfy either (but not both) of the two conditions, as shown in Figure 1.1.

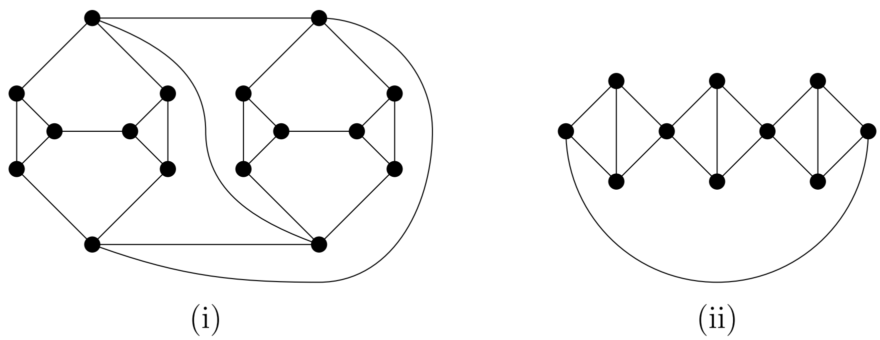

Figure 1.1: Two planar graphs with chromatic number four. The graph in (i) has no 4-cycles, while the graph in (ii) has no 5-cycles.

Let G be a graph and let $u, v \in V(G)$. We will let $dist_G(u, v)$ denote the distance between u and v in G, i.e. the length of the shortest path between the two vertices. We say the *square* of G, denoted G^2, is a new graph having the same vertex set as G, and where $uv \in E(G^2)$ if and only if $dist_G(u, v) \leq 2$. Finding a proper vertex coloring of G^2 is equivalent to finding a proper vertex coloring of G with the added condition that vertices at distance two from each other must receive distinct colors. It is readily apparent that any graph G with maximum degree Δ satisfies $\chi(G^2) \geq \Delta + 1$. This is because a Δ-vertex $u \in V(G)$ along with $N(u)$ will form a clique on $\Delta + 1$ vertices in G^2, thus all of these vertices must get different colors.

While much attention has been given to the problem of coloring planar graphs, it has only been recently that coloring the squares of these graphs has been seriously studied. One of the earliest instances of this was in [12], where the following conjecture was put forward.

Conjecture 1.2 (Wegner [12])**.** *Let G be a planar graph with maximum degree Δ. Then*

$$\chi(G^2) \leq \begin{cases} 7 & \textit{if } \Delta = 3 \\ \Delta + 5 & \textit{if } 4 \leq \Delta \leq 7 \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1 & \textit{if } \Delta \geq 8 \end{cases} .$$

The upper bounds given for $\Delta \geq 4$ would be sharp in that specific constructions were given where the bounds are attained. In particular, the general construction for $\Delta \geq 8$,

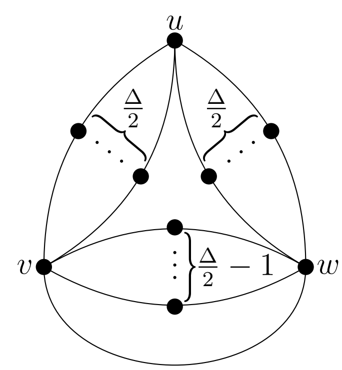

Figure 1.2: Wegner's construction for $\Delta \geq 8$, where Δ is even.

called a "fat triangle", is shown in Figure 1.2. In [10] it was shown that the upper bound conjectured by Wegner holds asymptotically, i.e. for a planar graph G with maximum degree Δ sufficiently high, $\chi(G^2) \leq \frac{3\Delta}{2}(1 + o(1))$. Even if we restrict to planar graphs without 3-cycles, the lower bound would not change substantially, since we could simply subdivide edge vw in the graph of Figure 1.2 to get a nearly identical fat triangle without any 3-cycles, which still requires $\frac{3\Delta}{2}$ colors to color its square. Hence in order to substantially lower the upper bound on $\chi(G^2)$, the given construction must be avoided, which in particular implies that 4-cycles should be forbidden.

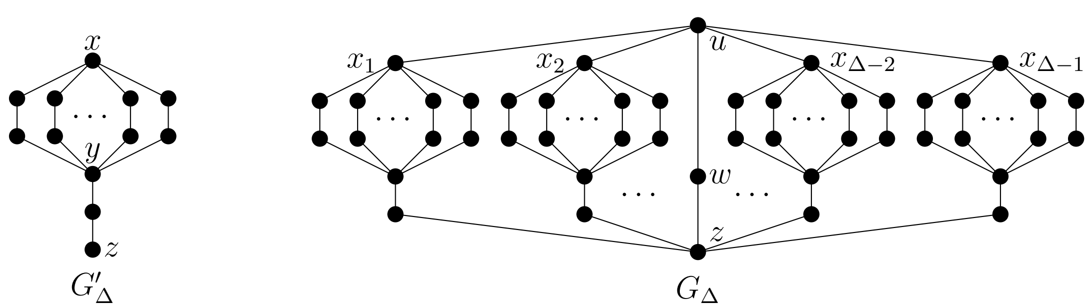

Figure 1.3: Constructions for a lower bound of $\Delta + 2$: in any $(\Delta + 1)$-coloring of the square of G'_Δ, the $(\Delta - 1)$-vertex x and the 1-vertex z cannot receive the same color. Because of this, no $(\Delta + 1)$-coloring of the square of G_Δ is possible, hence $\chi(G^2_\Delta) \geq \Delta + 2$.

The *girth* of a graph G is the length of the shortest cycle in G. In [3] it was shown that for a planar graph G of girth at least seven and with maximum degree $\Delta \geq 30$, $\chi(G^2) = \Delta + 1$. This provides not just an upper bound but true equality since $\Delta + 1$ is always a lower bound on the chromatic number of G^2. Later, in [8, 4], it was shown that if G is a planar graph of girth at least six and maximum degree $\Delta \geq 30$, then $\chi(G^2) \leq \Delta + 2$. This means that widening the class of graphs considered to include those with 6-cycles, we will only ever need at most one more color than the trivial lower bound. Furthermore, planar graphs with girth six and arbitrarily high maximum degree were constructed (see Figure 1.3) needing $\Delta + 2$ colors to color the square, meaning for at least some graphs, the given upper bound is sharp.

