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The Inner Power of a Graph

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

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We define a new graph operation called the *inner power of a graph*. The construction is similar to the direct power of graphs, except that factors are intertwined in such a way that certain structural properties of graphs are more clearly reflected in their inner powers. We investigate various properties of inner powers, such as connectivity, bipartiteness, and their interaction with the direct product. We explore possible connections between inner powers and the problem of cancellation over the direct product of graphs.

Preliminaries

In this thesis we construct and develop theory for a new graph operation which we will call the *inner power* of a graph. The inner power is similar to the direct power, and results concerning connectivity and bipartiteness of the two powers are in many ways comparable. Moreover the inner power distributes over the direct product. This yields potential applications to the problem of direct product cancellation. We explore this connection. Before proceeding, it is necessary to quickly review the relevant definitions and terminology.

1.1 Basic Definitions and Terminology

A *graph* is a pair $G = (V, E)$ of sets such that V is nonempty and finite and the elements of E are unordered pairs of the elements from V. If $E = \emptyset$, then G is *empty*. The elements of V are the *vertices* (a single element is a *vertex*) of G and the elements of E are the *edges* of G. It is convenient to henceforth denote an edge $e = (u, v)$ by uv (or vu). If $e = uv$ is an edge of a graph G, the vertices u and v are *adjacent vertices*, and e *adjoins* u and v. Vertices that are not adjacent are *nonadjacent*. A vertex with no adjacent vertices is an *isolated vertex*. A vertex v may be adjacent to itself, in which case the edge vv is a *loop*. For a given vertex v in the vertex set of a graph G, the set of all vertices in G adjacent to v is called the *neighborhood of* v, and is denoted $N(v)$.

A graph G may be described by means of a diagram where points represent the vertices and each edge $e = uv$ is represented by a line segment or curve joining the two points corresponding to vertices u and v.

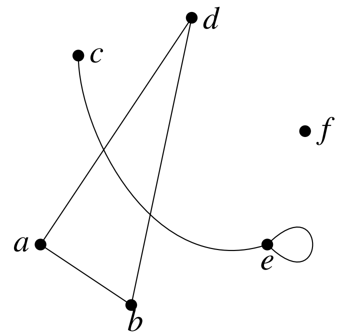

Figure 1.1: A graph on $V = \{a, b, c, d, e, f\}$ with edge set $E = \{ab, ad, bd, ce, ee\}$.

To specify the graph in consideration, it is conventional to denote the vertex set of a graph G by $V(G)$ and the edge set of G by $E(G)$.

A graph G is *isomorphic* to a graph H if there exists a injective mapping φ, called an *isomorphism*, from $V(G)$ onto $V(H)$ where $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. The relation "is isomorphic to" is an equivalence relation on graphs and will be denoted "$\cong$"; hence the statement "graph G is isomorphic to graph H" is equivalent to "$G \cong H$". If G is not isomorphic to H, then G is *nonisomorphic* to H, or, equivalently, $G \ncong H$. A general mapping φ from $V(G)$ into $V(H)$ for which $uv \in E(G)$ implies $\varphi(u)\varphi(v) \in E(H)$ is called a *homomorphism*.

If G and H are graphs with $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$, then we say G is a *subgraph* of H or H *contains* G, and write $G \subseteq H$ or $H \supseteq G$. It follows that if $G \subseteq H$ and $G \supseteq H$, then $G \cong H$. Let U be a subset of $V(G)$. For a given subset U of the vertex set of a graph G, the subgraph *induced by U in G* is the graph on U with the edge set that consists of all edges of G incident with two elements of U.

1.2 Walks, Connectivity, and Some Graph Classes

Let u and v be vertices in a graph G. A *u-v walk* W of G is a finite sequence

$$W : u = x_0, x_1, ..., x_k = v$$

of vertices in G where $x_i x_{i+1} \in E(G)$ for all $i \in \{0, ..., k-1\}$. The number k (the number of occurences of edges) is the *length* of W. If W has even (or odd) length, then W is *even* (or *odd*). A *trivial walk* is a walk of length zero, and a *nontrivial walk* is a walk that is not trivial.

A u-v walk is *closed* if $u = v$. A *u-v path* is a u-v walk of distinct vertices. A closed walk $v_0 v_1 ... v_{n-1} v_0$ where $v_0 v_1 ... v_{n-1}$ is a path is a *cycle*. An arbitrary path on n vertices is denoted P_n, and a cycle on n vertices is denoted C_n.

A graph G is *complete* if every distinct pair of vertices in $V(G)$ are adjacent. The complete graph on n vertices is denoted K_n.

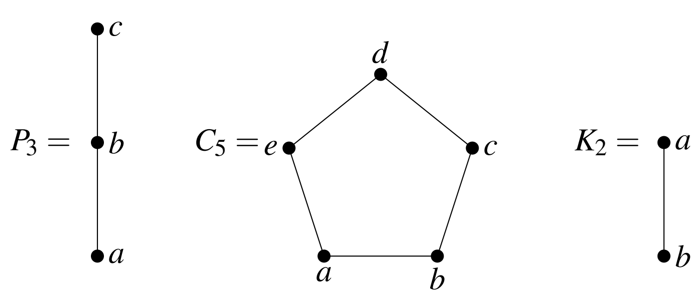

Figure 1.2: The path on three vertices, the cycle on five vertices, and the complete graph on two vertices are examples of some well-known classes of graphs.

A vertex u is *connected* to a vertex v in a graph G if there exists a u-v path in G, or, equivalently, if there exists a u-v walk in G. A graph is *connected* if every pair of its vertices are connected. If a graph is not connected, then it is *disconnected*.

A graph G is *bipartite* if $V(G)$ can be partitioned into two subsets V_1 and V_2, called *partite sets*, such that every element of $E(G)$ adjoins an element of V_1 with an element of V_2. The following theorem is a useful equivalency for bipartite graphs.

THEOREM 1.1. A graph G is bipartite if and only if G does not contain an odd cycle.

The proof of this result can be found in most introductory texts on graph theory, such as that by Chartrand [1].

1.3 The Direct Product

Operations defined on graphs are the primary focus of this thesis. In particular, we are interested in the direct product, and a new construction called the inner power, which is introduced in Chapter 2. The *direct product* of two graphs G and H, denoted $G \times H$, is defined on the Cartesian product of the vertex sets of the factors, $V(G) \times V(H)$. The edge set, $E(G \times H)$, is the set of all pairs $(a,b)(a',b')$ where $aa' \in E(G)$ and $bb' \in E(H)$. Figures 1.3 and 1.4 show some examples.

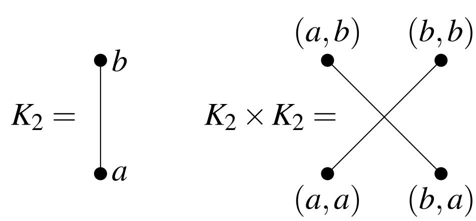

Figure 1.3: The complete graph on 2 vertices, K_2, and the direct product of K_2 with itself.

We can also describe the direct product of k graphs for any arbitrary positive integer k. Let $G_1, G_2, \ldots, G_k$ be graphs. Then $G_1 \times G_2 \times \ldots \times G_k$ is defined on the vertex set

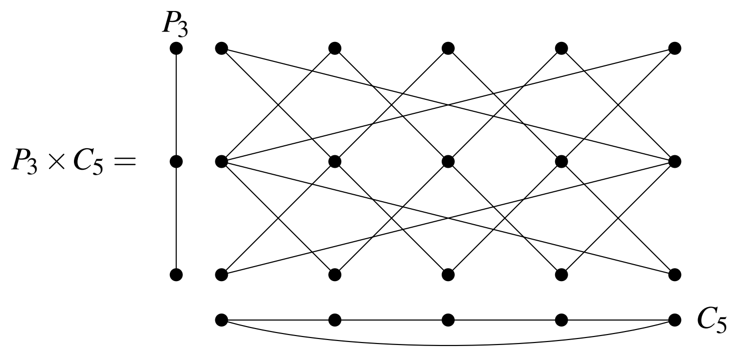

Figure 1.4: The direct product of the path on three vertices and the cycle on five vertices is shown above, with the factors on the left and bottom added for reference.

