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A Variational Approach to the First Boundary Value Problem for Navier Equation

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Navier 方程式の第1種境界値問題に対する変分法

新潟歯学部・物理　桜　岡　　充

齲歯の治療によって口腔内に装着された歯冠補綴物が，長期にわたって機能するには，その維持力の強さが問題であろう。これを規定する因子としては，支台歯の形や強度，接着セメントの強度，鋳造冠の良否や適合性，合着の状態等，非常に多岐にわたるであろう。支台歯の強度，合着セメントの強度に対する考察は筆者の及び得るところではなく，臨床医諸賢あるいは化学工学者等々の考察対象である。然るに，比較的程度のよい齲歯の補綴冠の場合，その脱落は合着セメント層の破壊によって生ずるものと思われる。

食物を咬み砕く時，苛酷な程大きな外力が補綴処置歯に加わる訳であるが，これが鋳造冠と歯質の硬さの違いのため，合着面に応力集中をひき起すであろう。この点の合着セメント層に，最大ずれ応力の面に沿って微細な初期破壊（クラック）が起り，これが拡がることによって冠の維持力が失われるのである。この微細な初期破壊が容易か否かこそが冠の維持力にとって極めて重要と思われるのである。そして，これを与える外力は咬合時に起り，咬合面にほぼ垂直として大きく相異することはないと思われる。それ故，この垂直外圧に対して応力集中をでき得る限り小さくするような支台歯の形が問題である。また，それを実現する補綴冠の材質も研究されるべきである。

この小論では，てはじめとして，長方形状の弾性体に外力が作用した時の応力を Rayleigh-Ritz の方法で求めて，適当な補綴冠の場合の 合着層に現れる 応力を図示する。但し，この小論の場合，理想的な等硬度補綴冠であるので応力集中現象は現れない。Nicholls 型の補綴冠（参考文献の10）に対して，（鋳造補綴冠）:（セメント）:（歯質）の硬度比と，応力集中の大きさの関係は次回報告の予定である。

A Variational Approach to the First Boundary Value Problem for Navier Equation

The approximate solution of the elastostatic problem of a rectangular region are derived in the case that tensile forces at the ends are distributed according to a biquadratic form. The solutions are determined by minimum energy principle under 12 adjustable parameters. They are applied to the retentivity of a dental crown restoration.

Sec. 1. Introduction

The two-dimensional elastostatic problem is to be finding two functions which satisfy the appropriate boundary conditons. One of these functions is the bi-harmonic funtion $\phi(x, y)$ called Airy stress function, and another is a harmonic function $\psi(x, y)$ which should be called displacement function. The stress components σ_x, σ_y and σ_{xy}, and the displace components u_x and u_y are expressed in terms of ϕ and ψ as

$$\sigma_x=\partial^2\phi/\partial^2 y,\ \sigma_y=\partial^2\phi/\partial x^2,\ \sigma_{xy}=-\partial^2\phi/\partial x\partial y, \tag{1}$$

$$u_x=[\partial\psi/\partial y-(1+\nu)\cdot\partial\phi/\partial x]\,/\,E, \tag{2a}$$

$$u_y=[\partial\psi/\partial x-(1+\nu)\cdot\partial\phi/\partial y]\,/\,E, \tag{2b}$$

in case of plane stress condition.*) Where ν and E denote the Poisson's ratio and the Young's modulus of the elastic material, respectively. The forms of stresses and displaccments given in Eqs. (1), (2a) and (2b) automatically satisfy the equations of equilibrium and Hooke's law. The determinantal conditions on the real stress-strain state are boundary conditions and the equations:

$$\nabla^4\phi=0, \tag{3a}$$

$$\nabla^2\psi=0,\ \partial^2\psi/\partial x\partial y=0, \tag{3b}$$

where ∇^2 denotes two-dimensional Laplacian, $\partial^2/\partial x^2+\partial^2/\partial y^2$.

