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Surreal Numbers

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This is to certify that the thesis prepared by Joshua Daniel Hostetler titled "Surreal Numbers" has been approved by his or her committee as satisfactory completion of the thesis requirement for the degree of Master of Science.

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

SURREAL NUMBERS

By Joshua Daniel Hostetler, Master of Science.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

Virginia Commonwealth University, 2012.

Director: Richard Hammack, Associate Professor, Department of Mathematics and Applied Mathematics.

The purpose of this thesis is to explore the Surreal Numbers from an elementary, constructivist point of view, with the intention of introducing the numbers in a palatable way for a broad audience with minimal background in any specific mathematical field. Created from two recursive definitions, the Surreal Numbers form a class that contains a copy of the real numbers, transfinite ordinals, and infinitesimals, combinations of these, and infinitely many numbers uniquely Surreal. Together with two binary operations, the surreal numbers form a field. The existence of the Surreal Numbers is proven, and the class is constructed from nothing, starting with the integers and dyadic rationals, continuing into the transfinite ordinals and the remaining real numbers, and culminating with the infinitesimals and uniquely surreal numbers. Several key concepts are proven regarding the ordering and containment properties of the numbers. The concept of a surreal continuum is introduced and demonstrated. The binary operations are explored and demonstrated, and field properties are proven, using many methods, including transfinite induction.

Disclaimer

Unless otherwise cited, the main definitions and concepts in this thesis can be found and verified in a combination of the sources listed in the bibliography (though with notational differences). The theorems and proofs, however, are all the original work of the author, even in cases where a proof is very similar to one from the source material.

The above statement notwithstanding, this thesis has been a solo project of the author. No outside assistance was received from any source at any time regarding the contents or development of this thesis, aside from stylistic suggestions from his thesis committee.

Foreword

A personal note, explaining the journey behind and the scope of this thesis:

When I first started studying the Surreal Numbers (ages ago), I picked up John Conway's On Numbers and Games [1], (commonly abbreviated "ONAG") with its roughly 67 (small) pages dedicated to Numbers (and the rest dedicated to Games), and I thought I had it made. Little did I know that those 67 pages would be so dense with complex material that it would take me over a year to even begin to truly understand it. Conway does a wonderful job of making everything seem so easy, with his disarmingly casual prose and notation, that it is very easy to be lulled into a false sense of security—at least, that is how it was for me. I read the book and did many exercises and believed I knew what was going on.

So, I decided I would write an amazing thesis that explored every single aspect of the Surreal Numbers in great detail, and that it would be an easy thing to do.

But, when I started trying to write, I realized that, even after so much work and exploration, I could not explain a thing. There were many roadblocks in my way (all limitations of my own) keeping me from "getting it."

Some were simple and easy to overcome, like my initial inability to understand that a statement requiring arithmetic on an element of the empty set was discardable. (I could not seem to wrap my head around that idea for a long time, because I was tempted to label such a statement as undefined, thereby rendering whatever original statement had invoked the operation undefined as well… the algebra teacher in me was holding back the mathematician in me.)

Once the minor roadblocks were overcome, the major ones presented themselves. The most difficult concept for me was transfinite induction. Conway uses transfinite induction to prove just about everything in ONAG. Transfinite induction is an extremely powerful proof tool that quite often allows for one-line proofs, because, unlike traditional induction, it often allows the prover to completely throw out the idea of an initial base case (on the Surreal Numbers, anyway). That is why there are only 67 pages. The problem was, because of the nature of transfinite induction, I often found myself suspicious of it because I was highly skeptical about the arguments used in conjunction with it.

In short, I just did not trust transfinite induction. I understood the reason it was valid, (that every number is “built” out of 0, though that alone doesn’t technically justify the use of transfinite induction), but when it came down to analyzing the arguments in proofs that relied on it, I often simply couldn’t see that it could be justifiably invoked because of them.

This proved to be a problem of incredible magnitude for me because, as I said, Conway uses it constantly. As a result, I found myself obsessively trying to convince myself that the proofs in ONAG were, in fact, valid. Of course, I trusted Conway’s judgment, and I assumed he knew what he was talking about, but that didn’t help me to understand, which I desperately needed to do before I could even think about writing a thesis on the subject (in spite of advice given to me by some of my mentors).

After quite a long time spent (night and day, regardless of what else was going on around me) obsessing over the ideas in Conway’s book, I decided to see if there was any way around the use of transfinite induction to prove the same concepts. In many cases, I was successful (as you will see soon enough). In many other cases, I was not. Luckily for me (and for you, perhaps), I eventually discovered that in almost all of the cases where I could not find a way around transfinite induction (most of the theorems involving binary operations), there was a way to (relatively) quickly and clearly see that its use was justified.

I also referred to other source material for insight. Eventually, each source had something within it that was enlightening for me. At first, however, the other sources available were just as difficult for me to understand, either because of vastly different notation that seemed unrelated to Conway's numbers (everyone seems to have their own... even the notation I use within is unique, based on a combination of some of the other notations used in the source material), or because of a very high level of rigor and technicality, particularly involving set theory and topology, which were, at least at first, a bit over my head.

Then, the philosopher Alain Badiou published his book, Number and Numbers [2]. Badiou, as expected (based on all the other source material), has his own unique notation style. He also has a very unique perspective on the Surreal Numbers. Badiou talks of a number as a cut in a continuum (analogous to Theorem 4.7 within), each one defined by its matter, form, and residue. All the while, Badiou relates the Surreal Numbers to philosophy, politics, and sociology in interesting and enlightening ways.

This proved to be just the thing I needed to understand what I was dealing with, and I felt ready to get back into writing.

Because of my initial struggle to understand this material, I decided it would be best for me to limit the scope of this thesis. Instead of doing a thorough investigation of every aspect of the Surreal Numbers, as I originally thought I would do, I decided to stick to the basics, in hopes of helping other interested parties to understand the subject.

Thus, what follows is as elementary a discussion on the Surreal Numbers as I could manage. It does not require a great deal of specialized background in any particular field of mathematics. Instead, I have tried to explain the basic concepts as thoroughly as I can without turning readers away due to prerequisite knowledge requirements. Many times, this elementary approach is accomplished at the expense of brevity, but I believe it is worthwhile to sacrifice brevity in lieu of comprehension. When higher order concepts are needed, I explain them as simply and completely as I can, hopefully easing the reader in so that nobody feels left behind.

Because of this goal of simplicity, there will be much left out, particularly full discussions on higher-order arithmetic operations. Also because of this goal, there are some key elements I will ask the reader to accept on faith, referring the reader to other sources for proof. (For instance, I will not prove that any part of the Surreal Numbers is isomorphic to any part of the Real Numbers, though, after a while, the reader will be asked to accept that it is so. Proving the relevant isomorphisms would be far outside the scope I have set for this work, as it would involve either a large amount of background information on set theory and topology, or, in instances where there is a way around using fields that are not under discussion, very many cases to prove and many pages of proof. I hope the reader is willing to accept these things as true on faith or refer to source material in order to be convinced of their truth, as using the concepts will be considerably helpful in maximizing understanding.)

I hope that I have accomplished my goal of making the Surreal Numbers at least slightly more accessible to the average reader. The subject is fascinating and rich, and it would be a shame not to share it with as many people as possible.

-JH

Contents

Introduction

Building on and combining the works of Dedekind, Cantor, and Von Neumann, J. H. Conway constructed a class of numbers containing the real numbers, infinite ordinals, infinitesimals, and arithmetic combinations of these. This class of numbers was first exposed to the world by a friend of Conway's, Donald Knuth, in his book Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness [3], a short work of fiction in which a couple, stranded on an island, discovers a stone on which the basic definitions of the numbers, along with some of their properties and hints at proofs, are etched.

The term "Surreal Numbers" was coined by Knuth for this book, and became the name this class is commonly given, despite how little the numbers have to do with surrealism (they have absolutely nothing to do with surrealism, actually). Conway, in his book On Numbers and Games, states that he prefers to omit the adjective "surreal" and refer instead to **No**, "the class of all numbers." (To avoid confusion in this document, the term "Surreal Numbers" will be henceforth reserved mostly for references to the entire class, while the term "number" will mostly be used when referring to individual elements of the class. Symbolically, the class of Surreal Numbers will be denoted by **No**.)

This document will explain and explore the Surreal Numbers and their basic algebraic properties.

Basic Definitions and Conventions

The Surreal Numbers are constructed using a set of two recursive definitions:

DEFINITION 2.1. Definition of "number":

Let L, R be two sets of numbers such that for all $x_L \in L, x_R \in R, x_L \not\geq x_R$. Then $\{\, L \,|\, R \,\}$ is a number. All numbers are created in this way.

DEFINITION 2.2. Let x, y be two numbers. Then $x \geq y$ (equivalently, $y \leq x$) if and only if ($\forall x_R \in R, x_R \not\leq y$ and $\forall y_L \in L, x \not\leq y_L$).

From this point forward, the following notational conventions will be used:

If $x = \{\, L \,|\, R \,\}$ is a typical number, x_L will represent a typical element of its left set and x_R will represent a typical element of its right set. The left and right sets of x will often be denoted X_L and X_R, respectively, when necessary. However, more often than not, the phrase $\forall x_L$ will be used to imply $\forall x_L \in X_L$. The right set will be implied similarly by its elements.

In general, curly brackets will not surround a number's left and right sets. For example, a number will appear as $\{\, a,b,c \,|\, d,e,f \,\}$, rather than $\{\, \{\, a,b,c\} \,|\, \{d,e,f\,\}\,\}$. Nested curly brackets will be reserved for cases where expanding a number within L or R is necessary. Additionally, if either set of a number is empty, nothing will appear on the empty side. So, a number $\{\varnothing \,|\, R\,\}$ will instead be written $\{\,|\, R\,\}$. (To be clear, L and R are not elements of $x = |\{L|R\}$ themselves. Instead, the elements of L are the elements of the left set of X, and the elements of R are the elements of the right set of X. For instance, if $L = \{1,2,3\}$, $R = \{4,5,6\}$, and $x = \{L|R\}$, then the elements of the left set of x are $1,2$, and 3, and the elements of the right set of x are $4,5$, and 6.)

The term "number" will generally be used to describe a single element of the Surreal Numbers, and in any case where a different number system might be in use, the system will be specified. The term "Surreal Numbers" will be used most often when referring to the entire class of Surreal Numbers. Symbolically, the class of Surreal Numbers will be signified by **No**, in keeping with the tradition of Conway (and others before him, for that matter, when referring to the class of ordinals).

Conway adopts the seemingly strange convention of using the terms "no" and "some" directly within expressions, as a shorthand denotation of nonexistence or existence of an element within satisfying the expression. For example, "no $x_L > x_R$" would mean "there does not exist an $x_L \in X_L$ and $x_R \in X_R$ such that $x_L > x_R$". This convention is surprisingly handy, and I will therefore adopt it as well.

Using these conventions, the definitions of numbers given above could be rewritten in the following way:

DEFINITION 2.1 Let L, R be two sets of numbers such that no $x_L \geq x_R$. Then $\{L|R\}$ is a number. All numbers are created in this way.

DEFINITION 2.2 Let x, y be two numbers. Then $x \geq y$ (equivalently $y \leq x$) and only if (no $x_R \leq y$ and $x \leq$ no y_L).

Some other basic definitions are needed before beginning to construct the surreal numbers. These will be discussed and explored more fully in later chapters.

DEFINITION 2.3. The statement $x = y$ will be taken to mean $x \geq y$ and $x \leq y$.

DEFINITION 2.4. The statement $x < y$ will be taken to mean $x \leq y$ and $x \ngeq y$.

DEFINITION 2.5. The statement $x > y$ will be taken to mean $y < x$.

DEFINITION 2.6. If $x = \{ x_L \mid x_R \}$, then $-x$ is defined as $\{ -x_R \mid -x_L \}$.

DEFINITION 2.7. If the left and right sets of x and y are identical, then x and y are said to be identical, rather than equal. This is denoted $x \equiv y$. (This is a distinction that Conway makes, but one which we will not often need.)

The basic definitions for the binary operations on surreal numbers follow. Although these may seem nonsensical at this point, they will be needed during the process of constructing the Surreal Numbers, and will be fully explored in a later chapter.