$V(G_1) \times V(G_2) \times ... \times V(G_k)$. The edge set is the set of all pairs $(x_1, x_2, ..., x_k)(x'_1, x'_2, ..., x'_k)$ where $x_i x'_i \in E(G_i)$ for each $i \in \{1, ..., k\}$. The n^{th} *direct power*, denoted G^n, of a graph G is the product of n copies of G.

The Inner Power: Definition and Properties

We have now developed enough basics to begin our discussion of our new graph operation, the inner power. We will first give a definition, then discuss some properties of the inner power.

2.1 Definition

The n^{th} *inner power* of a graph G, denoted $G^{(n)}$, is defined on the n^{th} Cartesian product of the vertex set of G. The edges are all pairs $(x_0,...,x_{n-1})(x'_0,...,x'_{n-1})$ where $x_i x'_{i-1} \in E(G)$ and $x_i x'_{i+1} \in E(G)$ for all $i \in \{0,...,n-1\}$, where arithmetic on the indices is performed modulo n for the sake of readability. Hence $x_0 x'_{n-1}, x_{n-1} x'_0 \in E(G)$.

For example, the second inner power, denoted $G^{(2)}$, of a graph G is defined on $V(G) \times V(G)$, and the edges are all pairs $(a,b)(a',b')$ where $ab' \in E(G)$ and $a'b \in E(G)$. Please take a moment and refer to Figures 1.3 and 2.1 to compare the second inner power with the second direct power. The third inner power, $G^{(3)}$, of a graph G is defined on the vertex set $V(G) \times V(G) \times V(G)$, and the edges are all pairs $(a,b,c)(a',b',c')$ where $ab', ac', ba', bc', ca', cb' \in E(G)$.

Figures 2.1 and 2.2 give some examples of inner powers.

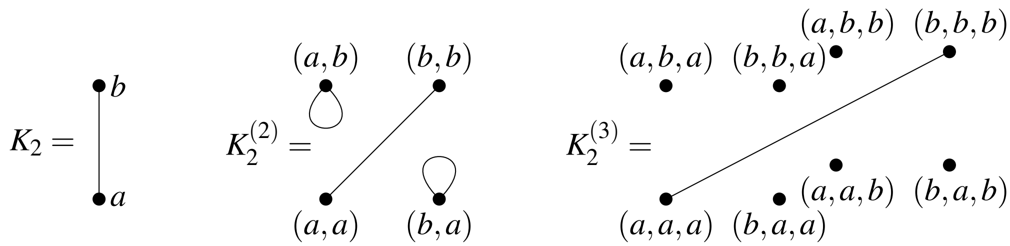

Figure 2.1: The complete graph on two vertices, K_2, the second inner power, $K_2^{(2)}$, and the third inner power, $K_2^{(3)}$.

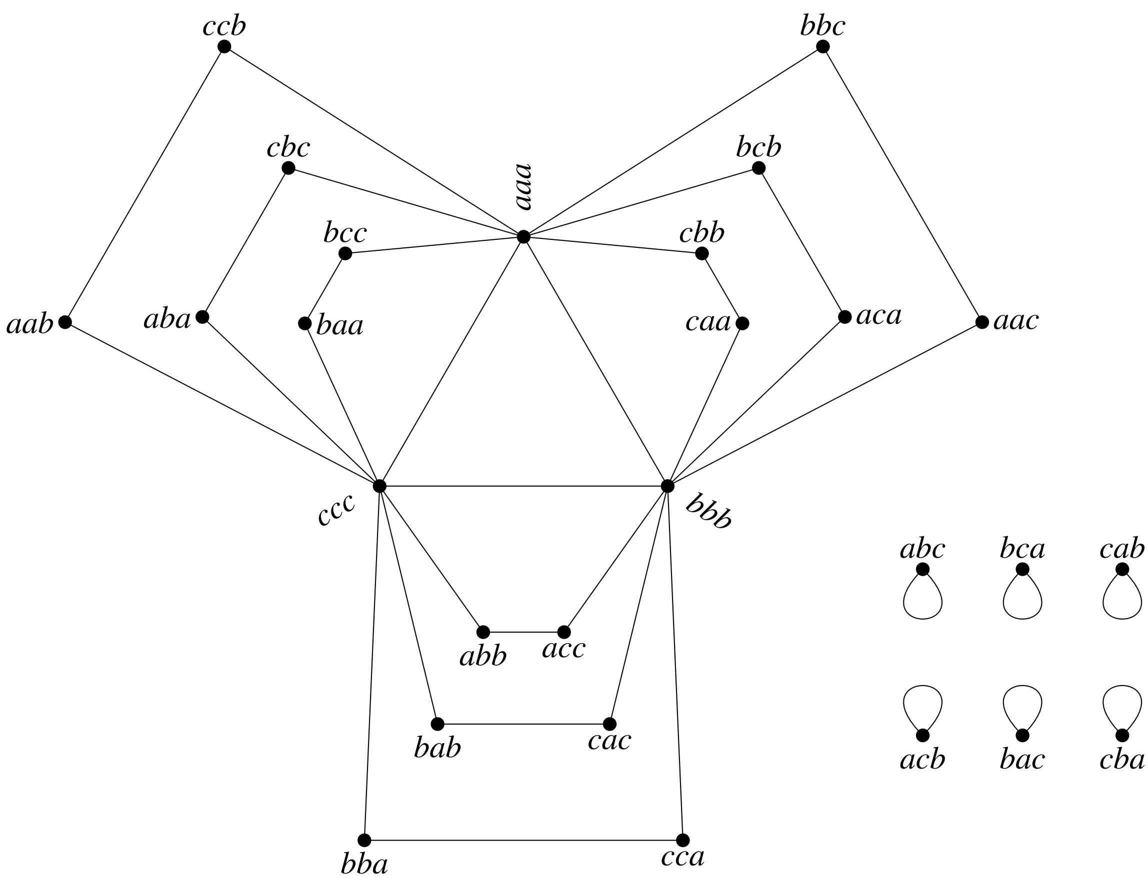

Figure 2.2: The third inner power, $K_3^{(3)}$, of K_3. For convenience, the vertices are labeled without parentheses or commas. Hence $abb = (a,b,b)$.

2.2 Isolated Vertices and Loops

Following are some various properties regarding inner powers. Isolated vertices and loops are two immediately recognizable characteristics in a graph. The proofs of the theorems regarding their existence shed light on the structure of the inner power of a graph. Additionally, these two properties are helpful when considering connectivity and bipartiteness in the following sections.

First we give two theorems regarding the existence of isolated points in a graph. The conditions for the existence of isolated points in inner powers of $k=1$ or 2 differs from those in general higher powers. We consider those two cases separately.

THEOREM 2.1. Let G be a graph and suppose that either $k=1$ or $k=2$. Then $G^{(k)}$ has an isolated vertex if and only if G has an isolated vertex.

Proof. The result follows trivially if $k=1$, since $G^{(1)}=G$. Assume $k=2$. Further, suppose that $G^{(k)}$ has an isolated vertex at some vertex $(x_1,x_2)\in V(G^{(2)})$. Then $(x_1,x_2)(y_1,y_2)\notin E(G^{(k)})$ for any $(y_1,y_2)\in V(G^{(k)})$. Then either x_1 is isolated or x_2 is isolated; otherwise, G has edges x_1y_2 and x_2y_1 for some $y_1,y_2\in V(G)$, which would imply $(x_1,x_2)(y_1,y_2)\in E(G^{(2)})$, a contradiction to our assumption. Hence G contains an isolated vertex.

Conversely, suppose instead that G has an isolated vertex at x_1. Then (x_1,x_1) is an isolated vertex in $G^{(k)}$. Therefore, $G^{(k)}$ has an isolated vertex if and only if G has an isolated vertex. $\square$

THEOREM 2.2. Let G be a graph and $k\geq 3$ be an integer. Then $G^{(k)}$ has an isolated vertex if and only if $N(c)\cap N(d)=\emptyset$ for some $c,d\in V(G)$.