Let us denote by $\phi(x, y)$, the actual Airy stress function for a deformed elastic

*) For the plain strain condition, these expressions are also true if, instead of E and ν, a modified Young's modulus $E'=E\,/\,(1-\nu^2)$ and modified Poisson's ratio $\nu'=\nu\,/\,(1-\nu)$ are substituted, respectively. At the moment, however, we write down the formulae for the state of plane stress condition.

body under a given condition. The elastic stain energy, i.e., the free energy accumlated in this deformed elastic body for unit thickness is

$$U=\frac{1}{2E}\int_V dx\,dy\,[\sigma_x^2+\sigma_y^2-2\nu\cdot\sigma_x\cdot\sigma_y+2(1+\nu)\cdot\sigma_{xy}^2],$$
$$=\frac{1}{2E}\int_V dx\,dy\Big[\Big(\frac{\partial^2\phi}{\partial x^2}\Big)^2+\Big(\frac{\partial^2\phi}{\partial y^2}\Big)^2-2\nu\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}+2(1+\nu)\Big(\frac{\partial^2\phi}{\partial x\cdot\partial y^2}\Big)^2\Big], \quad (4)$$

the intsegral being taken over the whole region of elastic body.

Let us consider the change in strain energy, $U\rightarrow U+\delta U$, caused by the functional variation on stress function:

$$\begin{cases}\phi(x,y)\rightarrow\phi(x,y)+\delta\phi(x,y),\\ \text{but, at the boundaries of the elastic body,}\\ \delta\phi(x,y)=\partial\delta\phi(x,y)/\partial x=\partial\delta\phi(x,y)/\partial y=0\end{cases}.$$

From Eq. (4), we can easily derive the following form for δU by partial integration:

$$\delta U=\frac{1}{E}\int_V dx\,dy\cdot\delta\phi(x,y)\cdot\nabla^4\phi(x,y)$$
$$+\frac{1}{2E}\int_V dx\,dy\Big[\Big(\frac{\partial^2\delta\phi}{\partial x^2}\Big)^2+\Big(\frac{\partial^2\delta\phi}{\partial y^2}\Big)^2-2\nu\cdot\frac{\partial^2\delta\phi}{\partial x^2}\cdot\frac{\partial^2\delta\phi}{\partial y^2}+2(1+\nu)\Big(\frac{\partial^2\delta\phi}{\partial x\,\partial y}\Big)^2\Big]. \quad (5)$$

The first term on the right hand side represents the first-order incrcment in strain energy and equals zero according to Eq. (3a). The second term is the second-order increment, and it is positive since ν is smaller than 1/2 for any kinds of elastic materials. It states merely that the general condition of stability of equilibrium also holds for the stress-strain state of an elastic boody, i.e., the free energy of a system takes minimum at the state of equilibrium.

Eq. (4), however, gives the essence of the variational method [1)-3)] of finding out the solution of an elastostatic problem. Under a given boundary condition, if we discovered a function which yields the minimum value of U in Eq. (4), we can identify it with the correct Airy stress function ϕ. However, we can not find out such a solution in general. For the practical usage of the principle, we take as a trial function on ϕ the finite number of integer functions, rational functions, trigonometric functions and so on, which contains a number of parameters in such that the boundary condition is satisfied. Evaluating U, the integrals on the right side of Eq. (4), with this trial function and varying the parameters until the value of U is a minimum, we reach the approximate solution of the problem under the given boundary condition. The more we take as the members of trial function, the closer the result is to the rigorous solution in precision. If we start with taking, as the members of trial function, the complete set of functions of the functional space corresponding to the region of elastic body, we arrive the rigorous solution, if possible. It is guaranteed by Eq. (4) in a word.

The above-stated is the outline of the method of finding the approximate solution by variational calculation. However we leave the precise argument concerened with the variational method to the superior literatures[4)5)].

Sec. 2. The Solution in Rectangular Region

Let us consider the case of a rectangular elastic body in tension, when the tensile forces at the ends are distributed according to biquadratic expression (See Fig.

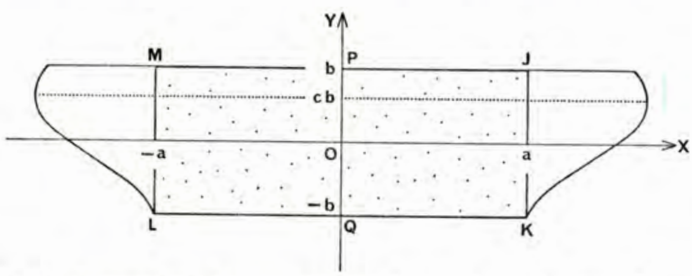

Fig. 1 Geometrical Assignment and distributions of external forces.