DEFINITION 2.8. Addition of surreal numbers is defined by

$$x + y = \{ x_L + y, x + y_L \mid x_R + y, x + y_R \}.$$

DEFINITION 2.9. Subtraction of surreal numbers is defined by

$$x - y = x + (-y).$$

DEFINITION 2.10. Multiplication of surreal numbers is defined by

$$xy = \left\{ x_L y + x y_L - x_L y_L,\ x_R y + x y_R - x_R y_R \mid x_L y + x y_R - x_L y_R,\ x_R y + x y_L - x_R y_L \right\}.$$

The usual alternative representations $x \cdot y$ and $x(y)$ will be used in place of xy on occasion, and where appropriate.

(Note: Division will not be defined here. Instead, it will be discussed in Chapter 9. Also note that the properties of addition and multiplication will be discussed in chapters 8 and 9, respectively. However, in examples prior to those chapters, some of those properties, particularly commutativity and associativity, will be assumed in order that the examples may go smoothly.)

Existence and Beginning Construction of the Surreal Numbers

At first glance, the definition of number given here, by referring to other numbers, may seem (as it first did to me) to presuppose and assert its own validity. Pondering the matter reveals no such assertion is at play after all. In fact, questioning the validity of the definition immediately begins the construction of the class.

If we assume that the definition is invalid, then there are no numbers, and thus a collection of numbers would necessarily have nothing in it, that is, the collection would be the empty set, $\varnothing$.

Using $\varnothing$ as the sole candidate for the left and right sets of $\{L|R\}$ gives us $\{\,|\,\}$. By virtue of having no elements in either set to compare, $\{\,|\,\}$ vacuously satisfies the comparison used in the definition (regardless of how such a comparison is done), and thus is itself a number, contradicting the assertion that the definition is invalid.

If we had assumed the existence of numbers defined in this way in the first place, we could still use $\varnothing$ as the left and right sets of a number-candidate, and verify the result as a number.

Conway names this number 0, that is, $0 = \{\,|\,\}$. This choice of names turns out to be an appropriate one, which will be demonstrated later.

Now that we have 0 defined, we can begin constructing other numbers. With 0 being the only candidate for a set element, we have a few choices for new numbers: $\{0|\,\}, \{\,|0\}$, and $\{0|0\}$.

The first two candidates above fit the definition of number. Since there are no elements in the right set of $\{0|\}$ to compare to the 0 in the left set, the comparison required in the definition of number is vacuously fulfilled. By similar argument, $\{\,|0\}$ is also a number.

With $\{0\,|\,0\}$, however, it has to be established whether or not $0 \geq 0$. It may seem obvious that $0 = 0$, but nothing can be taken for granted at this point.

Now, $0 = 0$ unless some $0_L \geq 0_R$ or some $0_R \geq 0_L$ Since there are no elements in either set to compare, this is satisfied vacuously, and so, as expected, $0 = 0$. Similarly, the lack of elements in the left and right sets of 0 means that $-0 = 0$ as well, (a fact that we will need very soon).

Since $0 = 0$, the expression $\{0\,|0\}$ does not fit the definition of number. This leaves us with just the two new numbers, $\{0|\}$, and $\{\,|0\}$, which we will name 1 and -1, respectively, for reasons that will become clear later.

In the next chapter, the construction process will be interrupted in order to establish some basic information about numbers in general, including some facts about the ordering of the numbers. For now, though, since these facts have not been established, and since nothing can be taken for granted, it seems a good idea to prove a few facts about the numbers we have so far. If nothing else, these exercises will serve as some concrete examples of the behavior of the numbers.

PROPOSITION 3.1. $-1 = -(1)$

Proof. Recall the definition of negation, $-x = \{-x_R\,|-x_L\}$.

Since $-(1) = -\{0|\} = \{\,|-0\}$,

and since it has already been established that $-0 = 0$,

$-(1) = \{\,|\,0\} = -1.$ □

PROPOSITION 3.2. $0 < 1$

Proof. This is true if $0 \leq 1$ and $0 \not\geq 1$.

By definition, $0 \leq 1$ unless some $0_L \geq 1$ or $0 \geq$ some 1_R.

Since $0_L = \varnothing$, the former is ruled out.

Likewise, since $1_R = \varnothing$, the latter is false as well.

Thus, $0 \leq 1$.

By definition, $0 \not\geq 1$ since $0 \in 1_L$ and $0 \leq 0$.

Therefore, $0 < 1$. □

The statements $-1 < 0$ and $-1 < 1$ can be proven using similar arguments.

We should test out arithmetic on these numbers before moving on. This will serve to show that the numbers and operations work like their real counterparts. It will also give us a basis for inductive proofs in later chapters. Please note when viewing these examples that the operations are assumed to be commutative, as they were designed to be (which can be verified by looking at the structure of the definitions). The properties of the operations will be explored and proven in a later chapter. Note also that if a term refers to the empty set, then it will be eliminated since there is nothing to operate on within the empty set.

EXAMPLE 3.3.

$$\begin{aligned}
0+0 &= \{0_L+0,\, 0+0_L \,|\, 0_R+0,\, 0+0_R\} \\
&= \{0_L+0 \,|\, 0_R+0\} \\
&= \{\,|\,\} \\
&= 0
\end{aligned}$$

EXAMPLE 3.4.

$$\begin{aligned} 0+1 &= \{0_L+1, 0+1_L \,|\, 0_R+1, 0+1_R\} \\ &= \{0+0 \,|\, \} \\ &= \{0 \,|\, \} \\ &= 1 \end{aligned}$$

EXAMPLE 3.5.

$$\begin{aligned} 0-1 &= \{0_L+(-1), 0+(-1)_L \,|\, 0_R+(-1), 0+(-1)_R\} \\ &= \{0-1_R \,|\, 0-1_L\} \\ &= \{\,|\, 0-0\} \\ &= \{\,|\, 0\} \\ &= -1 \end{aligned}$$

EXAMPLE 3.6.

$$\begin{aligned} 1-1 &= \{1_L+(-1), 1+(-1)_L \,|\, 1_R+(-1), 1+(-1)_R\} \\ &= \{0+(-1), 1-1_R \,|\, 1-1_L\} \\ &= \{-1 \,|\, 1-0\} \\ &= \{-1 \,|\, 1\} \\ &= 0 \end{aligned}$$

(Note: The last step in this example may seem to be a leap in logic, but it will be proven true in Theorem 4.7.)

EXAMPLE 3.7.

$$0\cdot 0=\left\{0_L\cdot 0+0\cdot 0_L-0_L\cdot 0_L, 0_R\cdot 0+0\cdot 0_R-0_R\cdot 0_R \,|\, 0_L\cdot 0+0\cdot 0_R-0_L\cdot 0_R, 0_R\cdot 0+0\cdot 0_L-0_R\cdot 0_L\right\}$$

$$=\{\,|\,\}$$

$$=0$$

Notice that because the left and right sets of 0 are both empty, every term in the product $0\cdot 0$ is eliminated. By the definition of multiplication, this should hold true for multiplying any number by 0. This property will be discussed again in chapter 9, but was worth mentioning now. The following example should drive the point home.

EXAMPLE 3.8.

$$0\cdot 1=\left\{0_L\cdot 1+0\cdot 1_L-0_L\cdot 1_L, 0_R\cdot 0+1\cdot 0_R-1_R\cdot 1_R \,|\, 0_L\cdot 1+0\cdot 1_R-0_L\cdot 1_R, 0_R\cdot 1+0\cdot 1_L-0_R\cdot 1_L\right\}$$

$$=\{\,|\,\}$$

$$=0$$

Now that we have (almost) exhaustively explored the numbers we have created so far, it is time to move on. Rather than jumping straight into constructing the rest of the numbers, the next chapter has been included to help the reader understand the numbers in general. This should help in understanding what number goes where, and why, when the construction continues.

Some General Properties of Surreal Numbers and Their Order

At this point, we will take a break from construction of the numbers in order to explore some of their basic properties. This is in an effort to allow a general understanding of the ordering of the Surreal Numbers, to help create and maintain a sense of logic during the construction process, especially in regards to the naming of the numbers.

Two essential facts will be established in this chapter: that each number is strictly greater than any given member of its left set and strictly less than any member of its right set, and that the ordering relations we have defined are transitive. There are some interesting consequences arising directly from these two facts, and some of those will be explored here as well.

(Note: Some preliminary work is needed for most of the proofs in the chapter to work. The first few theorems proven here were the key components to my personal understanding of the Surreal Numbers, and arose out of my own early limitations and skepticism regarding transfinite induction.)

THEOREM 4.1. Let $a \ngeq b$. Then $a = \{a_L \,|\, a_R, b\}$.

Proof. Since $a \ngeq b$, we know that some $b_L \geq a$ or $b \geq$ some a_R.

Let $x = \{a_L \,|\, a_R, b\}$.

By Definition (2.3), $x = a$ if $x \leq a$ and $x \geq a$.

By Definition (2.2), $x \leq a$ unless some $x_L \geq a$ or $x \geq$ some a_R.

Assume $x_L \geq a$ for some fixed x_L. Then no $a_L \geq x_L$. But $A_L = X_L$. This is a contradiction.

Thus, no $x_L \geq a$.

Assume $x \geq a_R$ for some fixed a_R. Then $a_R \geq$ no x_R. But $a_R \in X_R$. This is a contradiction.

Thus, $x \geq$ no a_R.

Since no $x_L \geq a$ and $x \geq$ no a_R, it follows that $x \leq a$.

By definition, $x \geq a$ unless some $a_L \geq x$ or $a \geq$ some x_R.

Assume $a_L \geq x$ for some fixed a_L. Then no $x_L \geq a_L$. But $X_L = A_L$. This is a contradiction.

Thus, no $a_L \geq x$.

Assume $a \geq$ some fixed x_R.

Then $a \geq b$, which is false, or $a \geq$ some a_R.

By definition, $a \geq a_R$ unless $a_R \geq a_R$, which is clearly true,

Thus, $a \geq$ no x_R.

Since no $a_L \geq x$ and $a \geq$ no x_R, it follows that $x \geq a$.

Since $x \leq a$ and $x \geq a$, we get $x = a$. Therefore $a = \{a_L \mid a_R, b\}$. □

THEOREM 4.2. Let $a \ngeq b$. Then $b = \{a, b_L \mid b_R\}$.

Proof. We are given $a \ngeq b$, so, by Definition (2.2), some $b_L \geq a$ or $b \geq$ some a_R.

Let $y = \{a, b_L \mid b_R\}$.

By definition, $y = b$ if $y \leq b$ and $y \geq b$.

By definition, $y \leq b$ unless some $y_L \geq b$ or $y \geq$ some b_R.

By definition, $y_L \geq b$ if $a \geq b$ or $b_L \geq b$.

We know $a \ngeq b$, so $y_L \geq b$ only if $b_L \geq b$, which would mean no $b_L \geq b_L$.

This is a contradiction.

Thus, no $y_L \geq b$.

By definition, $y \geq b_R$ unless $b_R \geq y_R$. But $Y_R = B_R$.

This is also a contradiction.

Thus $y \geq$ no b_R.

Since no $y_L \geq b$ and $y \geq$ no b_R, it follows that $y \leq b$.

By definition, $y \geq b$ unless some $b_L \geq y$ or $b \geq$ some y_R.

By definition, $b_L \geq y$ unless some $y_L \geq b_L$. But $B_L \subseteq Y_L$.

Thus, no $b_L \geq y$.

By definition, $b \geq y_R$ unless $y_R \geq$ some b_R. But $Y_R = B_R$.

Thus, $b \geq$ no y_R.

Therefore, $y \geq b$.

Since $y \leq b$ and $y \geq b$, we get $y = b$. Therefore $b = \{a, b_L \mid b_R\}$. □

These two theorems give us equivalent forms of the number b. In fact, there are many equivalent forms, as you can see by imagining using the above theorems to add members to the left and right sets at will. Because of this fact, the phrase "Let $x \in$ **No**," by virtue of not giving a specified form of x, allows us to jump between forms of x without consequence. Henceforth, then, if no particular form of a given number is specified, it is safe to assume any equivalent form of that number.

Using that concept leads us to the following:

THEOREM 4.3. If $a < b$ then $a \in B_L$.

Proof. By definition 2.4, $a < b$ implies $a \ngeq b$, and thus, by Theorem 4.2, $b = \{a, b_L \mid b_R\}$.

Since no particular form of b was specified as given, we can take $\{a, b_L\} = B_L$ in our given b. □

THEOREM 4.4. If $a > b$ then $a \in B_R$

Proof. By definition 2.5, $a > b$ implies $a \not\leq b$, and thus, by Theorem 4.1, $b = \{b_L \,|\, b_R, a\}$.

Since no particular form of b was specified as given, we can take $\{b_R, a\} = B_L$ in our given b. □

Theorems 4.1 and 4.2 lead directly to the first of the essential facts about numbers and their order: that a number is strictly greater than any member of its left set, and strictly less than any member of its right set.

