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A Lexicographic Product Cancellation Property for Digraphs

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College of Humanities and Sciences
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This is to certify that the thesis prepared by Kendall Lee Manion titled "A Lexicographic Product Cancellation Property for Digraphs" has been approved by his committee as satisfactory completion of the thesis requirement for the degree of Master of Science.

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A Lexicographic Product Cancellation Property for Digraphs

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Master of Science

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Richmond, Virginia
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Contents

Abstract

A LEXICOGRAPHIC PRODUCT CANCELLATION PROPERTY FOR DIGRAPHS

By Kendall Lee Manion, Master of Science.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

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There are four prominent product graphs in graph theory: Cartesian, strong, direct, and lexicographic. Of these four product graphs, the lexicographic product graph is the least studied. Lexicographic products are not commutative but still have some interesting properties. This paper begins with basic definitions of graph theory, including the definition of a graph, that are needed to understand theorems and proofs that come later. The paper then discusses the lexicographic product of digraphs, denoted $G \circ H$, for some digraphs G and H. The paper concludes by proving a cancellation property for the lexicographic product of digraphs G, H, A, and B: if $G \circ H \cong A \circ B$ and $|V(G)| = |V(A)|$, then $G \cong A$. It also proves additional cancellation properties for lexicographic product digraphs and the author hopes the final result will provide further insight into tournaments.

Chapter 1: Introduction

The study of graph theory began in the eighteenth century with the great mathematician Leonhard Euler's proof of the Köningsberg bridges problem. Once considered recreational mathematics, graph theory has today evolved to be an efficient tool for modeling problems in many different fields and its applications widely range from logistics, communication, data organization, flow of computation, social network analysis, and molecule structures in chemistry and physics. Graph theory features many operations that can be performed on graphs, including four prominent graph products: the Cartesian product, the strong product, the direct product, and the lexicographic product.

Each of these graph products has its own unique characteristics that make it interesting in its own right; however, the focus of this paper is on a cancellation property specific to the lexicographic product of digraphs. Before discussing cancellation of the lexicographic product of digraphs, many definitions, properties and propositions of graphs and digraphs will be introduced. We begin with some very basic information, followed by the definition of the lexicographic product of graphs, and stating some of its specific properties. We conclude with a cancellation property for the lexicographic product of digraphs.

1.1 Preliminaries

The basis of all graph theory is the definition of a graph. Informally, a graph is a collection of vertices, or nodes, along with a set of distinct 2-element subsets of vertices called edges. Edges are also referred to as lines or links. The precise definition of a graph is given below.

DEFINITION 1.1. A graph G, is a set V of objects called vertices (the singular is vertex) together with a possibly empty set E of distinct 2-element subsets of V called edges. The vertex set of G is denoted by $V(G)$ and the edge set is denoted by $E(G)$. The Order of a digraph D is the number of vertices in D and is denoted $|V(D)|$. The Size of a digraph D is the number of arcs in D and is denoted $|A(D)|$. Vertices of G are generally written as a single letter, such as $u, v \in V(G)$ and edges are written as vertex pairs such as $uv \in E(G)$.

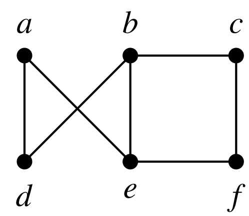

Figure 1.1: An Example of a Graph with $V(G) = \{a,b,c,d,e,f\}$ and $E(G) = \{ae,ad,bc,bd,be,cf,ef\}$.

The definition of a graph given in Definition 1.1 eliminates multiple edges. Given two vertices $u, v \in G$, the edges $uv, uv \in E(G)$ are said to be multiple since they connect the same two vertices. This paper also assumes that for any $u \in G$ $uu \notin E(G)$ for all $u \in V(G)$, that is the graph has no loops. Figure 1.2 and provides an example of a graph that is not simple because it contains loops. Figure 1.3 provides an example of a graph that is not simple because it has multiple edges. Figure 1.1 is an example of a simple graph.

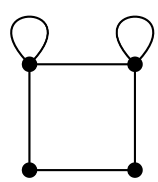

Figure 1.2: A graph that has loops.

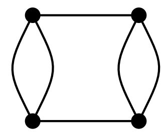

Figure 1.3: A graph that has multiple edges.

If edges have direction associated with them, the graph is called a directed graph, or digraph, or the elements in the 2-element subsets are ordered.

DEFINITION 1.2. A directed graph, or digraph D is a set of objects called vertices, together with a set of ordered pairs of distinct vertices of D called arcs, or directed edges. The set of vertices is denoted $V(D)$ and the set of arcs is denoted $A(D)$.

Recall that a simple graph is a graph without multiple uv-edges or loops. As with graphs, digraphs in this paper do not have multiple uv-arcs or loops. These digraphs, understood to be simple digraphs, are the focus of this paper. The concept of an underlying graph is useful with many topics related to digraphs. Definition 1.3 provides the definition of an underlying graph.

DEFINITION 1.3. For a digraph D, the underlying graph G is the graph with $V(D) = V(G)$ and $uv \in E(G)$ if $uv \in A(D)$ or $vu \in A(D)$. The direct edge uv, from u to v is called an arc.

1.2 Properties of Digraphs

Digraphs and graphs share many similar properties, however there are also some differences. If interested in similarities and differences between graphs and digraphs see *Chartrand, Lesniak, and Zhang* [1].

This section explores some properties of simple digraphs, which will enable us to investigate cancellation properties of lexicographic product of digraphs. For the rest of this paper, the term *digraph* explicitly implies a simple digraph unless otherwise stated.

Every digraph has certain characteristics. One of the characteristics of digraphs is the notion of its order, $|V(D)|$, and its size, $|A(D)|$. As discussed in Definition 1.1, a digraph's order is the number of vertices in its vertex set and the digraph's size is the number of arcs in its arc set. In Figure 1.4, $|V(D)| = 4$ and $|A(D)| = 6$.

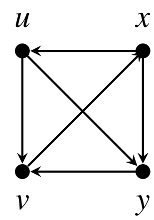

Figure 1.4: An illustration of a digraph D.

Definition 1.2 provides a foundation for understanding of digraphs. Figure 1.4 illustrates a digraph with $V(D) = \{u, v, x, y\}$ and $A(D) = \{xu, xy, yv, vx, uv, uy\}$.

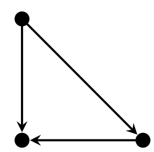

Figure 1.5: A subdigraph of the digraph D in Figure 1.4.

Given a digraph G, two new digraphs can be generated from G called the subdigraph and the subdigraph induced on S.

DEFINITION 1.4. A digraph H, is a subdigraph of a digraph G, if $V(H) \subseteq V(G)$ and $A(H) \subseteq A(G)$. Digraph H is a proper subdigraph of G if $V(H) \subset V(G)$ or $A(H) \subset A(G)$.

Given a digraph G, and a subset $S \subseteq V(G)$, the subdigraph induced on S, denoted $\langle S \rangle$, has vertex set S, and for any $x, y \in S$, $xy \in E(\langle S \rangle) \iff xy \in E(G)$.

In a digraph, two vertices are adjacent if there is an arc that connects them. For vertices $u, v \in V(G)$, the vertex u is said to be adjacent from the vertex v if $vu \in A(G)$ and the vertex

v is said to be adjacent to the vertex u if $vu \in A(G)$. Figure 1.4 illustrates adjacencies in a digraph. The vertex u is adjacent to v and the vertex v is adjacent from u. Figure 1.6 graphically shows the complete digraph K_3.