Proof. Suppose $G^{(k)}$ has an isolated vertex at $(v_1,\ldots,v_k)\in V(G^{(k)})$. Then

$$(v_1,\ldots,v_k)(u_1,\ldots,u_k)\notin E(G^{(k)})$$

for any $(u_1,...,u_k) \in V(G^{(k)})$. Suppose that $N(c) \cap N(d) \neq \emptyset$ for all $c,d \in V(G)$. Then for each pair $v_{(i-1)}, v_{(i+1)} \in V(G)$, there exists $a_i \in V(G)$ with

$$a_i \in N(v_{(i-1)}) \cap N(v_{(i+1)}).$$

But then $(a_1,...,a_k) \in V(G^{(k)})$ with $(v_1,...,v_k)(a_1,...,a_k) \in E(G^{(k)})$, a contradiction. Thus, $N(c) \cap N(d) = \emptyset$ for some $c,d \in V(G)$.

Conversely, suppose that $N(c) \cap N(d) = \emptyset$ for some $c,d \in V(G)$. Consider $(d,d,...,c) \in V(G^{(k)})$. This vertex has no adjacencies. To prove this, suppose that it does. Then $(d,d,...,c)(x_1,x_2,...,x_k) \in E(G^{(k)})$ for some $(x_1,x_2,...,x_k) \in V(G^{(k)})$. Hence $dx_1, cx_1 \in E(G)$. But then $x_1 \in N(c) \cap N(d)$, a contradiction to the assumption that $N(c) \cap N(d) = \emptyset$. Thus G contains an isolated vertex. $\square$

THEOREM 2.3. Let G be a graph and let k be a positive integer. Then $G^{(k)}$ has a loop on vertex $(x_1,...,x_k)$ if and only if $x_1,...,x_k,x_1$ is a closed walk in G.

Proof. Let G be a graph. Then $G^{(k)}$ has a loop on vertex $(x_1,...,x_k)$ only if

$$x_1x_2, x_2x_3, ..., x_kx_1 \in E(G).$$

Hence $x_1,...,x_k,x_1$ is a closed walk in G. Conversely, suppose $x_1,...,x_k,x_1$ is a closed walk in G. Then $x_ix_{i+1} \in E(G)$ and $x_ix_{i-1} \in E(G)$ for all $i \in \{1,...,n\}$, and thus $G^{(k)}$ has a loop on vertex $(x_1,...,x_k)$. Therefore $G^{(k)}$ has a loop on vertex $(x_1,...,x_k)$ if and only if $x_1,...,x_k,x_1$ is a closed walk in G. $\square$

2.3 Connectivity

The following theorems involving connectivity and the existence of paths in the inner power of a graph provide insight into the structure of the inner power. We first investigate connectivity in the second inner power of a graph. Connectivity in the second inner power of graphs has a simple characterization that does not extend to higher powers. Before giving this characterization, we first give two lemmas.

LEMMA 2.4. Let G be a connected graph containing at least one odd cycle. Then for any $u, v \in V(G)$, there exists an odd u-v walk and an even u-v walk in G.

Proof. Let G be a connected graph containing an odd cycle, $c_0, c_1, ..., c_{2k}, c_0$. Let $u, v \in V(G)$ be arbitrary. Since G is connected, there exists a u-v path and a u-c_0 path. Either the u-v path is even or it is odd.

Suppose the u-v path is even. Then the walk obtained adjoining the u-c_0 path, the cycle $c_0, c_1, ..., c_{2k}, c_0$, the c_0-u path, and the u-v path is odd. Suppose instead that the u-v path is odd. Then the walk obtained by adjoining the u-c_0 path, the cycle $c_0, c_1, ..., c_{2k}, c_0$, the c_0-u path, and the u-v path is even. This proves the desired result. $\square$

LEMMA 2.5. Let G be a graph and let $a, b, c, d \in V(G)$. Then an (a, b)-(c, d) walk exists in $G^{(2)}$ if and only if one of the following is true:

1. there is an even a-c walk and an even b-d walk, or
2. there is an odd a-d walk and an odd b-c walk.

Proof. Let G be a graph and let $a, b, c, d \in V(G)$. Suppose first that an (a, b)-(c, d) walk exists in $G^{(2)}$. Suppose that Statement 2 is false. Then there can not exist an odd walk $(a, b), (x_1, y_1), (x_2, y_2), ..., (c, d)$; otherwise $a, y_1, x_2, ..., d$ and $b, x_1, y_2, ..., c$ would define odd walks in G. Thus the walk $(a, b), (x_1, y_1), (x_2, y_2), ..., (c, d)$ is even. Then $a, y_1, x_2, ..., c$ and

$b, x_1, y_2, \ldots, d$ are even walks, and Statement 1 is true. Hence, either Statement 1 or Statement 2 are true.

To prove the converse, first suppose Statement 1 is true. Then there is an even walk, $a, a_1, a_2, a_3, \ldots, c$ and an even walk $b, b_1, b_2, b_3, \ldots, d$. From the adjacencies given in the walk by sequential vertices, the walk (a,b), $(b_1,a_1), (a_2,b_2), (b_3,a_3), \ldots, (c,d)$ is derived in $G^{(2)}$. Suppose instead that Statement 2 is true. Then there is an odd walk, $a, a_1, a_2, a_3, \ldots, d$ and an even walk $b, b_1, b_2, b_3, \ldots, c$. This, again, gives the walk $(a,b), (b_1,a_1), (a_2,b_2)$, $(b_3,a_3), \ldots, (c,d)$ in $G^{(2)}$. In either case, an (a,b)-(c,d) walk exists in $G^{(2)}$. □

Before giving our characterization of graphs with connected 2^{nd} inner powers, we give two examples of graphs, P_3 and a second graph ("P_3 with a loop"), and their powers to consider in Figures 2.3 and 2.4.

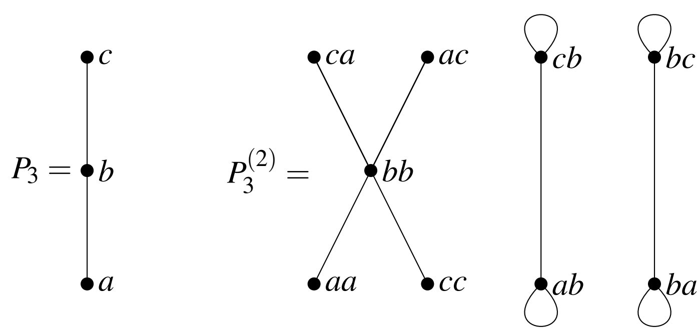

Figure 2.3: The path on three vertices, P_3, has a disconnected inner power.

THEOREM 2.6. Let G be a graph. Then $G^{(2)}$ is connected if and only if G is connected and G contains an odd cycle.

Proof. Let G be a graph. If $|V(G)| = 1$, then the result is trivial. Suppose $|V(G)| > 1$.

Now suppose $G^{(2)}$ is connected. To show G is connected, it is sufficient to show that a u-v walk exists for any two arbitrary vertices u and v in G. So, let u and v be arbitrary distinct (since an isolated vertex is itself a trivial walk) vertices in G.

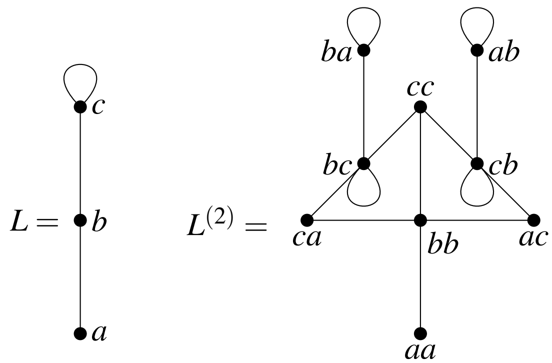

Figure 2.4: The graph, L, containing P_3 and an odd cycle has a connected inner power.

There must exist an edge in G; otherwise $G^{(2)}$ is empty, contradicting the assumption that $G^{(2)}$ is connected. Suppose there is an edge adjoining vertices a and b in G. Since $ab \in E(G)$, it follows that $(a,b)(a,b) \in E(G^{(2)})$. Thus there is an odd cycle (specifically, a loop) on vertex (a,b) in $G^{(2)}$.

By Lemma (2.4), there exists an even (u,v)-(v,u) walk in $G^{(2)}$. Lemma (2.5) then gives a u-v walk in G. Hence G is connected.