THEOREM 4.5. If $x = \{x_L \,|\, x_R\}$, then, for all $x_L \in L$, $x_L < x$.

Proof. Assume not. Then, by negation of Definition (2.4), some $x_L \not\leq x$ or $x_L \geq x$.

Assuming the first choice, fix $a \in X_L$ such that $a \not\leq x$.

By Theorem (4.1), $x = \{x_L \,|\, x_R, a\}$.

But, then $a \in X_L$ and $a \in X_R$, but this contradicts the definition of number, so $x_L \not\leq x$.

Thus, our assumption can only be true if $x_L \geq x$.

By Definition (2.2), $x_L \geq x$ unless some $x_L \geq x_L$, which is obviously true, so $x_L \not\geq x$.

Thus, we have $x_L \leq x$ and $x_L \not\geq x$, and therefore, $x_L < x$. □

THEOREM 4.6. If $x = \{x_L \,|\, x_R\}$, then $x < x_R$.

Proof. Assume not. Then, by negation of Definition (2.4), $x \not\leq$ some x_R or $x \geq$ some x_R.

Assuming the first choice, fix $b \in X_R$ such that $x \not\leq b$.

By Theorem 4.2, $x = \{b, x_L \,|\, x_R\}$.

But, then $b \in X_L$ and $b \in X_R$, contradicting the definition of number.

Thus, our assumption can only be true if $x \leq x_R$.

By Definition (2.2), $x \geq x_R$ unless some $x_R \geq x_R$, which is obviously true, so $x \not\geq x_R$.

Thus, we have $x \leq x_R$ and $x \not\geq x_R$, and therefore, $x < x_R$. □

These theorems have some important immediate consequences (too important to be deemed lemmas):

THEOREM 4.7. Let $x \in$ **No**, and let $A_< = \{a \in \textbf{No} : a < x\}$, and let $A_> = \{a \in \textbf{No} : a > x\}$. Then $x = \{A_< | A_>\}$.

Proof. By Theorem 4.3, $a < x$ implies $a \in X_L$. So, $A_< \subseteq X_L$.

By Theorem 4.5, $x_L < x$, so $x_L \in A_L$, and so $X_L \subseteq A_<$.

Thus, we have $A_< = X_L$.

Similarly, by Theorem 4.4, $a > x$ implies $a \in X_R$. So, $A_> \subseteq X_R$.

By Theorem 4.6, $x_R > x$, so $x_R \in A_L$, and so $X_R \subseteq A_<$.

Thus, we have $A_> = X_R$.

Therefore, $\{X_L | X_R\} = \{A_< | A_>\}$. □

Notice that this theorem sets up the concept that every number is essentially a cut in a continuum of numbers. (Thereby, this form of a number will henceforth be referred to as the "cut form" of the number.) This is an important concept in the understanding of the Surreal Numbers. (It is in fact the main idea Badiou uses in his book when talking about numbers.) As important as this is, describing a number in this way, while a perfectly fine thing to do, tells us very little about the value of a number. Each number has many different representations, as we've already seen. As we construct the rest of the numbers in the following chapters, we will develop a canonical form that will define each number using left and right sets that are basically stripped down to the bare minimum needed for defining the number.

There are a few more essential consequences of the previous theorems regarding properties of the ordering relations.

THEOREM 4.8. If $x \ngeq y$, then $x < y$.

Proof. Let $x \ngeq y$.

By Theorem 4.2, $x \in Y_L$.

By Theorem 4.5, $y_L < y$.

Therefore, $x < y$ □

THEOREM 4.9. (Transitivity of $<$)

If $a < b$ and $b < c$, then $a < c$.

Proof. Let $b < c$ and let $a < b$

By theorem 4.3, $a < b$ implies $a \in B_L$.

By definitions 2.2 and 2.4, $b < c$ implies no $b_L \geq c$

Thus $a \ngeq c$.

Therefore, by theorem 4.8, $a < c$.

□

THEOREM 4.10. (Transitivity of $\leq$)

If $a \leq b$ and $b \leq c$, then $a \leq c$.

Proof. Let $b \leq c$ and let $a \leq b$

By definition, $a \leq b$ implies no $a_L \geq b$, and so for all a_L, $a_L < b$, and so, by Theorem 4.3, $A_L \subseteq B_L$.

By definition, $b \leq c$ implies no $b_L \geq c$.

Thus, no $a_L \geq c$.

By definition, $b \leq c$ implies $b \geq$ no c_R, and so for all c_R, $b \ngeq c_R$, and so $C_R \subseteq B_R$.

Since $a \leq b$, we have $a \geq$ no b_R, and so $a \geq$ no c_R

So, we have no $a_L \geq c$ and $a \geq$ no c_R.

Therefore, $a \leq c$. □

A relation $\leq$ is a total order on a set (or class, in the case of **No**) if it exhibits antisymmetry, transitivity, and totality. Antisymmetry of $\leq$ over the Surreal Numbers is defined by: for all $a, b \in \mathbf{No}$, $a \leq b$ or $b \leq a$. Transitivity is defined by: if $a \leq b$ and $b \leq c$, then $a \leq c$. Totality is defined by: if $a \nleq b$ then $b \leq a$.

Notice that, by definition of $=$, $\leq$ is antisymmetric, and that we have just proven $\leq$ is transitive (by theorem 4.10) and total (by theorem 4.8). Thus, $\leq$ is a total order on the Surreal Number continuum.

Hopefully, this chapter has provided perspective on the placement of the numbers we are about to construct and how they relate to one another. This will greatly reduce the quantity of number candidates we need to consider when constructing, because we already will know at a glance which choices for left and right sets will not work and which will lead to numbers equivalent to already constructed numbers.

The Birthday Function, Integers, and Dyadic Numbers

During the construction process we have already started, each non-zero number can be constructed using as the elements of its left and right sets only previously constructed numbers. Whatever positive integer is created during a given iteration of the construction process is said to be the "birthday" of all the numbers created at that step of construction, and the iterations themselves are called "days". The numbers constructed prior to a number constructed on a given day are alternately referred to as "older," "simpler," or "more primitive" than the numbers created on the given day, depending on the context. These three terms can be taken as synonymous.

The number 0, recall, was created using only $\varnothing$ as its left and right sets. It is said to have been "born on day 0". The numbers 1 and -1 could only be constructed after day 0, and only had $\{0\}$ and $\varnothing$ to use as left and right sets. These are the only two numbers with birthday 1. The number 0, consequently, is considered "older" than 1 and -1.

A number's birthday gives a measurement of its primitivity and the "depth" of the embedded sets it contains. The number 0, as constructed, contains no sets of numbers, and so it has 0 depth. The number 1, on the other hand, contains 0 in its left set, which gives it depth 1 (i.e. $1 = \{0|\} = \{\{|\}\,|\,\}$, so there is only one level of embedded set in 1). The number 0, consequently, is considered "simpler" and "more primitive" than 1 and -1.

Another way of looking at this concept is that all numbers are built from other numbers, and so on, until eventually they are built from 0. (i.e $0 \subset 1 \subset 2 \ldots$). The number of steps in the chain between 0 and the number in question, gives you its birthday. (i.e. $0 \subset -1$, so, by virtue of there only being one link in this chain, -1's birthday is 1.)

We shall define the birthday function, $\mathfrak{B}$, as taking a number as its input and returning that number's birthday as its output.

DEFINITION 5.1. Let $\mathfrak{B}$:**No**$\rightarrow$**No** be defined by the following:

If $x \in$ **No** was born on day n, then $\mathfrak{B}(x) = n$.

The idea of birthdays is an important one, because it gives us a tool to use in defining a number's value. A given number's "value" is the name assigned the simplest equivalent form of that number. As construction of the numbers continues, we will be using only numbers born on days before the one we are working in, and the names assigned these numbers will have a certain pattern (which will be explained shortly). Furthermore, as we begin exploring arithmetic, especially when doing examples, it will be absolutely essential to be able to recall this pattern in order to work with the simplest form of each number. Some forms would prove cumbersome or even impossible to work with in conjunction with the definitions given for the arithmetic operations, so using the most streamlined version of the numbers available will become necessary. (As you will see after doing a few examples of this type, even this simplest form will lead to prohibitive amounts of work, and so we will have to employ shortcuts eventually to get around this.)

Furthermore, because each number is a cut in the Surreal Continuum, as justified by Theorems 4.7, each number $x = \{L|R\}$ is equivalent to the simplest number such that $x_L < x < x_R$. That is, given any form of a number $x = \{L|R\}$, the value of x is the name of $x' \in$ **No** such that ($\forall x_L \in L, x_R \in R$, $x_L < x' < x_R$ and for all $y \in$ **No** such that $x_L < y < x_R$ with $y \neq x'$, $\mathfrak{B}(x') < \mathfrak{B}(y)$). (For example, $\{-1\,|1\} = 0$, because there is no $x \neq 0$ such that $-1 < x < 1$ and $\mathfrak{B}(x) \leq \mathfrak{B}(0)$. In other words, $\{-1\,|1\} = 0$, because 0 is the most primitive number between -1 and 1; it has the lowest birthday of any number in that interval.)

Having introduced the concept of birthdays, we can now get back into the construction process.

Previously, we constructed -1, 0, and 1. We can now use all three in constructing additional numbers. The theorems in the previous chapter allow us to eliminate many possible number candidates immediately. The only possibilities for new numbers are:

$$\{\,|-1\,\}\,,\{-1\,|\,0\,\}\,,\{0\,|\,1\,\}\,,\text{and}\,\{\,1|\,\}$$

.

Using theorems from the previous chapter, we can see that since $-1<0<1$ and $0=\{\,|\,\}$, we have $0=\{-1\,|\,1\,\}$. Also notice that $\{-1\,|\,0,1\,\}=\{-1\,|\,0\,\}$ and that $\{-1,0\,|\,1\,\}=\{0\,|\,1\,\}$, so none of these needs to be considered as a potential new number. (Generally, these principles will be used without being mentioned to weed out bad number-candidates as we go. This time, they are being mentioned to provide an example of the process.)

In general, there are 2^n new numbers born on any given finite day n. (Note: When I say "2^n" here, I am referring to the real quantity 2^n. I do not mean to imply a surreal quantity, though there really is no distinction between the two at this point. I thought it best to clarify nonetheless.)

On day n, there is one new positive integer (n), greater than all the numbers born on prior days, one new negative integer ($-n$), less than all the numbers born on prior days, and one new number between pair of consecutive numbers born on prior days (not necessarily the same prior day). The phrase "pair of consecutive numbers born on prior days" means a pair $a,b\in\mathbf{No}$ such that $\mathfrak{B}(a)<n$, $\mathfrak{B}(b)<n$, $a<b$, and there does not exist $c\in\mathbf{No}$ such that $\mathfrak{B}(c)<n$ and $a<c<b$. Any such pair will define a new number $\{a\,|\,b\,\}$ with birthday

n. Any non-consecutive pair of numbers born before day n will not produce a new number at all, due to theorems (4.2) and (4.1).

(As long as we continue this process without allowing for infinite left or right sets, n corresponds to some $n \in \mathbb{N}$, and the numbers created on day n will be named to correspond to numbers in $\mathbb{R}$. More specifically, the new numbers will correspond to numbers in $\mathbb{Q}$. As we continue, the naming conventions given will be expressions in $\mathbb{Q}$ and should be interpreted as such, so that the names will correspond to those of numbers in $\mathbb{Q}$.)

The two integers created on day n, as has already been stated, are named n and $-n$. The rest are named according to the following convention:

If a, b are a pair of consecutive numbers born prior to day n, corresponding to $a, b \in \mathbb{Q}$, then $\{a \,|\, b\}$ shall be named to correspond to $\frac{a+b}{2} \in \mathbb{Q}$.

Using that convention, the numbers $\{\,|-1\}$, $\{-1\,|\,0\}$, $\{0\,|\,1\}$, and $\{\,1|\,\}$ are assigned the values -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2, respectively. These are all of the numbers born on day 2.

So, at the end of day 2, the following numbers exist: -2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1 and 2. These are the numbers we have to work with on day 3.

By the naming conventions just described, the new positive numbers on day 3 are: $3 = \{2|\,\}$, $\frac{3}{2} = \{1\,|\,2\}$, $\frac{3}{4} = \{\,\frac{1}{2}\,|\,1\}$, and $\frac{1}{4} = \{0\,|\,\frac{1}{2}\}$. The new negative numbers are defined using these numbers and the definition of negation, $-x = \{-x_R\,|-x_L\}$. For example, $-\frac{1}{4} = \{-\frac{1}{2}\,|\,0\}$.