1.). The boundary conditions are followings;

$$\text{for } x = \pm a, \left\{ \begin{array}{l} \sigma_x = \dfrac{15}{16\Sigma} \cdot \dfrac{F}{b} \cdot (1+\eta)^2 \cdot (1-\eta+2c)^2, \\ \sigma_{xy} = 0, \end{array} \right. \tag{6a}$$

and

$$\text{for } y = \pm b,\ \sigma_y = \sigma_{xy} = 0, \tag{6b}$$

where Σ and η denote $1+5c+10c^2$ and y/b, respectively. The external force amounts to F in all, and takes maximum at $y=cb$ or $\eta=c$. When F is negative, this is also possible to represents the situation in which the cempressive force at the end JK of a rectangular elastic body PJKQ acts as such illustrated, and the other end PQ makes contact with the perfectly rigid surface under the frictionless condition.

It should be noted, from Eq. (1), that for a first boundary problem, such as we have in the present case, the stress distribution does not depend on the elastic constants of the material and further calculations can therefore be simplified by taking Poisson's ratio ν as zero. Then the correct expression for the stress function is that satisfing the boundary conditions and making the strain energy

$$U = \frac{1}{2E} \int_V dx\, dy \left[\left(\frac{\partial^2 \phi}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)^2 + 2 \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right] \tag{7}$$

a minimum. If we apply variational calculus to determine the minimum of Eq. (7), we shall arrive at true stress function. Instead of this, from a practical point of view we shall use the trial function method for a satisfactory approximate solution of the problem. The trial function must be chosen so that the boundary conditions (6a) and (6b) are satisfied. Then, by taking into account the symmetry property with respect to y-axis, we take the form of the trial function as

$$\phi=(Fb/32\Sigma)\cdot\eta^2[\eta^4-6c\eta^3-5(1+2c-2c^2)\eta^2+20c(1+2c)\eta+15(1+2c)^2]$$
$$+\frac{b^2}{4}(1-\xi^2)^2(1-\eta^2)^2\Big\{A_1+2\xi^2A_2+2\eta^2A_3+\xi^4A_4+2\xi^2\eta^2A_5+\eta^4A_6 \qquad (8)$$
$$+\eta B_1+2\xi^2\eta B_2+2\eta^3B_3+\xi^4\eta B_4+2\xi^2\eta^3B_5+\eta^5B_6\Big\},$$

where ξ and η are x/a and y/b, respectively. A_i $(i=1-6)$ and B_i $(i=1-6)$ are constants to be determined for which the strain energy U takes minimum. Substituting the expression (8) into right side of Eq. (7), we find U as a second degree in A_i and B_i. Then A_i and B_i can be calculated from the conditions $\partial U/\partial A_i=\partial U/\partial B_i=0$. These yield the following two groups of equations:

$$\boldsymbol{\Gamma}\cdot\boldsymbol{A}=F\boldsymbol{\Lambda},\ \boldsymbol{\Gamma}_B\cdot\boldsymbol{B}=F\boldsymbol{\Lambda}_B, \qquad (9)$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}_B$ denote the column vectors defined as

$$\boldsymbol{A}=\begin{pmatrix}A_1\\A_2\\A_3\\A_4\\A_5\\A_6\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}B_1\\B_2\\B_3\\B_4\\B_5\\B_6\end{pmatrix},\quad \boldsymbol{\Lambda}=\frac{3\cdot5\cdot7\cdot11}{32\,\Sigma\,b}\begin{pmatrix}1+3.5c-3.5c^2\\2(1+3.5c-3.5c^2)/7\\c(1-c)\\(1+3.5c-3.5c^2)/7\\c(1-c)/7\\-(1-5.5c+5.5c^2)/33\end{pmatrix}$$

and

$$\boldsymbol{\Lambda}_B=\frac{3\cdot5\cdot7\cdot11}{32\,\Sigma\,b}c\begin{pmatrix}3/2\\3/7\\1\\1/14\\1/7\\5/22\end{pmatrix}.$$

And, $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}_B$ are 6 by 6 matrices those elements are

$$(\Gamma)_{11}=77(1+P^2)+44P,$$
$$(\Gamma)_{12}=(\Gamma)_{21}=2(7+11P^2),$$
$$(\Gamma)_{13}=(\Gamma)_{31}=2(11+7P^2),$$
$$(\Gamma)_{14}=(\Gamma)_{41}=21/13-4P/3+11P^2/3,$$
$$(\Gamma)_{15}=(\Gamma)_{51}=2(1+P^2),$$