To show that G contains an odd cycle, we will suppose that it does not and derive a contradiction. Suppose G does not contain an odd cycle. Let u and v be adjacent vertices of G. Since G contains no odd cycle, G is a bipartite graph. Adjancency implies u and v are elements of distinct partite sets, whence there can not exist either an even u-v path nor an odd u-u path. By Lemma (2.5), a (u,u)-(u,v) walk does not exist in $G^{(2)}$, contradicting the assumption that $G^{(2)}$ is a connected graph. It must be the case that G does not contain an odd cycle. Thus, $G^{(2)}$ is connected only if G is connected and G contains an odd cycle.

To prove the converse, suppose that G is connected and G contains an odd cycle. Let $(a,b),(c,d) \in G^{(2)}$, where $a,b,c,d \in V(G)$. By Lemma (2.4), there exists an even a-c walk and an even b-d walk. By Lemma (2.5), there exists an (a,b)-(c,d) walk. Therefore, $G^{(2)}$ is connected if G is connected and G contains an odd cycle, concluding the proof. $\square$

We now wish to investigate the connectivity of general inner powers of a graph. We begin by looking at the conditions for the existence of paths in these inner powers.

THEOREM 2.7. Let G be a graph and let k and n be positive integers. Then there is a walk of length n in $G^{(k)}$ with ends $\mathbf{v_1} = (v_0^1, v_1^1, ..., v_{k-1}^1)$ and $\mathbf{v_n} = (v_0^n, v_1^n, ..., v_{k-1}^n)$ in $V(G^{(k)})$ if and only if there is a homomorphism $\varphi : (P_n \times C_k) \longrightarrow G$ with $\varphi((i,1)) = v_i^1$ and $\varphi((i,n)) = v_i^n$ for all $i \in \{0, 1, ..., k-1\}$.

Proof. Let G be a graph. We start by proving the converse. Suppose there is a homomorphism $\varphi : P_n \times C_k \longrightarrow G$ for some positive integers n and k where $\phi((i,1)) = v_i^1$ and $\phi((i,n)) = v_i^n$ for all $i \in \{1, ..., n\}$. For each $1 \leq i \leq k$, let

$$\mathbf{v_i} = (\varphi((i,0)), \varphi((i,1)), ..., \varphi((i,k-1))) \in V(G^{(k)}),$$

where (i, j) refers to the vertex in the graph product $P_n \times C_k$ corresponding to the i^{th} vertex in P_n and the j^{th} vertex in C_k. Consider some arbitrary $i \in \{1, 2, ..., n\}$. Since $\varphi : P_n \times C_k \longrightarrow G$ is a homomorphism, $\varphi((i, j))\varphi((i+1, j+1)) \in E(G)$ and $\varphi((i, j))\varphi((i+1, j-1)) \in E(G)$ for all $j \in \{0, 1, ..., k-1\}$. Hence $\mathbf{v_i v_{i+1}} \in E(G^{(k)})$ for all $i \in \{1, 2, ..., n\}$. Therefore, the vertices $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ induce a walk, $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$, of length n in $G^{(k)}$.

Now suppose that there is a walk

$$\mathbf{v_1} = (v_0^1, v_1^1, ..., v_{k-1}^1), \mathbf{v_2} = (v_0^2, v_1^2, ..., v_{k-1}^2), ..., \mathbf{v_n} = (v_0^n, v_1^n, ..., v_{k-1}^n)$$

of length n in $G^{(k)}$. Then

$$v_i^j v_{i-1}^{j+1} \in E(G)$$

and

$$v_i^j v_{i+1}^{j+1} \in E(G)$$

for all $i \in \{0,...,k-1\}$ and all $j \in \{1,...,n\}$. Define $\phi : P_n \times C_k \longrightarrow G$ where $\phi(i,j) = v_i^j$. Then

$$(i,j)(i-1,j+1) \in E(P_n \times C_k)$$

and

$$(i,j)(i+1,j+1) \in E(P_n \times C_k)$$

for all $i \in \{0,...,k-1\}$ and all $j \in \{1,...,n\}$. Since $\phi(i,j) = v_i^j$, the function ϕ is a natural graph homomorphism that arises from the structure of the inner power. □

Recall that a graph is connected if every pair of vertices are connected. The following result falls as an immediate corollary of the previous theorem.

COROLLARY 2.8. Let G be a graph and let $k > 2$ be an integer. Then $G^{(k)}$ is connected if and only for every pair of vertices $(u_1,...,u_k)$ and $(v_1,...,v_k)$ in $V(G^{(k)})$ there exists a positive integer n and a homomorphism $\varphi : (P_n \times C_k) \longrightarrow G$ with $\varphi((i,1)) = u_i$ and $\varphi((i,n)) = v_i$ for all $i \in \{0,1,...,k-1\}$.

Proof. Let G be a graph and let $k > 2$ be an integer. By definition, $G^{(k)}$ is connected if and only if for every two vertices $(u_1,...,u_k)$ and $(v_1,...,v_k)$ in $V(G^{(k)})$ there exists a $(u_1,...,u_k)$-$(v_1,...,v_k)$ path. Hence the result follows directly from Theorem (2.7). □

2.4 Bipartiteness

We now discuss relationships between bipartiteness in a graph and the inner power.

THEOREM 2.9. Let G be a bipartite graph with $|V(G)| \geq 2$. Then $G^{(k)}$ is disconnected for all integers $k \geq 2$.

Proof. Let G be a bipartite graph with partite sets V_1 and V_2. If either partite set is empty, then G contains no edges and $G^{(k)}$ is certainly disconnected. Suppose V_1 and V_2 are nonempty and G contains at least one edge. Consider $(v,\ldots,v) \in V(G^{(k)})$ for some integer $k \geq 2$ and $v \in V_1$. Let W be a $(v,\ldots,v)$-$(x_1,\ldots,x_k)$ walk in $G^{(k)}$. Either W is even or it is odd.

Suppose that W is even. Then there is an even v-x_i walk in G for all $i \in \{1,\ldots,k\}$. Since G is bipartite, $x_i \in V_1$ for all $i \in \{1,\ldots,k\}$. Hence there exists no even $(v,\ldots,v)$-$(y_1,\ldots,y_k)$ walk in $G^{(k)}$ where $y_i \in V_2$ for some $i \in \{1,\ldots,k\}$.

Suppose instead that W is odd. Then there is an odd v-x_i walk in G for all $i \in \{1,\ldots,k\}$. Since G is bipartite, $x_i \in V_2$ for all $i \in \{1,\ldots,k\}$. Hence there exists no odd $(v,\ldots,v)$-$(y_1,\ldots,y_k)$ walk in $G^{(k)}$ where $y_i \in V_1$ for some $i \in \{1,\ldots,k\}$.

Thus, given a vertex $(y_1,\ldots,y_k)$ in $G^{(k)}$ where $y_i \in V_1$ and $y_j \in V_2$ for some $i, j \in \{1,\ldots,k\}$, there does not exist an even $(v,\ldots,v)$-$(y_1,\ldots,y_k)$ walk nor an odd $(v,\ldots,v)$-$(y_1,\ldots,y_k)$ walk. Since V_1 and V_2 are nonempty, such a vertex in $G^{(k)}$ is guaranteed to exist, and thus $G^{(k)}$ is disconnected $\square$

THEOREM 2.10. Let G be a nonempty graph and k be a positive integer. Then $G^{(k)}$ is bipartite if and only if G is bipartite and k is odd.

Proof. Let G be a nonempty graph and k be a positive integer. Furthermore, suppose $G^{(k)}$ is bipartite. It remains to be shown that k must be odd and G is bipartite. In hopes of a contradiction, suppose k is even. Since G is nonempty, there exists $a, b \in V(G)$ with $ab \in E(G)$. Consider $(v_1, v_2, \ldots, v_k) \in V(G^{(k)})$ where

$$v_i = \begin{cases} a & \text{if } i \text{ is odd} \\ b & \text{if } i \text{ is even} \end{cases}.$$

Then $v_{i-1}v_i, v_iv_{i+1} \in E(G)$ for all $i \in \{1,2,\ldots,k\}$, and hence $(v_1, v_2, \ldots, v_k)$ is self-adjacent, inducing a loop. But then $G^{(k)}$ is not bipartite, a contradiction. Thus, k is odd.