Notice that on any given day n, all of the numbers created correspond to rational numbers with denominator no more than 2^{n-1}, and that all the numbers created on or before day n correspond to rational numbers that can be expressed with denominator 2^{n-1}. As long as we continue allowing only finite sets of previously defined numbers in the left and right sets of newly constructed numbers, every number created corresponds to a dyadic rational, that is, each number corresponds to $\frac{p}{2^q}$, for $p, q \in \mathbb{Z}$. Also, so long as we continue in this manner, we can never create any number that doesn't correspond to an integer or a dyadic rational number.

We will rectify that limitation in the next chapter. Before we do, though, it would be good to demonstrate that these names are appropriately chosen so that these numbers behave as we might expect them to behave under arithmetic (presupposing we have a predisposition to rational arithmetic). A handful of examples follow. (Recall definitions (2.8) and (2.10) of addition and multiplication, respectively. Notice also that the binary operations are defined to be commutative, and will be treated as such (with proof given in later chapters). Note also that in the beginning, all the formalities will be followed, with shortcuts included as the process becomes more familiar. Also note that if an operation calls for a member of the empty set, there will be no resulting number (for example, the result of $\{0|1_R+1\}$ would be $\{0|\}$, because 1_R is empty).)

EXAMPLE 5.2.

$$\begin{aligned}1+1 &= \{1_L+1, 1+1_L \,|1_R+1, 1+1_R\} \\ &= \{0+1, 1+0|\} \\ &= \{1|\} \\ &= 2\end{aligned}$$

EXAMPLE 5.3.

$$\begin{aligned}1+\frac{1}{2} &= \left\{ 1_L+\frac{1}{2}, 1+\frac{1}{2}_L \,\middle|\, 1_R+\frac{1}{2}, 1+\frac{1}{2}_R \right\} \\ &= \left\{ 0+\frac{1}{2}, 1+0 \,\middle|\, 1+1 \right\} \\ &= \left\{ \frac{1}{2}, 1 \,\middle|\, 2 \right\} \\ &= \{1|2\} && \text{by (4.2)} \\ &= \frac{3}{2}\end{aligned}$$

EXAMPLE 5.4.

$$\begin{aligned}
\frac{1}{2}+\frac{1}{2} &= \left\{ \frac{1}{2_L}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2_L} \,\middle|\, \frac{1}{2_R}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2_R} \right\} \\
&= \left\{ 0+\frac{1}{2} \,\middle|\, 1+\frac{1}{2} \right\} \\
&= \left\{ \frac{1}{2} \,\middle|\, 1+\frac{1}{2} \right\} \\
&= \left\{ \frac{1}{2} \,\middle|\, \frac{3}{2} \right\} \\
&= 1
\end{aligned}$$

EXAMPLE 5.5.

$$\begin{aligned}
\frac{3}{2}+\frac{1}{2} &= \left\{ \frac{3}{2_L}+\frac{1}{2}, \frac{3}{2}+\frac{1}{2_L} \,\middle|\, \frac{3}{2_R}+\frac{1}{2}, \frac{3}{2}+\frac{1}{2_R} \right\} \\
&= \left\{ 1+\frac{1}{2}, \frac{3}{2}+0 \,\middle|\, 2+\frac{1}{2}, \frac{3}{2}+1 \right\} \\
&= \left\{ \frac{3}{2} \,\middle|\, \left\{ 2_L+\frac{1}{2}, 2+\frac{1}{2_L} \,\middle|\, 2_R+\frac{1}{2}, 2+\frac{1}{2_R} \right\}, \left\{ \frac{3}{2_L}+1, \frac{3}{2}+1_L \,\middle|\, \frac{3}{2_R}+1, \frac{3}{2}+1_R \right\} \right\} \\
&= \left\{ \frac{3}{2} \,\middle|\, \left\{ 1+\frac{1}{2}, 2+0 \,\middle|\, 2+1 \right\}, \left\{ 1+1, \frac{3}{2}+0 \,\middle|\, 2+1 \right\} \right\} \\
&= \left\{ \frac{3}{2} \,\middle|\, \left\{ \frac{3}{2}, 2 \,\middle|\, 3 \right\}, \left\{ 2, \frac{3}{2} \,\middle|\, 3 \right\} \right\} \\
&= \left\{ \frac{3}{2} \,\middle|\, \frac{5}{2} \right\} \\
&= 2
\end{aligned}$$

EXAMPLE 5.6.

$$\begin{aligned}
2\cdot 2 &= \{2_L\cdot 2+2\cdot 2_L-2_L\cdot 2_L, 2_R\cdot 2+2\cdot 2_R-2_R\cdot 2_R \,|\, 2_L\cdot 2+2\cdot 2_R-2_L\cdot 2_R, 2_R\cdot 2+2\cdot 2_L-2_R\cdot 2_L\}\\
&= \{1\cdot 2+2\cdot 1-1\cdot 1|\}\\
&= \{2+2-1|\}\\
&= \{\{2_L+2|2_R+2\}-1|\}\\
&= \{\{1+2|\}-1|\}\\
&= \{\{\{1_L+2\,|1_R+2\}|\}-1|\}\\
&= \{\{\{0+2|\}|\}-1|\}\\
&= \{\{\{2|\}|\}-1|\}\\
&= \{\{3|\}-1|\}\\
&= \{4-1|\}\\
&= \{4+(-1)|\}\\
&= \{\{4_L-1,4-1_R|4_R-1,4-1_L\}|\}\\
&= \{\{3-1|4-0\}|\}\\
&= \{\{\{3_L-1,3-1_R|3_R-1,3-1_L\}|4\}|\}\\
&= \{\{\{2-1|3-0\}|4\}|\}\\
&= \{\{\{1|3\}|4\}|\}\\
&= \{\{2|4\}|\}\\
&= \{3|\}\\
&= 4
\end{aligned}$$

(Notice that toward the end of that demonstration, the pattern that emerged when repeatedly subtracting 1 was finally assumed. This makes a great case for transfinite induction. This example was included to demonstrate just how quickly, using even the simplest form of the numbers, arithmetic becomes exceptionally cumbersome. From this point forward, in simplifying further examples, it will be taken on faith that calculations work as we would like them to work. Unfortunately, this is an example of a concept that, within the scope of this thesis, cannot be proven. The Surreal Numbers that correspond to the Real Numbers are isomorphic to the Real Numbers as we know them, but proving that is the case requires one of two things that make it impossible to do here: a very extensive knowledge of set theory and topology, or the space and time to break a more elementary proof into many, many cases within this document. Since neither of those is available here, we will need to accept it without proof, with the caveat that proofs can be found in the source material, specifically Alling's book [5], should one be needed.)

Day ω (and Beyond)

At this point in the construction process, we have created all of the dyadic rationals, including the integers, but there is so much more to do. In order to get beyond the dyadic rationals, we have to allow for infinite left and right sets in the numbers we construct.

The first such number we will create will take the form $\{L|R\} = \{0,1,2,\ldots|\}$. This certainly fits the definition of number, because no $x_L \geq x_R$ and all the members of L and R are numbers. This infinite ordinal number is given the name ω, that is, $\omega = \{0,1,2,\ldots|\}$, or, more simply, $\omega = \{\mathbb{N}|\}$. Of course, that means we have reached Day ω, which turns out to be a very fruitful day.

On Day ω, as usual, two numbers are created on the extremes of our continuum, ω and $-\omega$. Just as before, all previously created numbers are strictly between these two, that is, for all $n \in \mathbb{N}$, $-\omega < n < \omega$. (This should be easy enough to prove using the definition of $\geq$ that I will leave it as an exercise for the reader, probably a quick mental exercise at this point.)

Day ω is special because it is the day when all the rest of the Real Numbers are created. By allowing infinite left and right sets, the possibilities of potential numbers become endless, as sets can now be sequential, and the new numbers created from such sets (under the proper circumstances, of course) can now be defined as the limits of those sequences.

For example, if we use $\left\{\frac{m}{2^n} \in \mathbf{No} : m,n \in \mathbb{Z}^+ \text{ with } m = \left\lfloor \frac{2^n}{3} \right\rfloor\right\}$ as the left set and $\left\{\frac{m}{2^n} \in \mathbf{No} : m,n \in \mathbb{Z}^+ \text{ with } m = \left\lceil \frac{2^n}{3} \right\rceil\right\}$ as the right set, that is, let $x = \left\{0, \frac{1}{4}, \frac{5}{16}, \ldots \middle| \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \ldots\right\}$, the result is $x = \frac{1}{3}$.

Because these are infinite sets, checking this notion by evaluating $3x$ or $x+x+x$ is a very tedious task indeed. The reader is invited to try it. Instead, it can be verified by assuming the dyadic rationals in **No** are isomorphic to the dyadic rationals in $\mathbb{R}$, and considering the notion that no $x_L \geq \frac{1}{3}$, no $x_R \leq \frac{1}{3}$, and verifying that $\sum_{n=1}^{\infty} \frac{1}{2^{2i}} = \frac{1}{3}$ and $\frac{1}{2} - \sum_{i=1}^{\infty} \frac{1}{2^{2i+1}} = \frac{1}{3}$.

For all $n \in \mathbb{N}$, finding a multiplicative inverse is fairly easy to do intuitively by letting the left and right sets be all the positive dyadic rationals less than $\frac{1}{n}$ and all the positive dyadic rationals greater than $\frac{1}{n}$, respectively (using Real Number ordering). The result is accurate, yet cumbersome. There are other methods of coming up with a streamlined version. One is to create the left and right sets based on sequence of dyadic rationals with the desired limit, if one comes to mind.

The method I created to help me understand is the one used above in creating $\frac{1}{3}$. That is, for all $x \in \mathbb{N}$, let $\frac{1}{x}_L = \left\{ \frac{m}{2^n} \in \mathbf{No} : m,n \in \mathbb{Z}^+ \text{ with } m = \left\lfloor \frac{2^n}{x} \right\rfloor \right\}$, (where $\left\lfloor \frac{2^n}{x} \right\rfloor$ represents the floor of $\frac{2^n}{x}$), and let $\frac{1}{x}_R = \left\{ \frac{m}{2^n} \in \mathbf{No} : m,n \in \mathbb{Z}^+ \text{ with } m = \left\lceil \frac{2^n}{x} \right\rceil \right\}$, (where $\left\lceil \frac{2^n}{x} \right\rceil$ represents the ceiling of $\frac{2^n}{x}$). Then $\frac{1}{x} = \left\{ \frac{1}{x}_L | \frac{1}{x}_R \right\}$. The idea here is that we want the series to include the maximum $m \in \mathbb{Z}$ for each $n \in \mathbb{N}$ such that $x \cdot \frac{m}{2^n} < 1$ in the left set and the minimum $m \in \mathbb{Z}$ for each $n \in \mathbb{N}$ such that $x \cdot \frac{m}{2^n} > 1$ in the right set. In this way, we can guarantee the sequence in the left set approaches $\frac{1}{x}$ from the left and the sequence in the right set approaches $\frac{1}{x}$ from the right. We limit ourselves to positive integers for this procedure, realizing that $\frac{1}{-x} = -(\frac{1}{x})$, so that if we need the multiplicative inverse of a negative integer, we can just negate the multiplicative inverse of the positive integer with the same birthday. The fact that the dyadic rationals are dense in $\mathbb{Q}$ together with the fact that $\frac{1}{n} \cdot m = \frac{m}{n}$ allows us to do this for all $\frac{m}{n} \in \mathbb{Q}$.

Of course, completing the rational numbers is not the entirety of Day ω. Irrational numbers can be created with sequences as left and right sets as well, so on Day ω, all of $\mathbb{R}$ is created.

In addition to the Reals, still other numbers are created on Day ω.

For example, $\frac{1}{\omega} = \left\{0 \middle| \frac{1}{2^n} : n \in \mathbb{N}\right\}$ is the smallest positive number that can be created out of earlier numbers. Similar constructions exist that create infinitesimals on either side of any other given number. For example, the number $\frac{137}{128} + \frac{1}{\omega}$ would be constructed to be $\left\{\frac{137}{128} \middle| \frac{137}{128} + \frac{1}{2^n} : n \in \mathbb{N}\right\}$, and so on.

After Day ω, still more numbers can be made using the numbers from previous days. Day $\omega+1$ gives us $\omega+1 = \{0,1,2,\ldots,\omega|\}$, and $-\omega-1 = \{\,|-1,-2,-3,-\omega\}$. So far, anyone who has studied Cantor will have recognized all of the numbers we have constructed, but now we can also create numbers that are unique to the Surreal Numbers, which in other systems would have been deemed nonsense.