DEFINITION 1.5. The complete digraph K_n is the digraph on n vertices, for which every pair of vertices are adjacent, that is, if $u, v \in V(K_n)$ then both uv and vu belong to $A(K_n)$.

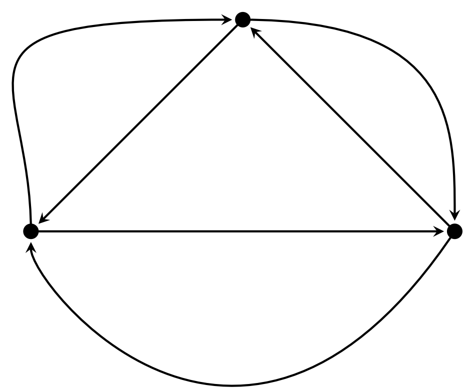

Figure 1.6: The complete digraph K_3.

One of the common digraphs (and graphs) one can work with is called a path, which is defined in Definition 1.6.

DEFINITION 1.6. A Path is a digraph P_n for some integer n whose vertices can be labeled $v_1, v_2, \ldots, v_n$ and $E(P) = \{v_i v_j : j = i+1\}$, where $(1 \leq i \leq |V(P_n)| - 1)$.

A path is a specific type of digraph, but it can also be a subdigraph. For vertices $u, v \in V(D)$ a u, v-path is a series of arcs that connect vertex u to vertex v. In a graph, a u, v-path and a v, u-path are equivalent. However, in digraphs, a u, v-path is not the same as a v, u-path. If a u, v-path exists a v, u-path does not necessarily exist. Definition 1.7 gives the

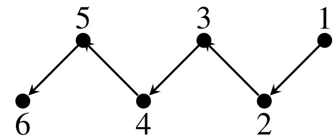

Figure 1.7: Path on Six Vertices (P_6)

formal definition and Figure 1.8 provides an example of digraph with a u,v-path that does not contain a v,u-path. The direction in which a path is traversed is important in digraphs since the orientation of an arc begins at one vertex and ends at another. If the direction is reversed it goes against the orientation. In Figure 1.8, a u,v-path traverses u,x,v. However, a v,x,u-path does not exist.

DEFINITION 1.7. If $u,v \in V(D)$ for some digraph D, a u,v-path is defined to be a series of arcs that connect the vertex u with the vertex v in D.

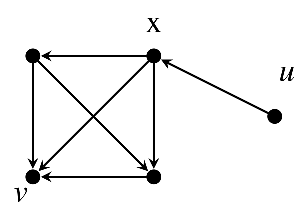

Figure 1.8: A digraph with a u,v-Path but no v,u-Path.

Now that u,v-paths have been defined, we investigate the concept of connectedness. Two vertices u and v are said to be connected if there is a u,v-path. A graph G is said to be connected if every two vertices of G are connected. A digraph D is said to be weakly connected if the underlying graph is connected. Figure 1.8 is an example of a weakly connected digraph, since the underlying graph is connected.

A digraph D is said to be strongly connected if for every $u,v \in V(D)$ there exists both a u,v and a v,u path. The digraph in Figure 1.4 is a strongly connected digraph.

DEFINITION 1.8. A digraph D is said to be strongly connected, if for every $u,v \in V(D)$ both a u,v and a v,u path exist.

Chapter 2: Homomorphisms and an Introduction to Lexicographic Product Digraphs

Chapter 1 covered basic concepts in graph theory necessary to explore the lexicographic product of digraphs. This chapter covers more specific material including homomorphisms and isomorphisms between digraphs and an introduction to the lexicographic product of digraphs.

2.1 Injective Homomorphisms

We can define functions on graphs or digraphs. One such function is a homomorphism. A homomorphism is an adjacency preserving mapping from a digraph G to a digraph H, defined as follows:

DEFINITION 2.1. A function $\phi : V(G) \to V(H)$ is a homomorphism if $uv \in A(G)$ implies $\phi(u)\phi(v) \in A(H)$, for all $u, v \in V(G)$. A homomorphism ϕ is injective if ϕ is injective as a map on sets.

DEFINITION 2.2. Two digraphs G and H are isomorphic, written $G \cong H$ if there exists a bijective function $\theta : V(G) \to V(H)$ such that $uv \in A(G) \iff \theta(u)\theta(v) \in A(H)$.

Assume there are two digraphs, G and H, and that G has an injective homomorphism into H, and that H has an injective homomorphism into G. Theorem 2.3 shows that G is then isomorphic to H.

THEOREM 2.3. If there are injective homomorphisms $\alpha : G \to H$ and $\beta : H \to G$, then $G \cong H$.

Proof. Let $\alpha : G \to H$ and $\beta : H \to G$ be injective homomorphisms. Since α and β are both injective as maps on sets, $|V(G)| = |V(H)|$. Note: α^{-1} does not necessarily equal β. Since α is an injective homomorphism, $|A(G)| \leq |A(H)|$, and β is an injective homomorphism, $|A(H)| \leq |A(G)|$, thus $|A(G)| = |A(H)|$. Recall Definition 2.2. We know that α is injective and that if $uv \in A(G)$ then $\alpha(u)\alpha(v) \in A(H)$. We need to show that if $\alpha(u)\alpha(v) \in A(H)$, then $uv \in A(G)$. Suppose $\alpha(u)\alpha(v) \in A(H)$ but $uv \notin A(G)$. Since H has an arc $\alpha(u)\alpha(v)$ and $uv \notin A(G)$ this implies $|A(G)| < |A(H)|$. But since β is an injective homomorphism, $|A(H)| \leq |A(G)|$. This is a contradiction. Hence $G \cong H$. $\square$

2.2 The Lexicographic Product of Digraphs and Its Size

This section provides basic information about the lexicographic product of digraphs. First, we define the lexicographic product of digraphs and then define an H-layer of a lexicographic product of digraphs, which is also called a fiber. Each of these definitions can be found in *Hammack, Imrich, and Klavžar* [6].

DEFINITION 2.4. The lexicographic product $G \circ H$ of digraphs G and H is defined as:

$$V(G \circ H) = \{(g,h) | g \in V(G), h \in V(H)\},$$

$$A(G \circ H) = \{(g,h)(g',h') \,|\, gg' \in A(G) \text{ or } g = g' \text{and } hh' \in A(H)\}.$$

The lexicographic product is associative, that is, for digraphs G, H, and K, $(G \circ H) \circ K = G \circ (H \circ K)$. However, it is not necessarily commutative. We illustrate the lexicographic product of digraphs using the disjoint union of two digraphs. A disjoint union is the operation on two or more digraphs where each digraph is drawn, but no arcs are adjacent to a vertex from another digraph. Figures 2.1 and 2.2 illustrate a lexicographic product of two digraphs that are not commutative.