Again in hopes of a contradiction, suppose G is not bipartite. Then G contains an odd cycle $v_0v_1...v_{2n}v_0$ where $v_i \in V(G)$ for all $i \in \{0,1,...,2n\}$. Consider

$$\mathbf{v_0} = (v_0,...,v_0)$$

$$\mathbf{v_1} = (v_1,...,v_1)$$

$$\vdots$$

$$\mathbf{v_{2n}} = (v_{2n},...,v_{2n}).$$

These $2n+1$ vertices induce an odd cycle, $\mathbf{v_0v_1}$ ... $\mathbf{v_{2n}v_0}$, in $G^{(k)}$. But then $G^{(k)}$ is not bipartite, a contradiction. Thus, G is bipartite.

Conversely, suppose k is odd and G is bipartite. It remains to be shown that $G^{(k)}$ is bipartite. Partition $V(G)$ into partite sets V_1 and V_2. Partition $V(G^{(k)})$ into the following three subsets:

$$V_{(V_1)} = \{(v_1,v_2,...,v_k) : v_i \in V_1 \text{ for all } i \in \{1,2,...,k\}\}$$

$$V_{(V_2)} = \{(v_1,v_2,...,v_k) : v_i \in V_2 \text{ for all } i \in \{1,2,...,k\}\}$$

$$V_{else} = V(G^{(k)}) \setminus (V_{(V_1)} \cup V_{(V_2)})$$

Suppose $\mathbf{x} = (x_1,x_2,...,x_k)$ and $\mathbf{y} = (y_1,y_2,...,y_k)$ are adjacent vertices in $G^{(k)}$. Since $x_1 \in V(G)$ and since G is bipartite, either $x_1 \in V_1$ or $x_1 \in V_2$. Suppose $x_1 \in V_1$. By the definition of the inner power,

$$x_1,y_2,x_3,y_4,...,y_{k-1},x_k,y_1,x_2,...,y_{k-2},x_{k-1},y_k$$

is a path. In general, x_i and x_j are ends of a path of even length for all $i, j \in \{1, ..., k\}$, and x_i and y_j are ends of a path of odd length for all $i, j \in \{1, ..., k\}$. Hence $\mathbf{x} \in V_{(V_1)}$ and $\mathbf{y} \in V_{(V_2)}$. If instead $x_1 \in V_2$, then it may similarly be shown that $\mathbf{x} \in V_{(V_2)}$ and $\mathbf{y} \in V_{(V_1)}$. Thus two vertices in $G^{(k)}$ are adjacent if and only if one vertex is in $V_{(V_1)}$ and one vertex is in $V_{(V_2)}$. Then $V_{(V_1)}$ and $V_{(V_2)}$ induce a bipartite subgraph of $G^{(k)}$, and V_{else} is a set of isolated vertices in $G^{(k)}$. Union the vertices of V_{else} with either $V_{(V_1)}$ or $V_{(V_2)}$ in an arbitrary fashion. Since no two vertices within any one of the resulting sets are adjacent, then $G^{(k)}$ is bipartite.

Therefore, $G^{(k)}$ is bipartite if and only if G is bipartite and k is odd. □

2.5 Inner Powers over Direct Products

We will now show that inner powers have distributive properties over the direct product. This fact will be important in the next chapter.

THEOREM 2.11. Let G and H be graphs, and let k be a positive integer. Then $(G \times H)^{(k)} \cong G^{(k)} \times H^{(k)}$.

Proof. Let G and H be graphs, and k be a positive integer. Define $\varphi : V((G \times H)^{(k)}) \to V(G^{(k)} \times H^{(k)})$ by

$$\varphi(((x_1, y_1), (x_2, y_2), ..., (x_k, y_k))) = ((x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k)).$$

To show φ is an isomorphism, it remains to show that φ is injective, surjective, and preserves adjacencies and nonadjacencies. To show φ preserves adjacencies and nonadjacencies, suppose

$$((x_1, y_1), ..., (x_k, y_k))((u_1, v_1), ..., (u_k, v_k)) \in E((G \times H)^{(k)}).$$

Then $(x_i, y_i)(u_{i+1}, v_{i+1}), (x_i, y_i)(u_{i-1}, v_{i-1}) \in E(G \times H)$ for all $i \in \{1, 2, ..., k\}$, hence $x_i u_{i+1}, x_i u_{i-1} \in E(G)$ and $y_i v_{i+1}, y_i v_{i-1} \in E(H)$. Then

$$(x_1, x_2, ..., x_k)(u_1, u_2, ..., u_k) \in E(G^{(k)})$$

and

$$(y_1, y_2, ..., y_k)(v_1, v_2, ..., v_k) \in E(H^{(k)}),$$

and thus

$$((x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k))((u_1, u_2, ..., u_k), (v_1, v_2, ..., v_k)) \in E(G^{(k)} \times H^{(k)}).$$

Since the above implications can be reversed, φ preserves both adjacencies and nonadjacencies.

To show φ is injective, suppose that

$$\varphi(((x_1, y_1), ..., (x_k, y_k))) = \varphi(((u_1, v_1), ..., (u_k, v_k))).$$

Then $((x_1, ..., x_k), (y_1, ..., y_k)) = ((u_1, ..., u_k), (v_1, ..., v_k))$, hence $(x_1, ..., x_k) = (u_1, ..., u_k)$ and $(y_1, ..., y_k) = (v_1, ..., v_k)$, whence $x_i = u_i$ and $y_i = v_i$ for all $i \in \{1, ..., k\}$. Then $(x_i, y_i) = (u_i, v_i)$ for all $i \in \{1, ..., k\}$, and thus $((x_1, y_1), ..., (x_k, y_k)) = ((u_1, v_1), ..., (u_k, v_k))$. As hoped, φ is injective.

To show φ is surjective, let $((x_1, ..., x_k), (y_1, ..., y_k)) \in V(G^{(k)} \times H^{(k)})$. Then $(x_1, ..., x_k) \in G^{(k)}$ and $(y_1, ..., y_k) \in H^{(k)}$, hence $x_i \in V(G)$ and $y_i \in V(H)$ for all $i \in \{1, ..., k\}$, whence $(x_i, y_i) \in V(G \times H)$, and thus $((x_1, y_1), ..., (x_k, y_k)) \in V((G \times H)^{(k)})$ with

$$\varphi(((x_1, y_1), ..., (x_k, y_k))) = ((x_1, y_1), ..., (x_k, y_k)).$$

Then φ is surjective.

Therefore φ is an isomorphism, and $(G \times H)^{(k)} \cong G^{(k)} \times H^{(k)}$. $\square$

The property $(G \times H)^{(k)} \cong G^{(k)} \times H^{(k)}$ suggests that the inner power may have applications to questions inolving the direct product. In the next chapter we explore possible connections to the problem of direct product cancellation.

Applications to Cancellation in the Direct Product

A primary motivation in our investigation of the inner power has been to provide some insight into the direct product of graphs. In particular, we hope that the inner power can be used to derive an intuitive proof for the following theorem. The current proof is due to Lovász [5] and is quite lengthy and complex.

THEOREM 3.1. If G, H, and K are graphs and K contains an odd cycle, then $G \times K \cong H \times K$ if and only if $G \cong H$.

We elaborate on this idea in the following sections.

3.1 More Preliminaries

In this section, we derive a theorem that will be instrumental to the alternate proof of Theorem 3.1. The following definitions, theorems, and proofs are either motivated by or paraphrased from a text by Hell and Nešetřil [4].

Recall that a *homomorphism of G to H* is a mapping φ from $V(G)$ into $V(H)$ such that if $uv \in E(G)$, then $\varphi(u)\varphi(v) \in E(H)$. The set of all homomorphisms from G into H is denoted $\mathrm{Hom}(G,H)$, and the number of homorphisms from G into H is denoted $\hom(G,H)$. The number of injective homomorphisms from G into H is denoted $\mathrm{inj}(G,H)$.

Suppose $\theta = \{S_1, S_2, \ldots, S_n\}$ is a partition of $V(G)$. The *quotient, G/θ, of G by θ* is a graph on the vertex set $V(G/\theta) = \{1, 2, \ldots, n\}$. The edge set is the set of all pairs (i, j) where $uv \in V(G)$ for some $u \in S_i$ and $v \in S_j$.

The following is a useful identity involving the number of homomorphisms.