For instance, $\omega-1$, which in other systems would have been equal to ω. But here, it has a unique value: $\omega-1 = \{\mathbb{N}|\,\omega\}$. Our familiarity with the patterns of arithmetic can easily verify that $(\omega-1)+1 = \omega$. It is interesting to note that $\omega-1$ is strictly between the Reals and ω, by theorems 4.5 and 4.6. Numbers with this property are unique to **No**.

Here are some other interesting examples of numbers that have no counterparts outside of the Surreal Numbers: $\omega + \frac{1}{\omega} = \left\{\omega \middle| \omega + \frac{1}{2^k} : k \in \mathbb{N}\right\}$, $\frac{\omega}{2} = \{1,2,3,\ldots|\,\omega,\omega-1,\omega-2,\ldots\}$, and $\sqrt{\omega} = \left\{0,1,2,3,\ldots \middle| \omega, \frac{\omega}{2}, \frac{\omega}{4}, \frac{\omega}{8}\cdots\right\}$. (The last three examples were borrowed from Conway, [1].)

After creating all of this, it may occur to the reader that maybe it would be good to see what happens by creating a number like $\check{x} = \{x < y \in \mathbf{No}\,|z \geq y \in \mathbf{No}\}$. (It certainly occurred to me one day, and the result was frightening. I thought I had just broken the Surreal Numbers irreparably… and after so many years!)

It would seem that $\{x < y \in \mathbf{No}\,|z \geq y \in \mathbf{No}\}$ certainly satisfies the definition of a number. But this number would have to be strictly between the elements in its left and right sets, that is $\check{x}_L < \check{x} < \check{x}_R$ for all $\check{x}_L \in \check{X}_L, \check{x}_R \in \check{X}_R$, which is nonsensical, because if $\check{x} <$ all $\check{x}_R$, then $\check{x} < y$, which, by the way we defined $\check{x}$, places it firmly in its own left set, contradicting Theorem 4.5.

Constructions of this sort are called “gaps” in **No**. Conway is rather noncommittal in ONAG [1] as to whether or not these gaps are actually numbers or not. In one sentence on page 37 of ONAG, he talks about a "sequence of numbers and gaps," implying he does not consider them numbers, but no further explanation is given regarding the question.

Conway declares that some gaps have special importance, and so he names them. They are defined as: $\mathbf{On} = \{\mathbf{No}|\}$ (the gap at the end of the number line), $\frac{1}{\mathbf{On}} = \{0|x > 0 \in \mathbf{No}\}$ (the gap between 0 and the positive Surreal Numbers), ∞ (the gap between the real numbers and the positive infinite ordinals), and $\frac{1}{\infty}$ (the gap between the infinitesimals and the positive Reals).[1]

It seems to me that gaps violate the definition of number by necessarily being members of their own left or right sets, and so I do not regard them as numbers. Gaps may be a very interesting topic for further exploration, but not here.

Transfinite Induction on **No**

Transfinite induction in general is a method for proving statements consisting of elements from well ordered sets, that is, a totally ordered set, every nonempty subset of which has a least element. Generally, the procedure involves demonstrating that the statement is true on the least element, then assuming it must be true for all elements $\leq$ some element, and then demonstrating that it must be true for elements greater than that. [8]

In **No**, each x is itself a well ordered set under containment ($\subset$, rather than $\leq$) with least element 0. Since each number can be constructed using only older numbers, each number $x \in$ **No** is a chain of length $\mathfrak{B}(x)+1$ with least element 0 (actually, a chain of chains of length $\leq \mathfrak{B}(x)+1$ with least element 0). That is, if $\subset$ is defined by $x \subset y$ if $x \in y_L$ or $x \in y_R$, each number x consists of chains in the form $0 \subset a \subset b \subset \ldots \subset (x_L)_L \subset x_L \subset x$. (For example, $\frac{5}{2} = \{2|3\}$. It consists of two chains: $0 \subset 1 \subset 2 \subset \frac{5}{2}$ and $0 \subset 1 \subset 2 \subset 3 \subset \frac{5}{2}$.)

Because each number consists of chains containing least element 0, we can assume that what is true about 0 must be true for all numbers born on or before some birthday (possibly 0), and then demonstrate that the property holds for younger numbers. If we can demonstrate that a property holds for numbers born after an arbitrary birthday, then we have proven that it holds for all numbers.

With transfinite induction on **No**, the base case is usually true vacuously, thereby essentially allowing it to be ignored. In theory, if R is a relation between numbers in **No**, we can verify $\mathrm{R}(x,y)$ by verifying $\mathrm{R}(x_L,y)$, $\mathrm{R}(x_R,y)$, $\mathrm{R}(x,y_L)$, $\mathrm{R}(x,y_R)$, and, if necessary

$\mathrm{R}(x_L, y_L)$, $\mathrm{R}(x_R, y_L)$, $\mathrm{R}(x_L, y_R)$, and $\mathrm{R}(x_R, y_R)$. Notice that the last sentence implies repeating verification attempts until an actual verification is completed. This is not a problem because of the recursive nature of the definition of number. Since every non-zero number is built from sets of sets, eventually containing 0, which is built of nothing, one attempt to verify that a relation holds "a level down," as it were, often results in an argument that is essentially a repetition of the original question at hand.

For instance, in a transfinite induction proof of Theorem 4.5, one might, after some manipulations, realize that $x_L < x$ if and only if $(x_L)_L < x_L$, for all $(x_L)_L$ that exist, and so, one could conclude that, this means $x_L < x$ if and only if $((x_L)_L)_L < (x_L)_L$, for all $((x_L)_L)_L$ that exist and so on, with the process ending only upon reaching a number with $\varnothing$ as its left set, at which point, the original theorem becomes vacuously true. So, once the prover runs into the first similar argument, the prover can invoke transfinite induction, and be done with it.

Essentially, this reduces most every problem to one of comparisons to the empty set, which can make proofs very quick and simple. However, from speaking to others, and from my own experience, transfinite induction can be very confusing.

Conway uses it to prove just about everything in ONAG. Personally, I have tried to avoid it as much as possible, especially in the beginning of this project. But, in certain instances, when a proof is just starting, there will be a crystal clear indication that transfinite induction is exactly the way to go.

For instance, in the next chapter, we will prove that addition is commutative. Almost immediately, it becomes apparent that x and y commute under addition if x and y_R (and all the other pairs of this sort) commute under addition. This immediately leads to the realization that this argument will continue to reappear as we dig down a little further. Eventually, of course, the more levels we descend into a number, the closer we get to the $\varnothing$ at the bottom. At each step, we either reach another level to ask the same question about, or we reach $\varnothing$, at which point our original question becomes vacuously true.

This is an example of when transfinite induction is clearly in order for finishing the proof. (For a more concrete example of this, see Theorem 8.1 and the comments that follow it at the beginning of the next chapter.)

(A thought: If you are having trouble with the concept of transfinite induction still, the following analogy might help. Think of each number as being like a Russian nesting doll, with each number containing either more numbers inside it or nothing at all, much in the same way each doll contains either another doll or nothing. If a question is asked about your doll that can only be answered by asking the same question about all the dolls it contains, then you would have to keep opening dolls until you reach the point where there are no more dolls inside, at which point you could declare the proposition true about all of the dolls inside (since there aren't any). Alternately, knowing you would eventually reach that point in advance, you could have declared the proposition true as soon as you realized that the question could only be answered by asking it again about each internal doll, knowing that if anyone doubted you, you could challenge them to look for themselves. That is essentially the idea behind transfinite induction. If your doll contained infinitely more dolls inside, the latter choice would be your only choice, necessitating your use of transfinite induction, and making your challenge to your doubters a very cruel one indeed. This is why we are forced to use transfinite induction in some proofs. Sometimes, there is just no way around it.)

Algebraic Properties of Addition

The goal of this chapter is to prove several properties of addition on the Surreal Numbers, with the ultimate goal of showing that $(\mathbf{No},+)$ exhibits the properties an Abelian group. Definitions of properties and the requirements for an Abelian group can be found in [6].

Recall the definition of addition: $x+y=\{x_L+y, x+y_L\,|\,x_R+y, x+y_R\}$. Using this definition and what we know so far about numbers in general, we will now explore the properties of addition (may of which we have been assuming until now). Note: Most of these proofs will require the use of transfinite induction.

THEOREM 8.1. (The Additive Identity for **No**) For all $x \in$**No**, $0+x=x+0=x$.

Proof. Let $x \in$ **No**. Then, $0+x=\{0_L+x, 0+x_L\,|\,0_R+x, 0+x_R\}=\{0+x_L\,|\,0+y_R\}$.

Since $0+x=x$ if $0+x_L=x_L$ and $0+x_R=x_R$, we can use transfinite induction and conclude that $0+x=x$.

Likewise, $x+0=\{x+0_L, x_L+0\,|\,x+0_R, x_R+0\}=\{x_L+0\,|\,x_R+0\}$.

Since $x+0=x$ if ($x_L+0=x_L$ and $x_R+0=x_R$), we can use transfinite induction and conclude that $0+x=x$. □

Hopefully, the above proof makes the idea of transfinite induction clear. If we knew every element of every left and right set of x, and every element of each of those, and so on, each would eventually (perhaps after infinite iterations) have an empty left or right set, and therefore have vanishing terms upon adding 0, using the definition of addition. Realizing that, we can see that eventually, every term would vanish, leaving us with, say, $0+x=x$ only if $0+a=a$, for all a in $\varnothing$, which is vacuously true.

THEOREM 8.2. Let $x,y \in \mathbf{No}$ such that $x>y$. Then (i) $x-y>0$, and (ii) $y-x<0$.

Proof. Proof of (i):

Recall that $-y=\{-y_R|-y_L\}$.

So, $x-y=\{x_L-y, x-y_R \,|\, x_R-y, x-y_L\}$.

To show $x-y>0$, it is sufficient to show that some $(x-y)_L>0$.

Thus, if ($\exists x_L$ such that $x_L-y>0$ or $\exists y_R$ such that $x-y_R>0$), then $x-y>0$.

Since $x>y$, $\{y|x\}$ is a number, and $x>\{y|x\}>y$.

Assuming the cut form of x and y (so that the left and right sets of each can be assumed non-empty), and by Theorem 4.2, $\{y|x\}\in X_L$. Likewise, by Theorem 4.1, $\{y|x\}\in Y_R$.

Thus, there exists x_L such that $x_L>y$, (namely, $\{y|x\}$). Also, there exists y_R such that $x>y_R$, (again, $\{y|x\}$).

Since x and y were arbitrarily chosen, we have ($x-y>0$ when $x>y$) if ($x_L-y>0$ when $x_L>y$) (and we have shown such an x_L does exist).

Also, though the above is actually sufficient in itself, we have ($x-y>0$ when $x>y$) if ($x-y_R>0$ when $x>y_R$) (and such a y_R does exist).

Both statements allow us to invoke transfinite induction to declare them true.

Thus, $x-y>0$ whenever $x>y$.

Proof of (ii):

Similarly, we have $y-x<0$ if there exists $(y-x)_R<0$.

Since $y - x = \{y - x_R, y_L - x \,|\, y - x_L, y_R - x\}$, we have $y - x < 0$ if ($\exists$ some x_L such that $y - x_L < 0$ or $\exists$ some y_R such that $y_R - x < 0$).

Again, assuming the cut form of x and y, we have $y < \{y|x\} < x$, and by Theorems 4.2 and 4.1, this means $\{y|x\} \in Y_R$ and $\{y|x\} \in X_L$. So, there does exist such and x_L and y_R such that $x_L > y$ and $x > y_R$ (both namely $\{y|x\}$).

So, again we have ($y - x < 0$ when $x > y$) if ($y - x_L < 0$ when $x_L > y$) or ($y_R - x < 0$ when $x > y_R$).

Again, by transfinite induction, both conditions are true, and so $y - x < 0$ when $x > y$. □

THEOREM 8.3. (Additive Inverses in **No**)

For all $x \in \mathbf{No}$, $x - x = 0$.

Proof. Let $x \in \mathbf{No}$. Recall that $-x = \{-x_R | -x_L\}$.

So, $x - x = \{x - x_R, x_L - x \,|\, x - x_L, x_R - x\}$.

Since $x_R > x$, by Theorem 8.2, we have $x - x_R < 0$ and $x_R - x > 0$.