$(\Gamma)_{16}=(\Gamma)_{61}=11/3-4P/3+21P^2/13,$

$(\Gamma)_{22}=4\,(21/13+4P+33P^2),$

$(\Gamma)_{23}=(\Gamma)_{32}=4\,(1+P^2),$

$(\Gamma)_{24}=(\Gamma)_{42}=2\,(21/13+56P/13+53P^2)/3,$

$(\Gamma)_{25}=(\Gamma)_{52}=12\,(1/13+P^2),$

$(\Gamma)_{26}=(\Gamma)_{62}=2\,(1+9P^2/13)/3,$

$(\Gamma)_{33}=4\,(33+4P+21P^2/13),$

$(\Gamma)_{34}=(\Gamma)_{43}=2\,(9/13+P^2)/3,$

$(\Gamma)_{35}=(\Gamma)_{53}=12\,(1+P^2/13),$

$(\Gamma)_{36}=(\Gamma)_{63}=2\,(53+56P/13+21P^2/13)/3,$

$(\Gamma)_{44}=(49/17+12P+643P^2/3)/13,$

$(\Gamma)_{45}=(\Gamma)_{54}=2\,(1/13+53P^2/33),$

$(\Gamma)_{46}=(\Gamma)_{64}=(1+52P/99+P^2)/13,$

$(\Gamma)_{55}=36\,(1+P^2)/13+16P/11,$

$(\Gamma)_{56}=(\Gamma)_{65}=2\,(53/33+P^2/13),$

$(\Gamma)_{66}=(643/3+12P+49P^2/17)/13,$

and

$(\Gamma_B)_{11}=55+44P/3+7P^2,$

$(\Gamma_B)_{12}=(\Gamma_B)_{21}=10+2P^2,$

$(\Gamma_B)_{13}=(\Gamma_B)_{31}=110/3+16P/3+42P^2/13,$

$(\Gamma_B)_{14}=(\Gamma_B)_{41}=15/13-4P/9+P^2/3,$

$(\Gamma_B)_{15}=(\Gamma_B)_{51}=10/3+6P^2/13,$

$(\Gamma_B)_{16}=(\Gamma_B)_{61}=25/3+20P/39+7P^2/13,$

$(\Gamma_B)_{22}=60/13+16P/3+12P^2,$

$(\Gamma_B)_{23}=(\Gamma_B)_{32}=20/3+12P^2/13,$

$(\Gamma_B)_{24}=(\Gamma_B)_{42}=10/13+112P/117+106P^2/33,$

$(\Gamma_B)_{25}=(\Gamma_B)_{52}=20/13+32P/33+36P^2/13,$

$(\Gamma_B)_{26}=(\Gamma_B)_{62}=50/33+2P^2/13,$

$(\Gamma_B)_{33}=4\,(65/3+68P/39+7P^2/13),$

$(\Gamma_B)_{34}=(\Gamma_B)_{43}=10/13-16P/99+2P^2/13,$

$(\Gamma_B)_{35}=(_B)_{53}=260/33+4P^2/13,$

$(\Gamma_B)_{36}=(\Gamma_B)_{63}=(1150+64P+294P^2/17)/39,$

$(\Gamma_B)_{44}=(35/17+4P+643P^2/33)/13,$

$(\Gamma_B)_{45}=(\Gamma_B)_{54}=(10/3+224P/99+106P^2/11)/13,$

$(\Gamma_B)_{46}=(\Gamma_B)_{64}=(25/11-20P/99+P^2/3)/13,$

$(\Gamma_B)_{55}=4\,(65/11+68P/33+3P^2)/13,$

$(\Gamma_B)_{56}=(\Gamma_B)_{65}=(1150/33+14P^2/17)/13,$

$(\Gamma_B)_{66}=521/39+28P/51+441P^2/4199,$

where P is b^2/a^2.

If we take for instance $a=4b$, we find, from Eq. (9),

$$\text{for } c=0, \quad \boldsymbol{A}=\frac{F}{b}\begin{pmatrix} 0.387 \\ 0.210 \\ -0.078 \\ 2.520 \\ -0.010 \\ -0.001 \end{pmatrix} \text{ and } \boldsymbol{B}=0,$$

$$\text{for } c=0.4, \ \boldsymbol{A}=\frac{F}{b}\begin{pmatrix} 0.151 \\ 0.084 \\ -0.017 \\ 0.101 \\ -0.002 \\ 0.000 \end{pmatrix} \text{ and } \boldsymbol{B}=\frac{F}{b}\begin{pmatrix} 0.070 \\ -0.021 \\ -0.001 \\ 0.877 \\ 0.008 \\ 0.000 \end{pmatrix},$$

and

$$\text{for } c=0.8, \ \boldsymbol{A}=\frac{F}{b}\begin{pmatrix} 0.052 \\ 0.029 \\ -0.007 \\ 0.345 \\ -0.001 \\ 0.000 \end{pmatrix} \text{ and } \boldsymbol{B}=\frac{F}{b}\begin{pmatrix} 0.057 \\ -0.017 \\ -0.001 \\ 0.708 \\ 0.007 \\ 0.000 \end{pmatrix}.$$