Chapter 2

Results

2.1 Statement and Supporting Lemmas

As seen in the previous chapter, when coloring the square of planar graphs, we can achieve an upper bound that is linear in Δ by only considering graphs that do not contain certain structures. To reiterate from before, when 3-cycles, 4-cycles, and 5-cycles are all forbidden, this upper bound is $\Delta + 2$. If we then widen the class of graphs under consideration, it can be expected that more colors may be needed to color the square of a given graph in the class. In [6], the class was expanded to include all planar graphs without any 4-cycles or 5-cycles (but where 3-cycles may be present). For a graph G in this class having maximum degree $\Delta \geq 9$, it was shown that $\chi(G^2) \leq \Delta + 5$. We improve on this upper bound, giving us the following result.

Main Theorem. *Let G be a planar graph with maximum degree $\Delta \geq 32$ that contains no 4-cycles and no 5-cycles. Then $\chi(G^2) \leq \Delta + 3$.*

To prove this, we use the Discharging Method, a powerful tool that has been used in graph theory for over 100 years, including in the original proof of the Four Color Theorem. Discharging is a form of counting argument used to prove various structural results about graphs. In a discharging argument, charge is assigned to elements in a graph (e.g. the vertices), and then is moved around (but never created or destroyed) according to some specially tailored rules. By assuming that certain structures or configurations do not exist in the graph, one can reach a contradiction to some global hypothesis, and therefore conclude that the graph must contain one of the given configurations.

Though the structural results of a discharging argument can stand on their own, they are often used to prove that all graphs in some family $\mathcal{G}$ have a property P. A configuration C is chosen that cannot exist in a minimal counterexample to this claim, i.e. if $G \in \mathcal{G}$ such that every proper subgraph of G has property P, and C appears in G, then G has property P as well. Such configurations are said to be *reducible* for the property at hand. Once reducible configurations have been found, discharging can be used to show that a minimal counterexample to a desired claim must contain a reducible configuration, and therefore a counterexample cannot exist, hence the claim must be true. An in-depth exploration of the discharging method is given in [7].

When thinking about coloring the square of a graph G, it is useful to consider the 2-neighborhood of a vertex. Let $u \in V(G)$, and let $N^2(u)$ denote the 2-neighborhood of u, i.e. the set of all vertices at distance at most two to u in G. When we are coloring the vertices of G and come to u, we must avoid using any color that has already been used in $N^2(u)$. Note that

$$|N^2(u)| \leq d_G(u) + \sum_{v \in N_G(u)} (d_G(v) - 1) = \sum_{v \in N_G(u)} d_G(v).$$

If $|N^2(u)| \leq \Delta + 2$, then even when all the vertices in $N^2(u)$ are colored before u, we can still assign u a color when we have $\Delta + 3$ total colors to choose from. More generally, if we can guarantee that at the time u gets colored, at most $\Delta + 2$ vertices in $N^2(u)$ have already been colored, then we can find a viable color for u out of $\Delta + 3$ total colors. This leads to the following reducibility lemma.

Basic Reducibility Lemma. *Let G be a graph with maximum degree Δ, and let $u \in V(G)$ such that $|N^2(u)| \leq \Delta+2$. If $(G-u)^2 \cong G^2-u$ and $\chi((G-u)^2) \leq \Delta+3$, then $\chi(G^2) \leq \Delta+3$.*

Proof. Fix a proper $(\Delta+3)$-coloring ϕ of $(G-u)^2$. Since $(G-u)^2 \cong G^2 - u$, it follows that whether or not two vertices are adjacent in G^2 is unaffected by the presence of u, hence we can extend ϕ to be a proper $(\Delta+3)$-coloring of G^2 by simply choosing a color for u. By assumption, there are at most $\Delta+2$ vertices in the 2-neighborhood of u, and thus at most $\Delta+2$ colors forbidden for u. Since there are $\Delta+3$ colors to choose from, at least one viable color remains, and the coloring can be extended. $\square$

This Lemma is most useful in the following specific cases, which are illustrated in Figure 2.1. Note that here and throughout the paper, a vertex that is drawn as a filled circle has all of its incident edges drawn, while a vertex that is drawn as an empty box may have other incident edges that are not shown.

Corollary. *Let G be a graph with maximum degree Δ. If $u \in V(G)$ is (i) a 1-vertex or (ii) a 2-vertex on a 3-cycle uv_1v_2 such that $d(v_1)+d(v_2) \leq \Delta+4$, and $\chi((G-u)^2) \leq \Delta+3$, then $\chi(G^2) \leq \Delta+3$.*

Proof. A 1-vertex has a 2-neighborhood of size at most Δ, and a 2-vertex as described has a 2-neighborhood of size at most $\Delta+2$. In each case, no edges in G^2 arise from two vertices being connected at distance 2 through u, so $(G-u)^2 \cong G^2 - u$. $\square$

Figure 2.1: Basic Reducibility cases: In (i) $|N^2(u)| \leq \Delta$, so u can always be colored, thus $\delta(G) \geq 2$ for a minimal counterexample. In (ii) $|N^2(u)| \leq \Delta+2$ if $d(v_1)+d(v_2) \leq \Delta+4$, making this configuration reducible.

This corollary implies that a minimal counterexample G to the Main Theorem must have $\delta(G) \geq 2$. We can extend the idea behind the Basic Reducibility Lemma to give another, even stronger reducibility lemma.