LEMMA 3.2. For any graphs G and H, it follows that

$$\hom(G,H) = \sum_{\theta \in P} \text{inj}(G/\theta, H),$$

where P is the set of all partitions of $V(G)$.

Proof. Given a homomorphism $f : G \longrightarrow H$, we may associate with it a partition

$$\theta = \{f^{-1}(h) : h \in V(H)\}$$

of $V(G)$. Group together all homomorphisms f with the same partition θ. This collection corresponds to the set of all injective homomorphisms from G/θ to H. Hence the total number of homorphisms is equal to the sum of all injective homomorphisms from G/θ to H over every possible partition θ of $V(G)$. In other words, $\hom(G,H) = \sum_{\theta \in P} \text{inj}(G/\theta, H)$. □

We also find that the number of homomorphisms into a product of graphs can be related to the number of homomorphisms into the factors. Before proving this result, we give a useful definition. Given a direct product $G_1 \times G_2 \times ... \times G_n$ of n graphs, the *projection*

$$\pi_{G_i} : V(G_1 \times G_2 \times ... \times G_n) \longrightarrow V(G_i)$$

of $G_1 \times G_2 \times ... \times G_n$ into G_i, for $i \in \{1, ..., n\}$, is a mapping where $\pi_{G_i}((g_1, g_2, ..., g_n)) = g_i$. We observe that projections are homomorphisms since, for each $i \in \{1, ..., n\}$, if

$$(x_1, ..., x_n)(x_1', ..., x_n') \in E(G_1 \times ... \times G_n),$$

then

$$\pi_{G_i}((x_1, ..., x_n)(x_1', ..., x_n')) = x_i x_i' \in E(G_i).$$

Now we are ready for the theorem (first proved by Lovász in [5]).

LEMMA 3.3. For any graphs X, G, and H, we have

$$\hom(X, G\times H) = \hom(X,G)\cdot\hom(X,H).$$

Proof. Consider the sets $\mathrm{Hom}(X,G\times H)$ and $\mathrm{Hom}(X,G)\times\mathrm{Hom}(X,H)$. These sets have cardinalities $\hom(X,G\times H)$ and $\hom(X,G)\cdot\hom(X,H)$ respectively. If we show that there exists a bijection between these sets, then the equality in the claim must hold.

Define $\beta : \mathrm{Hom}(X,G\times H) \longrightarrow \mathrm{Hom}(X,G)\times\mathrm{Hom}(X,H)$ so that $\beta(f) = (\pi_G\circ f, \pi_H\circ f)$ for all $f\in\mathrm{Hom}(X,G\times H)$. This function is well-defined since $\pi_G\circ f$ is a homomorphism from a graph into G and $\pi_H\circ f$ is a homomorphism from a graph into H. It remains to show that β is a bijection.

To show that β is surjective, suppose $(\mu,\lambda)\in\mathrm{Hom}(X,G)\times\mathrm{Hom}(X,H)$. Define $\Phi : X \longrightarrow G\times H$ so that $\Phi(v) = (\mu(v),\lambda(v))$ for all $v\in X$. To show that Φ is a member of the domain of β, we must verify that Φ is indeed a homomorphism. Suppose that $uv\in E(X)$. Then $\mu(u)\mu(v)\in E(G)$ and $\lambda(u)\lambda(v)\in E(H)$, since $\mu\in\mathrm{Hom}(X,G)$ and $\lambda\in\mathrm{Hom}(X,H)$. Then $(\mu(u),\lambda(u))(\mu(u),\lambda(v))\in E(G\times H)$ and thus $\Phi(u)\Phi(v)\in E(G\times H)$. In other words, $\Phi\in\mathrm{Hom}(X,G\times H)$. Since

$$\beta(\Phi) = (\pi_G\circ\Phi, \pi_H\circ\Phi) = (\mu,\lambda),$$

we conclude that β is indeed surjective.

To show that β is injective, suppose that $\beta(\Phi) = \beta(\Psi)$ for some $\Phi,\Psi\in\mathrm{Hom}(X,G\times H)$. Since $\Phi,\Psi\in\mathrm{Hom}(X,G\times H)$, we can express them componentwise, so $\Phi(x) = (\Phi_G(x),\Phi_H(x))$ and $\Psi(x) = (\Psi_G(x),\Psi_H(x))$. To show that $\Phi=\Psi$, it suffices to show that $\Phi(x)=\Psi(x)$ for all vertices $x\in V(X)$. With this in mind we let

$x \in V(X)$ be arbitrary. Then

$$\begin{aligned}
\Phi(x) &= (\Phi_G(x), \Phi_H(x)) \\
&= (\pi_G \circ \Phi(x), \pi_H \circ \Phi(x)) \\
&= \beta(\Phi)(x) \\
&= \beta(\Psi)(x) \\
&= (\pi_G \circ \Psi(x), \pi_H \circ \Psi(x)) \\
&= (\Psi_G(x), \Psi_H(x)) \\
&= \Psi(x).
\end{aligned}$$

Then β is a bijection, and therefore $\hom(X, G \times H) = \hom(X,G) \cdot \hom(X,H)$. □

It is possible to find an enumeration, $X_1, X_2, \ldots$ of all nonisomorphic graphs. The *Lovász vector* of a graph G is the sequence $\{n_i\}$ where $n_i = \hom(X_i, G)$. Interestingly, not only can the existence of a graph be determined given the Lovász vector, but a graph can be uniquely constructed given the Lovász vector. This follows from the next theorem (proved in [5]).

THEOREM 3.4. Given graphs G and H, then $G \cong H$ if and only if $\hom(X,G) = \hom(X,H)$ for every graph X.

Proof. Let G and H be graphs. If $G \cong H$, then surely $\hom(X,G) = \hom(X,H)$ for every graph X. Suppose conversely that $\hom(X,G) = \hom(X,H)$ for every graph X. To show $G \cong H$, it will suffice to show that $\mathrm{inj}(X,G) = \mathrm{inj}(X,H)$ for all graphs X. This would imply that $\mathrm{inj}(G,H) = \mathrm{inj}(G,G) \neq 0$ and $\mathrm{inj}(H,G) = \mathrm{inj}(H,H) \neq 0$, hence implying that there exist injective homomorphisms from G into H and from H into G, and thus that $G \cong H$.

We proceed by induction on the number of vertices in X. Suppose X has exactly one vertex. Then any homomorphism of X is injective, and thus $\text{inj}(X,G) = \text{hom}(X,G) = \text{hom}(X,H) = \text{inj}(X,H)$. Hence we have established a basis for induction.

Now suppose that $\text{inj}(X,G) = \text{inj}(X,H)$ for all graphs X with fewer than n vertices. We assumed that $\text{hom}(X,G) = \text{hom}(X,H)$. By Lemma 3.2, we thus have

$$\sum_{\theta \in P} \text{inj}(X/\theta, G) = \sum_{\theta \in P} \text{inj}(X/\theta, H),$$

where P is the set of all partitions of $V(G)$. Let t be the trivial partition of $V(X)$ into singletons. Then $G/t \cong G$, so

$$\text{inj}(X,G) + \sum_{\theta \in P-t} \text{inj}(X/\theta, G) = \text{inj}(X,H) + \sum_{\theta \in P-t} \text{inj}(X/\theta, H). \tag{3.1}$$

If θ is not the partition of $V(X)$ into singletons, then X/θ is a graph with fewer than n vertices. Hence, by the inductive hypothesis,

$$\sum_{\theta \in P-t} \text{inj}(X/\theta, G) = \sum_{\theta \in P-t} \text{inj}(X/\theta, H). \tag{3.2}$$

Thus by Equations 3.1 and 3.2, $\text{inj}(X,G) = \text{inj}(X,H)$. By the reasoning given above, we have $G \cong H$, and therefore, by induction, the claim is shown to be true. □

The previous results in this section were given so that we could derive the next theorem, which will be useful in our proof of Theorem 3.1. Notice that this theorem is a weaker result than Theorem 3.1.

THEOREM 3.5. Let G, H, and K be graphs where K contains a loop. Then $G \times K \cong H \times K$ if and only if $G \cong H$.