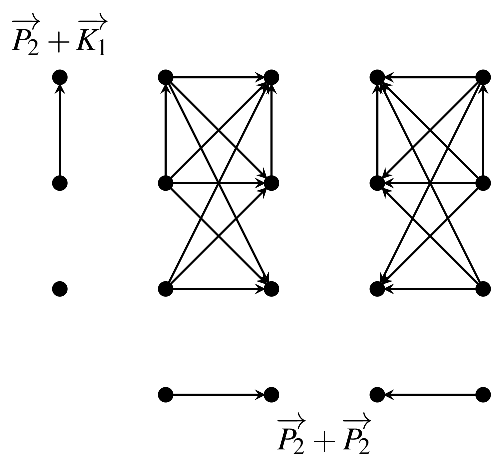

Figure 2.1: The lexicographic product of two disjoint union digraphs.

DEFINITION 2.5. Given a vertex $h = (h_1, h_2, \ldots, h_k)$ of the product

$$H = H_1 \circ H_2 \circ \cdots \circ H_k,$$

the H_i-layer through $h = (h_1, h_2, \ldots, h_k)$ is the induced subdigraph

$$H_i^h = \langle \{x \in V(H) | p_j(x) = h_j \, for \, j \neq i\} \rangle$$

$$= \langle \{(h_1, h_2, \ldots, x_i \ldots h_k) \,|\, x_i \in V(H_i)\} \rangle,$$

where p_j is a projection map defined as $p_i = (x_1, x_2, \ldots, x_k) = x_i$.

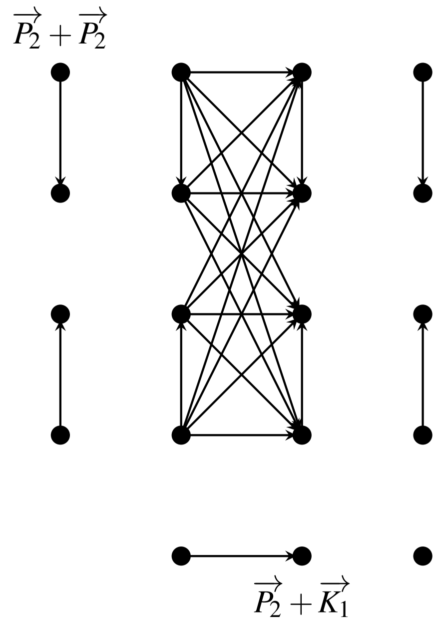

Figure 2.2: The lexicographic product of two disjoint union digraphs.

Note: In a lexicographic product $G \circ H$, there are precisely $|V(G)|$ H-layers.

DEFINITION 2.6. Given a digraph H_i and $uv \in A(H_i)$ and a product graph $H_1 \circ H_2$ and $hh' \in A(H_1 \circ H_2)$ where $h = (x_1, x_2)$ and $h' = (x_1', x_2')$, then the fiber-over-the-arc uv is defined as:

$$A(H_i)_{uv} = \{hh' \in A(H_1 \circ H_2) | \pi_i(h)\pi_i(h') \in A(H_i)\},$$

where π_i is a projection mapping.

Note: In a lexicographic product $G \circ H$, there are precisely $|A(G)|$ fibers-over-the-arcs. Also, note that the disjoint union of arcs of the fibers-over-the-arcs and the H_i-layers make the totality of the arcs of $G \circ H$.

Given digraphs G and H the size of $G \circ H$ can be calculated from the order and size of both G and H, as follows:

PROPOSITION 2.7. If $G \circ H$ is a Lexicographic Product, then

$$|A(G \circ H)| = |V(H)|^2 \cdot |A(G)| + |V(G)| \cdot |A(H)|.$$

Proof. Let $G \circ H$ be a lexicographic product of digraphs. (Recall Definition 2.4.) By definition of the lexicographic products of digraphs, $(g,h)(g'h') \in A(G \circ H)$ if either $g = g'$ and $hh' \in A(H)$ or $gg' \in A(G)$.

Suppose $g = g'$ and $hh' \in A(H)$. Notice when $g = g'$ an H-layer is created (Definition 2.5). Each H-layer has $|A(H)|$ arcs. And there are $|V(G)|$ H-layers in $G \circ H$. Therefore, there are $|V(G)| \cdot |A(H)|$ arcs in all H-layers of $G \circ H$.

Now suppose $gg' \in A(G)$. By definition of the lexicographic product of digraphs, if $gg' \in A(G)$ then there is an arc between each vertex in adjacent H-layers. Recall, each H-layer has $|V(H)|$ vertices. So there are $|V(H)|$ arcs from each vertex in adjacent H-layers and there are $|V(H)|$ vertices in each H-layer. So there are $|V(H)|^2$ arcs in each fiber-over-the-arc. There are $|A(G)|$ fibers in $G \circ H$, so the sum of the number of all arcs in every fiber of $G \circ H$ is $|V(H)|^2 \cdot |A(G)|$. Hence $|A(G \circ H)| = |V(H)|^2 \cdot |A(G)| + |V(G)| \cdot |A(H)|$. $\square$

In order to show that Proposition 2.7 holds, refer to Figure 2.1. If you count the arcs in each fiber-over-the-arc it is nine, notice that nine is the square of the number of vertices in $\overrightarrow{P_2} + \overrightarrow{K_1}$ and that there are two fibers-over-the-arcs, which is the same as the size of $\overrightarrow{P_2} + \overrightarrow{P_2}$. Now observe each H-layer has exactly one arc, which equals $|A(\overrightarrow{P_2} + \overrightarrow{K_1})|$ and as noted earlier there are $|V(G)|$ H-layers in a lexicographic product of digraphs. The result holds.

Note: $(\overrightarrow{P_2} + \overrightarrow{K_1}) \circ (\overrightarrow{P_2} + \overrightarrow{P_2}) \neq (\overrightarrow{P_2} + \overrightarrow{P_2}) \circ (\overrightarrow{P_2} + \overrightarrow{K_1})$, that is, the lexicographic product of digraphs is not necessarily commutative, as illustrated in Figures 2.1 and 2.2.

Chapter 3: Lexicographic Product Digraph Cancellation

In Chapter 2 we defined the lexicographic product of digraphs and showed that it is not necessarily commutative. This chapter explores cancellation properties of lexicographic product digraphs.

3.1 Number of Injective Homomorphisms

We begin this chapter the definition of a weak homomorphism and continue with the definition of the number of homomorphisms and injective homomorphisms between two digraphs.

DEFINITION 3.1. A weak homomorphism $\phi : G \to H$, where G and H are digraphs, is a map $\phi : V(G) \to V(H)$ for which $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(H)$ or $\phi(u) = \phi(v)$.

Note that every lexicographic product of digraphs has a weak homomorphism, $\phi : G \circ H \to G$ by mapping each H-layer to one vertex of G. Specifically, $\phi(g,h) = g$ for every $(g,h) \in G \circ H$.

DEFINITION 3.2. If G and H are digraphs, then $\hom(G,H)$ is the number of homomorphisms $\theta : G \to H$, and $\mathrm{inj}(G,H)$ is the number of injective homomorphisms $\psi : G \to H$.

DEFINITION 3.3. Suppose we have an arbitrary partition Ω of $V(G)$, with nonempty sets $S_i, i \in I$. We define the quotient of G by Ω to be the digraph G/Ω with vertex set Ω, and $S_iS_j \in E(G/\Omega)$ if some $uv \in E(G)$ has $u \in S_i$ and $v \in S_j$.

The number of injective homomorphisms from a digraph X into a digraph $G \circ H$ can be counted as well. Theorem 3.4 provides a formula that determines the number of injective homomorphisms from a digraph X to the product $G \circ H$.