Also, since $x_L < x$, we have $x_L - x < 0$ and $x - x_L > 0$.

Thus, we have all $(x - x)_L < 0$ and all $(x - x)_R > 0$.

So, by Theorem 4.7, we have $x - x = 0$. □

THEOREM 8.4. (Commutativity of Addition in **No**)

For all $x, y \in \mathbf{No}$, $x + y = y + x$

Proof. Let $x, y \in \mathbf{No}$.

Since, by (8.1), we know this is true for $x = 0$ and $y = 0$, we can assume $x_L + y = y + x_L$, $x_R + y = y + x_R$, $y_L + x = x + y_L$, $y_R + x = x + y_R$, for the purpose of using transfinite induction.

We have $x+y=\{x_L+y, x+y_L \mid x_R+y, x+y_R\}$, and $y+x=\{y_L+x, y+x_L \mid y_R+x, y+x_R\}$, by the definition of addition.

Thus, we have $x+y=y+x$ only if $x_L+y=y+x_L$, and $x+y_L=y_L+x$, and $x_R+y=y+x_R$, and $x+y_R=y_R+x$.

In other words, x and y commute only if (x commutes with any elements of the left and right sets of y, and y commutes with any elements of the left and right sets of x).

By transfinite induction, this is true.

Therefore, $x+y=y+x$. □

THEOREM 8.5. (Associativity of Addition in **No**)

For all $a,b,c \in \mathbf{No}$, $(a+b)+c=a+(b+c)$.

Proof. For all x, let x' denote $x' \in \{x_L, x_R\}$.

We know that $(0+b)+c=0+(b+c)$ and $(a+0)+c=a+(0+c)$, and $(a+b)+0=a+(b+0)$.

Thus, we can assume $(a'+b)+c=a'+(b+c)$ and $(a+b')+c=a+(b'+c)$ and $(a+b)+c'=a+(b+c')$, for the purpose of using transfinite induction.

Let $a,b,c \in \mathbf{No}$. Then, $(a+b)+c=\{a_L+b, a+b_L \mid a_R+b, a+b_R\}+c=$
$\{(a_L+b)+c, (a+b_L)+c, (a+b)+c_L \mid (a_R+b)+c, (a+b_R)+c, (a+b)+c_R\}$, and

$a+(b+c)=a+\{b_L+c, b+c_L \mid b_R+c, b+c_R\}=$
$\{a_L+(b+c), a+(b_L+c), a+(b+c_L) \mid a_R+(b+c), a+(b_R+c), a+(b+c_R)\}$.

So, if all of the following are true, then $(a+b)+c = a+(b+c)$:

$$(a_L + b) + c = a_L + (b+c) \text{ and } (a_R + b) + c = a_R + (b+c),$$
$$(a + b_L) + c = a + (b_L + c) \text{ and } (a + b_R) + c = a + (b_R + c), \text{ and}$$
$$(a+b) + c_L = a + (b + c_L) \text{ and } (a+b) + c_R = a + (b + c_R).$$

Thus, we can use transfinite induction to conclude that $(a+b)+c = a+(b+c)$. □

THEOREM 8.6. Let $a, b, c \in \mathbf{No}$ such that $a < b$. Then $a + c < b + c$.

Proof. Recall that $a+c = \{a_L + c, a + c_L \,|\, a_R + c, a + c_R\}$, and $b+c = \{b_L + c, b + c_L \,|\, b_R + c, b + c_R\}$.

We need to show that $(a+c) \not\geq (b+c)$, that is, some $(b+c)_L \geq (a+c)$ or $(b+c) \geq$ some $(a+c)_R$. In other words, $a+c < b+c$ if $b_L + c \geq (a+c)$ or $b + c_L \geq (a+c)$ or $(b+c) \geq a_R + c$ or $(b+c) \geq a + c_R$.

Thus, it would be sufficient to show that $a + c < b_L + c$, for some b_L, or $a_R + c < b + c$, for some a_R.

Since $a < b$, we know that $\{a \,|\, b\}$ is a number and that $a < \{a \,|\, b\} < b$.

Assuming the cut form of a and b, this means $\{a \,|\, b\} \in A_R$ and $\{a \,|\, b\} \in B_L$.

Because of this, there does exist an a_R such that $a_R < b$ and there does exist a b_L such that $a < b_L$.

So, we have $a + c < b + c$ when $a < b$ if ($a + c < b_L + c$ when $a < b_L$) or ($a_R + c < b + c$, when $a_R < b$).

Thus, we can use transfinite induction and conclude that $a + c < b + c$ whenever $a < b$. □

THEOREM 8.7. Let $a, b \in \mathbf{No}$. Then $a+b \in \mathbf{No}$

Proof. Recall that $a+b = \{a_L + b, a + b_L \,|\, a_R + b, a + b_R\}$.

Recall also that $a+b \in \mathbf{No}$ if no $(a+b)_L \geq (a+b)_R$. So, we need to show that

(i) $a_L + b < a_R + b$,

(ii) $a_L + b < a + b_R$,

(iii) $a + b_L < a_R + b$, and

(iv) $a + b_L < a + b_R$.

We know that $a_L < a < a_R$ and $b_L < b < b_R$, and so, by Theorem 8.6, $a_L + b < a_R + b$. So, (i) is confirmed. Likewise, also by Theorem 8.6, $a_L + b < a + b < a + b_R$, and so (ii) is also confirmed. Similarly, $a + b_L < a + b < a_R + b$, and $a + b_L < a + b_R$, so (iii) and (iv) are also confirmed. Thus, $a+b \in \mathbf{No}$. □

Taking the above properties together, we have **No** exhibiting all of the properties of an Abelian group under addition. That is, all of the following are true: all elements of **No** have additive inverses, addition on **No** is associative, addition on **No** is commutative, and **No** is closed under addition.

Technically, since **No** is a proper class (essentially, every element of **No** is an ordered pair of members of the power set of **No**, so **No** is not considered a set itself [7]) it is not appropriate to actually call **No** a group, but we have demonstrated that it exhibits all of the properties of an Abelian group under addition. This will be important to remember in the next chapter.

Algebraic Properties of Multiplication, **No** is a Field

The goal of this chapter is to prove several properties of multiplication on the Surreal Numbers, with the ultimate goal to show that $(\mathbf{No}, +, \times)$ exhibits the properties of a field. Definitions of properties and the requirements for a field can be found in [6]. They are also listed at the end of this chapter.

Recall the definition of multiplication:

$$xy = \{x_L y + xy_L - x_L y_L, x_R y + xy_R - x_R y_R \mid x_L y + xy_R - x_L y_R, x_R y + xy_L - x_R y_L\}.$$

THEOREM 9.1. Let $a \in \mathbf{No}$. Then $0 \cdot a = a \cdot 0 = 0$

Proof. In the definition of multiplication, each member of the left and right set of the product of two numbers refers to an element of either the left or right set of each of the numbers being multiplied together. Since $0 = \{|\}$, this means that there are no elements in the left or right sets of the products $0 \cdot a$ and $a \cdot 0$. Thus, $0 \cdot a = a \cdot 0 = 0$. □

THEOREM 9.2. (The Multiplicative Identity for **No**)

Let $a \in \mathbf{No}$. Then $1 \cdot a = a \cdot 1 = a$

Proof. Recall the definition of multiplication, and that $1 = \{0|\}$. Then,

$$\begin{aligned} 1 \cdot a &= \{1_L \cdot a + 1 \cdot a_L - 1_L \cdot a_L, 1_R \cdot a + 1 \cdot a_R - 1_R \cdot a_R \,|\, 1_L \cdot a + 1 \cdot a_R - 1_L \cdot a_R, 1_R \cdot a + 1 \cdot a_L - 1_R \cdot a_L\} \\ &= \{0 \cdot a + 1 \cdot a_L - 0 \cdot a_L \,|\, 0 \cdot a + 1 \cdot a_R - 0 \cdot a_R\} \\ &= \{1 \cdot a_L \,|\, 1 \cdot a_R\}. \end{aligned}$$

So, we have $1 \cdot a = a$ if ($1 \cdot a_L = a_L$ and $1 \cdot a_R = a_R$).

Thus, by transfinite induction, $1 \cdot a = a$.

Similarly,

$$\begin{aligned} a \cdot 1 &= \{a_L \cdot 1 + a \cdot 1_L - a_L \cdot 1_L, a_R \cdot 1 + a \cdot 1_R - a_R \cdot 1_R \,|\, a_L \cdot 1 + a \cdot 1_R - a_L \cdot 1_R, a_R \cdot 1 + a \cdot 1_L - a_R \cdot 1_L\} \\ &= \{a_L \cdot 1 + a \cdot 0 - a_L \cdot 0 \,|\, a_R \cdot 1 + a \cdot 0 - a_R \cdot 0\} \\ &= \{a_L \cdot 1 \,|\, a_R \cdot 1\}. \end{aligned}$$

So, $a \cdot 1 = a$ if ($a_L \cdot 1 = a_L$ and $a_R \cdot 1 = a_R$).

Thus, by transfinite induction, $a \cdot 1 = a$. □

THEOREM 9.3. Let $a \in \mathbf{No}$. Then $-a = -1 \cdot a$.

Proof. Recall the definition of negation, $-a = \{-a_R \,|\, -a_L\}$. Also recall $-1 = \{|0\}$.

Then, since the terms referring to $(-1)_L$ don't really exist, and since $(-1)_R = 0$ we have

$$\begin{aligned} -1 \cdot a &= \{0 \cdot a + (-1) \cdot a_R - 0 \cdot a_R \,|\, 0 \cdot a + (-1) \cdot a_L - 0 \cdot a_L\} \\ &= \{(-1) \cdot a_R \,|\, (-1) \cdot a_L\}. \end{aligned}$$

So, $(-a = -1 \cdot a)$ if $(-a_R = -1 \cdot a_R$ and $-a_L = -1 \cdot a_L)$.

Therefore, by transfinite induction, $-a = -1 \cdot a$. □

THEOREM 9.4. Let $a \in \mathbf{No}$. Then $-(-a) = a$.

Proof. By definition of $-x$, $-(-a) = \{(-a)_R \,|\, (-a)_L\} = \{a_L \,|\, a_R\} = a$. □

THEOREM 9.5. (Commutativity of Multiplication on **No**)

Let $x, y \in \mathbf{No}$. Then $xy = yx$.

Proof. (Note: Because of the sheer size of the elements of these sets, they will be listed separately, instead of within set brackets.)

Because $0 \cdot y = y \cdot 0$, for all $y \in \mathbf{No}$, we can assume that, for purposes of transfinite induction, for all $x' \in \{x_L, x_R\}$ and $z \in \mathbf{No}$, $x'z = zx'$.

The left set of xy contains elements of two forms: $x_L y + xy_L - x_L y_L$, and $x_R y + xy_R - x_R y_R$.

The left set of yx contains elements of two forms: $y_L x + yx_L - y_L x_L$, and $y_R x + yx_R - y_R x_R$.

Because addition is commutative, we can rearrange the elements of the left set of yx and rewrite them as follows: $yx_L + y_L x - y_L x_L$, and $yx_R + y_R x - y_R x_R$.

Consequently, if $x_L y + xy_L - x_L y_L = yx_L + y_L x - y_L x_L$ and $x_R y + xy_R - x_R y_R = yx_R + y_R x - y_R x_R$,then $(xy)_L = (yx)_L$.

The right set of xy contains elements of two forms: $x_L y + xy_R - x_L y_R$, and $x_R y + xy_L - x_R y_L$.

The right set of yx contains elements of two forms: $y_L x + yx_R - y_L x_R$, and $y_R x + yx_L - y_R x_L$.

Because addition is commutative, we can rearrange the elements of the right set of yx and rewrite them as follows: $yx_R + y_L x - y_L x_R$, and $yx_L + y_R x - y_R x_L$.

Consequently, if $x_L y + xy_R - x_L y_R = yx_L + y_R x - y_R x_L$ and $x_R y + xy_L - x_R y_L = yx_R + y_L x - y_L x_R$, then $(xy)_R = (yx)_R$.

The rearrangement we performed helps us see that each member of $(xy)_L$ corresponds to a member of $(yx)_L$ with corresponding terms matching the requirement that $x'y = yx'$

or $xy' = y'x$, for all $x' \in \{x_L, x_R\}$, $y' \in \{y_L, y_R\}$. This allows us to conclude, by transfinite induction, that $xy = yx$.

□

THEOREM 9.6. (Distributivity of Multiplication over Addition on **No**)

Let $a, b, c \in \mathbf{No}$. Then $a(b+c) = ab + ac$.