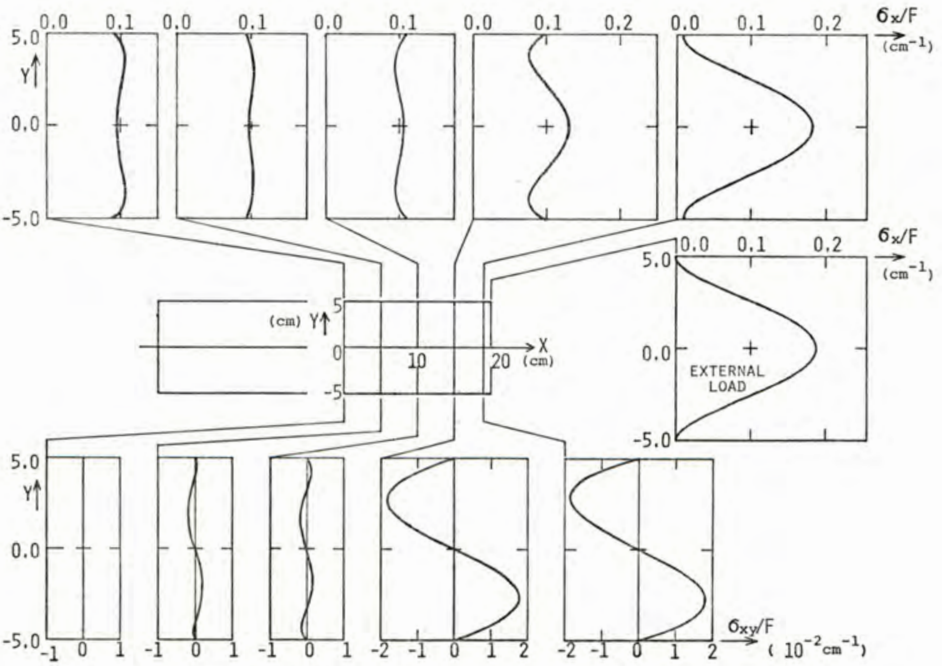

Fig. 2 The stress distributions which arise on the several vertical cross sections.

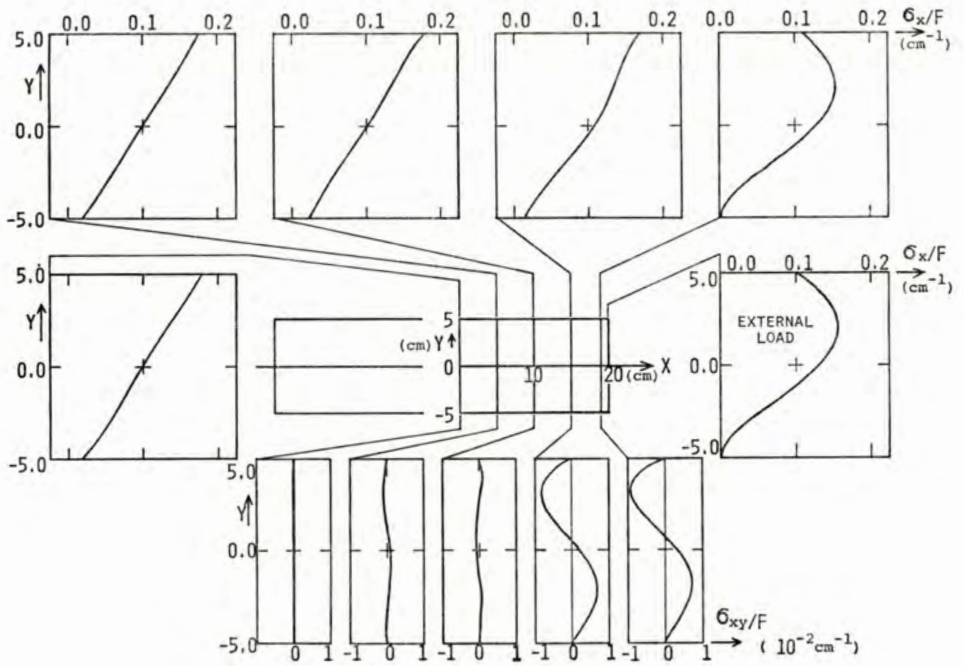

Fig. 3 The same as Fig. 2.