THEOREM 3.4. If X, G, and H are digraphs, then

$$\operatorname{inj}(X, G \circ H) = \sum_{\Omega \in \mathcal{P}(V(X))} \left(\operatorname{inj}(X/\Omega, G) \cdot \prod_{\theta \in \Omega} \operatorname{inj}(\langle \theta \rangle, H) \right),$$

where $\mathcal{P}(V(X))$ is the set of all partitions on $V(X)$.

Proof. Let X, G, and H be digraphs. Let $\operatorname{Inj}(X, G \circ H)$ be the set of injective homomorphisms from X to $G \circ H$, and $w : G \circ H \to G$ be the weak homomorphism where $w(g,h) = g$ for all $(g,h) \in V(G \circ H)$. Let z the projection from $G \circ H$ defined by $z(g,h) = h$ for all $(g,h) \in V(G \circ H)$.

Any function f gives rise to a partition of $V(X)$ defined as

$$\Omega_f = \{(wf)^{-1}(wf(x)) : x \in V(X)\}.$$

Even though f is not an injective homomorphism from X to G, f can be regarded as an injective homomorphism to G from the quotient X/Ω_f. This is true as follows:

Given such f, define the function $r : X/\Omega_f \to G$, where
$r(U) = f(U) \text{for} U \subseteq \Omega_f$. We will show that r is an injective homomorphism.

Assume there exists an $x_1, x_2 \in X/\Omega_f$, such that $r(x_1) = r(x_2)$ and $x_1 \neq x_2$. Recall, Ω_f partitions $V(X)$ by collecting all $x \in V(X)$ such that $f(x) = g \in V(G)$. Since $x_1 \neq x_2$, these sets are different in Ω_f, but if $r(x_1) = r(x_2)$ then $f(x_1) = f(x_2)$ which contradicts the definition of Ω_f, and hence r is injective.

Now we need to show the function, r, is an homomorphism. Recall, $r(U) = \phi(U)$ and ϕ is an homomorphism. Since r is defined by the preimage of the homomorphism ϕ, if $x_1x_2 \in A(X/\Omega_\phi)$ then $r(x_1)r(x_2)$ must be in $A(G)$ or r contradicts its definition. Therefore, r is an injective homomorphism.

We already defined $\mathrm{Inj}(X, G \circ H)$ to be the set of injective homomorphisms from X to $G \circ H$, formally $\mathrm{Inj}(X, G \circ H) = \{\phi \mid \phi : X \to G \circ H \text{ is an injective homomorphism}\}$, and $|\mathrm{Inj}(X, G \circ H)| = \mathrm{inj}(X, G \circ H)$. The partition Ω_ϕ is one element of $\mathcal{P}(V(X))$.

Next, define the set Δ to be

$$\Delta = \bigcup_{\Omega \in \mathcal{P}(V(X))} \left(\mathrm{Inj}(X/\Omega, G) \times \prod_{\theta_i \in \Omega} \mathrm{Inj}(\langle \theta_i \rangle, H) \right),$$

where $\langle \theta_i \rangle$ is the subdigraph induced on $\theta_i \subseteq V(X)$. By definition of the cardinality of a Cartesian product,

$$|\Delta| = \sum_{\Omega \in \mathcal{P}(V(X))} \left(\mathrm{inj}(X/\Omega, G) \prod_{\theta_i \in \Omega} \mathrm{inj}(\langle \theta_i \rangle, H) \right).$$

The theorem follows once we show that there is a bijection $\Psi : \mathrm{Inj}(X, G \circ H) \to \Delta$. Consider $\Psi : \mathrm{Inj}(X, G \circ H) \to \Delta$ such that for $f \in \mathrm{Inj}(X, G \circ H)$, $f \mapsto (w \circ f, z \circ f, z \circ f, \ldots, z \circ f_{k_\Omega})$. Additionally, this f maps to

$$\Delta_{\Omega_f} = \mathrm{Inj}(X/\Omega_f, G) \prod_{\theta_i} \mathrm{Inj}(\langle \theta_i \rangle, H).$$

We need to show that this Ψ is bijective.

We use proof by contradiction to show that Ψ is injective. Consider $f, g \in \mathrm{Inj}(X, G \circ H)$ where $f \neq g$, and assume that $\Psi(f) = \Psi(g)$. Let $x \in V(X)$. Two cases follow.

Case 1: The function $f(x)$ maps to fiber H_1 of $G \circ H$ and $g(x)$maps to fiber H_2 of $G \circ H$. If each go to a different fiber, then $\Psi(f) \neq \Psi(g)$ since $\Psi(f), \Psi(g)$ would map to different $w \circ f$. Therefore, in Case 1 if $\Psi(f) = \Psi(g)$ then $f = g$.

Case 2: The functions $f(x)$ and $g(x)$ map to the same fiber of $G \circ H$. Since $f \neq g$, $f(x) \neq g(x)$, $f(x)$ and $g(x)$ map to different vertices in that fiber of $G \circ H$. Then $\Psi(f)$ and $\Psi(g)$ would not be equal since $\Psi(f)$ would map x to a different $(z \circ f)_i$ than $\Psi(g)$ since both map to the same fiber but different vertices. Then in Case 2, if $\Psi(f) = \Psi(g)$ then $f = g$. Therefore Ψ is injective.

Next, we must show that Ψ is onto. Consider the element $(f, f_1, f_2, \ldots, f_k) \in \Delta_\Omega$, where each f_i maps to $z \circ f$. The domains of $(f_1, f_2, \ldots, f_k)$ can be written $\Omega = \{\theta_1, \theta_2, \ldots, \theta_{k_\Omega}\}$. Recall $f : X/\Omega \to G$, where $\Omega = \{\theta_1, \theta_2, \ldots, \theta_k\}$, and $f_i : \theta_i \to H$.

Using this information, a function $g \in \mathrm{Inj}(X, G \circ H)$ can be defined as $g : X \to G \circ H$ defined by

$$g(x) = (f(\theta_i), f_i(x)),$$

where $x \in \theta_i$. We will show that g is an injective homomorphism and then show $\Psi(g) = (f, f_1, f_2, \ldots, f_k)$.

We will show g is injective using proof by contradiction. Assume g is not injective, then there exists $x_1, x_2 \in \theta_i$, such that $x_1 \neq x_2$ but $g(x_1) = g(x_2)$. If $g(x_1) = g(x_2)$, then $(f(\theta_i), f_i(x_1)) = (f(\theta_i), f_i(x_2))$ for all i. Recall that θ_i is the domain for f_i. If $x_1 \neq x_2$, then x_1 maps to a different fiber than x_2. If $g(x_1) = g(x_2)$, then x_1 and x_2 both map to the same fiber, but by the construction of θ_i $x_1 = x_2$, which is a contradiction. Therefore g is injective.

Next, we will show g is an homomorphism. Assume $x_1, x_2 \in \theta_i$ and $x_1 x_2 \notin A(X)$, but $g(x_1)g(x_2) \in A(G \circ H)$. Then $(f(\theta_i), f_i(x_1))(f(\theta_i, f_i(x_2))) \in A(G \circ H)$. Recall that $\theta_i \in \Omega$ and that the quotient partition generated is injective to G. Therefore, if

$(f(\theta_i), f_i(x_1))(f(\theta_i, f_i(x_2))) \in A(G \circ H)$, then the preimage of g on the quotient is defined to be $wg^{-1}(g')$, where $g' \in V(G)$. Earlier, this was established to be a homomorphism, which is a contradiction. Therefore g is a homomorphism. To be more precise g is an injective homomorphism.