Proof. If $G \cong H$, then certainly $G \times K \cong H \times K$. Suppose that $G \times K \cong H \times K$. Then for an arbitrary graphs X, $\hom(X, G \times K) = \hom(X, H \times K)$. By Lemma 3.3, we have $\hom(X,G) \cdot \hom(X,K) = \hom(X,H) \cdot \hom(X,K)$. Since K contains a loop, it follows that $\hom(X,K) \neq 0$, since the mapping of all the vertices of X into the vertex of K with a loop is a homomorphism. Thus $\hom(X,G) = \hom(X,H)$. Since X was an arbitrary graph, Theorem 3.4 gives us $G \cong H$. □

Homomorphisms can also be used to prove the following result involving direct powers, which will be mentioned in the next section.

THEOREM 3.6. Let G and H be graphs and let k be a positive integer. Then $G^k \cong H^k$ if and only if $G \cong H$.

Proof. Let G and H be graphs and let k be a positive integer. Certainly if $G \cong H$, then $G^k \cong H^k$. Suppose that $G^k \cong H^k$. By Theorem 3.4, $\hom(X, G^k) = \hom(X, H^k)$ for all graphs X. By Lemma 3.3, $\hom(X,G)^k = \hom(X,H)^k$ for all graphs X. Since hom is defined over the nonnegative integers, $\hom(X,G) = \hom(X,H)$ for all graphs X, and therefore $G \cong H$, completing the proof. □

3.2 Cancellation of Inner Powers

As shown in Theorem 3.6, it is true that for any positive integer k, $G^k \cong H^k$ if and only $G \cong H$. This result regarding direct powers naturally leads to an analagous question regarding inner powers. Suppose that $G^{(k)} \cong H^{(k)}$ for some positive integer k. Is it necessarily true that $G \cong H$?

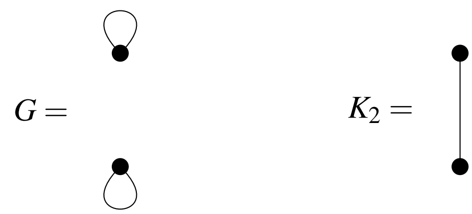

Figure 3.1: Nonisomorphic graphs G and K_2.

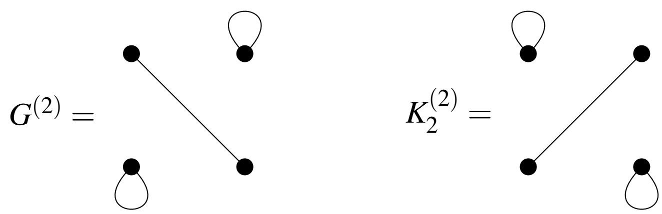

Figure 3.2: The second inner powers, $G^{(2)}$ and $K_2^{(2)}$.

In Figures 3.1 and 3.2, G and K_2 are examples of nonisomorphic graphs with isomorphic inner powers, thus proving that the answer to the question above is "no." In fact, the graphs G and K_2 are counterexamples for any even positive integer k.

THEOREM 3.7. Let G be the graph with exactly two vertices and two loops. Then $G^{(k)} \cong K_2^{(k)}$ for all positive even integers k.

Proof. Let G be the graph with two distinct vertices a and b and two loops aa and bb. Let K_2 be the complete graph on two vertices c and d with edge cd. Let k be an even integer. Define $\varphi : V(G)^k \longrightarrow V(K_2)^k$ by $\varphi((x_1, x_2, \ldots, x_k)) = (\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_k(x_k))$ where

$$\varphi_i(x_i) = \begin{cases} c & \text{if either } x_i = a \text{ and } i \text{ is odd or } x_i = b \text{ and } i \text{ is even} \\ d & \text{if either } x_i = b \text{ and } i \text{ is odd or } x_i = a \text{ and } i \text{ is even} \end{cases}.$$

It is quickly evident that φ is a bijection. To show that φ is an isomorphism, it remains to show that φ preserves both adjacencies and nonadjacencies.

There are exactly two loops, on vertices $(a,a,...,a,a)$ and $(b,b,...,b,b)$, in G, since the only closed walks of length k in G are $a,a,...,a,a$ and $b,b,...,b,b$. Analagously, there are exactly two loops, on vertices $(c,d,...,c,d)$ and $(d,c,...,d,c)$, in K_2, since the only closed walks of length k in K_2 are $c,d,...,c,d$ and $d,c,...,d,c$. Since $\varphi((a,a,...,a,a)) = (c,d,...,c,d)$ and $\varphi((b,b,...,b,b)) = (d,c,...,d,c)$, it follows that φ preserves these adjacencies.

There is an edge, $(a,b,...,a,b)(b,a,...,b,a)$, that is not a loop in $G^{(k)}$. There is also an edge, $(c,c,...,c,c)(d,d,...,d,d)$, that is not a loop in $K_2^{(k)}$. Since $\varphi((a,b,...,a,b)) = (c,c,...,c,c)$ and $\varphi((b,a,..,b,a)) = (d,d,...,d,d)$, it follows that φ preserves this adjacency.

All vertices in $G^{(k)}$ other than $(a,a,...,a,a)$, $(b,b,...,b,b)$, $(a,b,...,a,b)$, and $(b,a,...,b,a)$ are isolated. More precisely, any vertex that is not isolated is of the form $(x_1,x_2,...,x_k)$ where $x_1 = x_3 = ... = x_{k-1}$ and $x_2 = x_4 = ... = x_k$, since an adjacency with a vertex $(y_1,y_2,...,y_k)$ relies on the existence of an even path $x_i,y_{i+1},...,y_{j-1},x_i$ for all i and j even and for all i and j odd. Similarly, all vertices in $K_2^{(k)}$ other than $(c,c,...,c,c)$, $(d,d,...,d,d)$, $(c,d,...,c,d)$, and $(d,c,...,d,c)$ are isolated. Hence, since φ preserves adjacencies and nonadjacencies, it is an isomorphism. $\square$

These particular nonisomorphic graphs are not unique counterexamples, as shown in Figures 3.3 and 3.4.

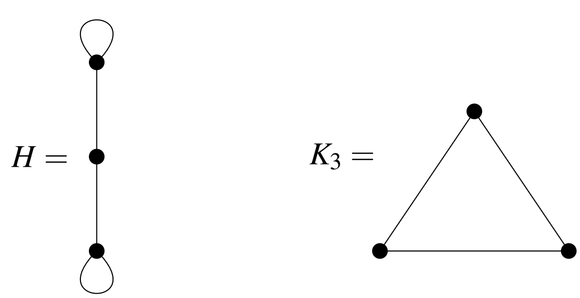

Figure 3.3: Nonisomorphic graphs H and K_3.

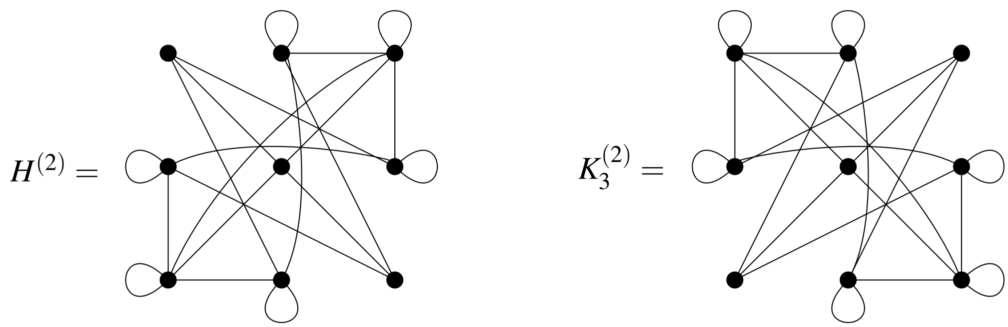

Figure 3.4: The second inner powers of H and K_3 are isomorphic (notice that these graphs are each vertical reflections of the other).

We can show that Theorem 3.7 does not extend to all integral powers k. Suppose k is odd. Since G contains a loop, $G^{(k)}$ contains a loop. However, by Theorem 2.10, $K_2^{(k)}$ is bipartite. Thus $G^{(k)}$ and $K_2^{(k)}$ are not isomorphic. This gives evidence that $G^{(k)} \cong H^{(k)}$ might imply $G \cong H$ when k is odd. We have not been able to prove this conjecture. However, we provide what may be a partial proof.