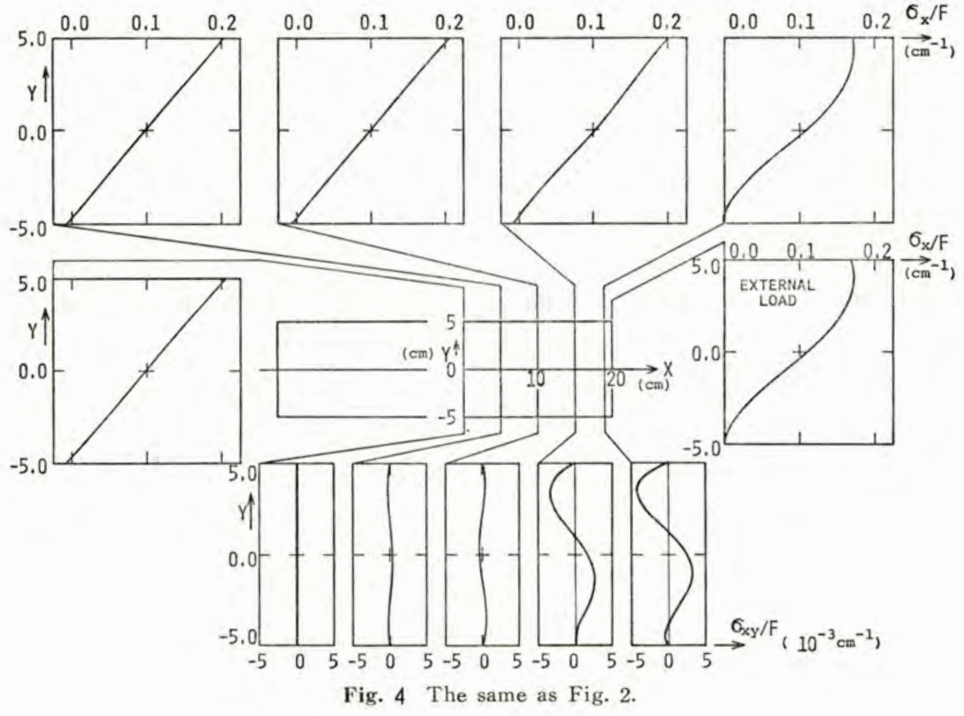

Fig. 4 The same as Fig. 2.

Substituting these solutions into right hand side of Eq. (7), we can immediately obtain the stresses due to each external load from Eq. (1). In Figs. 2, 3 and 4, the stress distributions which arise on the several vertical cross sections are shown schematically in the case of $a=4b=20$ cm. From these figures, we can say that the stress distribution almost reaches Saint-Venant limit at the position of which distance from the end JK equals to the width of elastic material.

To deal with the other distributions of external forces which act on the edges $x=\pm a$, we have only to change the form of the first line of the right side in expression (8). Only the column vectors, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}_B$, have to be changed. $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}_B$ do not change. This is true even for oblique external forces if they amount to being in equilibrium in all.

Sec. 3. Application to Prosthetic Dentistry [6)-13)] and Remarks.

While one is eating, various external loads act on teeth. This external force propagates as stress inside crown restration, layer of cement agent, tooth structure, periodontium and also maxillary bone. The maximum external load may be realized at the time when he crush a tough food with the teeth. This is a compressive force almost normal to occlusal surface. The first blow to the retentivity of a crown restoration may be fine crack in the cement layer at the point of stress concentration. Such failure in the cement will be initiated at the points where induced stresses exceed the recorded stresses causing failure. If a fine crack were formed in cement layer, the stress concentrations would no longer exist there since the load-resisting qualities are lost. Then it is natural to consider that this stress concentration would move, and as a result, cause the crack to grow larger. Following on such process, the socondary blow to restations due to various kinds of external loads including tensile force causes fracture of cement layer, which ultimately causes a loss of retention. In the light of such a consideration, one can say that it is significant for the problem of crown retention that the analysis of induced stresses in the cement layer under vertical compressive loading.