Now we must show that $\Psi(g) = (f, f_1, f_2, \ldots, f_k)$.

$$\Psi(g) = ((f(\theta_i), f_i(x)), (f(\theta_i), f_1(x)), (f(\theta_i), f_2(x)), \ldots, (f(\theta_i), f_k(x))).$$

The first part, $(f(\theta_i), f_i(x))$ can be written as $w \circ f$ since it will be mapped to G. Recall that each θ_i is a domain for f_i, so it can be rewritten as $z \circ f_i$.. Now, we have $\Psi(g) = (w \circ f, z \circ f_1, z \circ f_2, \ldots, z \circ f_k) = (f, f_1, f_2, \ldots f_k)$. Therefore, the total number of injective homomorphisms from X to $G \circ H$ is

$$\operatorname{inj}(X, G \circ H) = \sum_{\Omega_f \in \mathcal{P}(V(X))} \left(\operatorname{inj}(X/\Omega_f, G) \cdot \prod_{\theta \in \Omega_f} \operatorname{inj}(\langle \theta \rangle, H) \right).$$

□

Theorem 3.4 provides a base step for the cancellation properties we need to establish. This result is used to show that for digraphs G, H, and K that if $G \circ H \cong G \circ K$ then $H \cong K$ as well as to show that if $G \circ H \cong K \circ H$ then $G \cong K$.

THEOREM 3.5. If G, H, and K are digraphs and $G \circ H \cong G \circ K$, then $H \cong K$.

Proof. Let X, G, H, and K be digraphs and $G \circ H \cong G \circ K$. We need to show $H \cong K$. This will be accomplished by showing $\operatorname{inj}(X, H) = \operatorname{inj}(X, K)$ for all X by induction on $|V(X)|$.

If $|V(X)| = 1$, then $\operatorname{inj}(X, H) = |V(H)| = |V(K)| = \operatorname{inj}(X, K)$ since $|V(H)| = |V(K)|$ by properties of isomorphic digraphs and lexicographic product digraphs (the order of $G \circ H$ equals the order of $G \circ K$).

Now assume that $\operatorname{inj}(X,H) = \operatorname{inj}(X,K)$ whenever $|V(X)| < N$, for some integer N. We need to show that $\operatorname{inj}(X,H) = \operatorname{inj}(X,K)$ when $|V(X)| = N$. By Theorem 3.4 we have the following:

$$\operatorname{inj}(X, G \circ H) = \sum_{\Omega \in \mathcal{P}(V(X))} \operatorname{inj}(X/\Omega, G) \cdot \prod_{\theta \in \Omega} \operatorname{inj}(\langle \theta \rangle, H), \tag{3.1}$$

where $\mathcal{P}(V(X))$ is the set of all partitions of the vertex set of X, Ω is an element of this partition set, and θ is one element of Ω. Similarly,

$$\operatorname{inj}(X, G \circ K) = \sum_{\Omega \in \mathcal{P}(V(X))} \operatorname{inj}(X/\Omega, G) \cdot \prod_{\theta \in \Omega} \operatorname{inj}(\langle \theta \rangle, K). \tag{3.2}$$

Because $G \circ H \cong G \circ K$, it follows that $\operatorname{inj}(X, G \circ H) = \operatorname{inj}(X, G \circ K)$ and by Equation 3.1 and Equation 3.2

$$\sum_{\Omega \in \mathcal{P}(V(X))} \operatorname{inj}(X/\Omega, G) \cdot \prod_{\theta \in \Omega} \operatorname{inj}(\langle \theta \rangle, H) = \sum_{\Omega \in \mathcal{P}(V(X))} \operatorname{inj}(X/\Omega, G) \cdot \prod_{\theta \in \Omega} \operatorname{inj}(\langle \theta \rangle, K). \tag{3.3}$$

Therefore,

$$0 = \sum_{\Omega \in \mathcal{P}(V(X))} \operatorname{inj}(X/\Omega, G) \cdot \left(\prod_{\theta \in \Omega} \operatorname{inj}(\langle \theta \rangle, H) - \prod_{\theta \in \Omega} \operatorname{inj}(\langle \theta \rangle, K) \right). \tag{3.4}$$

By the induction hypothesis, whenever the cardinality of θ is less than N it follows that $\operatorname{inj}(\langle \theta \rangle, H) = \operatorname{inj}(\langle \theta \rangle, K)$ and every term in Equation 3.4 is zero except for $\Omega = \{V(X)\}$. Additionally, $|\theta| \geq N$ when $\Omega = V(X)$. If $\Omega = \{V(X)\}$ then there is only one element of the partition and Equation 3.4 can be reduced to:

$$0 = \operatorname{inj}(X/X, G) \cdot (\operatorname{inj}(X,H) - \operatorname{inj}(X,K)). \tag{3.5}$$

Since $\text{inj}(X/X,G) \neq 0$ then it follows that,

$$0 = \text{inj}(X,H) - \text{inj}(X,K).$$

Therefore, $\text{inj}(X,H) = \text{inj}(X,K)$. So, when $|V(X)| = N$, $\text{inj}(X,H) = \text{inj}(X,K)$, implying that $\text{inj}(X,H) = \text{inj}(X,K)$ for all graphs X.

If $X = H$ then $\text{inj}(H,H) = \text{inj}(H,K)$ and $\text{inj}(H,K) > 1$. Similarly, if $X = K$ then $\text{inj}(K,K) = \text{inj}(K,H)$ and $\text{inj}(K,H) > 1$. By Theorem 2.3, $H \cong K$.

Therefore, if $G \circ H \cong G \circ K$, then $H \cong K$, $\square$

THEOREM 3.6. If G, H, and K are digraphs and $G \circ H \cong K \circ H$, then $G \cong K$.

Proof. Let X, G, H, and K be digraphs and $G \circ H \cong K \circ H$. We need to show $G \cong K$, which will be accomplished by showing $\text{inj}(X,G) = \text{inj}(X,K)$ for all X by induction on $|V(X)|$.

If $|V(X)| = 1$, then $\text{inj}(X,G) = |V(G)| = |V(K)| = \text{inj}(X,K)$ since $|V(G)| = |V(K)|$ by properties of isomorphic digraphs and lexicographic product digraphs (the order of $G \circ H$ equals the order of $G \circ K$).