Our argument relies on two theorems due to Lovász [5]. Proofs of the these two theorems will not be given, but we will remark that the proof to the first theorem is hard, and that the second theorem follows from the first.

THEOREM 3.8. If $A \times C \cong B \times C$, then there is an isomorphism $\Phi : V(A \times C) \longrightarrow V(B \times C)$ where $\Phi(a,c) = (\Phi(a,c),c)$.

THEOREM 3.9. If C contains an odd cycle, then $A \times C \cong B \times C$ if and only if $A \cong B$.

From these two theorems, we argue the following. Our theorem would follow as a corollary.

CONJECTURE 3.10. For graphs G and H, $G \times C_k \cong H \times C_k$ if and only if $G^{(k)} \cong H^{(k)}$.

Our argument goes as follows. Suppose $G \times C_k \cong H \times C_k$. By Theorem 3.8, we can choose some isomorphism $\Phi : V(G \times C_k) \longrightarrow V(H \times C_k)$ where $\Phi((g,i)) = (\varphi(g,i),i)$.

Define $\Psi : V(G^{(k)}) \longrightarrow V(H^{(k)})$ where

$$\Psi((x_0, x_1, ..., x_{k-1})) = (\varphi(g,0), \varphi(g,1), ..., \varphi(g,k-1)).$$

To show that Ψ is an isomorphism, we first show that Ψ is injective.

To show that Ψ is injective, notice that

$$\begin{aligned}
& \Psi((x_0, ..., x_{k-1})) = \Psi((y_0, ..., y_{k-1})) \\
\Rightarrow \quad & (\varphi(x_0,0), \varphi(x_1,1), ..., \varphi(x_{k-1},k-1)) = (\varphi(y_0,0), \varphi(y_1,1), ..., \varphi(y_{k-1},k-1)) \\
\Rightarrow \quad & \varphi(x_i,i) = \varphi(y_i,i) \text{ for all } i \in \{0, ..., k-1\} \\
\Rightarrow \quad & (\varphi(x_i,i),i) = (\varphi(y_i,i),i) \text{ for all } i \in \{0, ..., k-1\} \\
\Rightarrow \quad & \Phi(x_i,i) = \Phi(x_i,i) \text{ for all } i \in \{0, ..., k-1\} \\
\Rightarrow \quad & x_i = y_i \text{ for all } i \in \{0, ..., k-1\}.
\end{aligned}$$

Now we show that Ψ preserves both adjacencies and nonadjacencies. Notice that

$$\begin{aligned}
& (x_0, ..., x_{k-1})(y_0, ..., y_{k-1}) \in E(G^{(k)}) \\
\Leftrightarrow \quad & x_i y_{i\pm 1} \in E(G) \text{ for all } i \in \{0, .., k-1\} \\
\Leftrightarrow \quad & (x_i,i)(y_{i\pm 1}, i\pm 1) \in E(G \times C_k) \text{ for all } i \in \{0, .., k-1\} \\
\Leftrightarrow \quad & \Psi(x_i,i)\Psi(y_{i\pm 1}, i\pm 1) \in E(H \times C_k) \text{ for all } i \in \{0, .., k-1\} \\
\Leftrightarrow \quad & (\varphi(x_i,i),i)(\varphi(y_{i\pm 1}, i\pm 1), i\pm 1) \in E(H \times C_k) \text{ for all } i \in \{0, .., k-1\} \\
\Leftrightarrow \quad & \varphi(x_i,i)\varphi(y_{i\pm 1}, i\pm 1) \in E(H) \text{ for all } i \in \{0, .., k-1\} \\
\Leftrightarrow \quad & (\varphi(x_0,0), \varphi(x_1,1), ..., \varphi(x_{k-1},k-1))(\varphi(y_0,0), \varphi(y_1,1), ..., \varphi(y_{k-1},k-1)) \in E(H^{(k)}) \\
\Leftrightarrow \quad & \Psi(x_0, x_1, ..., x_{k-1})\Psi(y_0, y_1, ..., y_{k-1}) \in E(H^{(k)}).
\end{aligned}$$

Since Ψ is an isomorphism preserving adjacencies and nonadjacencies, Ψ is an isomorphism, and thus $G^{(k)} \cong H^{(k)}$.

Conversely, suppose $G^{(k)} \cong H^{(k)}$. Choose some isomorphism $\Phi : V(G^{(k)}) \longrightarrow V(H^{(k)})$ where $\Phi((x_0, x_1, ..., x_k)) = (\phi_0(x_0, ..., x_{k-1}), \phi_1(x_0, ..., x_{k-1}), ..., \phi_{k-1}(x_0, ..., x_{k-1}))$. Define a map $\Psi : V(G \times C_k) \longrightarrow V(H \times C_k)$ where $\Psi(x, i) = (\phi_i(x, x, ..., x), i)$. We want to show that Ψ is an isomorphism.

We show that Ψ preserves both adjacencies and nonadjacencies. Notice that

$$
\begin{aligned}
& (x,i)(y,i \pm 1) \in E(G \times C_k) \\
\Leftrightarrow \quad & x, y \in E(G) \\
\Leftrightarrow \quad & (x,x,...,x)(y,y,...,y) \in E(G^{(k)}) \\
\Leftrightarrow \quad & \Phi(x,x,...,x)\Phi(y,y,...,y) \in E(H^{(k)}) \\
\Leftrightarrow \quad & (\varphi_0(x,x,...,x), ..., \varphi_{k-1}(x,x,...,x))(\varphi_0(y,y,...,y), ..., \varphi_{k-1}(y,y,...,y)) \in E(H^{(k)}) \\
\Leftrightarrow \quad & \varphi_i(x,x,...,x)\varphi_{i\pm1}(y,y,...,y) \in E(H) \text{ for all } i \in \{0, ..., k-1\} \\
\Leftrightarrow \quad & (\varphi_i(x,x,...,x), i)(\varphi_{i\pm1}(y,y,...,y), i \pm 1) \in E(H \times C_k) \text{ for all } i \in \{0, ..., k-1\} \\
\Leftrightarrow \quad & \Psi(x,i)\Psi(y,i \pm 1) \in E(H \times C_k)
\end{aligned}
$$

If we could show that Ψ is injective, we would have a proof of Conjecture 3.10. The next result is necessary for our proof of Theorem 3.1, but since it relies on Conjecture 3.10, we list it as a conjecture.

CONJECTURE 3.11. Let G and H be graphs, and let k be a positive odd integer. Then $G^{(k)} \cong H^{(k)}$ if and only if $G \cong H$.

Our argument goes as follows. If Conjecture 3.10 is true, then $G^{(k)} \cong H^{(k)}$ if and only if $G \times C_k \cong H \times C_k$. Since C_k contains an odd cycle, we have by Theorem 3.9 that $G \times C_k \cong H \times C_k$ if and only if $G \cong H$. By transitivity, then $G^{(k)} \cong H^{(k)}$ if and only if $G \cong H$.

3.3 Inner Powers and Direct Product Cancellation

We now provide our argument for Theorem 3.1, reiterated below:

THEOREM 3.12. If K contains an odd cycle, then $G \times K \cong H \times K$ if and only if $G \cong H$.

Certainly if $G \cong H$ then $G \times K \cong H \times K$. Suppose conversely that $G \times K \cong H \times K$ and that K is nonbipartite. Then K has a cycle of odd length k. Raising each side of the isomorphism to the k^{th} inner power, we find that

$$(G \times K)^{(k)} \cong G^{(k)} \times H^{(k)}.$$

By Theorem 2.11,

$$G^{(k)} \times K^{(k)} \cong H^{(k)} \times K^{(k)}.$$

Since k is the length of a closed walk in K, then, by Theorem 2.3, $K^{(k)}$ contains a loop. Hence, by Theorem 3.5, it follows that $G^{(k)} \cong H^{(k)}$. Finally, since k is odd, we would have, if Conjecture 3.11 is true, that $G \cong H$.

As mentioned, the original motivation of the inner power was to provide a proof of this theorem. We would like to be able to provide a proof that does not rely on Theorem 3.8, due to the length and complexity of its proof. It is the opinion of the author that a proof of Theorem 3.1 involving inner powers, and that is not lengthy and complicated, exists, and we invite the reader to join us in searching.

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Vita

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