Proof. (Note: Because of the sheer size of the elements of these sets, they will be listed separately, instead of within set brackets.)

Because $0(b+c) = 0b + 0c$, $a(0+c) = a \cdot 0 + ac$, and $a(b+0) = ab + a \cdot 0$, we can assume, for purposes of transfinite induction, that for all $a' \in \{a_L, a_R\}, b' \in \{b_L, b_R\}$, $c' \in \{c_L, c_R\}$,

$a'(b+c) = a'b + a'c$, $a(b'+c) = ab' + ac$, $a(b+c') = ab + ac'$

The left set of $(b+c)$ contains $b_L + c$ and $b + c_L$.

The right set of $(b+c)$ contains $b_R + c$ and $b + c_R$.

The left set of $a(b+c)$ contains elements of the following forms:

$a_L(b+c) + a(b+c)_L - a_L(b+c)_L$ and $a_R(b+c) + a(b+c)_R - a_R(b+c)_R$.

Substituting left and right elements of $(b+c)$ into $[a(b+c)]_L$ gives us elements of the following forms:

$a_L(b+c) + a(b_L+c) - a_L(b_L+c)$,

$a_L(b+c) + a(b+c_L) - a_L(b+c_L)$,

$a_R(b+c) + a(b_R+c) - a_R(b_R+c)$, and

$a_R(b+c) + a(b+c_R) - a_R(b+c_R)$.

The right set of $a(b+c)$ contains elements of the following forms:

$a_L(b+c) + a(b+c)_R - a_L(b+c)_R$ and $a_R(b+c) + a(b+c)_L - a_R(b+c)_L$.

Substituting left and right elements of $(b+c)$ into $[a(b+c)]_R$ gives us elements of the following forms:

$a_L(b+c)+a(b_R+c)-a_L(b_R+c)$,

$a_L(b+c)+a(b+c_R)-a_L(b+c_R)$,

$a_R(b+c)+a(b_L+c)-a_R(b_L+c)$, and

$a_R(b+c)+a(b+c_L)-a_R(b+c_L)$.

The left set of ab contains elements $a_Lb+ab_L-a_Lb_L$ and $a_Rb+ab_R-a_Rb_R$.

The right set of ab contains elements $a_Lb+ab_R-a_Lb_R$ and $a_Rb+ab_L-a_Rb_L$.

The left set of ac contains elements $a_Lc+ac_L-a_Lc_L$ and $a_Rc+ac_R-a_Rc_R$.

The right set of ac contains elements $a_Lc+ac_R-a_Lc_R$ and $a_Rc+ac_L-a_Rc_L$.

The left set of $ab+ac$ contains elements $(ab)_L+(ac)$ and $(ab)+(ac)_L$.

The right set of $ab+ac$ contains elements $(ab)_R+(ac)$ and $(ab)+(ac)_R$.

Substituting left and right elements of ab and ac into elements from the left set of $ab+ac$ gives us elements of the following forms:

$(a_Lb+ab_L-a_Lb_L)+(ac)$,

$(a_Rb+ab_R-a_Rb_R)+(ac)$,

$(ab)+(a_Lc+ac_L-a_Lc_L)$, and

$(ab)+(a_Rc+ac_R-a_Rc_R)$.

Substituting left and right elements of ab and ac into elements from the right set of $ab+ac$ gives us elements of the following forms:

$(a_Lb+ab_R-a_Lb_R)+(ac)$,

$(a_Rb+ab_L-a_Rb_L)+(ac)$,

$(ab)+(a_Lc+ac_R-a_Lc_R)$, and

$(ab)+(a_Rc+ac_L-a_Rc_L)$.

To each $(ab+ac)_L$ or $(ab+ac)_R$, we can creatively add and subtract a term without changing the element's value, and then rearrange the terms to match the distribution requirements of a corresponding $[a(b+c)]_L$ or $[a(b+c)]_L$.

This gives us elements in $(ab+ac)_L$ of the following forms (with the new added and subtracted terms in brackets for clarity):

$a_L b+[a_L c]+ab_L+(ac)-a_L b_L-[a_L c]$,

$a_R b+[a_R c]+ab_R+(ac)-a_R b_R-[a_R c]$,

$[a_L b]+a_L c+(ab)+ac_L-[a_L b]-a_L c_L$, and

$[a_R b]+a_R c+(ab)+ac_R-[a_R b]-a_R c_R$.

This also gives us elements in $(ab+ac)_R$ with the following forms:

$a_L b+[a_L c]+ab_R+(ac)-a_L b_R-[a_L c]$,

$a_R b+[a_R c]+ab_L+(ac)-a_R b_L+(ac)-[a_R c]$,

$[a_L b]+a_L c+(ab)+ac_R-[a_L b]-a_L c_R$, and

$[a_R b]+a_R c+(ab)+ac_L-[a_R b]-a_R c_L$.

After all of that, we have $a(b+c)=ab+ac$ if all of the following are true:

$$a_L(b+c)+a(b_L+c)-a_L(b_L+c)=(a_L b+[a_L c]+ab_L+ac-a_L b_L-[a_L c]) \tag{9.1}$$

$$a_L(b+c)+a(b+c_L)-a_L(b+c_L)=[a_L b]+a_L c+(ab)+ac_L-[a_L b]-a_L c_L \tag{9.2}$$

$$a_R(b+c)+a(b_R+c)-a_R(b_R+c)=(a_R b+[a_R c]+ab_R+ac-a_R b_R)-[a_R c] \tag{9.3}$$

$$a_R(b+c)+a(b+c_R)-a_R(b+c_R)=[a_R b]+a_R c+(ab)+ac_R-[a_R b]-a_R c_R \tag{9.4}$$

$$a_L(b+c)+a(b_R+c)-a_L(b_R+c)=a_L b+[a_L c]+ab_R+ac-a_L b_R-[a_L c] \tag{9.5}$$

$$a_L(b+c)+a(b+c_R)-a_L(b+c_R)=[a_L b]+a_L c+(ab)+ac_R-[a_L b]-a_L c_R \tag{9.6}$$

$$a_R(b+c)+a(b_L+c)-a_R(b_L+c)=a_R b+[a_R c]+ab_L+ac-a_R b_L+(ac)-[a_R c] \tag{9.7}$$

$$a_R(b+c)+a(b+c_L)-a_R(b+c_L)=[a_R b]+a_R c+(ab)+ac_L-[a_R b]-a_R c_L \tag{9.8}$$

Let $x \in \{a,b,c\}$, x' denote $x' \in \{x,x_L,x_R\}$.

What we have, then, by comparing corresponding terms in each equation above, is that (9.1) will be true if $a'(b'+c') = a'b' + a'c'$, with at least one of the following true: $a' \neq a$, $b' \neq b$, or $c' \neq c$. Thus, these corresponding terms all satisfy the requirements in our assumption.

By transfinite induction, therefore, $a(b+c) = ab+ac$. □

THEOREM 9.7. (Associativity of Multiplication in **No**)

Let $a,b,c \in$ **No**. Then $a(bc) = (ab)c$.

Proof. (Note: Because of the sheer size of the elements of these sets, they will be listed separately, instead of within set brackets.)

Since for all $a,b,c \in$ **No**, $(0 \cdot b)c = 0(bc)$, $(a \cdot 0)c = a(0 \cdot c)$, and $(ab) \cdot 0 = a(b \cdot 0$, we can assume that for all $a' \in \{a_L,a_R\}, b' \in \{b_L,b_R\}, c' \in \{c_L,c_R\}$, $(a'b)c = a'(bc)$, $(ab')c = a(b'c)$, and $(ab)c' = a(bc')$, for the purposes of transfinite induction.

The left set of (bc) contains $b_Lc + bc_L - b_Lc_L$ and $b_Rc + bc_R - b_Rc_R$.

The right set of (bc) contains $b_Lc + bc_R - b_Lc_R$, and $b_Rc + bc_L - b_Rc_L$.

The left set of $a(bc)$ contains $a_L(bc) + a(bc)_L - a_L(bc)_L$ and $a_R(bc) + a(bc)_R - a_R(bc)_R$.

The right set of $a(bc)$ contains $a_L(bc) + a(bc)_R - a_L(bc)_R$, and $a_R(bc) + a(bc)_L - a_R(bc)_L$.

By substituting elements from $(bc)_L$ and $(bc)_R$ into those of $[a(bc)]_L$, we get the following:

$a_L(bc) + a(b_Lc + bc_L - b_Lc_L) - a_L(b_Lc + bc_L - b_Lc_L)$,

$a_L(bc) + a(b_Rc + bc_R - b_Rc_R) - a_L(b_Rc + bc_R - b_Rc_R)$,

$a_R(bc) + a(b_Lc + bc_R - b_Lc_R) - a_R(b_Lc + bc_R - b_Lc_R)$, and

$a_R(bc) + a(b_Rc + bc_L - b_Rc_L) - a_R(b_Rc + bc_L - b_Rc_L)$.

By substituting elements from $(bc)_L$ and $(bc)_R$ into those of $[a(bc)]_R$, we get the following:

$a_L(bc)+a(b_Lc+bc_R-b_Lc_R)-a_L(b_Lc+bc_R-b_Lc_R)$,

$a_L(bc)+a(b_Rc+bc_L-b_Rc_L)-a_L(b_Rc+bc_L-b_Rc_L)$,

$a_R(bc)+a(b_Lc+bc_L-b_Lc_L)-a_R(b_Lc+bc_L-b_Lc_L)$, and

$a_R(bc)+a(b_Rc+bc_R-b_Rc_R)-a_R(b_Rc+bc_R-b_Rc_R)$.

The left set of (ab) contains $a_Lb+ab_L-a_Lb_L$ and $a_Rb+ab_R-a_Rb_R$.

The right set of (ab) contains $a_Lb+ab_R-a_Lb_R$, and $a_Rb+ab_L-a_Rb_L$.

The left set of $(ab)c$ contains $(ab)_Lc+(ab)c_L-(ab)_Lc_L$, and $(ab)_Rc+(ab)c_R-(ab)_Rc_R$.

The right set of $(ab)c$ contains $(ab)_Lc+(ab)c_R-(ab)_Lc_R$, and $(ab)_Rc+(ab)c_L-(ab)_Rc_L$.

By substituting elements from $(ab)_L$ and $(ab)_R$ into those of $[(ab)c]_L$, we get the following:

$(a_Lb+ab_L-a_Lb_L)c+(ab)c_L-(a_Lb+ab_L-a_Lb_L)c_L$,

$(a_Rb+ab_R-a_Rb_R)c+(ab)c_L-(a_Rb+ab_R-a_Rb_R)c_L$,

$(a_Lb+ab_R-a_Lb_R)c+(ab)c_R-(a_Lb+ab_R-a_Lb_R)c_R$, and

$(a_Rb+ab_L-a_Rb_L)c+(ab)c_R-(a_Rb+ab_L-a_Rb_L)c_R$.

By substituting elements from $(ab)_L$ and $(ab)_R$ into those of $[(ab)c]_R$, we get the following:

$(a_Lb+ab_L-a_Lb_L)c+(ab)c_R-(a_Lb+ab_L-a_Lb_L)c_R$,

$(a_Rb+ab_R-a_Rb_R)c+(ab)c_R-(a_Rb+ab_R-a_Rb_R)c_R$,

$(a_Lb+ab_R-a_Lb_R)c+(ab)c_L-(a_Lb+ab_R-a_Lb_R)c_L$, and

$(a_Rb+ab_L-a_Rb_L)c+(ab)c_L-(a_Rb+ab_L-a_Rb_L)c_L$.

Using the distributive property established in (9.6), the elements in $[(ab)c]_L$ are:

$(a_Lb)c+(ab_L)c-(a_Lb_L)c+(ab)c_L-(a_Lb)c_L-(ab_L)c_L+(a_Lb_L)c_L$,

$(a_Lb)c+(ab_R)c-(a_Lb_R)c+(ab)c_R-(a_Lb)c_R-(ab_R)c_R+(a_Lb_R)c_R$,

$(a_Rb)c+(ab_L)c-(a_Rb_L)c+(ab)c_R-(a_Rb)c_R-(ab_L)c_R+(a_Rb_L)c_R$, and

$(a_R bc) + (ab_R)c - (a_R b_R)c + (ab)c_L - (a_R b)c_L - (ab_R)c_L + (a_R b_R)c_L$.