Main Reducibility Lemma. *Let G be a graph with maximum degree Δ such that for every proper subgraph H of G, $\chi(H^2) \leq \Delta + 3$. If there is a sequence $S = \{v_1, \ldots, v_n\}$ of distinct vertices in $V(G)$ such that $E(G[S]) \neq \emptyset$, and $|N^2(v_i) \setminus \{v_{i+1}, \ldots, v_n\}| \leq \Delta + 2$ for $1 \leq i \leq n$, then $\chi(G^2) \leq \Delta + 3$.*

Proof. Let $e \in E(G[S])$. Since $G - e$ is a proper subgraph of G, we can fix a proper $(\Delta+3)$-coloring ϕ of $(G-e)^2$. Note that since $e \in E(G[S])$, two vertices in $V(G) \setminus S$ are adjacent in G^2 if and only if they are adjacent in $(G-e)^2$. Now we can modify ϕ to be a proper $(\Delta + 3)$-coloring of G^2 by uncoloring the vertices in S, and then greedily recoloring them in their given order (i.e. starting with v_1 and ending with v_n). By assumption, each vertex in S will have no more than $\Delta + 2$ neighbors in its 2-neighborhood that have already been colored, hence it will have no more than this number of colors forbidden. Since there are $\Delta + 3$ colors to choose from, at least one viable color must remain, so the coloring can be extended. □

Whenever this lemma is invoked, we will always list the sequence S in the appropriate order. While this result holds in general, it will be most often used in the following form:

Corollary. *Let G be a graph with maximum degree Δ such that for every proper subgraph H of G, $\chi(H^2) \leq \Delta + 3$. If u and v are adjacent vertices in G such that $|N^2(u)| \leq \Delta + 3$ and $|N^2(v)| \leq \Delta + 2$, then $\chi(G^2) \leq \Delta + 3$.*

Proof. This corollary follows directly from the lemma where $S = \{u, v\}$. □

In section 2.2.2 we will often know that $|N^2(u)| \geq \Delta + k$ for some vertex u and integer k, and seek to show that u receives sufficient charge. The following lemma proves useful for this end.

Concavity Lemma. *Let $f(x) = 1 - \frac{4}{x}$, considered on some interval $[a, \infty)$ where $a > 0$. If $x_1, \ldots, x_n$ are to be chosen in $[a, \infty)$ such that $\sum_{i=1}^n x_i = C$ for some fixed constant C, then the minimum value of $\sum_{i=1}^n f(x_i)$ is achieved when $x_1 = \ldots = x_{n-1} = a$ and $x_n = C - a(n-1)$.*

Proof. It suffices to show that $f(x_1) + f(x_2) \geq f(a) + f(x_1 + x_2 - a)$ for all $x_1, x_2 \in [a, \infty)$, since we can then proceed by induction on the number of x_i that are not equal to a.

Assume without loss of generality that $x_1 \leq x_2$, and let $t = x_1 - a$. Since f is concave, its derivative is decreasing, and can be bounded at a point by left and right secants there, giving:

$$\frac{f(x_2 + t) - f(x_2)}{t} \leq f'(x_2) \leq f'(x_1) \leq \frac{f(x_1) - f(x_1 - t)}{t}.$$

Clearing denominators and rearranging terms gives $f(x_2 + t) + f(x_1 - t) \leq f(x_1) + f(x_2)$. But this is equivalent to $f(x_1 + x_2 - a) + f(a) \leq f(x_1) + f(x_2)$, as was desired. $\square$

We have stated the Concavity Lemma in terms of the function $f(x) = 1 - \frac{4}{x} = \frac{x-4}{x}$ since this is how we apply it hereafter. However, the same reasoning used above shows that the result holds for any concave, differentiable, strictly increasing function.

2.2 Proof of the Main Theorem via Discharging

As stated above, in order to prove the Main Theorem, we will employ the Discharging Method.

Proof of the Main Theorem. Assume that the Main Theorem is not true, and let G be a minimal counterexample. Since G is assumed to be a counterexample, it must be a planar graph with maximum degree $\Delta \geq 32$ that contains no 4-cycles and no 5-cycles such that $\chi(G^2) \geq \Delta + 4$. Since G is furthermore assumed to be a *minimal* counterexample, it must further be that for any proper subgraph H of G, $\chi(H^2) \leq \Delta + 3$. We can assume that G is connected, since otherwise we could color the components of G individually, violating the minimality of G. Now we fix a plane embedding of G, and assign initial charges to each vertex and face:

$$ch(x) = d(x) - 4 \text{ for every } x \in V(G) \cup F(G).$$

Euler's formula states that if G is a plane graph, then $|V(G)| - |E(G)| + |F(G)| = 2$. We can use this to calculate the sum of initial charges as follows:

$$\sum_{x \in V(G) \cup F(G)} ch(x) = \sum_{v \in V(G)} d(v) - 4 + \sum_{f \in F(G)} d(f) - 4$$

$$= (2|E(G)| - 4|V(G)|) + (2|E(G)| - 4|F(G)|) = -4(|V(G)| - |E(G)| + |F(G)|) = -8.$$

Now we redistribute charge via the four discharging rules outlined in section 2.2.1, giving a final charge function ch^*. Since G is a minimal counterexample, it must not contain any configurations that are reducible for being $(\Delta + 3)$-colorable. We use the absence of such configurations to show in section 2.2.2 that each face and vertex finishes with nonnegative final charge, giving the following contradiction:

$$-8 = \sum_{x \in V(G) \cup F(G)} ch(x) = \sum_{x \in V(G) \cup F(G)} ch^*(x) \geq 0.$$

Hence no such minimal counterexample can exist, thus the Main Theorem is true. $\square$

2.2.1 Discharging Rules

The following four discharging rules are applied to the elements of G successively, in phases. Examples of these rules are illustrated in Figure 2.2.

R1: Each 6^+-face gives charge $\frac{1}{3}$ to each incident edge. If such an edge e is incident to a 3-face f, then e gives this charge to f. Otherwise, e splits this charge evenly between any 3^--endpoints it has, or else splits it evenly between both endpoints if both have degree at least 4. [1]

[1] Edges only ever act as a charge carrier between faces and other faces or vertices. Outside of this phase, edges always have zero charge.

R2: Each 5^+-vertex v splits its initial charge evenly among its lower-degree neighbors if $d(v) < 10$, or among its lower-or-equal-degree neighbors if $d(v) \geq 10$. [2]

R3: Let u be a 4^+-vertex on a 3-face uvw and suppose u receives some charge c during R2 from v. If w is a 2-vertex, then u passes charge c on to w. If instead w is a 3-vertex with a 2-neighbor whose other neighbor has degree less than Δ, then u passes charge $\min\{c, \frac{1}{2}\}$ on to w. [3]

R4: If a 3^+-vertex has positive charge after R1-R3, it splits this charge among its neighbors with negative charge, such that a 3-vertex gives charge at most $\frac{4}{15}$ to another 3-vertex, and otherwise all charge splits evenly.