The fact that the thickness of cement agent between tooth structure and the crown restoration is ordinarily smaller than 50 micrometers lead us the conclusion that the effect of the thickness of cement is not so serious, and can be neglected. Thus we could adopt the scheme of direct contact between the tooth structure and the crown restoration. In particular, as regards the tooth structure, it is never isotropic and besides is not homogenious. For simplicity, however, we assume that dentine consists of homogenious isotropic material of which solidity equals to that of restoration. Moreover we take two dimensional idealization and simplify the problem by taking the sape of tooth rectangular.

For the two cases of external pressure distributions, the stresses arising on the surface between tooth structure and the two examples of the prosthetic crown are illustrated in Figs. 5~7.

The present arguments however, are rather oversimplified. Our simplification that the tooth structure, the prosthetics and also the cement layer have the same elastic constants primarily discards the possibility of stress concentration. Then, on the present framework, further argument about the retentivity of the crown restration have no meaning.

The stress analysis taking into the real shape of tooth has been carried out by Takahashi and coworkers [11) 13)]. They studied energetically stresses arising in the cement layer between tooth structure and the crown restoration by taking mandibular second premolar as an example by making use of finite element method. The behavior of restoring tooth has been studied under the several kinds of external force. Generally speaking, however, these analysis, as a matter of course, have the limit of utility. The first molar, for example, has undespisable individual variations on the geometrical structures and the elastic properties. Even though we could neglect

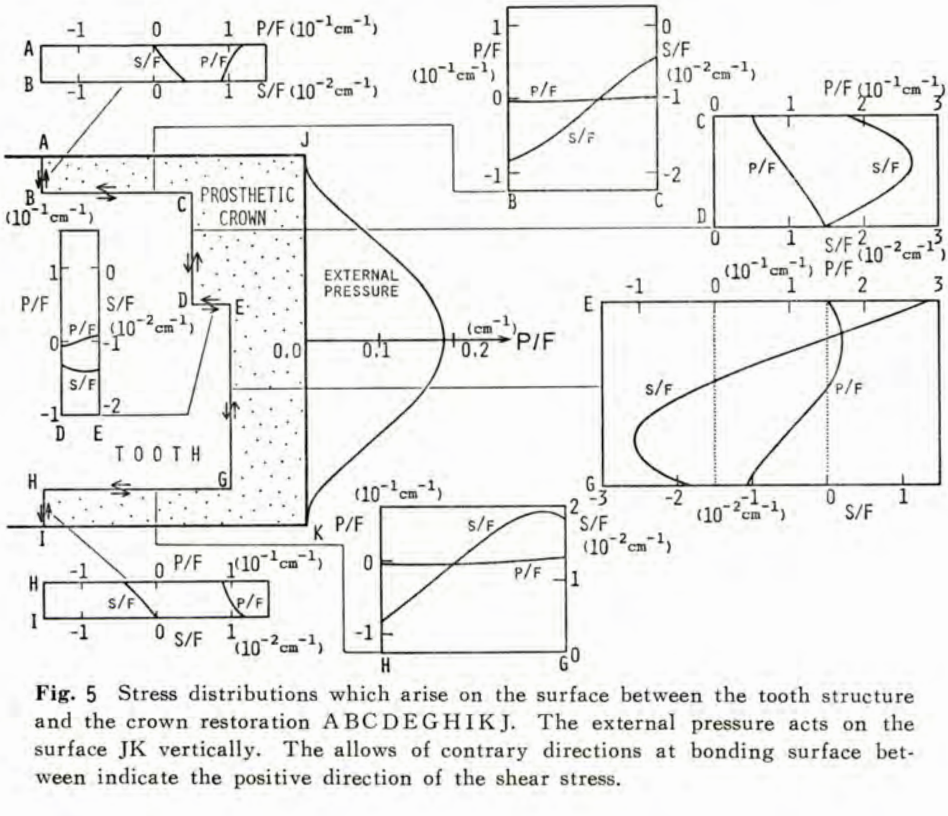

Fig. 5 Stress distributions which arise on the surface between the tooth structure and the crown restoration ABCDEGHIKJ. The external pressure acts on the surface JK vertically. The allows of contrary directions at bonding surface between indicate the positive direction of the shear stress.