Now assume that $\text{inj}(X,G) = \text{inj}(X,K)$ whenever $|V(X)| < N$, for some integer N. We need to show that $\text{inj}(X,G) = \text{inj}(X,K)$ when $|V(X)| = N$. By Theorem 3.4 we have the following:

$$\text{inj}(X, G \circ H) = \sum_{\Omega \in \mathcal{P}(V(X))} \text{inj}(X/\Omega, G) \cdot \prod_{\theta \in \Omega} \text{inj}(\langle \theta \rangle, H), \tag{3.6}$$

where $\mathcal{P}(V(X))$ is the set of all partitions of the vertex set of X, Ω is an element of this partition set, and θ is one element of Ω. Similarly,

$$\text{inj}(X, K \circ H) = \sum_{\Omega \in \mathcal{P}(V(X))} \text{inj}(X/\Omega, K) \cdot \prod_{\theta \in \Omega} \text{inj}(\langle \theta \rangle, H). \tag{3.7}$$

Because $G \circ H \cong K \circ H$ it follows that $\operatorname{inj}(X, G \circ H) = \operatorname{inj}(X, K \circ H)$ and by Equations (3.6) and (3.7).

$$0 = \sum_{\Omega \in \mathcal{P}(V(X))} (\operatorname{inj}(X/\Omega, G) - \operatorname{inj}(X/\Omega, K)) \cdot \left(\prod_{\theta \in \Omega} \operatorname{inj}(\langle \theta \rangle, H) \right). \tag{3.8}$$

By the induction hypothesis whenever the cardinality of X/Ω is less than N it follows that $\operatorname{inj}(X/\Omega, G) = \operatorname{inj}(X/\Omega, K)$. Furthermore, whenever $|x/\Omega| \geq N$ the partition of $V(X)$ is one element. Then there is one element in Ω and $\operatorname{inj}(\langle \theta \rangle, H) = |V(H)| > 0$, so $\operatorname{inj}(X/\Omega, G) - \operatorname{inj}(X/\Omega, K) = 0$ for all graphs X.

If $X = K$, then $\operatorname{inj}(K, G) = \operatorname{inj}(K, K)$ and $\operatorname{inj}(K, G) \geq 1$. Similarly, if $X = G$, then $\operatorname{inj}(G, G) = \operatorname{inj}(G, K)$ and $\operatorname{inj}(G, K) \geq 1$. By Theorem 2.3 if $G \circ H \cong K \circ H$ then $G \cong K$. □

3.2 More on the Cancellation Property of Lexicographic Product of Digraphs

Section 3.1 provided some cancellation properties of lexicographic products of digraphs. In this section we provide more such cancellation properties. The section starts off with a very important definition, externally related vertex sets. This definition is extremely useful in each proof in this section.

DEFINITION 3.7. Let X and G be digraphs with $X \subseteq G$. The subdigraph X is said to be externally related if for every $u \in V(G) - V(X)$ the following conditions hold.

1. Any $u \in V(G) - V(X)$ is either adjacent to every $x \in V(X)$ or is not adjacent to any $x \in V(X)$,

2. Any $u \in V(G) - V(X)$ is either adjacent from every $x \in V(X)$ or is not adjacent from any $x \in V(X)$.

Lemma 3.8 shows that every lexicographic product digraph $G \circ H$ contains at least $|V(G)|$ externally related subdigraphs.

LEMMA 3.8. If G and H are digraphs and $G \circ H$ is their lexicographic product, then any H-layer of $G \circ H$ is externally related.

Proof. Let G and H be digraphs. Show any H-layer of $G \circ H$ is externally related. This will be shown using proof by contradiction.

Suppose H_i is an H-layer in $G \circ H$ where the weak homomorphism ϕ maps every vertex of H_i to $g_i \in V(G)$. Let $(g_j, h_n), (g_j, h_m) \in A(H_i\text{-layer}) \subset A(G \circ H)$ and let $(g_j, h_l) \in A(G \circ H)$ where $i \neq j$. Figure 3.1 illustrates the different H-layers. Figures 3.2 and 3.3 have stripped Figure 3.1 of all fibers to illustrate the two different cases below.

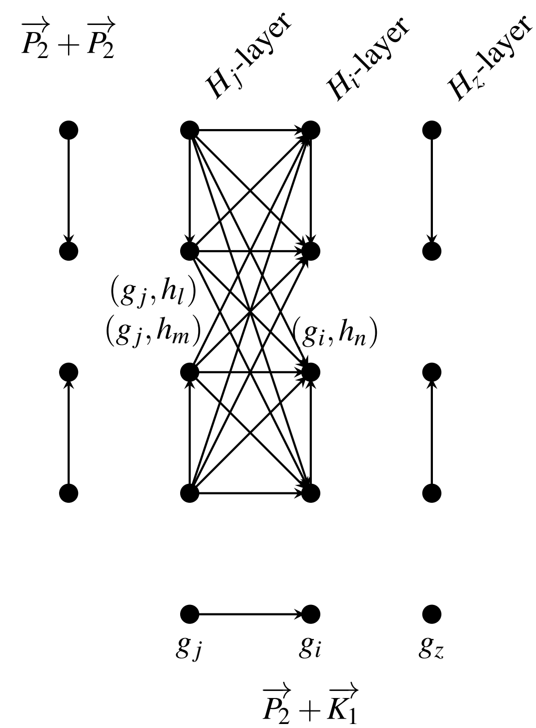

Figure 3.1: Illustration of each H-layer in $G \circ H$.

Two cases follow:

Case 1: The arc $g_i g_j \in A(G)$. There exists a $(g_i, h_n)(g_j, h_m) \in A(G \circ H)$ but $(g_i, h_n)(g_j, h_l) \notin A(G \circ H)$ for $i \neq j$. Then (g_j, h_m) and (g_j, h_l) are in a different H-layer than (g_i,h_n). Since $i \neq j,k$ then $(g_j,h_m),(g_k,h_l) \in V(G \circ H) - V(H_i)$. This contradicts the definition of the lexicographic product, and therefore if any vertex from another H-layer is adjacent from H_i then they all must be.

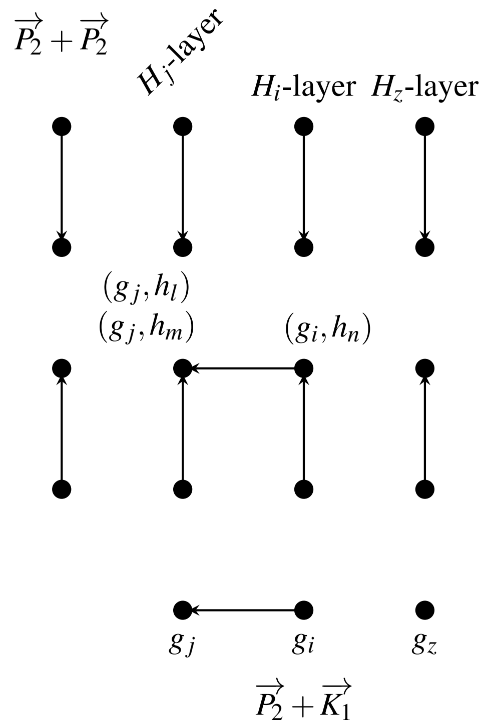

Figure 3.2: Showing $(g_ig_j) \in V(G)$ and $(g_i,h_n)(g_j,h_m) \in A(G \circ H)$ but $(g_i,h_n)(g_j,h_l) \notin A(G \circ H)$.

Case 2: The arc $g_jg_i \in A(G)$. There exists a $(g_j,h_m)(g_i,h_n) \in A(G \circ H)$ but $(g_j,h_l)(g_i,h_n) \notin A(G \circ H)$ for $i \neq j$. Then this contradicts the definition of lexicographic product digraphs, and therefore if any vertex is adjacent to H_i then all are adjacent to H_i.