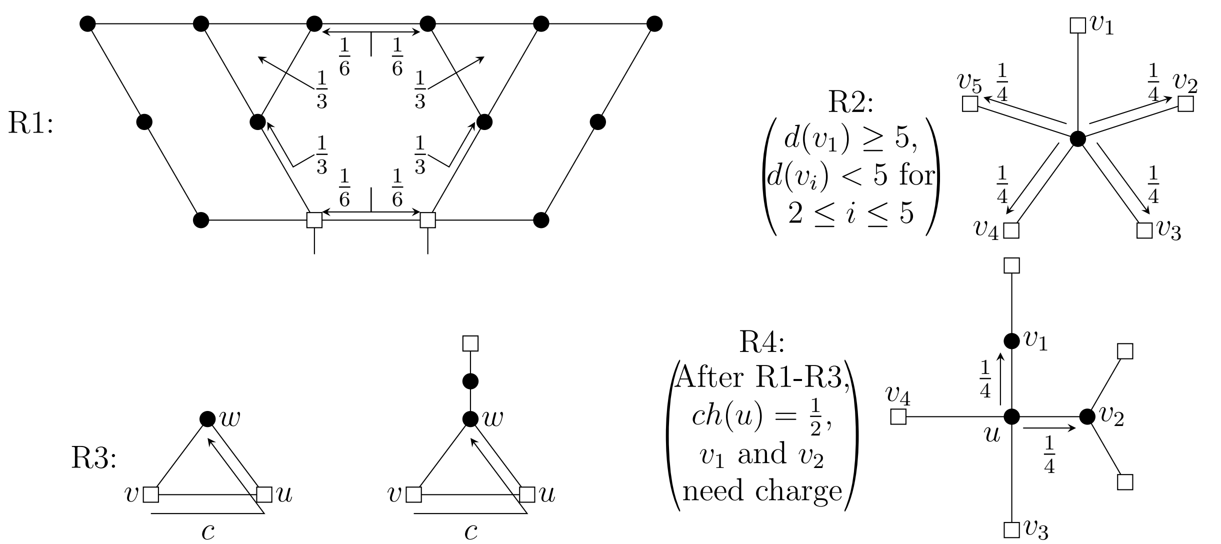

Figure 2.2: Illustrations of how charge is redistributed in the four discharging rules.

2.2.2 Discharging Analysis

As stated above, we now show that $ch^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$. It turns out that this is easy for everything except 3-vertices and 2-vertices, which require more detailed analysis.

[2] The distinction between lower-degree and lower-or-equal degree is only necessary to prevent a single problematic case from disrupting the analysis, and can for the most part be safely ignored.

[3] This rule does not frequently come into play; most charge passes via R1, R2, and R4.

Faces and High-Degree Vertices

First we show that all faces end with nonnegative final charge. Each 6^+-face f starts with charge $d(f)-4$ and gives away charge $\frac{d(f)}{3}$, thus f ends with $ch^*(f) = \frac{2d(f)}{3} - 4$, which is nonnegative since $d(f) \geq 6$. A 3-face cannot be adjacent to another 3-face since 4-cycles are forbidden. Since G has no 4-cycles or 5-cycles, each 3-face g must be adjacent to a 6^+-face on each of its edges. Each such 6^+-face passes charge $\frac{1}{3}$ to g via their common edge, so $ch^*(g) = 3 - 4 + 3(\frac{1}{3}) = 0$.

Each 4^+-vertex v starts out with nonnegative initial charge, and by the design of the discharging rules never gives away more than it currently has (i.e. its "positive balance"), so $ch^*(v) \geq 0$. Now we must verify that all 2-vertices and 3-vertices end with nonnegative final charge as well, which will complete the analysis.

3-vertices

First consider a 3-vertex u that is not incident to any 3-faces. The three faces meeting at u must all be 6^+-faces, and thus each will give total charge $\frac{2}{3}$ to two of the edges incident to u. Even if all of u's neighbors are 3^--vertices, u will receive at least half of this charge, and hence end with $ch^*(u) \geq 3 - 4 + 3(\frac{1}{3}) = 0$.

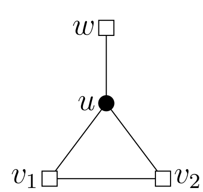

Figure 2.3: The 3-vertex u on a 3-face under consideration.

Now consider a 3-vertex u on a 3-face uv_1v_2 whose third neighbor is w, as shown in Figure 2.3. Note that since v_1 and v_2 are adjacent, $|N^2(u)| \leq d(w) + d(v_1) + d(v_2) - 2$. The two faces incident to u other than the 3-face must be 6^+-faces, and hence will give total charge $\frac{2}{3}$ to the edge uwvia R1. If $d(w) \geq 4$, then all of this charge will pass to u, while if $d(w) \leq 3$, then u will receive charge $\frac{1}{3}$ from this edge.

If $d(v_i) = 2$ for any i, then v_i is reducible under the Basic Reducibility Lemma. Alternately, if $d(v_i) \geq 12$ for some i, then u will receive charge at least $\frac{12-4}{12} = \frac{2}{3}$ from v_i via R2, and since uw sends u charge at least $\frac{1}{3}$ via R1, $ch^*(u) \geq 3 - 4 + \frac{2}{3} + \frac{1}{3} = 0$. Hence we can assume $3 \leq d(v_i) \leq 11$ for $i \in \{1, 2\}$. Also, if $d(v_1) + d(v_2) \geq 16$, then by the Concavity Lemma u receives at least as much charge as when one of the v_i's is a 12-vertex, which, as just shown, ensures that $ch^*(u) \geq 0$. Thus we will assume $d(v_1) + d(v_2) \leq 15$. Now we consider what happens to u based on the degree of w.

Case $d(w) \geq 6$: Here u receives charge at least $\frac{6-4}{6} = \frac{1}{3}$ from w via R2, as well as charge $\frac{2}{3}$ from uw, thus ends with $ch^*(u) \geq 3 - 4 + \frac{2}{3} + \frac{1}{3} = 0$.

Case $d(w) = 2$: Since $|N^2(u)| \leq d(v_1) + d(v_2) \leq 15$ and $|N^2(w)| \leq \Delta + 3$, when $\Delta \geq 13$ this configuration is reducible under the Main Reducibility Lemma.