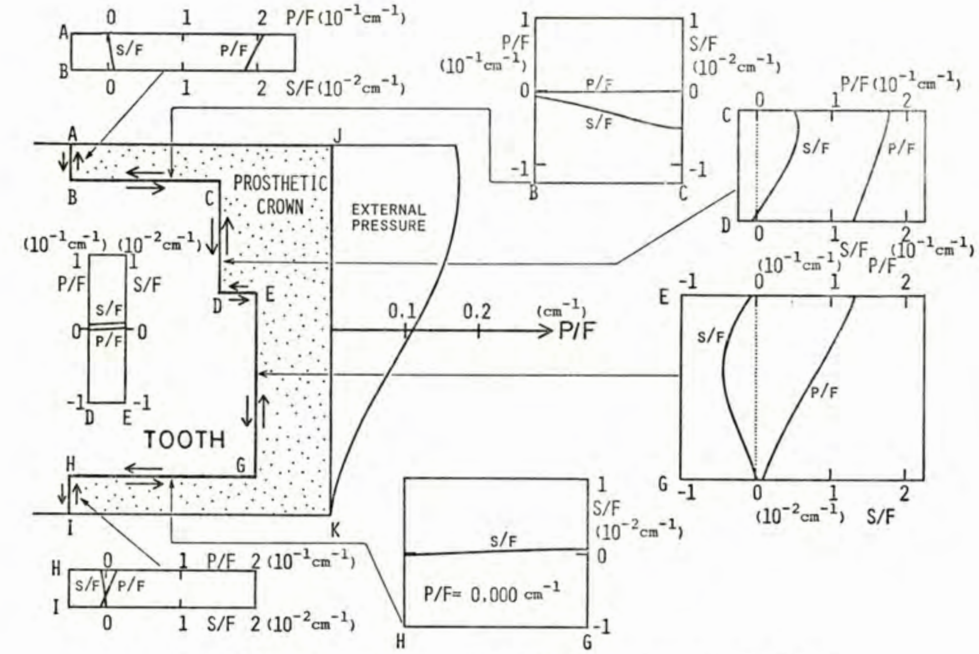

Fig. 6 Stress distributions. The details are the same as Fig. 5.

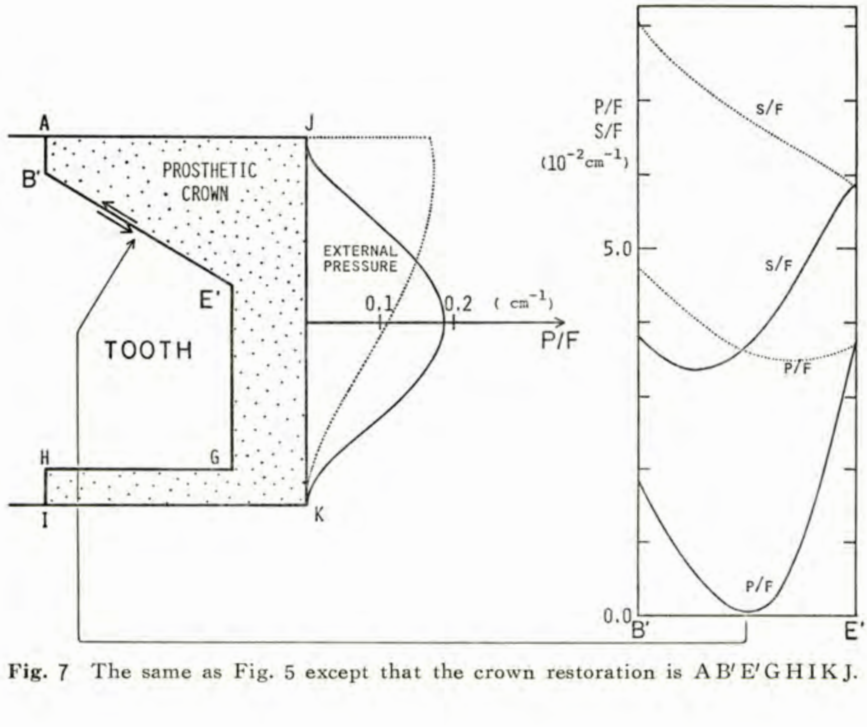

Fig. 7 The same as Fig. 5 except that the crown restoration is AB′E′GHIKJ.

such variations, tooth structure is inhomogenious and anisotropic. Then the arguments about the actual restoring tooth are restricted in practice. But it is emphasized that these sorts of studings on the retentivity of a crown restoration are greatly significant. The final aim appears to yield a rough but systematic grasp on the limit of retention of a restoration. These may have great utility on the clinical dental treatment.

In forthcoming paper, the auther intends to report the relationship between the magnitude of the stress concentration and the ratio of solidity of crown restoration to that of tooth structure.

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