Using the distributive property, the elements in $[a(bc)]_L$ are:

$a_L(bc) + a(b_L c) + a(bc_L) - a(b_L c_L) - a_L(b_L c) - a_L(bc_L) + a_L(b_L c_L)$,

$a_L(bc) + a(b_R c) + a(bc_R) - a(b_R c_R) - a_L(b_R c) + -a_L(bc_R) + a_L(b_R c_R)$,

$a_R(bc) + a(b_L c) + a(bc_R) - a(b_L c_R) - a_R(b_L c) + -a_R(bc_R) + a_R(b_L c_R)$, and

$a_R(bc) + a(b_R c) + a(bc_L) - a(b_R c_L) - a_R(b_R c) + -a_R(bc_L) + a_R(b_R c_L)$.

By distributing, the elements in $[(ab)c]_R$ are:

$(a_L b)c + (ab_L)c - (a_L b_L)c + (ab)c_R - (a_L b)c_R - (ab_L)c_R + (a_L b_L)c_R$,

$(a_R b)c + (ab_R)c - (a_R b_R)c + (ab)c_R - (a_R b)c_R - (ab_R)c_R + (a_R b_R)c_R$,

$(a_L b)c + (ab_R)c - (a_L b_R)c + (ab)c_L - (a_L b)c_L - (ab_R)c_L + (a_L b_R)c_L$, and

$(a_R b)c + (ab_L)c - (a_R b_L)c + (ab)c_L - (a_R b)c_L - (ab_L)c_L + (a_R b_L)c_L$.

Also, by distributing, the elements in $[a(bc)]_R$ are:

$a_L(bc) + a(b_L c) + a(bc_R) - a(b_L c_R) - a_L(b_L c) - a_L(bc_R) + a_L(b_L c_R)$,

$a_L(bc) + a(b_R c) + a(bc_L) - a(b_R c_L) - a_L(b_R c) - a_L(bc_L) + a_L(b_R c_L)$,

$a_R(bc) + a(b_L c) + a(bc_L)a(b_L c_L) - a_R(b_L c) - a_R(bc_L) + a_R(b_L c_L)$, and

$a_R(bc) + a(b_R c) + a(bc_R) - a(b_R c_R) - a_R(b_R c) - a_R(bc_R) + a_R(b_R c_R)$.

For all $x \in \{a,b,c\}$, let x' denote $x' \in \{x, x_L, x_R\}$.

By careful examination, we can see that each of the members of $[a(bc)]_L$ corresponds to a member of $[a(bc)]_R$ consisting of corresponding terms that match the requirement $(a'b')c' = a'(b'c')$, where at least one of the three numbers in the term is a member of the left or right sets of a, b, or c, and satisfying the assumption we originally made. Thus, each is true and, by transfinite induction, $[(ab)c]_L = [a(bc)]_L$.

Similar comparisons show that, by transfinite induction, $[(ab)c]_L = [a(bc)]_R$.

Since each set in $(ab)c$ is equal to its corresponding set in $a(bc)$, we conclude that $(ab)c = a(bc)$.

□

Regarding multiplicative inverses:

It has already been established that, for all $x \in \mathbf{No}$, a multiplicative inverse $\frac{1}{x}$ exists and that it is defined by $\frac{1}{x} = y$, where $xy = 1$ (See page 28). During the previous discussion, we discussed an algorithm for constructing $\frac{1}{x}$. That method may or may not work suitably for all $x \in \mathbf{No}$. (To see what I mean, think about using that method to construct $\frac{1}{\omega - 3}$. If you are very clever, you may just pull it off, but it will likely be rather difficult.)

Luckily, Conway has created a recursive definition of $\frac{1}{x}$ that will work for any $x \in \mathbf{No}$. It is constructed in a very unusual way, and will require some explanation.

DEFINITION 9.8. Let $x \in \mathbf{No}$ such that $x > 0$. There exists unique $y \in \mathbf{No}$, such that $xy = 1$. It is defined as follows:

$$y = \left\{ 0, \frac{1 + (x_R - x)y_L}{x_R}, \frac{1 + (x_R - x)y_R}{x_L} \,\middle|\, \frac{1 + (x_L - x)y_L}{x_L}, \frac{1 + (x_R - x)y_R}{x_R} \right\}.$$

Conway goes on to point out that the definition of y refers directly to y_L and y_R, admitting that it might seem strange. He explains that y_L and y_R are considered typical "older" elements of the left and right sets of y and that newer elements are defined in terms of the old ones.

Then, in a footnote on page 21 of ONAG, he gives an example of the use of this definition, which is being included here verbatim (with slight changes in notation):

> "To see how this definition works, take $x = \{0, 2|\} = 3$. Then there is no x_R and the only x_L is 2, so $x_L - x = -1$ and the formula for y becomes $y = \left\{0, \frac{1}{2}(1 - y_R) \,\middle|\, \frac{1}{2}(1 - y_L)\right\}$. The initial value $y_L = 0$ gives us $\frac{1}{2}(1 - 0) = \frac{1}{2}$ for a new y_R, whence $\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$ as a y_L, then $\frac{1}{2}(1 - \frac{1}{4}) = \frac{3}{8}$ for a y_R, and so on, yielding $y = \left\{0, \frac{1}{4}, \frac{5}{16}, \ldots \,\middle|\, \frac{1}{2}, \frac{3}{8} \ldots\right\}$, which certainly looks like $\frac{1}{3}$." [1]

This definition for the multiplicative inverse of a number is certainly inventive, and I could not hope to improve upon it by trying to describe it, which is why it was included as it appears in Conway's book. I should add that he then goes on immediately to prove that this definition does indeed provide an actual multiplicative inverse for each number in **No**, and refer the reader to the book in order to see that proof, should the reader indeed wish to see it.

According to Dummit and Foote in [6], a field $(S,+,\times)$ is defined as a set S, together with two binary operations, addition $(+)$ and multiplication $(\times)$, satisfying all of the following conditions:

(i) $(S,+)$ is an abelian group,

(ii) $\times$ is associative,

(iii) $\times$ distributes over $+$, for all $a,b,c \in S$, that is,

$a \times (b+c) = a \times b + a \times c$ for all $a,b,c \in S$,

(iv) multiplication is commutative,

(v) a multiplicative identity exists in S, that is,

there exists $1 \in S$ such that$1 \neq 0$ and $1 \times a = a \times 1 = a$ for all $a \in S$,

(vi) for all $x \in S$, there exists $y \in S$ such that $x \times y = 1$.

We have seen that **No** exhibits all of these properties under addition and multiplication.

Technically, **No** is a proper class, not a set, and so rather than referring to $(\mathbf{No},+,\cdot)$ as a field, technically speaking, we should refer to it as a proper class with field properties. However, this distinction is truly just a technicality (and one Conway apparently considers quite unjust).

Obviously, there is much more to explore. Definitions for higher arithmetic operations have been created by Conway and others. Exponents, radicals, and logarithms have all been defined, and many other topics have been given a thorough investigation. If your interest is piqued, please refer to the source material for more advanced material, or continue experimenting and see what you can discover.

The purpose of this was just to give a basic, yet thorough introduction to the Surreal Numbers and their basic properties. Hopefully, that has been accomplished.

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Vita

Josh Hostetler was born in the mid-seventies in Richmond, Virginia, where he grew up a statistical anomaly and something of a jack-of-all-trades.

In the beginning, his elementary school was piloting an at-your-own-pace program. By the end of first grade, he had exceeded the elementary-school level mathematics and language skills curricula. But then he moved…

The new school did not have such a program. His second grade teacher vehemently expressed her refusal to believe or act on the fact that he was, indeed, past the second-grade level in any subject.

So, Josh stagnated for many years, bored and disgruntled.

He realized he had an affinity for mathematics during recess in the third or fourth grade, when some of the other kids would insist on challenging him with verbal assaults like, "I bet you don't know what 5 times 1,000,000 is…" Though such challenges did not really require any mathematical excellence to quell, they did spawn an interest in the subject for Josh.

Between sixth and seventh grades, at the beach, he read James Gleick's Chaos: Making a New Science[9] cover to cover, understanding very little of it, as it did not include passages explaining what variables were or the like. Regardless, he was too fascinated by the book to do beach things.

After that, there was more stagnation for many years.

In the seventh grade, spawned by a Social Studies lesson about the electoral college, Josh began occasionally mismatching his socks as a symbolic (albeit somewhat obtuse) gesture that was surprisingly effective in expressing an utter disinterest in participating in the norms of a society that lies to its children.

In eighth grade, Josh was not allowed to take algebra because of his seventh-grade refusal to do homework to practice things he had learned seven years before. In pre-algebra class one day, he noticed a pattern regarding squaring the sum of two numbers. Excited, he wrote down an expression of the pattern, using ? as his variable (since he still had not been introduced to them by anyone), and he took it to show his teacher. Her response was horrid, yet typical coming from one one of his teachers:

"Oh Josh… That's just algebra… Go sit down."

Irritating as this teacher's attitudinally disinterested and potentially educationally stifling statement was, it gave Josh the right to claim for the remainder of his days that he had discovered algebra.

In ninth grade, the sock thing became permanent, and he joined the math team, which was undefeated all year.

In tenth grade, in geometry class, Josh noticed a pattern involving right triangles, which he painstakingly wrote up as a theorem and then proved. Though the theorem turned out to be nothing more than a very simple corollary to Pythagorean Theorem, thereby rendering it small beans compared to the teenage discoveries of the great mathematicians in history, it gained him awe and respect of a handful of teachers at Henrico High School. One of those teachers even gave him his first computer, a then fourteen-year-old Franklin Ace 1200, free of charge, which he obsessively programmed to do all kinds of crazy things. The teacher was impressed.

During the same period, one of Josh's math teachers (there were two, as his guidance counselor had realized the error in his having not taken algebra two years before, and allowed him to double up his math classes in tenth grade) expressed attitudinally disinterested and potentially educationally stifling statements in the form of answers to his questions in class. (For example: “Where did quadratic formula come from?” “Oh, I don’t know Josh… Some old guy made it up hundreds of years ago.” Flippancy is always a helpful teaching aid…) As a consequence, while he got a perfect score in Geometry that year, he only got a C in Algebra II.

Despite this, the math team remained undefeated during tenth grade, and Josh received an award for placing in the top twenty percent at the VCTM - VCU Statewide Mathematics Contest. Some people were impressed.

In eleventh grade, Josh seriously studied acting at the Center for the Arts, started taking Japanese, and became captain of the math team, which remained undefeated. As a project for Japanese class that year, he created a simple yet fully functional Japanese word processor on his Franklin. A few people were impressed.

In twelfth grade, Josh was still captain of the math team, became president of the Henrico High chapter of the National Thespian Society, and worked at a video store to pay for gas and car insurance (which was horrifyingly expensive due to his demographic and corporate use of actuaries in policy-making, despite his impeccable driving record and statistical anomaly status), and the SAT (which, in his household was optional, and therefore his responsibility).

The math team remained undefeated.

At the end of the year, Josh was given the Japanese Award, for having the highest grade in Japanese class among anyone in the state (122%), and the Math Award, which had been specially created for him, due to that now unimpressive theorem and the continued undefeated status of math team, by the teacher who had given him his computer. She was clearly impressed.

After high school, Josh went to VCU, where he floated around without declaring a major for a handful of years. He did not study math at all.

He took classes in a wide variety of disciplines, and started making music under the moniker "Antmanmusic". Eventually, he decided to major in filmmaking and film history. He made three films during his undergraduate college career (two of which won awards and one of which traveled the East Coast in a touring festival) and graduated with a BGS in film studies.

Then he got a job in the advertising industry which he quickly lost in a wave of layoffs that occurred due to a sudden lack of business that was spawned by the 2000 presidential election result. For the next five years, laid off and unable to make a film due to the expense, he worked in various jobs, mostly as a temp, for not very much money at all and with sporadic, unpredictable hours and job opportunities. Antmanmusic continued during this period, making music that was often mathematical in nature and created using algorithmic composition methods.

This reinvigorated an interest in studying math in Josh. He checked out books about sequences (particularly the Fibonacci sequence) and elementary group theory. Eventually, this renewed interest led him to use his alumnus status to gain entry to seek advice from Dr. Wood at VCU about the possibility of attending graduate school for mathematics. By the end of that meeting, Josh had inexplicably been admitted provisionally with a scholarship to pay for a year of undergraduate math courses before he could enter graduate school.

Many years later, he finished this thesis.

Incidentally, Josh has twelve brothers and sisters (from at least five different sets of parents… probably six, but who can be sure?), a huge, mostly local, close and supportive extended family. He has a profound case of ADD/ADHD-Combined type, which went undiagnosed until he was 32. He drinks exorbitant amounts of coffee, smokes like a chimney, and still mismatches his socks.