Case $d(w) \in \{3, 4, 5\}$: We will show that u receives charge at least $\frac{1}{2}$ total from w and the edge uw, and at least $\frac{1}{4}$ from each of v_1 and v_2. This ensures that $ch^*(u) \geq 3 - 4 + \frac{1}{2} + 2(\frac{1}{4}) = 0$. First consider the charge from w and uw: if $d(w) \geq 4$, then as mentioned above, all $\frac{2}{3}$ of the charge that passes through uw will go to u, and $\frac{2}{3} > \frac{1}{2}$. Otherwise, if $d(w) = 3$, then u receives $\frac{1}{3}$ from uw, and so needs at least $\frac{1}{6}$ more from w for this total to reach $\frac{1}{2}$.

Let x_1 and x_2 denote the neighbors of w other than u. Since $\{u, w\}$ is not reducible, the Main Reducibility Lemma implies that $d(x_1) + d(x_2) \geq \Delta + 1$. Now the Concavity Lemma implies that w will have the least charge to give to u via R4 when $d(x_1) = \Delta - 1$ and $d(x_2) = 2$. If w does not lie on a 3-face, then it receives charge $3(\frac{1}{3})$ from its three incident edges via R1, making its charge nonnegative. Now the additional charge of $\frac{(\Delta-1)-4}{\Delta-1}$ from x will be split at most two ways. When $\Delta \geq 7$, this ensures that u gets an additional charge of at least $\frac{1}{6}$ from w.

Suppose instead that w does lie on a 3-face. Now we know that $d(x_2) \geq 3$, since a 2-vertex on a 3-face with a 3-neighbor is reducible according to the Basic Reducibility Lemma. Now if $d(x_2) \geq 4$, then x_2 always has nonnegative charge and thus never needs to receive charge. If $d(x_2) = 3$, then x_2 will receive charge at least $\frac{1}{3}$ from its incident edge not on the 3-face, and at least $\frac{2}{3}$ from x as long as $d(x) \geq 12$, meaning it will not need any charge from w. Thus, whatever the degree of x_2, vertex w will not have to give any charge to x_2 via R4. As long as $\Delta \geq 25$, this will ensure w gets charge $\frac{1}{3} + \frac{5}{6}$ via R1 and R2, and thus can give charge $\frac{1}{6}$ to u via R4. Hence we have shown that u always gets charge at least $\frac{1}{2}$ from w and the edge uw.

Now we show that u receives charge at least $\frac{1}{4}$ from v_1 and, by symmetry, also from v_2. If $d(v_1) \geq 6$, then v_1 gives charge at least $\frac{1}{3}$ to u via R2, and $\frac{1}{3} > \frac{1}{4}$. Otherwise assume $d(v_1) \leq 5$. Recall that $|N^2(u)| \leq d(w) + d(v_1) + d(v_2) - 2 \leq 18$. If $\{u, v_1\}$ is not reducible under the Main Reducibility Lemma, then $|N^2(v_1)| \geq \Delta + 4$, i.e. v_1 has at least one high-degree neighbor z. If $d(v_1) = 5$, then v_1 will split its charge at most four ways in R2, meaning it gives charge at least $\frac{1}{4}$ to u, as desired. If instead $d(v_1) \in \{3, 4\}$, then v_1 has no excess charge to give to u initially, but will be able to give the needed charge via R4. Note that by the same reasoning used above, since $\{u, v_2\}$ is not reducible under the Main Reducibility Lemma, v_2 must have a high-degree neighbor as well. This means that v_1 will never need to give charge to v_2 via R4: v_2 only ever needs to receive charge if it is a 3-vertex, and in such a case, it will receive all the charge it needs from its high-degree neighbor and incident edge off of the 3-face.

In the case that $d(v_1) = 3$, v_1's neighbor z not on the 3-face must have degree at least $\Delta - 8$. When $\Delta \geq 18$, this ensures that v_1 gets charge at least $\frac{3}{5} + \frac{2}{3}$ from z and the edge $v_1 z$. Thus v_1 will be able to pass charge at least $\frac{4}{15} > \frac{1}{4}$ to u.

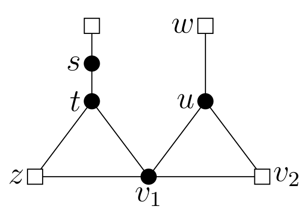

Figure 2.4: This configuration, where R3 would apply, is reducible by the Main Reducibility Lemma.

If instead $d(v_1) = 4$, then v_1 will have to split any excess charge it receives at most two ways via R4 (since neither z nor v_2 will need charge). Let t be v_1's neighbor other than u, v_2, and z, and note that v_1 will only send charge to t via R4 if $d(t) < 4$. By the Concavity Lemma, v_1 will receive the least charge when $d(z) = \Delta - 4$, $d(v_2) = 4$, and $d(t) = 3$. If v_1zt is not a 3-face, then v_1 will receive at least charge $\frac{(\Delta-4)-4}{\Delta-4} + \frac{1}{3}$ from z and the edge v_1z. When $\Delta \geq 9$, this lets v_1 get charge at least $\frac{8}{15}$, meaning it passes at least $\frac{4}{15} > \frac{1}{4}$ to u via R4.

If instead v_1zt is a 3-face, then we note that t cannot be a 2-vertex, since this would be reducible. Also, t cannot be a 3-vertex with a 2-neighbor s, where the other neighbor of s has degree less than Δ, because this also would be reducible under the Main Reducibility Lemma (using the vertex sequence $S = \{t, s, u\}$ and coloring t first), as shown in Figure 2.4. Since these are the only times when R3 can apply, we conclude that this rule is not used here. Hence v_1 gets charge at least $\frac{(\Delta-4)-4}{\Delta-4}$ from z, which it can then send at least half of to u. As long as $\Delta \geq 12$, this means v_1 sends at least $\frac{1}{4}$ to u as desired.