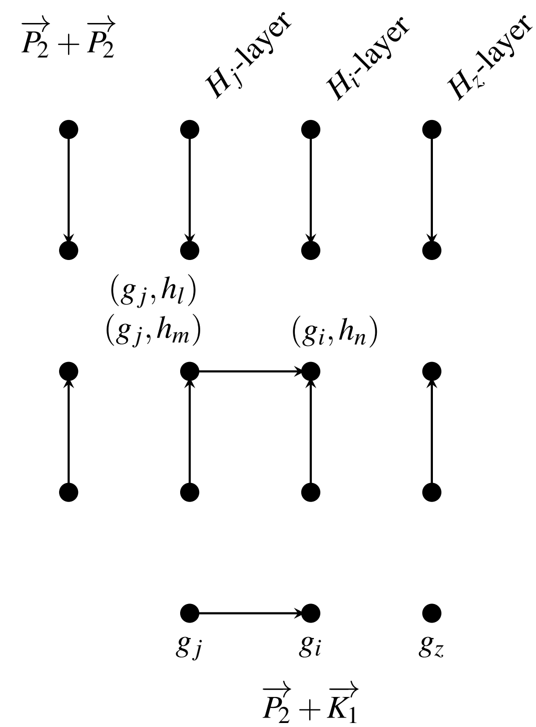

Figure 3.3: Showing $(g_jg_i) \in V(G)$ and $(g_j,h_m)(g_j,h_n) \in A(G \circ H)$ but $(g_j,h_l)(g_i,h_n) \notin A(G \circ H)$.

Therefore H_i is externally related. $\square$

Next, we show that if X, $Y \subseteq G$ are both externally related and each are complete then $X \cap Y \neq \phi$ and $X \cup Y$ is externally related (Proposition 3.9). Following Proposition 3.9, Proposition 3.10 will show that if $X \cap Y \neq \phi$ and each are totally disconnected, then $X \cup Y$ is totally disconnected. Proposition 3.11 will show that if $X \subseteq G \circ H$ then $\mathcal{P}_G(X)$, the projection of X onto G, is externally related in G. Proposition 3.12 will show that if $X \subseteq G$ is externally related then the preimage of the projection of X onto G is externally related.

PROPOSITION 3.9. If X, $Y \subseteq G$ are externally related and each is complete and $X \cap Y \neq \phi$, then $X \cup Y$ is externally related and complete.

Proof. Let X, Y, and G be digraphs and X,$Y \subseteq G$ and both X and Y are externally related and complete. Also, $X \cap Y \neq \phi$. We will show $X \cup Y$ is externally related and complete. First we will show that $X \cup Y$ is externally related and then show it is complete.

Consider $V(X \cup Y) = \{g \in V(X \cup Y) : g \in V(X) \, \text{or} \, g \in V(Y)\}$.
If $X \cap Y \neq \phi$, then there exists at least one $g_1 \in V(X \cap Y)$. Assume $X \cup Y$ is neither externally related nor complete.

Then there exists one $g_1 g_2 \in A(G)$ such that $g_1 \in V(X \cup Y)$ and $g_2 \notin V(X \cup Y)$. Assume, without loss of generality, that g_2 is only adjacent to $g_1 \in V(X \cup Y)$. By definition of $V(X \cup Y)$, $g_1 \in V(X)$ or $g_1 \in V(Y)$. Since X is externally related $g_1 \notin V(X)$ and since Y is externally related $g_1 \notin V(Y)$. Since X and Y are externally related, this is a contradiction and either $g_1 g_2 \notin A(G)$ or $g_1 \notin V(X \cup Y)$. Thus $X \cup Y$ is externally related.

Since $X \cap Y \neq \phi$ then there exists at least one $g \in V(X \cup Y)$ such that $g \in V(X)$ and $g \in V(Y)$. Consider $x \in V(X \cup Y)$, where $x \in V(X)$ but $x \notin V(Y)$. Since X and Y are complete by definition of externally related and complete digraphs, every $v \in V(X)$ is either adjacent to every $h \in V(G) - V(X)$ or nonadjacent to every $h \in V(G) - V(X)$. Or every $v \in V(X)$ is either adjacent from every $h \in V(G) - V(X)$ or nonadjacent from every $h \in V(G) - V(X)$. Recall $g \in V(X \cap Y)$ and so $g \in V(X \cup Y)$ as well. Recall, X is complete. Since X is complete, $xg, gx \in A(X)$ and hence also $xg, gx \in A(X \cup Y)$. The vertex $x \notin V(Y)$. Recall that Y is also complete. Then there exists $y \in V(Y)$ such that $y \notin V(G)$ but $yg, gy \in A(Y)$ and hence $yg, gy \in A(X \cup Y)$. Note, that $x \in V(X)$ and $y \in V(Y)$ are both adjacent to and adjacent from $g \in V(X \cap Y)$.

Recall that we assumed that $X \cup Y$ is not complete. Now consider that X is externally related and $X \cap Y \neq \phi$. Then $g \in V(X)$ is adjacent to and adjacent from every vertex in $V(Y)$ since X is externally related. Then $X \cup Y$ is complete. Similarly, the second condition of externally related could be followed to show that $V(X \cup Y)$ is complete.

Therefore if X, Y, and G are digraphs with $X, Y \subseteq G$ and both X and Y are externally related and complete with $X \cap Y \neq \phi$, then $X \cup Y$ is externally related and complete. □

PROPOSITION 3.10. If X, Y, and G are digraphs and $X, Y \subseteq G$ and are externally related and each is totally disconnected and $X \cap Y \neq \phi$, then $X \cup Y$ is totally disconnected.

Proof. Let X, Y, and G be digraphs and $X, Y \subseteq G$. Also X and Y are externally related and totally disconnected with $X \cap Y \neq \phi$. We need to show $X \cup Y$ is totally disconnected.

Let $g_1, g_2 \in V(G) - V(X \cup Y)$. Then $g_1, g_2 \notin V(X)$ and $g_1, g_2 \notin V(Y)$. Let $x, y, v \in V(X \cup Y)$, where $x, v \in V(X)$ and $y, v \in V(Y)$. Assume $X \cup Y$ is not externally related. Then there exists a $v \in V(X \cup Y)$ such that $vg_1 \in A(G)$ but not adjacent to any other $v \in V(X \cup Y)$. Recall that $X, Y \subseteq G$ are externally related. Then $v \notin V(X)$ and $v \notin V(Y)$ since both are totally disconnected. So, $X \cup Y$ is externally related.

Now, assume $X \cup Y$ is not totally disconnected. Then there exists $x \in V(X)$ and a $y \in V(Y)$ such that $xy \in A(X \cup Y)$. Recall that $X \cap Y \neq \phi$. Then there exists a $v \in V(X \cap Y)$. Since X and Y are externally related then for any $g \in V(G) - V(X)$ every $x \in V(X)$ is not adjacent to or adjacent from every $g \in V(G) - V(X)$. Consider $y \in V(X \cup Y)$ and $y \notin V(X)$. Since Y is totally disconnected, $y \in V(Y)$ and $y \in V(G) - V(X)$ is not adjacent to or adjacent from any other vertex in Y. Since $v \in V(X \cap Y)$ v is not adjacent to or adjacent from y, but $v \in V(X)$. Hence $X \cup Y$ is totally disconnected. □

PROPOSITION 3.11. If $X \subseteq G \circ H$ is externally related, then the projection of X onto G, $P_G(X)$, is externally related in G.

Proof. Let X, G, and H be digraphs and $X \subseteq G \circ H$ and X is externally related. Show $P_G(X)$ is externally related in G. This will be accomplished using proof by contradiction.