2-vertices

2-vertex on a 3-face: First consider a 2-vertex u on a 3-face uv_1v_2, as depicted in Figure 2.5. By the Basic Reducibility Lemma, this is reducible unless $d(v_1) + d(v_2) \geq \Delta + 5$. By the Concavity Lemma, we know that u receives at least as much charge as if $d(v_1) = \Delta$ and $d(v_2) = 5$. Now u will receive charge at least $\frac{\Delta-4}{\Delta} + \frac{1}{4}$ via R2. However, v_2 also receives charge $\frac{\Delta-4}{\Delta}$ from v_1 via R2, and the conditions are met for R3, so v_2 will pass this charge along to u. Hence in total u receives charge at least $2(\frac{\Delta-4}{\Delta}) + \frac{1}{4}$. When $\Delta \geq 32$, u will end with $ch^*(u) \geq 2 - 4 + 2(\frac{32-4}{32}) + \frac{1}{4} = 0$.

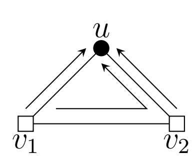

Figure 2.5: A 2-vertex on a 3-face receives charge via R2 and R3.

2-vertex with one high-degree neighbor: Now we will assume that the 2-vertex u, with neighbors v_1 and v_2, does not lie on a 3-face. Note that if $d(v_i) = 2$ for some $i \in \{1, 2\}$, then $\{u, v_i\}$ is reducible under the Main Reducibility Lemma. Hence we assume that $d(v_1) \geq 3$ and $d(v_2) \geq 3$. Suppose $d(v_1) \geq \Delta - 2$; now u receives charge $\frac{2}{3}$ through the edge uv_1 via R1 and $\frac{(\Delta-2)-4}{\Delta-2}$ from v_1 via R2. If $d(v_2) \geq 4$, then u also gets $\frac{2}{3}$ through the edge uv_2 via R1, and so ends with final charge at least $2 - 4 + 2(\frac{2}{3}) + \frac{(\Delta-2)-4}{\Delta-2}$, which is nonnegative when $\Delta \geq 14$.

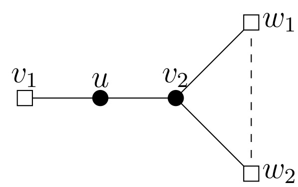

Figure 2.6: A 2-vertex u with a neighbor v_1 such that $d(v_1) \geq \Delta - 2$.

Otherwise, suppose $d(v_2) = 3$, where v_2's other neighbors are w_1 and w_2, as pictured in Figure 2.6. Note that v_2 and u each receive charge $\frac{1}{3}$ from the edge uv_2 via R1. Now $\{u, v_2\}$ is reducible under the Main Reducibility Lemma unless $|N^2(v_2)| \geq \Delta + 3$. Suppose that v_2 lies on a 3-face, which implies $d(w_1) + d(w_2) \geq \Delta + 3$. By the Concavity Lemma, v_2 receives at least as much charge as when $d(w_1) = \Delta - 1$ and $d(w_2) = 4$. Hence after R2, v_2 has charge at least $3 - 4 + \frac{1}{3} + \frac{(\Delta-1)-4}{\Delta-1}$. When $\Delta \geq 26$, this ensures that v_2 has charge at least $-1 + \frac{1}{3} + \frac{21}{25} > \frac{1}{6}$ after R2, which it passes to u via R4. (Note that w_2 does not receive charge from v_2 via R4: since $v_2w_1w_2$ is a 3-face, $d(w_2) > 2$. Further, if $d(w_2) = 3$, then w_2 will receive enough charge from w_1 and its incident edge off of the 3-face.) Hence $ch^*(u) \geq 2 - 4 + \frac{2}{3} + \frac{1}{3} + \frac{(26-2)-4}{26-2} + \frac{1}{6} = 0$.

So suppose instead that v_2 does not lie on a 3-face. Now $|N^2(v_2)| \geq \Delta + 3$, implying that $d(w_1) + d(w_2) \geq \Delta + 1$. Again using the Concavity Lemma, we can assume that $d(w_1) \geq \Delta - 3$. Now v_2 gets charge at least $\frac{1}{3}$ from each of the edges uv_2 and v_2w_2, and $\frac{2}{3}$ from the edge v_2w_1 via R1, which already puts its total charge at $3 - 4 + \frac{4}{3} = \frac{1}{3}$. Now v_2 splits its positive charge at most two ways (giving to u and possibly w_2) via R4. Since v_2

has charge at least $\frac{1}{3}$ after R1, it will give charge at least $\frac{1}{6}$ to u via R4. As shown above, when $\Delta \geq 26$ this ensures that $ch^*(u) \geq 0$, as desired.

Hereafter we assume that $d(v_1) \leq \Delta - 3$ and $d(v_2) \leq \Delta - 3$. We show that u must receive total charge at least 1 from edge uv_1 and vertex v_1; by symmetry the same is true of edge uv_2 and vertex v_2. This ensures that u ends with final charge at least $2 - 4 + 1 + 1 = 0$, as desired. If $d(v_1) \geq 6$, then u gets charge $\frac{2}{3}$ from uv_1 via R1 and charge $\frac{d(v_1)-4}{d(v_1)} \geq \frac{6-4}{6} = \frac{1}{3}$ from v_1 via R2. This gives u the charge of 1 from v_1's side as needed, so henceforth we assume $d(v_1) \leq 5$.

2-vertex with a 3-neighbor: Suppose $d(v_1) = 3$ where the other neighbors of v_1 are w_1 and w_2, such that $d(w_1) \geq d(w_2)$. Now u receives charge $\frac{1}{3}$ from the edge uv_1 via R1, meaning it needs to get $\frac{2}{3}$ from v_1 via R4. First suppose that v_1 does not lie on a 3-face. Since $d(v_2) \leq \Delta - 3$, we apply the Main Reducibility Lemma with $S = \{v_1, u\}$, unless $d(w_1) + d(w_2) \geq \Delta + 2$. Likewise, if $d(w_2) = 2$, then we simply take $S = \{v_1, w_2, u\}$.