Assume $P_G(X)$ is not externally related in G. Then there exists a $g \in P_G(X)$ and $g_1, g_2 \in V(G)$ such that $gg_1 \in A(G)$ but $gg_2 \notin A(G)$. Since $X \subseteq G \circ H$ and is externally related if $gg_1 \in A(G)$ and $gg_2 \notin A(G)$ then $(g,h)(g_2, j) \notin A(G \circ H)$ for some $h, j \in V(H)$.

Let $(g,m) \in V(X)$. Recall that $X \subseteq G \circ H$. Since X is externally related, if $(g,h) \in A(G \circ H)$, then for all $(g_l, h_i) \in A(X) \subseteq A(G \circ H)$ must be either adjacent to or adjacent from every element of an H-layer or nonadjacent to or nonadjacent from every vertex of an H-layer for all $(g,h) \in V(G \circ H) - V(X)$. If $P_G(X)$ is not externally related, then there exists a $P_G((g,m))g_2 \in A(G)$, but $P_G((g,m))g_3 \notin A(G))$ where $g_2, g_3 \notin P_G(X)$. Since X is externally related in $G \circ H$, if $P_G(g,m)g_2$ is adjacent to or adjacent from in G, then the preimage must be adjacent to or adjacent from all $V(G \circ H) - V(X)$ and likewise the preimage of $P_G(g,m)g_3$ are not adjacent to nor adjacent from the preimage of $P_G(X)$. Since $(g_2, h_i), (g_3, h_j)$ are not in the preimage, and each H-layer is externally related, this contradicts the fact that X is externally related in $G \circ H$. Therefore $P_G(X)$ must be externally related. $\square$

PROPOSITION 3.12. If $X \subseteq G$ is externally related, then $P_G^{-1}(X)$ is externally related.

Proof. Let X and G be digraphs with $X \subseteq G$. The subdigraph X is also externally related. Show $P_G^{-1}(X)$ is externally related. Let $h, g \in V(X)$ Then $P_G(X)$ and $P_G(X)$ both exist. Since X is externally related, all vertices in the set $V(G) - V(X)$ are either adjacent to or not adjacent to all vertices in $V(X)$ and either adjacent from or not adjacent from all vertices in $V(X)$. The preimage of the projection is X itself so it is externally related. $\square$

Several useful propositions have been proven using Definition 3.7. Recall the definition of strongly connected (Definition 1.8) for Proposition 3.13 to show that if $X \subseteq G \circ H$ and it is strongly connected and externally related with $|V(X)| \leq |V(H)|$ then the projection of X onto G is externally related and complete.

PROPOSITION 3.13. Let $X \subseteq G \circ H$ and assume it is strongly connected, externally related, and $|V(X)| \leq |V(H)|$. Then $P_G(X)$ is externally related and complete.

Proof. Let X,G, and H be digraphs and $X \subseteq G \circ H$ as well as externally related. The order of X is less than or equal to the order of H. We need to show that $P_G(X)$ is externally related and complete. Recall that by Proposition 3.11, $P_G(X)$ is externally related. So all that is needed is to show is that $P_G(X)$ is complete.

Let $(g_1, h_1) \in V(X)$ and $P_G(x) = g_1$ where $g_1 \in V(G)$. Since X is strongly connected, there must be a neighbor $(g_2, h_2) \in V(X)$ of (g_1, g_2). The neighbor (g_2, h_2) is either in the same H-layer of $G \circ H$ or is in an adjacent H-layer.

Case 1. Assume that $V(X) \subseteq V(H_i)$, that is X is a subset of an H-layer of $G \circ H$. Then $P_G(x) = \{g_1\}$ and $P_G(X)$ is complete.

Case 2. Assume that $V(X)$ is not a subset of an H-layer of $G \circ H$. Then there exists at least one $(g_1, h_1)(g_2, h_2) \in A(G \circ H)$ where $g_1 \neq g_2$. Since $P_G(X)$ is a projection and $X \subseteq G \circ H$ $(g_1 g_2) \in A(G)$. Since X is strongly connected, a $(g_1, h_1)(g_2, h_2)$-path exists and also a $(g_2, h_2)(g_1, h_1)$-path exists. So (g_1, h_1) has an out degree and in degree of at least one. Since $P_G(X)$ is externally related, we know that any vertex adjacent to or adjacent from g_1 is also adjacent to or adjacent from g_2 respectively.

Now, assume that $P_G(X)$ is not complete. Then there exists a $g_1, g_2, g_3 \in V(G)$, where $g_3 \notin P_G(X)$ but $g_1, g_2 \in P_G(X)$ and $g_1 g_3 \in A(P_G(X)$ but $g_2 g_3 \notin P_G(X)$. Then the H-layer defined by (g_1, h_i) is adjacent to (g_3, h_i) but (g_2, h_i) is not adjacent to (g_3, h_i). This contradicts the given fact that X is externally related since $(g_1, h_1), (g_2, h_2) \in V(X)$. Therefore $P_G(X)$ is complete. $\square$

Theorem 3.14 ties all the work together to show that if $G \circ H \cong A \circ B$ and $|V(G)| = |V(A)|$, then $G \cong A$.

THEOREM 3.14. If G, H, A, and B are digraphs and $G \circ H \cong A \circ B$ and $|V(G)| = |V(A)|$ and H is strongly connected, then $K_l \circ H \cong K_l \circ B$, for some l and $G \cong A$.

Proof. Let G, H, A, and B be digraphs with $G \circ H \cong A \circ B$ and $|V(G)| = |V(A)|$ and H is strongly connected. We want to show that $K_l \circ H \cong K_l \circ B$, for some l. (Note: by properties of lexicographic product of digraphs and isomorphisms $|V(H)| = |V(B)|$.) Define $\alpha : A \circ B$ as the projection $\alpha((a,b)) = a$ for $a \in V(A)$ and $(a,b) \in V(A \circ B)$. Additionally define $\gamma : G \circ H$ as the projection $\gamma((g,h) = g)$ for $g \in V(G)$ and $(g,h) \in V(G \circ H)$.

Define Φ_i to be the function from $G \circ H$ to $A \circ B$ from a fixed H-layer of $G \circ H$, that is $\Phi(g_i,h)$ for a fixed $g_i \in V(G)$ and for all $h \in V(H)$. Note that Φ_i can be written as $\Phi(\gamma^{-1}(g_i))$. Recall that H-layers of a lexicographic product digraph are externally related. Then by Proposition 3.13, Φ_i is externally related in $A \circ B$ and by Proposition 3.11 the projection of Φ_i onto A is externally related in A and by Proposition 3.13 the projection of Φ_i onto A is also complete, say K_p.

Since $K_p = \alpha(\Phi(\gamma^{-1}(g_i))) \subseteq A$ is externally related and complete, by Proposition 3.10 $\alpha^{-1}(K_p)$ is externally related in $A \circ B$. Again, by Proposition 3.10 $\Phi^{-1}(\alpha^{-1}(K_p))$ is externally related in $G \circ H$. Then by Proposition 3.13 and Proposition 3.11 $\gamma(\Phi^{-1}(\alpha^{-1}(K_p))$ is externally related and complete. Either this subset of $V(G) = K_p$ or its order is larger than K_p. If it equals K_p then $K_l \circ H \cong K_l \circ B$. If it is not, then take the preimage of this new set of vertices in $G \circ H$ and repeat the process until you have $K_n \subseteq G$ and $K_n \subseteq A$, then by Theorem 3.5 $H \cong B$ and hence $A \cong G$. $\square$

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Vita

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