Hence we assume $d(w_2) \geq 3$. If $d(w_2) \geq 4$, then v_1 receives charge $\frac{2}{3}$ from both of the edges v_1w_1 and v_1w_2, along with $\frac{1}{3}$ from the edge uv_1 via R1. This means that after R1 alone, v_1 will have charge $3-4+\frac{1}{3}+2(\frac{2}{3}) = \frac{2}{3}$, which it can then send to u via R4 as needed. So suppose instead $d(w_2) = 3$, which implies $d(w_1) \geq \Delta - 1$. Now v_1 gets charge at least $\frac{4}{3}$ via R1 ($\frac{1}{3}$ each from edges uv_1 and v_1w_2, and $\frac{2}{3}$ from edge v_1w_1) and $\frac{(\Delta-1)-4}{\Delta-1}$ from w_1 via R2. When $\Delta \geq 11$, this ensures that v_1 has charge at least $3 - 4 + \frac{4}{3} + \frac{(11-1)-4}{11-1} = \frac{14}{15}$ after R2. Since v_1 gives no more charge than $\frac{4}{15}$ to w_2 via R4, it can give at least $\frac{10}{15} = \frac{2}{3}$ to u via R4 as needed. So u gets charge at least 1 from v_1 and uv_1.

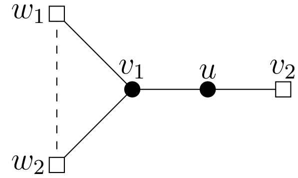

Figure 2.7: A 2-vertex u with a 3-neighbor v_1.

Now suppose instead that v_1 does lie on a 3-face. If we cannot apply the Main Reducibility Lemma with $S = \{v_1, u\}$, then $d(w_1) + d(w_2) \geq \Delta + 4$. By the Concavity Lemma, v_1 receives at least as much charge as when $d(w_1) = \Delta$ and $d(w_2) = 4$. Thus v_1 receives charge $\frac{1}{3}$ from edge uv_1 via R1, and further receives charge at least $\frac{\Delta-4}{\Delta}$ from w_1 via R2. Additionally, w_2 receives at least $\frac{\Delta-4}{\Delta}$ from w_1 via R2, and the criteria are met for R3; when $\Delta \geq 8$, this means w_2 will pass charge $\frac{1}{2}$ to v_1. Hence after R3, v_1 has charge at least $3-4+\frac{1}{3}+\frac{1}{2}+\frac{\Delta-4}{\Delta}$. When $\Delta \geq 24$, this means v_1 will have charge at least $-\frac{1}{6}+(\frac{24-4}{24}) = \frac{2}{3}$ that it can pass to u via R4 as needed.

2-vertex with a 4-neighbor: Now suppose $d(v_1) = 4$. In this case, u receives charge $\frac{2}{3}$ from edge uv_1 via R1, and hence only needs to get charge $\frac{1}{3}$ more from v_1 via R4. We can apply the Main Reducibility Lemma with $S = \{u, v_1\}$ unless $|N^2(v_1)| \geq \Delta + 4$, which means the degree sum of the neighbors of v_1 other than u is at least $\Delta + 2$. The least charge will pass from v_1 to u via R4 when v_1 has as many 3^--neighbors as possible, hence we will assume v_1 has two 3^--neighbors w_1 and w_2 and one high-degree neighbor z, as shown in Figure 2.8.

By the Concavity Lemma, v_1 receives at least as much charge via R2 as if $d(w_1) = d(w_2) = 3$ and $d(z) = \Delta - 4$. If v_1 and z do not lie on a common 3-face, then v_1 receives charge $\frac{1}{3}$ from edge v_1z via R1. When $\Delta \geq 16$, v_1 receives charge at least $\frac{(16-4)-4}{16-4} = \frac{2}{3}$ from z via R2, giving v_1 a total charge of at least 1 after R2. Since v_1 splits its charge at most three ways, it will pass charge at least $\frac{1}{3}$ to u via R4, as needed.

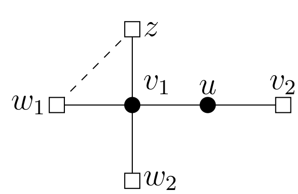

Figure 2.8: A 2-vertex u with a 4-neighbor v_1, where v_1 has a high-degree neighbor z.

Instead, assume v_1zw_1 is a 3-face. By the Basic Reducibility Lemma, we know w_1 cannot be a 2-vertex, so instead assume $d(w_1) = 3$, and let x be the third neighbor of w_1 besides v_1

and z. Now w_1 receives charge at least $\frac{1}{3}$ from edge w_1x via R1 and, since $\Delta \geq 16$, receives charge at least $\frac{(16-4)-4}{16-4} = \frac{2}{3}$ from z via R2. Hence w_1 has nonnegative charge after R2, and thus does not need charge from v_1 via R4, meaning v_1 only splits its positive charge at most two ways.

Now v_1 also receives charge at least $\frac{2}{3}$ from z via R1. If $d(x) = 2$ and the other neighbor of x has degree less than Δ, then the sequence $S = \{w_1, x, u\}$ is reducible under the Main Reducibility Lemma. If instead $d(x) \geq 3$, or $d(x) = 2$ and the other neighbor of x has degree Δ, then the conditions for R3 are not met, which means v_1 keeps its charge from z until R4. Splitting at most two ways, v_1 can give charge at least $\frac{1}{3}$ to u via R4, which is all u still required.

2-vertex with a 5-neighbor: Finally, suppose $d(v_1) = 5$. Similar to above, u will receive charge $\frac{2}{3}$ from edge uv_1 via R1. Also, we can apply the Main Reducibility Lemma with $S = \{u, v_1\}$ unless $|N^2(v_1)| \geq \Delta + 4$. Hence v_1 has at least one high-degree neighbor, and since it starts with initial charge $5 - 4 = 1$, it will pass charge $\frac{1}{4}$ to u via R2. Thus in order for u to receive charge at least 1 from v_1 and the edge uv_1, it only needs to get charge $\frac{1}{12}$ more from v_1 via R4.

Let z denote the highest-degree neighbor of v_1, and denote its other neighbors by w_1, w_2, and w_3. If v_1 and z are not together on a 3-face, then v_1 will receive charge $\frac{1}{3}$ from edge v_1z via R1, and will not lose this charge prior to R4. Thus in R4, v_1 has charge at least $\frac{1}{3}$ which it splits at most four ways, meaning it sends charge at least $\frac{1}{12}$ to u, as needed. Otherwise, assume that v_1zw_1 is a 3-face. Now since $|N^2(v_1)| \geq \Delta + 4$, we have $d(z) + d(w_1) + d(w_2) + d(w_3) \geq \Delta + 4$; by the Concavity Lemma, v_1 receives at least as much charge via R1 and R2 as if $d(z) = \Delta - 8$ and $d(w_1) = d(w_2) = d(w_3) = 4$.

Suppose $d(w_1) = 2$. This configuration is not immediately reducible under the Basic Reducibility Lemma or the Main Reducibility Lemma, but is in fact reducible using a hybrid of the two approaches. If we delete the vertex w_1 as in the Basic Reducibility Lemma, we can

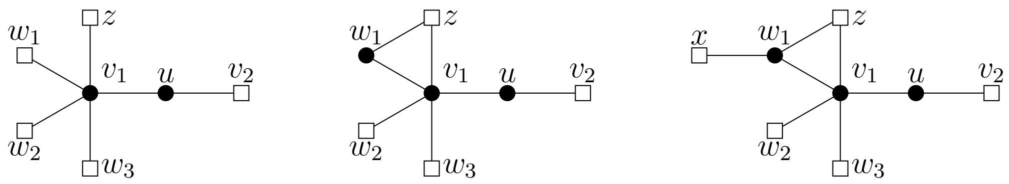

Figure 2.9: Cases where a 2-vertex u has a 5-neighbor v_1.

get a coloring of the square of the smaller graph. Now adding w_1 back in, we can uncolor u, leaving w_1 with at most $(\Delta+5-2)-1 = \Delta+2$ colored vertices in its 2-neighborhood. Thus we can choose a good color for w_1, and since $|N^2(u)| \leq d(v_1)+d(v_2) \leq (\Delta-3)+5 = \Delta+2$, we will always be able to choose a good color for u, hence this configuration is reducible.

Now assume $d(w_1) \geq 3$. If $d(w_1) \geq 4$ then whatever charge v_1 gets from z via R2 it keeps until R4. When $\Delta \geq 14$, this means v_1 receives charge at least $\frac{(14-8)-4}{14-8} = \frac{1}{3}$ in R2, and splits it at most three ways in R4, meaning it gives u charge at least $\frac{1}{9} > \frac{1}{12}$. So instead suppose $d(w_1) = 3$, and let x be the other neighbor of w_1. If the criteria for R3 are not met (i.e. $d(x) \geq 3$ or $d(x) = 2$ and the other neighbor of x has degree Δ), then v_1 keeps any charge it receives from z via R2 until R4. Thus as before, v_1 still gets charge at least $\frac{1}{3}$ since $\Delta \geq 14$, and splitting at most four ways will give charge $\frac{1}{12}$ to u via R4 as needed.

If instead $d(x) = 2$ and the other neighbor of x has degree at most $\Delta - 1$, then v_1 passes some charge that it gets from z via R2 to w_1 via R3. Since $\Delta \geq 24$, v_1 receives charge at least $\frac{(24-8)-4}{24-8} = \frac{3}{4}$ from z via R2. Now v_1 gives charge $\frac{1}{2}$ to w_1 via R3, leaving it with charge $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Since w_1 gets charge at least $\frac{1}{3}$ from the edge w_1x via R1, $\frac{3}{4}$ from z via R2, and $\frac{1}{2}$ from v_1 via R3, it has nonnegative charge, and thus needs no charge from v_1 via R4. Hence v_1 splits its remaining $\frac{1}{4}$ charge at most three ways, meaning it gives charge at least $\frac{1}{12}$ to u via R4 as needed.

2.3 Conclusion

In summary, we have shown that for every face $f \in F(G)$ we have $ch^*(f) \geq 0$, and for every vertex $v \in V(G)$ we have $ch^*(v) \geq 0$. This means that the final charges of all elements in the graph sum to a nonnegative value. Recall, however, that the initial charges summed to -8, and charge was only ever moved around, hence the initial charge sum and the final charge sum must be equal. This is a contradiction, and so we know that the minimal counterexample G must not exist, thus no counterexample can exist, and the Main Theorem is true.

An alternative way of understanding the result is the following: Suppose G is a planar graph containing no 4-cycles or 5-cycles such that $\Delta = \Delta(G) \geq 32$. Then the discharging argument shows that G must contain some reducible configuration, either from the Basic Reducibility Lemma, the Main Reducibility Lemma, or the hybrid configuration encountered at the end of section 2.2.2. Since these configurations are reducible, we know we can remove them and get a good $(\Delta + 3)$-coloring of the square of the smaller graph, and then extend this coloring to G^2 without using any additional colors, thus $\chi(G^2) \leq \Delta + 3$.

The way we showed that the coloring could be extended to each of our reducible configurations involved a simple argument about the number of forbidden colors at each step. That is, whenever we colored one of the final vertices v, we showed that at most $\Delta + 2$ vertices in the 2-neighborhood of v were already colored, and thus at most $\Delta + 2$ colors must not be used for v. Since there were $\Delta + 3$ total colors to work with, this guaranteed that the coloring could be extended. Since the reducibility arguments never addressed the actual colors being used, then the above approach and the main result can be extended to a stronger form of coloring called *list coloring*.

In list coloring, each vertex in a graph is given its own list L of possible colors, and the goal is to find a proper vertex coloring of the graph where each vertex uses a color from its own list. We say a graph is *k-choosable* if we can always find such a good coloring whenever the lists assigned to each vertex all have k colors. The *list chromatic number* of a graph G, denoted $\chi_\ell(G)$, is the least value k such that G is k-choosable. It is always the case that

$\chi(G) \leq \chi_\ell(G)$, since a valid list assignment is to give the same list to every vertex.

Now the above argument shows that that our reducible configuration work for G^2 to be $(\Delta + 3)$-choosable as well as $(\Delta + 3)$-colorable. Suppose every vertex starts with a list of $\Delta + 3$ allowable colors. Since we can build up the coloring from a smaller graph by choosing an ordering for the vertices so that there are at most $\Delta + 2$ forbidden colors at each step, then there will always be at least one good color remaining in the list of the vertex being colored, and the coloring can be extended. As before, the discharging argument used shows that each planar graph G of maximum degree $\Delta \geq 32$ that has no 4-cycles or 5-cycles must contain one of these reducible configurations, and so we can conclude that $\chi_\ell(G^2) \leq \Delta + 3$, which is a stronger result than the Main Theorem.

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Vita

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